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NONLINEAR EQUATIONS INVOLVING NONPOSITIVE DEFINITE LINEAR OPERATORS VIA VARIATIONAL METHODS

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ABSTRACT. In this paper we establish, via variational methods, some existence results for nonlinear equations of the type $u = K\mathbf{f}(u)$, where $K : L^{q_0}(\Omega) \to L^{p_0}(\Omega)$ is linear and $\mathbf{f} : L^p(\Omega) \to L^q(\Omega)$ is a superposition operator with $p_0 > p > 2$, $p^{-1} + q^{-1} = 1$ and $p_0^{-1} + q_0^{-1} = 1$. Then we apply these results to study a Hammerstein equation and a nonresonant nonlinear Fredholm integral equation. Our approach allows us to deal with nonpositive definite kernels. This is a novelty for the application of variational methods when coercivity fails to hold.

1. Preliminaries and basic definitions. Throughout this paper p_0, q_0 are two real numbers with $p_0 > 2$ and $1/p_0 + 1/q_0 = 1$, $\Omega \subseteq \mathbf{R}^N$ is a bounded Lebesgue measurable set, $K : L^{q_0}(\Omega) \to L^{p_0}(\Omega)$ is a completely continuous linear operator and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function. Consider p and q, with $2 < p < p_0$ and $1/p + 1/q = 1$; we suppose that $\mathbf{f}(u) \in L^q(\Omega)$, for every $u \in L^p(\Omega)$, where $\mathbf{f}(u) = f(\cdot, u(\cdot))$ denotes the superposition operator associated to f. Moreover, $\|\cdot\|_m$ will denote the usual norm in $L^m(\Omega)$ for $m \geq 1$. We are interested in finding solutions in $L^p(\Omega)$ to the following equation

$$u = K\mathbf{f}(u). \tag{1.1}$$

Equation (1.1) has been studied by several authors. One of the reasons which leads us to consider equation (1.1) concerns the fact that various boundary value problems for differential equations can be reduced to an integral equation like (1.1). For example, we refer to the case in which the operator K is defined by

$$K(u)(\cdot) = \int_\Omega k(\cdot, y)u(y)\, dy \tag{1.2}$$

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for $u \in L^q(\Omega)$. The function k is Green's function of the differential operator. In this case equation (1.1) is a Hammerstein integral equation, and k is the kernel of the operator K. A natural way to solve equations (1.1) is by fixed point methods, the solutions of (1.1) being the fixed points of the operator $K\mathbf{f}$. In literature, many authors have used such an approach, see, for instance, [**3, 6, 7**] and the references therein. The compactness of the operator $K\mathbf{f}$ plays a key role in these kinds of results.

Another way to solve (1.1) is by variational methods. The solution of (1.1), under suitable conditions, turns out to be the critical points of a differentiable functional. We point out that, actually, variational methods are less used to solve equation (1.1) than fixed point methods. However, among the most recent papers which use a variational approach, we can cite [**1, 2, 8**], we refer the reader also to the references therein. An incentive in using variational methods was given by a variational principle established by Ricceri in [**9**]. On this principle are based, in particular, the papers [**1, 2**]. In all of the cited papers, K is assumed to be positive definite. This implies that all the eigenvalues of K are positive.

Recall that, in general, if V is a real Hilbert space and $K : V \to V$ is a linear operator, an eigenvalue of K is a real number σ satisfying

$$K\phi = \sigma\phi$$

for some $\phi \in V \setminus \{0\}$. In this case, ϕ is said to be an eigenvector associated to σ.

K is said to be symmetric if it satisfies

$$(Ku, v) = (u, Kv) \tag{1.3}$$

for every $u, v \in V$.

K is positive definite, if

$$(Ku, u) > 0 \tag{1.4}$$

for all $u \in V \setminus \{0\}$. As it is well known, these conditions imply that K is self-adjoint. Moreover, when $V = L^2(\Omega)$, K has the following splitting representation

$$K = HH^* \tag{1.5}$$

where $H : L^2(\Omega) \to L^2(\Omega)$ is continuous and linear and H^* is the adjoint of H. Hence, we have

$$\int_\Omega H(u)v\,dx = \int_\Omega uH^*(v)\,dx, \tag{1.6}$$

for all $u, v \in L^2(\Omega)$.

When K acts from $L^{q_0}(\Omega)$ into $L^{p_0}(\Omega)$, by standard results, it can be split as in (1.5). More precisely, for each $q_0 < q \le 2$, set $p = q/(q-1)$; there exists H, which acts from $L^2(\Omega)$ into $L^p(\Omega)$ and H^*, acting from $L^q(\Omega)$ into $L^2(\Omega)$, such that (1.5) and (1.6), for all $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, both hold. Furthermore, if K is compact, then operators H, H^* are also compact, see Theorem 4.4 [**4**, p. 59].

From now on, we denote by F the primitive of f:

$$F(x,\xi) = \int_0^\xi f(x,t)\,dt \quad \text{for} \quad (x,t) \in \Omega \times \mathbf{R}.$$

From what is said above, since K is compact, it is easily seen that the following functional

$$\Psi(v) = \frac{1}{2}\,\|v\|_2^2 - \int_\Omega F(x, H(v)(x))\,dx$$

is sequentially lower weakly semi-continuous and continuously Gâteaux differentiable in $L^2(\Omega)$. Moreover, if $v \in L^2(\Omega)$ is a critical point of Ψ, then $u = H(v) \in L^p(\Omega)$ is a solution of equation (1.1), see [**4**, pp. 304–305].

We observe that, if the linear operator K has only a finite number of negative eigenvalues in $L^2(\Omega)$, then a variational approach is also possible to study equation (1.1). Actually, this situation has been considered in [**4**, Chapter VI]. We briefly summarize the arguments exploited there.

Assume K is self-adjoint compact and has a finite set E of negative eigenvalues. Let V_1 be the linear hull in $L^2(\Omega)$ of the eigenfunctions of K corresponding to the eigenvalues in E, and let V_2 be the orthogonal complement of V_1 in $L^2(\Omega)$. Let K_+ the positive definite linear operator associated to K, see [**4**, Chapter I]. K_+ turns out to be completely continuous and self-adjoint. So, due to the properties recalled above, K_+ splits as follows

$$K_+ = H_+ H_+^* \tag{1.7}$$

with $H_+ : L^2(\Omega) \to L^p(\Omega)$ and $H_+^* : L^q(\Omega) \to L^2(\Omega)$ linear and completely continuous operators satisfying (1.6), for all $u \in L^p(\Omega)$, $v \in L^q(\Omega)$. We put $u_i = P_{V_i}(u)$, where P_{V_i} denotes the projection of $L^2(\Omega)$ on V_i, $i = 1, 2$. Then, define

$$J(u) = u_1 - u_2$$

for all $u \in L^2(\Omega)$.

The following functional

$$\Phi(v) = -\frac{1}{2}\int_\Omega J(u)(x)u(x)\,dx - \int_\Omega F(x, H_+(v)(x))\,dx \tag{1.8}$$

turns out to be sequentially weakly lower semi-continuous and continuously Gâteaux differentiable in $L^2(\Omega)$. Moreover, it is easy to see that, if $v \in L^2(\Omega)$ is a critical point of Φ, then $u = H_+(v)$ is a solution of (1.1). Since every local minimum of Φ is in particular a critical point, a way to find a solution to (1,1) is to assume a suitable growth condition on F in order that Φ be coercive. Indeed, in this case Φ admits a global minimum. As proved in [**4**, Theorem 1.5, p. 312], a growth condition on the nonlinearity for the coercivity of Φ is the following one

$$F(x,\xi) \le -a\xi^2 + b(x)|\xi|^m + c(x) \tag{1.9}$$

where $0 < m < 2$, $b \in L^{2/(2-m)}(\Omega)$, $c \in L^1(\Omega)$ and $a|\lambda_{-1}| > 1$ with $\lambda_{-1} = \max E$ if $E \ne \varnothing$ (if $E = \varnothing$ then one can choose $a = 0$). To the best of our knowledge, this theorem seems to be the only result in literature where variational methods are used without assuming K positive definite.

The aim of this paper is to give a contribution in this direction. We will establish two existence results for operator equation (1.1), where K has a finite number of negative eigenvalues, assuming on the nonlinearity a weaker growth condition than (1.9). This leads us to treat the cases in which the coercivity of Φ does not hold. Finally, we apply our result to solve a Hammerstein integral equation and a nonresonant nonlinear Fredholm integral equation.

2. Main results. In this section and in the next one, in order to simplify the notation, we make use of these conventions

$$\frac{p}{p-m} = \infty, \quad \text{when} \quad m = p$$

and $0^0 = 1$.

Now we can state our first result:

Theorem 2.1. *Let K be a self-adjoint, completely continuous linear operator having a finite set E of negative eigenvalues. Set $\lambda_{-1} = \max E$ if $E \neq \varnothing$. Suppose that there exists $a \in \mathbf{R}$, with $a > 1/|\lambda_{-1}|$ when $E \neq \varnothing$ and $a = 0$ otherwise, such that*

$$\alpha) \quad \inf_{r>0} \sup_{\|u\|_2=1} \int_\Omega \left(\frac{F(x, rH_+(u)(x))}{r^2} + a(H_+(u)(x))^2 \right) dx < \frac{1}{2}\,\rho,$$

where $\rho = a|\lambda_{-1}| - 1/a|\lambda_{-1}| + 1$ if $E \neq \varnothing$ and $\rho = 1$ otherwise. Then, equation (1.1) *has at least a solution in $L^p(\Omega)$.*

Before the proof, we point out that Theorem 2.1 improves [**4**, Theorem 1.5, p. 312]. In fact, the growth condition (1.9), with $0 < m < 2$, implies α). The case in which $2 \leq m \leq p$ can also be considered as is shown by the following corollary.

Corollary 2.1. *Assume that the function F satisfies* (1.9) *with $2 \leq m \leq p$, $b \in L^{p/(p-m)}(\Omega)$ and $c \in L^1(\Omega)$. Put*

$$b_0 = \sup_{\|u\|_2=1} \int_\Omega b(x)|H_+(u)(x)|^m \, dx$$

and $c_0 = \int_\Omega c(x)\,dx$. If

$$m(b_0)^{2/m} \left(\frac{m-2}{2c_0} \right)^{(2/m)-1} < \rho,$$

where a and ρ are as in Theorem 2.1*, then equation* (1.1) *has at least a solution in $L^p(\Omega)$.*

Remark 2.1. For the applications of the previous corollary, it is useful to exploit the following upper estimate of the number b_0 which can be proved by exploiting the embedding of $L^p(\Omega)$ in $L^2(\Omega)$ and the splitting representation of K_+:

$$b_0 \leq |\Omega|^{m(p-2/2p)} \|b\|_{p/(p-m)} |\lambda_1|^{m/2},$$

where λ_1 is the greatest positive eigenvalue of K and $|\Omega|$ is the Lebesgue measure of Ω.

Proof of Theorem 2.1. As was shown in the introduction, when $E \neq \varnothing$, the thesis is achieved if we prove that the functional Φ, defined by (1.8), admits a local minimum. At first, we rewrite Φ as follows

$$\Phi(u) = \frac{1}{2}\,\|u\|_2^2 - \|u_1\|_2^2 - \int_\Omega \left(\int_0^{H_+(u)(x)} f(x,t)\,dt \right) dx$$

for $u \in L^2(\Omega)$ (recall that $u_1 = P_{V_1}(u)$). Since V_1 is of finite dimension and H_+ is a completely continuous linear operator, it is easy to check that the functional

$$F(u) = \|u_1\|^2 + \int_\Omega \left(\int_0^{H_+(u)(x)} f(x,t)\,dt \right) dx$$

is sequentially weakly continuous in $L^2(\Omega)$ and so weakly continuous on every bounded subset of $L^2(\Omega)$ by Eberlein-Smulian theorem. This implies that

$$\sup_{\|u\|_2 \leq r} F(u) = \sup_{\|u\|_2 = r} F(u) \tag{2.10}$$

for all $r > 0$. Due to [**9**, Theorem 2.5], it is enough to show that

$$\inf_{r>0} \inf_{\|u\|_2<r} \frac{\sup_{\|v\|_2 \leq r} F(v) - F(u)}{r^2 - \|u\|_2^2} < \frac{1}{2}, \tag{2.11}$$

in order to assure the existence of a local minimum for Φ. Define

$$\mu(t) = \sup_{\|v\|_2 \le t} F(v)$$

for all $t \ge 0$; it is easy to see that

$$\inf_{r>0} \left(\mu(r) - \frac{1}{2} r^2 \right) < 0, \tag{2.12}$$

implies inequality (2.11).

Let $\alpha \in]0, 1[$, and put $F_\alpha = \{u \in L^2(\Omega) : \|u_1\|^2 \ge (1-\alpha)\|u_2\|^2\}$. Consider any $u \in F_\alpha$ with $\|u\|_2 = r$. By [**4**, Lemma 1.2, p. 308; also pp. 309–310], we have

$$\|H_+(u)\|^2 \ge |\lambda_{-1}| \frac{1-\alpha}{2-\alpha} \|u\|^2.$$

Consequently, one has

$$\|u_1\|^2 - a\|H_+(u)\|_2^2 \le r^2 \left(1 - |\lambda_{-1}| a \frac{1-\alpha}{2-\alpha} \right). \tag{2.13}$$

Now, suppose $u \in L^2(\Omega) \setminus F_\alpha$ and $\|u\|_2 = r$. Then, one has

$$r^2 = \|u\|_2^2 = \|u_1\|_2^2 + \|u_2\|_2^2 \ge \|u_1\|_2^2 \left(1 + \frac{1}{1-\alpha} \right)$$

from which it follows that

$$\|u_1\|_2^2 - a\|H_+(u)\|_2^2 \le \frac{1-\alpha}{2-\alpha} r^2. \tag{2.14}$$

From (2.13) and (2.14), we get

(2.15)

$$\begin{aligned} \|u_1\|_2^2 - a\|H_+(u)\|_2^2 &\le r^2 \inf_{\alpha \in]0,1[} \max \left\{ \frac{1-\alpha}{2-\alpha}, 1 - |\lambda_{-1}| a \frac{1-\alpha}{2-\alpha} \right\} \\ &= \frac{r^2}{1 + a|\lambda_{-1}|} \end{aligned}$$

for all $u \in L^2(\Omega)$, with $\|u\|_2 = r$. At this point, taking into account (2.10), (2.15) and condition 2) part i), we obtain

$$\begin{aligned}\mu(r) \le &\sup_{\|u\|_2=r} (\|u_1\|_2^2 - a\|H_+(u)\|_2^2) \\ &+ \sup_{\|u\|_2=r}\left(\int_\Omega \left(F(x, H_+(u)(x)) + a(H_+(u)(x))^2\right) dx\right) \\ \le &\frac{r^2}{1+a|\lambda_{-1}|} + \sup_{\|u\|_2=1}\int_\Omega \left(F(x, rH_+(u)(x)) + ar^2(H_+(u)(x))^2\right);\end{aligned}$$

hence,

$$\begin{aligned}\mu(r) - \frac{1}{2}r^2 \le &-\frac{r^2}{2}\cdot\frac{a|\lambda_{-1}|-1}{a|\lambda_{-1}|+1} \\ &+ \sup_{\|u\|_2=1}\int_\Omega \left(F(x, rH_+(u)(x)) + ar^2(H_+(u)(x))^2\right) dx.\end{aligned}$$

By the previous inequality and condition α) we obtain (2.12). In the case $E = \varnothing$, we can repeat the previous proof with $V_1 = \{0\}$. □

Theorem 2.1 does not exclude that the solution to (1.1) may be trivial when $f(x, 0) = 0$, for almost all $x \in \Omega$. On the other hand, we obtained such a solution as a local minimum of the functional Φ. Exploiting this further information it is possible to guarantee that the solution is not identically null. This fact is shown by the following result:

Theorem 2.2. *Suppose that all the hypotheses of Theorem* 2.1 *are satisfied and $f(x, 0) = 0$, for almost all $x \in \Omega$. Moreover, assume that one of the following conditions hold*:

i) *$E \ne \varnothing$, and there exist $M, N > 0$ with $M < 1/(2|\lambda_{-1}|)$ such that $F(x, \xi) \ge -M\xi^2 - N\xi^p$ for almost all $x \in \Omega$ and $\xi \in \mathbf{R}$.*

ii) *$E = \varnothing$, and there exist $M, N > 0$ with $M < 1/(2\lambda_1)$, where λ_1 is the first eigenvalue of K such that $F(x, \xi) \ge M\xi^2 - N\xi^p$ for almost all $x \in \Omega$ and $\xi \in \mathbf{R}$.*

Then, equation (1.1) *has at least a nontrivial solution in $L^p(\Omega)$.*

Proof. We suppose $E \ne \varnothing$. When $E = \varnothing$ the proof is the same with some straightforward modifications. We have already observed that, by using [**9**, Theorem 2.5], we obtain the existence of a local minimum v for the functional Φ. This result assures that v is in fact a global minimum for the restriction of Φ to a ball centered in 0 with radius $r > 0$. We claim that $v \neq 0$. To see this, it suffices to show that $\inf_{\|w\|_2<r} \Phi(w) < 0$. Let φ_{-1} be an eigenfunction corresponding to the eigenvalue λ_{-1}, and let $\varepsilon > 0$. Then we have

$$\begin{aligned}
\Phi(\varepsilon\varphi_{-1}) &= -\frac{\varepsilon^2}{2}\|\varphi_{-1}\|_2^2 - \int_\Omega F(x, H_+(\varepsilon\varphi_{-1})(x))\,dx \\
&\le -\frac{\varepsilon^2}{2}\|\varphi_{-1}\|_2^2 + M\varepsilon^2\int_\Omega |H_+(\varphi_{-1})|^2\,dx \\
&\quad + N\varepsilon^p\int_\Omega |H_+(\varphi_{-1})|^p\,dx \\
&= \frac{\varepsilon^2}{2}\Big(\|\varphi_{-1}\|^2(-1+2M\lambda_{-1}) + 2N\varepsilon^{p-2}\int_\Omega |H_+(\varphi_{-1})|^p\,dx\Big).
\end{aligned}$$

Consequently, if ε is small enough, we have $\|\varepsilon\varphi_{-1}\|_2 < r$ and $\Phi(\varepsilon\varphi_{-1}) < 0$. Hence, $\inf_{\|w\|_2<r} \Phi(w) < 0$ and so $v \neq 0$. Now, put $u = H_+(v)$, u is a solution of equation (1.1). Then, to complete the proof we have to show that $u \neq 0$. Arguing by contradiction, suppose $u = 0$. Since v is a critical point of Φ, we have $\Phi'(v)(u) = 0$ for all $u \in L^2(\Omega)$, that is,

$$\begin{aligned}
\Phi'(v)(u) &= \int_\Omega v(x)u(x)\,dx - 2\int_\Omega v_1(x)u_1(x)\,dx \\
&\quad - \int_\Omega f(x, H_+(v)(x))H_+(u)(x)\,dx \\
&= \int_\Omega v(x)u(x)\,dx - 2\int_\Omega v_1(x)u_1(x)\,dx = 0
\end{aligned}$$

for all $u \in L^2(\Omega)$. In particular, choosing $u = v$ in the previous equality, we obtain

$$\int_\Omega |v(x)|^2\,dx = 2\int_\Omega |v_1(x)|^2\,dx.$$

Hence,

$$\Phi(v) = \frac{1}{2}\int_\Omega |v(x)|^2\,dx - \int_\Omega |v_1(x)|^2\,dx = 0,$$

against the fact that $\Phi(v) = \inf_{\|w\|_2<r} \Phi(w) < 0$. $\square$

3. Application to nonlinear Hammerstein and Fredholm integral equation. In this section we apply Theorems 2.1 and 2.2 to solve the nonlinear Hammerstein and Fredholm integral equations. As was stated in the introduction, a nonlinear Hammerstein integral equation is obtained from (1.1) when K is given by (1.2). So, to apply the main result we have to impose conditions on the kernel k in order that K is compact and self-adjoint. As it is well known, such conditions are the following ones, see [**4**]:

$$k(x,y)=k(y,x) \quad \text{for almost all} \quad x,y\in\Omega \tag{3.16}$$

$$\int_{\Omega\times\Omega} |k(x,y)|^{p_0}\,dx\,dy<+\infty \tag{3.17}$$

for some $p_0>p$. So, we have the following theorem:

Theorem 3.1. *Let $k:\Omega\times\Omega\to\mathbf{R}$ measurable and satisfying* (3.16) *and* (3.17). *Let K, defined by* (1.2), *have a finite set E of negative eigenvalues and satisfy condition α) of Theorem* 2.1. *Then, the following nonlinear Hammerstein integral equation*

$$u(x)=\int_\Omega k(x,y)f(y,u(y))\,dy$$

has at least a solution in $L^p(\Omega)$ which is nontrivial if, in addition, F satisfies the conditions of Theorem 2.2.

Now consider the following nonlinear Fredholm integral equation

$$u(x)=\gamma\int_\Omega k(x,y)u(y)\,dy+\int_\Omega k(x,y)f(y,u(y))\,dy, \tag{P_γ}$$

where $\gamma\geq 0$ is a real number and $k:\Omega\times\Omega\to\mathbf{R}$ is measurable and satisfying (3.16) and (3.17). Moreover, we suppose that the operator K defined by (1.2) is positive definite, that is, in this case,

$$\int_\Omega\left(\int_\Omega k(x,y)v(x)v(y)\,dx\right)dy>0. \tag{3.18}$$

for every $v\in L^2(\Omega)\setminus\{0\}$.

For every $i \in \mathbf{N}$, λ_i denotes the ith eigenvalue of the linear operator $K : L^2(\Omega) \to L^2(\Omega)$

$$Kv(x) = \int_\Omega k(x,y)v(y)\,dy.$$

Then, we set $\mu_i = 1/\lambda_i$, for each $i \in \mathbf{N}$ and $\mu_0 = -\infty$.

In these settings our result is as follows:

Theorem 3.2. *Suppose that $\mu_{i-1} < \gamma < \mu_i$ for some $i \in \mathbf{N}$. Set*

$$\bar{\gamma} = \min\{\gamma - \mu_{i-1}, \mu_i - \gamma\}.$$

We assume F satisfying (1.9) with $m \in]0,p]$, $a \in [0,+\infty[\cap]\gamma - \mu_1, +\infty[$, $b \in L^{p/(p-m)}(\Omega)$ and $c \in L^1(\Omega)$. Moreover, when $m \geq 2$, we also assume

$$\frac{m}{\bar{\gamma}^{m/2}}\;|\Omega|^{m(p-2/2p)}\;\|b\|_{p/(p-m)}\left(\frac{m-2}{2\|c\|_1}\right)^{(2/m)-1} < \left|\frac{a-(\mu_1-\gamma)}{a+(\mu_1-\gamma)}\right|.$$

Then (P_γ) has at least a solution in $L^p(\Omega)$ which is nontrivial if, in addition, F satisfies the conditions of Theorem 2.2.

Proof. Note that, since $\gamma \neq \mu_i$ for every $i \in \mathbf{N}$, the operator $(I-\gamma K)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is linear and bounded. Then $(I-\gamma K)^{-1}K$ turns out to be a completely continuous linear and self-adjoint operator because it inherits these properties from K [**5**, Theorem 16.1.1].

Consequently, (P_γ) is equivalent to the following operator equation

$$u = (I-\gamma K)^{-1}K\,\mathbf{f}\,u. \tag{3.19}$$

It is easy to check that, for every $2 \leq p \leq p_0$,

$$((I-\gamma K)^{-1}K)(L^q(\Omega)) \subseteq L^p(\Omega), \quad \text{with} \quad q = p/(p-1).$$

The eigenvalues σ_i of $((I-\gamma K)^{-1}K)$ are related to those of K by

$$\sigma_i = \frac{1}{\mu_i - \gamma} \quad \text{for} \quad i \in \mathbf{N}.$$

Hence, $(I - \gamma K)^{-1}K$ has a finite number of negative eigenvalues. The least in absolute magnitude of such negative eigenvalues is σ_1. The greatest positive eigenvalue is $1/\bar{\gamma}$. At this point, the conclusion follows by Theorems 2.1 and 2.2, taking into account Corollary 2.1 and Remark 2.1. $\square$

Remark 3.1. Theorem 3.1 is directly comparable to [**5**, Theorem 16.2.4]. Indeed, from this latter one, an existence result for a solution in $L^2(\Omega)$ of equation (P_γ) can be obtained. However, the fixed point method, used there, does not allow to find natural conditions on F, like i) and ii) in Theorem 2.2, in order to exclude that such a solution is trivial when $f(x, 0) = 0$ for almost all $x \in \Omega$.

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