JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 18, Number 4, Winter 2006

LAGRANGE INTERPOLATION AND BOUNDARY-VALUE PROBLEMS

AHMED I. ZAYED AND CHANG EON SHIN

ABSTRACT. It is known that some boundary-value problems give rise to Lagrange-type interpolation series that can be used to reconstruct entire functions from their samples at the eigenvalues of any such problem. Such boundary-value problems, of which regular Sturm Liouville boundary-value problems are prototype, are said to have the Lagrange-type interpolation property.

It was conjectured that any regular, self-adjoint, eigenvalue problem associated with nth order linear differential operator with simple eigenvalues has the Lagrange-type interpolation property. In 1994, P.L. Butzer and G. Schöttler proved that conjecture, but a year later Annaby pointed out an error in their paper and gave an alternative proof which is not only imprecise, but also deals with a very special case in which the problem is assumed to be one dimensional.

The aim of this article is to fill the gaps in these papers by providing an alternative proof.

1. Introduction.

1.1. *Sampling theorems and Lagrange interpolation.* The Whittaker-Shannon-Kotel'nikov (WSK) sampling theorem plays an important role not only in harmonic analysis and approximation theory, but also in communication engineering. The theorem can be stated as follows:

Theorem 1.1 (Whittaker-Shannon-Kotel'nikov). *If a function f is band-limited to $[-\sigma, \sigma]$, i.e., it is representable as*

$$f(t) = \int_{-\sigma}^{\sigma} e^{-ixt} g(x)\, dx \quad t \in \mathbf{R}, \tag{1.1}$$

Key words and phrases. Self-adjoint boundary-value problems, Lagrange interpolation, Shannon sampling theorem, Kramer sampling theorem.
Received by the editors on August 1, 2005, and in revised form on April 18, 2006.

for some function $g \in L^2(-\sigma, \sigma)$, then f can be reconstructed from its samples, $f(k\pi/\sigma)$. The construction formula is

$$f(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k\pi}{\sigma}\right) \frac{\sin(\sigma t - k\pi)}{(\sigma t - k\pi)} \quad t \in \mathbf{R}, \tag{1.2}$$

the series being absolutely and uniformly convergent on $\mathbf{R}$. *See, e.g.,* [**22**, p. 16].

The points $\{t_k = k\pi/\sigma\}$ are called the sampling points, and the functions

$$S_k(t) = \frac{\sin \sigma(t - t_k)}{\sigma(t - t_k)} = \operatorname{sinc}(\sigma(t - t_k)/\pi),$$

where

$$\operatorname{sinc}(z) = \begin{cases} \sin \pi z/(\pi z) & z \neq 0 \\ 1 & z = 0, \end{cases}$$

are called the sampling functions. The series in equation (1.2) can be put in the form

$$f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{G(t)}{(t - t_k)G'(t_k)}, \tag{1.3}$$

where $t_k = k\pi/\sigma$ and $G(t) = \sin \sigma t = \sigma t \prod_{k=1}^{\infty} \left(1 - t^2/t_k^2\right)$.

Recall from Hadamard's factorization theorem [**18**, p. 24] that if $f(z)$ is an entire function of order ρ, and $\{z_n\}_{n=1}^{\infty}$ is its nonzero zeros, i.e., $z_n \neq 0$ for all n, and p is the smallest integer for which the series $\sum_{n=1}^{\infty} 1/|z_n|^{p+1}$ converges, then

$$f(z) = z^m e^{P(z)} \prod_{n=1}^{\infty} G\left(\frac{z}{z_n}; p\right),$$

where $G(u;p) = (1 - u) \exp\left(u + u^2/2 + \cdots + u^p/p\right)$, $G(u;0) = (1 - u)$, m is the multiplicity of the zero at the origin and $P(z)$ is a polynomial of degree not exceeding ρ.

We shall call the product $\prod_{n=1}^{\infty} G(z/z_n; p)$ the *canonical product* of the $\{z_n\}_{n=1}^{\infty}$. As a special case, we have $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - z^2/n^2\right)$.

The fact that formula (1.3) resembles Lagrange interpolation formula prompts us to call any series of the form

$$\sum_k f(t_k) \frac{G(t)}{G'(t_k)(t-t_k)} \tag{1.4}$$

a *Lagrange-type interpolation series*, where $G(t)$ is an entire function whose zeros are located exactly at the points $\{t_k\}$. The points $\{t_k\}$ will be called the sampling points and the functions

$$G_k(t) = \frac{G(t)}{G'(t_k)(t-t_k)}, \tag{1.5}$$

will be called the sampling functions. The range of k is usually either $k = 0, 1, 2, \dots$, or $k = 1, 2, \dots$, or $k = 0, \pm 1, \pm 2, \dots$. The value t_0 is often reserved for $t_0 = 0$.

1.2 *Kramer's sampling theorem and boundary-value problems.* One of the interesting generalizations of the WSK sampling theorem is Kramer's sampling theorem, which was introduced by Kramer in 1959 [**17**]. It provides an important link between sampling theorems and boundary-value problems.

Theorem 1.2 (Kramer). *Let there exist a function $K(x,t)$ continuous in t such that $K(x,t) \in L^2(I)$ as a function in x for every real number t. Assume that there exists a sequence of real numbers $\{t_n\}_{n\in\mathbf{Z}}$ such that $\{K(x,t_n)\}_{n\in\mathbf{Z}}$ is a complete orthogonal family in $L^2(I)$ for some finite interval $I = [a,b]$. Then, for any function of the form*

$$f(t) = \int_a^b F(x)\overline{K}(x,t)\,dx = \langle F, K\rangle, \tag{1.6}$$

with $F \in L^2(I)$, we have

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) S_n^*(t), \tag{1.7}$$

where

$$S_n^*(t) = \frac{\int_a^b \overline{K}(x,t)K(x,t_n)\,dx}{\int_a^b |K(x,t_n)|^2\,dx}. \tag{1.8}$$

If there exists a nonnegative function $g(x) \in L^1(I)$ such that $|K(x,t)|^2 \leq g(x)$, for almost all $x \in I$ and all t, then the series converges uniformly on compact subsets of $\mathbf{R}$.

The WSK sampling theorem is a special case of Kramer's sampling theorem because, if we take $I = [-\sigma, \sigma]$, $K(x,t) = e^{ixt}$, and $t_n = (n\pi)/\sigma$, it is easy to see that $\left\{e^{it_n x}\right\}_{n=-\infty}^{\infty}$ is a complete orthogonal set in $L^2(I)$, and in addition, $S_n^*(t) = \operatorname{sinc}\left(\sigma(t-t_n)/\pi\right)$. Hence equations (1.6) and (1.7) reduce to (1.1) and (1.2).

Kramer [**17**] noted that the kernel function $K(x,t)$ and the sampling points $\{t_n\}_{n\in\mathbf{Z}}$ may be found from certain boundary-value problems. More precisely, let the differential operator L be defined by

$$L = p_0(x)\,\frac{d^n}{dx^n} + \cdots + p_{n-1}(x)\,\frac{d}{dx} + p_n(x), \quad x \in I,$$

where $p_k(x)$ is a complex-valued function with $n-k$ continuous derivatives, $k = 0, 1, \dots, n$, for any $x \in I = [a,b]$, and $p_0(x) \neq 0$ for any $x \in [a,b]$, with $-\infty \leq a < b \leq \infty$. The adjoint operator L^* is defined as

$$L^* g = (-1)^n \,\frac{d^n}{dx^n}\left(\overline{p}_0 g\right) + (-1)^{n-1}\,\frac{d^{n-1}}{dx^{n-1}}\left(\overline{p}_1 g\right) + \cdots + \overline{p}_n g.$$

The operator L is said to be formally *self-adjoint* if $L = L^*$. If the coefficient functions p_k, $k = 0, 1, \dots, n$, are real-valued, then it is easy to see that if L is self-adjoint, then n is even.

Let $U_j(y) = 0$, $j = 1, \dots, n$, be linearly independent homogeneous boundary conditions of the form

$$U_j(y) = \sum_{k=1}^{n} \left(\alpha_{j,k} y^{(k-1)}(a) + \beta_{j,k} y^{(k-1)}(b)\right), \quad j = 1, 2, \dots, n.$$

To any such system of boundary conditions, there exists an associated system of boundary conditions, known as the *adjoint system*, and if the two systems are equivalent, we say that they are self-adjoint [**11**, p. 287]. More precise formulation of the problem and its properties will be given in the next section.

The boundary-value problem

$$Ly = -ty, \quad x \in I, \tag{1.9}$$

$$U_j(y) = 0, \quad j = 1, \dots, n, \tag{1.10}$$

is said to be *self-adjoint* if the differential operator and the boundary conditions are self-adjoint and is said to be *regular* if L is regular. It is known that the eigenfunctions of a regular, self-adjoint boundary-value problem are complete in $L^2(I)$, [**20**, p. 82]. If $U_1, \dots, U_n$ are linear forms, A is a nonsingular matrix and $[W_1, \dots, W_n]^T = A\,[U_1, \dots, U_n]^T$, then the boundary conditions $W_1(y) = 0, \dots, W_n(y) = 0$ are equivalent to $U_1(y) = 0, \dots, U_n(y) = 0$.

Kramer observed that if the regular, self-adjoint boundary-value problem (1.9) and (1.10) possesses a function $\phi(x,t)$ that generates the eigenfunctions of the problem $\{\phi_n(x)\}$ when the eigenvalue parameter t is replaced by the eigenvalues$\{t_n\}$, i.e., $\phi(x,t_n) = \phi_n(x)$, then one can take the sampling points to be $\{t_n\}$ and the kernel function $K(x,t)$ to be $\phi(x,t)$.

Two points should be noted here:

i) The function $\phi(x,t)$ is not unique because one can add to it any entire function that vanishes at the eigenvalues,

ii) The function $K(x,t)$ does not always arise from boundary-value problems; one such example was given by Kak [**16**], who derived the Walsh sampling theorem as a special case of Kramer's.

It is customary in the theory of boundary-value problems to denote the eigenvalue parameter by λ and the eigenvalues by λ_n; therefore, from now on we shall denote the sampling points by λ_n whenever the sampling expansion is associated with a boundary-value problem. Kramer's observation leads us to the following definition.

Definition 1. We say that a boundary-value problem has the Kramer property if it possesses a function $\phi(x,\lambda)$, entire in λ, that satisfies $L\phi(x,\lambda) = -\lambda\phi(x,\lambda)$ and generates the eigenfunctions of the problem $\{\phi_n(x)\}$ when the parameter λ is replaced by the eigenvalues $\{\lambda_n\}$, i.e., $\phi(x,\lambda_n) = \varphi_n(x)$.

Under what conditions the boundary-value problem (1.9)–(1.10) has the Kramer property is still an open question. Some partial but important answers have been obtained in recent years and on which we shall elaborate a little more later. It had been conjectured that regular, self-adjoint boundary-value problems with simple eigenvalues have the Kramer property. In [**9**], Butzer and Schöttler proved that conjecture, but almost a year later Annaby [**1**] was the first to point out an error in [**9**] and gave an alternative proof which is not only imprecise, but also deals with a very special case in which he assumes that the problem is one-dimensional. For the definition of one-dimensional problems, see Definition 5 below.

The aim of this article is to explain the flaws in the aforementioned papers, and then provide an alternative proof.

1.3 *Boundary-value problems and Lagrange-type interpolation.* Although the connection between sampling theorems and boundary-value problems has been the focus of many research papers in the last few years [**3, 4, 7–9, 12, 13, 19, 21, 25**] in this section we shall focus more on the connection between boundary-value problems and Lagrange-type interpolation.

The series in (1.7) and (1.8) does not resemble, and in general, is not a Lagrange-type interpolation series since it cannot always be put in the form (1.4). Nevertheless, if the sampling points and functions are obtained from the self-adjoint boundary-value problem (1.9)–(1.10), then (1.7) can be brought closer to the Lagrange-type interpolation series, provided that the problem has the Kramer property. For, if $\phi(x,\lambda)$ is a function that generates the eigenfunctions of the problem, then

$$L\phi(x,\lambda) = -\lambda\phi(x,\lambda), \quad \text{and} \quad L\phi_n(x) = -\lambda_n\phi_n(x),$$

and hence,

$$\int_a^b \left[\overline{\phi}(x,\lambda)L\phi_n(x) - \phi_n(x)\overline{L\phi(x,\lambda)}\right]\,dx \\ = (\overline{\lambda} - \lambda_n)\int_a^b \overline{\phi}(x,\lambda)\phi_n(x)\,dx.$$

But on the other hand, by Lagrange's identity [**11**, p. 80] for differential operators, we have for any two functions $u(x)$ and $v(x)$ in $C^n[a,b]$,

$$\overline{v}(x)Lu(x) - u(x)\overline{Lv}(x) = \frac{d}{dx}\,[u(x)\,,\,v(x)]\,,$$

where

$$[u,v]\,(x) = \sum_{m=1}^{n}\sum_{j+k=m-1}(-1)^j u^{(k)}(x)\left(p_{n-m}(x)\overline{v}(x)\right)^{(j)}, \quad j,k \geq 0.$$

Therefore,

$$G_k(\lambda) = (\overline{\lambda} - \lambda_k)\int_a^b \overline{\phi}(x,\lambda)\phi_k(x)\,dx = [\phi_k,\,\phi]\,(b) - [\phi_k,\,\phi]\,(a). \tag{1.11}$$

Since in almost all cases of interest the operator L is real, to simplify the notation we shall assume from now on that this is indeed the case. Because $\phi(x,\lambda)$ is an entire function in λ, $G_k(\lambda)$ is also entire. Clearly, $G_k(\lambda_k) = 0$, and moreover, $G_k(\lambda_m) = 0$ if $k \neq m$ by the orthogonality of the eigenfunctions $\{\phi_k(x)\}$. Hence, $G_k(\lambda_m) = 0$ for all m. Differentiating $G_k(\lambda)$ leads to

$$G_k'(\lambda) = (\lambda - \lambda_k)\int_a^b \frac{\partial\phi(x,\lambda)}{\partial\lambda}\,\phi_k(x)\,dx + \int_a^b \phi(x,\lambda)\phi_k(x)\,dx,$$

and by setting $\lambda = \lambda_k$ we obtain

$$G_k'(\lambda_k) = \|\phi_k\|^2. \tag{1.12}$$

Hence, by combining (1.11), (1.12) and (1.8), we obtain

$$S_k^*(\lambda) = \frac{G_k(\lambda)}{(\lambda - \lambda_k)G_k'(\lambda_k)},$$

which, in turn, upon its substitution in (1.7), leads to

$$f(\lambda) = \sum_k f(\lambda_k)\,\frac{G_k(\lambda)}{(\lambda - \lambda_k)G_k'(\lambda_k)}. \tag{1.13}$$

Equation (1.13) is similar, but not exactly the same as (1.4) since, in (1.4) all the functions, $G_k(\lambda)$, are the same and equal to a function

$G(\lambda)$, which, without loss of generality, may be taken as the canonical product of its zeros.

In fact, all the functions, $G_k(\lambda)$, which have zeros at $\{\lambda_m\}$, do not have to be equal to obtain a form similar to (1.4). If we further assume that

$$G_k(\lambda) = a_k G(\lambda), \tag{1.14}$$

where a_k is a constant independent of λ, and

$$G(\lambda) = \begin{cases} \prod_j (1 - (\lambda/(\lambda_j))) & \text{if zero is not an eigenvalue,} \\ \lambda \prod_j (1 - (\lambda/(\lambda_j))) & \text{if zero is an eigenvalue,} \end{cases}$$

then $G_k'(\lambda_k) = a_k G'(\lambda_k)$ and the series (1.13) becomes a Lagrange-type interpolation series

$$f(\lambda) = \sum_k f(\lambda_k) \frac{G(\lambda)}{(\lambda - \lambda_k) G'(\lambda_k)}. \tag{1.15}$$

The infinite products converge absolutely for differential operators of orders $n \geq 2$ because $\lambda_j = O(j^n)$ for large j. For first order differential operators, the products may either diverge or converge but not necessarily absolutely. In either case, we can introduce a convergence factor as in Hadamard's theorem to obtain an entire function $G(\lambda)$ that vanishes exactly at the eigenvalues and nowhere else.

The above discussion motivates the following definition.

Definition 2. We say that a boundary-value problem possesses the Lagrange-type interpolation property if it has the Kramer property and its associated sampling series is a Lagrange-type interpolation series as in (1.15).

Recall that the set of all eigenfunctions which belong to the same eigenvalue is a finite-dimensional vector space of dimension less than or equal to n (n being the order of the differential operator). The dimension of this space is called the geometric multiplicity of this eigenvalue.

Definition 3. An eigenvalue λ is said to be geometrically simple if it has exactly one linearly independent eigenfunction. We shall say that a boundary-value problem is geometrically simple if all its eigenvalues are geometrically simple.

Let us denote the set of all regular, self-adjoint boundary-value problems (1.9)–(1.10) by $\mathbf{B}$, the set of all those that have the Kramer property by $\mathbf{K}$, and the set of all those that have the Lagrange-type interpolation property by $\mathbf{L}$ and the set of all those that are geometrically simple by $\mathbf{S}$. From the definitions, it follows that $\mathbf{L} \subset \mathbf{K}$. Theorem 1.3 below shows that $\mathbf{L}$ is not empty. Therefore, we have the following inclusions: $\mathbf{L} \subset \mathbf{K} \subset \mathbf{S} \subset \mathbf{B}$. $\mathbf{S}$ is a proper subset of $\mathbf{B}$ since the boundary-value problem:

$$y'' = -\lambda y, \quad 0 < x < \pi,$$

with periodic boundary conditions $y(0) = y(\pi)$, $y'(0) = y'(\pi)$, is regular self-adjoint, but all its eigenvalues, except for $\lambda = 0$, are double. In fact, the eigenvalues are $(2n)^2$ and the corresponding eigenfunctions are $\cos 2nx$ and $\sin 2nx$. We note that it is possible to obtain some kind of sampling theorem for this problem, but it is not of the simple form given by Kramer's theorem. What kind of boundary-value problem does belong to $\mathbf{L}$? We shall answer this question in the remaining part of this article.

The first result on the relationship between boundary-value problems and Lagrange-type interpolation was obtained by Zayed, Butzer and Hinsen in [**26**] (see also [**23**] for more general results), where it was shown that the general Sturm-Liouville problem of the type

$$y'' - q(x)y = -\lambda y, \quad x \in [a, b], \tag{1.16}$$

$$\cos\alpha\, y(a) + \sin\alpha\, y'(a) = 0, \tag{1.17}$$

$$\cos\beta\, y(b) + \sin\beta\, y'(b) = 0, \tag{1.18}$$

where $\alpha, \beta \in \mathbf{R}$ and $q \in C[a, b]$ has the Lagrange-type interpolation property, i.e., regular Sturm-Liouville problems belong to the class $\mathbf{L}$. More precisely, the following theorem was proved.

Theorem 1.3. *Consider the Sturm-Liouville problem* (1.16)–(1.18). *Let* $\Phi(x, \lambda)$ *be the solution of* (1.16) *satisfying the initial condition*

$$y(a) = \sin\alpha \quad \text{and} \quad y'(a) = -\cos\alpha. \tag{1.19}$$

Let $\{\lambda_k\}_k$ denote the eigenvalues of the problem (1.16)–(1.18)*, where λ_0 is reserved for the eigenvalue zero. Let f be represented in the form*

$$f(\lambda) = \int_a^b g(x)\Phi(x,\lambda)\,dx, \quad \lambda \in \mathbf{R},$$

for some $g(x) \in L^2(a,b)$. Then f is an entire function of exponential type in $t = \sqrt{\lambda}$ that admits the sampling representation

$$f(\lambda) = \sum_{k=1}^{\infty} f(\lambda_k)\,\frac{G(\lambda)}{G'(\lambda_k)(\lambda-\lambda_k)}, \quad \textit{if zero is not an eigenvalue,}$$

and

$$f(\lambda) = \sum_{k=0}^{\infty} f(\lambda_k)\,\frac{G(\lambda)}{G'(\lambda_k)(\lambda-\lambda_k)}, \quad \textit{if zero is an eigenvalue,}$$

where $G(\lambda)$ is an entire function having zeros only at $\{\lambda_k\}$, and without loss of generality, $G(\lambda)$ can be taken as the canonical product of its zeros, i.e.,

$$G(\lambda) = \begin{cases} \prod_j (1-(\lambda/(\lambda_j))) & \textit{if zero is not an eigenvalue,} \\ \lambda\prod_j (1-(\lambda/(\lambda_j))) & \textit{if zero is an eigenvalue.} \end{cases}$$

The series converges uniformly on each compact subset of $\mathbf{R}$.

In the setting of Kramer's theorem, the kernel function $K(x,\lambda) = \Phi(x,\lambda)$ because it is easily seen that $\Phi(x,\lambda)$ generates the eigenfunctions when λ is replaced by λ_k.

The above theorem has been extended to more general boundary-value problems. It was conjectured that regular, self-adjoint, boundary-value problems with algebraically simple eigenvalues have the Lagrange-type interpolation property. For the definition of an algebraically simple eigenvalue, see Definition 4 below. This conjecture was erroneously proved by Butzer and Schöttler in [**9**]. In [**1, 2**] Annaby gave an alternative proof which is not only imprecise, but also deals with a very special case.

The aim of this article is to fill the gaps in the aforementioned papers by providing an alternative proof. The rest of the article is organized as follows. In Section 2, we introduce the basic notation and terminology. To explain the flaw in the argument given in [**9**], we need to briefly introduce Butzer and Schöttler's proof and construction of the kernel function $K(x,\lambda)$, and then give a counterexample to their construction in Section 3. In Section 4, we discuss Annaby's proof and then provide an alternative proof. An example is provided in Section 5.

2. Preliminaries. Consider the following nth order eigenvalue problem consisting of the differential equation

(2.1)

$$Ly := p_0(x)y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + \cdots + p_n(x)y(x) = -\lambda y(x),$$
$$-\infty < a \le x \le b < \infty,$$

with the boundary conditions

$$U_i(y) := \sum_{k=1}^{n} \left\{ \alpha_{ik}\ y^{(k-1)}(a) + \beta_{ik}\ y^{(k-1)}(b) \right\} = 0, \qquad i = 1, \dots, n, \tag{2.2}$$

where $p_k \in C^{n-k}([a,b]),\ k = 0, \dots, n$ and $\alpha_{ik},\ \beta_{ik}$ are constants.

We assume that the problem is regular, that is, $p_0(x) \neq 0$ for any $x \in [a,b]$, and that the boundary conditions $U_i,\ i = 1, \dots, n$, are linearly independent.

For $u, v \in C^n([a,b])$, Green's formula [**11**, p. 86] holds:

$$\int_a^b [\bar{v}L(u) - u\overline{L^*(v)}]\,dx = [u(x)\,,\,v(x)]_a^b\,, \tag{2.3}$$

where

$$[u(x)\,,\,v(x)] = \sum_{k=1}^{n} \sum_{p+q=k-1} (-1)^p u^{(q)}(x)(p_{n-k}(x)\,\bar{v}(x))^{(p)}, \tag{2.4}$$

and L^* is the adjoint differential operator

$$L^*(y) = \sum_{k=0}^{n} (-1)^k (\bar{p}_{n-k}\,y)^{(k)}.$$

The boundary conditions $\{U_j\}_{j=1}^{n}$ can be completed to a linear independent system $\{U_j\}_{j=1}^{2n}$ [**11**, p. 287] and Green's formula can be written as [**11**, p. 288], [**20**, p. 9]: For $u, v \in C^n([a,b])$

$$\int_a^b [\bar{v}L(u) - u\overline{L^*}(v)]\,dx = U_1V_{2n} + U_2V_{2n-1} + \cdots + U_{2n}V_1, \tag{2.5}$$

where V_j are linear forms in $v(a), \dots, v^{(n-1)}(a),\ v(b), \dots, v^{(n-1)}(b)$. The boundary conditions

$$V_j(y) = 0, \quad j = 1, \dots, n,$$

are called the *adjoint boundary conditions*. The nth order problem is said to be *self-adjoint* if and only if $L = L^*$, and the boundary conditions are self-adjoint, that is, each system $\{y(a), \dots, y^{(n-1)}(a), y(b), \dots, y^{(n-1)}(b)\}$ satisfying $U_j(y) = 0$, $j = 1, \dots, n$, also fulfills the adjoint boundary conditions $V_j(y) = 0$, $j = 1, \dots, n$, and vice versa.

Let $\phi(x,\lambda)$ and $\phi(x,\tilde{\lambda})$ be solutions of (2.1), i.e., $L\phi(x,\lambda) = -\lambda\phi(x,\lambda)$ and $L\phi(x,\tilde{\lambda}) = -\tilde{\lambda}\phi(x,\tilde{\lambda})$. If the differential operator L is real and the problem (2.1)–(2.2) is self-adjoint, then (2.5) will take the form

$$\begin{aligned}(\lambda - \tilde{\lambda}) \int_a^b \phi(x,\lambda)\phi(x,\tilde{\lambda})\,dx \\ &= U_1\left(\phi(x,\lambda)\right)V_{2n}\left(\phi(x,\tilde{\lambda})\right) \\ &\quad + U_2\left(\phi(x,\lambda)\right)V_{2n-1}\left(\phi(x,\tilde{\lambda})\right) + \cdots \\ &\quad + U_{2n}\left(\phi(x,\lambda)\right)V_1\left(\phi(x,\tilde{\lambda})\right).\end{aligned} \tag{2.6}$$

We denote by y_i the solution of (2.1) satisfying

$$y_i^{(k-1)}(a,\lambda) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k, \end{cases} (= \delta_{ik}), \quad i, k = 1, \dots, n. \tag{2.7}$$

Then the set $\{y_1(x,\lambda), \dots, y_n(x,\lambda)\}$ forms a fundamental system of solutions of (2.1) so that for any solution $y(x,\lambda)$ of (2.1), there exist constants, $\alpha_1, \dots, \alpha_n$, such that

$$y(x,\lambda) = \sum_{i=1}^{n} \alpha_i\, y_i(x,\lambda).$$

Consider the $n \times n$ matrix

$$A(\lambda) = (U_i(y_j))_{i,j}, \tag{2.8}$$

and let $\Delta(\lambda)$ be its determinant

$$\Delta(\lambda) = \det A(\lambda), \tag{2.9}$$

which is called *the characteristic determinant*. Then $\tilde{\lambda}$ is an eigenvalue of (2.1) and (2.2) if and only if $\Delta(\tilde{\lambda}) = 0$ [**20**, p. 15]. Let us denote the set of all eigenvalues of the problem (2.1) and (2.2) by $EV = \{\lambda_m\}_m$. An eigenvalue $\tilde{\lambda}$ may be a multiple zero of $\Delta(\lambda)$. Its multiplicity as a zero is called the algebraic multiplicity of that eigenvalue. Now we introduce the following definition.

Definition 4. An eigenvalue λ is said to be algebraically simple if it is a simple zero of $\Delta(\lambda)$. We shall say that a boundary-value problem is algebraically simple if all its eigenvalues are algebraically simple.

The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity ([**20**, p. 15]). Thus, if $\tilde{\lambda}$ is a simple zero of $\Delta(\lambda)$, its geometric multiplicity is one, i.e., an algebraically simple eigenvalue is also geometrically simple.

For regular, self-adjoint, boundary-value problems, the geometric multiplicity of an eigenvalue is equal to its algebraic multiplicity. This can be seen from the following facts: 1) the algebraic multiplicity of an eigenvalue is equal to its geometric multiplicity plus the number of associated functions corresponding to that eigenvalue, see [**20**, Theorem VI, Section 2.3]); hence, the algebraic multiplicity of an eigenvalue is equal to its geometric multiplicity if the eigenvalue has no associated functions, 2) The eigenvalue problem (1.9)–(1.10) is equivalent to a homogeneous Fredholm integral equation whose kernel is the Green's function of the problem and such integral equation is known to have no associated functions in the sense of Naimark because the integral operator is a self-adjoint, compact operator. Therefore, from now on, and without loss of generality, by a *simple* eigenvalue, we mean an algebraically simple eigenvalue.

Counter-example. In this section, we introduce Butzer and Schöttler's result [**10**] and outline their proof and then conclude the section with a counterexample to their construction.

The first main step of their proof is the construction of the kernel function $\phi(x,\lambda)$. To this end, the authors defined $\phi(x,\lambda)$ by

$$\phi(x,\lambda) = \begin{vmatrix} y_1(x,\lambda) & \cdots & y_n(x,\lambda) \\ U_1(y_1) & \cdots & U_1(y_n) \\ \vdots & & \vdots \\ U_{n-1}(y_1) & \cdots & U_{n-1}(y_n) \end{vmatrix}. \tag{3.1}$$

Since $\phi(x,\lambda)$ is a linear combination of the fundamental solutions $y_i(x,\lambda)$; $i = 1, 2, \dots, n$, it is also a solution of the differential equation (2.1). Moreover, it is an entire function in λ because the y_i's are entire functions in λ. It is also evident that $U_i(\phi(x,\lambda)) = 0$ for $i = 1, 2, \dots, n-1$. Moreover, in view of (2.9), $U_n(\phi(x,\lambda)) = 0$ if and only if λ is an eigenvalue. Thus, at an eigenvalue $\lambda = \lambda_n$, $\phi(x,\lambda_n)$ is an eigenfunction, i.e., $\phi(x,\lambda)$ is a function that generates the eigenfunctions of the problem. Since the ordering of the boundary conditions is arbitrary, there are essentially n such choices for $\phi(x,\lambda)$. Butzer and Schöttler's theorem can be stated as follows.

Theorem 3.1. *Consider the regular, self-adjoint eigenvalue problem of nth order given by* (2.1) *and* (2.2). *Assume that all the eigenvalues are simple and denote the set of eigenvalues by EV. If F is representable in the form*

$$F(\lambda) = \int_a^b \phi(x,\lambda)\tilde{g}(x)\,dx, \quad \lambda \in \mathbf{R},$$

for some $\tilde{g} \in L^2(a,b)$, then F is an entire function that admits the sampling representation

$$F(\lambda) = \sum_{\lambda_k \in EV} F(\lambda_k)\,\frac{\beta(\lambda)G(\lambda)}{\beta(\lambda_k)G'(\lambda_k)(\lambda-\lambda_k)}, \tag{3.2}$$

where $\beta(\lambda)$ is an entire function with no zeros and

$$G(\lambda) = \begin{cases} \prod_{\lambda_k \in EV}(1-(\lambda/(\lambda_k))) & \text{if } 0 \notin EV, \\ \lambda \prod_{\substack{\lambda_k \in EV \\ \lambda_k \neq 0}}(1-(\lambda/(\lambda_k))) & \text{if } 0 \in EV \end{cases} \cdot$$

The series converges uniformly on each compact subset of $\mathbf{R}$.

In particular, if we set $\varphi(x,\lambda) = \phi(x,\lambda)/\beta(\lambda)$, *then for any function* f *of the form*

$$f(\lambda) = \int_a^b \varphi(x,\lambda) g(x)\, dx, \quad \lambda \in \mathbf{R},$$

for some $g \in L^2(a,b)$, *we have the sampling representation*

$$f(\lambda) = \sum_{\lambda_k \in EV} f(\lambda_k) \frac{G(\lambda)}{G'(\lambda_k)(\lambda - \lambda_k)},$$

where the series converges uniformly on each compact subset of $\mathbf{R}$.

To derive (3.2), the authors used Green's formula (2.6) and (1.11) to write

$$\begin{aligned} G_k(\lambda) &= (\lambda - \lambda_k) \int_a^b \phi(x,\lambda)\phi_k(x)\, dx \\ &= U_1(\phi(x,\lambda))\, V_{2n}(\phi_k(x)) + U_2(\phi(x,\lambda))\, V_{2n-1}(\phi_k(x)) + \cdots \\ &\quad + U_{2n}(\phi(x,\lambda))\, V_1(\phi_k(x)). \end{aligned}$$

Because the problem is self-adjoint, $\phi_k(x)$ satisfies all the boundary conditions $V_1, \dots, V_n$ and, in addition, because $\phi(x,\lambda)$ satisfies the first $n-1$ conditions, $U_1, \dots, U_{n-1}$, the last equation reduces to

$$G_k(\lambda) = U_n(\phi(x,\lambda))\, V_{n+1}(\phi_k(x)).$$

The authors then showed that $G_k(\lambda)$ has no zeros other than the eigenvalues, and hence they could write it, in view of Hadamard's factorization theorem, in the form $G_k(\lambda) = a_k \beta(\lambda) G(\lambda)$ for some constant a_k; see the comments preceding Definition 2.

Before we give a counterexample to the construction given in (3.1), let us observe that for problems associated with second order differential operators, there are two choices for $\phi(x,\lambda)$,

$$\phi_1(x,\lambda) = \begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) \\ U_2(y_1) & U_2(y_2) \end{vmatrix}, \tag{3.3}$$

and

$$\phi_2(x,\lambda) = \begin{vmatrix} y_1(x,\lambda) & y_2(x,\lambda) \\ U_1(y_1) & U_1(y_2) \end{vmatrix}. \tag{3.4}$$

In the following counterexample we show that for a second order, regular, self-adjoint, boundary-value problem, neither ϕ_1 nor ϕ_2 can generate the eigenfunctions.

Example 3.1. Consider the regular boundary-value problem:

$$y'' = -\lambda\, y, \quad 0 \le x \le \pi, \tag{3.5}$$

with

$$U_1(y) = \frac{y(0) - y(\pi)}{\pi} + y'(0) = 0 \tag{3.6}$$

and

$$U_2(y) = \frac{2(y(0) + y(\pi))}{\pi} + y'(0) + y'(\pi) = 0. \tag{3.7}$$

First, we show that the above problem is self-adjoint. Suppose that a twice continuously differentiable function y satisfies the boundary conditions (3.6). Then y satisfies

$$y'(0) = \frac{y(\pi) - y(0)}{\pi} \tag{3.8}$$

and by (3.7) we have

$$y'(\pi) = \frac{1}{\pi}\,(-y(0) - 3y(\pi)). \tag{3.9}$$

Let u, v be twice continuously differentiable functions satisfying the boundary conditions (3.6). In view of (3.8) and (3.9), it follows that

$$\int_0^\pi (u''(x)v(x) - u(x)v''(x))\, dx = 0.$$

Hence, the above system is self-adjoint.

Let y_j be the solution of the differential equation (3.5) satisfying

$$y_j^{(k-1)}(0) = \delta_{jk}, \quad j, k = 0, 1,$$

and $s = \sqrt{\lambda}$. Then

$$y_1(x, s^2) = \cos sx \quad \text{and} \quad y_2(x, s^2) = \frac{\sin sx}{s}.$$

Let

$$A(\lambda) = \begin{pmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2) \end{pmatrix}.$$

By simple computations, we have

(3.10)

$$A(\lambda) = \begin{pmatrix} \dfrac{1}{\pi} - \left(\dfrac{\cos s\pi}{\pi}\right) & -\left(\dfrac{\sin s\pi}{s\pi}\right) + 1 \\ \dfrac{2}{\pi} + \left(\dfrac{2\cos s\pi}{\pi}\right) - s \sin s\pi & \left(\dfrac{2\sin s\pi}{s\pi}\right) + 1 + \cos s\pi \end{pmatrix}.$$

Hence,

$$\Delta(\lambda) = \frac{-2\pi s - 2\pi s \cos \pi s + (4 + \pi^2 s^2) \sin \pi s}{\pi^2 s}, \tag{3.11}$$

and it easily follows that the eigenvalues are the solutions of the equation

$$\cos \frac{\pi s}{2} \left[-2\pi s \cos \frac{\pi s}{2} + (4 + \pi^2 s^2) \sin \frac{\pi s}{2} \right] = 0$$

or, equivalently, they are the solutions of the equations

$$\cos \frac{\pi s}{2} = 0, \quad \text{and} \quad \tan \frac{\pi s}{2} = \frac{2\pi s}{4 + \pi^2 s^2}, \quad s \geq 0.$$

There are two sequences of eigenvalues

$$\lambda_{2n} = (2n+1)^2, \quad \text{and} \quad \lambda_{2n+1} = s_n^2, \quad n = 0, 1, 2, \dots,$$

where s_n is a solution of the equation $\tan((\pi s)/2) = (2\pi s)/(4 + \pi^2 s^2)$, $s \geq 0$.

We next show that the eigenvalues are simple. Since $\Delta(0) = 0$ and $\lim_{\lambda\to 0}(\Delta(\lambda/\lambda)) = (4\pi)/3$, zero is a simple eigenvalue. Let $\lambda^* > 0$ be an eigenvalue. Suppose that the multiplicity of λ^* is greater than 1. Then $(\partial/\partial s)\Delta(\lambda)|_{\lambda=\lambda^*} = 0$. Differentiating (3.11), we have

(3.12)

$$(\partial/\partial s)\Delta(\lambda)|_{\lambda=\lambda^*} = \frac{\pi s^*(4+\pi^2 s^{*2})\cos\pi s^* + (-4+3\pi^2 s^{*2})\sin\pi s^*}{\pi^2 s^{*2}},$$

where $s^* = \sqrt{\lambda^*}$. Since $\Delta(\lambda^*) = 0$ and $(\partial/\partial s)\Delta(\lambda)|_{\lambda=\lambda^*} = 0$, from (3.11) and (3.12) we have

$$\sin\pi s^* = \frac{2\pi s^*(4+\pi^2 s^{*2})}{8+14\pi^2 s^{*2}+\pi^4 s^{*4}} \quad \text{and} \quad \cos\pi s^* = \frac{2(4-3\pi^2 s^{*2})}{8+14\pi^2 s^{*2}+\pi^4 s^{*4}}.$$

Therefore, setting $\pi s^* = \sqrt{x} = \pi\sqrt{\lambda^*}$, we have

$$\begin{aligned}\sin^2\sqrt{x} + \cos^2\sqrt{x} &= \frac{4(16-8x+17x^2+x^3)}{(8+14x+x^2)^2}\\ &= \frac{64-32x+68x^2+4x^3}{64+224x+212x^2+28x^3+x^4} = 1.\end{aligned}$$

But the only solutions of the equation

$$\frac{64-32x+68x^2+4x^3}{64+224x+212x^2+28x^3+x^4} = 1$$

are $x = 0, -4, -4, -16$, which is a contradiction since $\lambda^* > 0$. Hence λ^* is a simple eigenvalue.

In view of (3.10), we have

$$A(0) = \begin{pmatrix} 0 & 0 \\ 4/\pi & 4 \end{pmatrix} \quad \text{and} \quad A(1) = \begin{pmatrix} 2/\pi & 1 \\ 0 & 0 \end{pmatrix},$$

and since $\Delta(0) = 0$ and $\Delta(1) = 0$, 0 and 1 are eigenvalues of the regular, self-adjoint, eigenvalue value problem (3.5)–(3.7). The corresponding eigenfunctions are

$$1 - \frac{1}{\pi}\,x, \quad \text{and} \quad \cos x - \frac{2}{\pi}\,\sin x,$$

respectively. Therefore, by (3.3) and (3.4), we have

(3.13)
$$\phi_1(x,\lambda) = \begin{vmatrix} \cos sx & (\sin sx)/s \\ 2/\pi + (2\cos s\pi)/\pi - s\sin s\pi & (2\sin s\pi)/(s\pi) + 1 + \cos s\pi \end{vmatrix},$$

and

(3.14)
$$\phi_2(x,\lambda) = \begin{vmatrix} \cos sx & (\sin sx)/s \\ 1/\pi - (\cos s\pi)/\pi & -(\sin s\pi)/(s\pi) + 1 \end{vmatrix}.$$

It is now easy to see that $\phi_1(x,1)$ and $\phi_2(x,0)$ are identically zero, i.e.,
$$\phi_1(x,1) \equiv 0 \quad \text{and} \quad \phi_2(x,0) \equiv 0.$$

Hence, neither $\phi_1(x,\lambda)$ nor $\phi_2(x,\lambda)$ can be a kernel function producing all the eigenfunctions of the problem because they vanish identically at some eigenvalues.

4. A kernel function and the sampling theorem. In this section we give an alternate proof to Theorem 3.1. First, we give an alternate construction of the kernel function, and then prove the sampling theorem.

Consider the nth order, regular, self-adjoint, eigenvalue problem (2.1) and (2.2). Denote the set of all its eigenvalues, which is a subset of $\mathbf{R}$, by $EV = \{\lambda_m\}_m$. The index m may run over the integers or a subset thereof. For each $\lambda \in \mathbf{R}$, denote by $y_i(x,\lambda)$ the solution of (2.1) satisfying (2.7). It is known [**20**, p. 14], that for any $1 \le i \le n$, y_i is an entire function in λ. For $k = 1, \dots, n$, we define the functions

$$\phi_k(x,\lambda) = \begin{vmatrix} y_1(x,\lambda) & \cdots & y_n(x,\lambda) \\ U_1(y_1) & \cdots & U_1(y_n) \\ \vdots & \vdots & \vdots \\ U_{k-1}(y_1) & \cdots & U_{k-1}(y_n) \\ U_{k+1}(y_1) & \cdots & U_{k+1}(y_n) \\ \vdots & \vdots & \vdots \\ U_n(y_1) & \cdots & U_n(y_n) \end{vmatrix}.$$

Then each ϕ_k is a solution of (2.1) and an entire function in λ because it is a linear combination of the fundamental solutions. We seek a solution, $\phi(x,\lambda)$, of (2.1) with the following properties:

(1) $\phi(x,\lambda)$ is an entire function in λ,

(2) for each eigenvalue λ_m, $\phi(x,\lambda_m)$ is an eigenfunction,

(3) for each $\lambda \in \mathbf{R}$, $\phi(x,\lambda)$ does not vanish identically.

It should be noted that Annaby in [**1**, Theorem 1] assumes implicitly that, if a function $f(\lambda)$ vanishes at, say $\{\alpha_n\}$, and a function $g(\lambda)$ vanishes at, say $\{\beta_n\}$, then one can find a constant c such that $f(\lambda) + cg(\lambda)$ does not vanish for any real λ. This, however, is not true in general as can be seen from the example: $f(\lambda) = (\cos\lambda)e^{\lambda}$ and $g(\lambda) = \sin\lambda$.

In the next theorem we construct a kernel function $\phi(x,\lambda)$ that satisfies (2.1) and conditions (1)–(3) above. The proof uses special properties of the determinants defining ϕ_ν.

Theorem 4.1. *Consider the regular, self-adjoint, eigenvalue problem* (2.1) *and* (2.2). *Assume that all the eigenvalues are simple. Then there exist* $\beta_1, \dots, \beta_n$ *in* $\mathbf{R}$ *such that*

$$\phi(x,\lambda) = \sum_{i=1}^{n} \beta_i\, \phi_i(x,\lambda) \tag{4.1}$$

satisfies (2.1) *and conditions* (1)–(3) *above.*

Proof. Let

$$A(\lambda) = \begin{pmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \vdots & \vdots & \vdots \\ U_n(y_1) & \cdots & U_n(y_n) \end{pmatrix} \tag{4.2}$$

and

$$\Delta(\lambda) = |A(\lambda)|, \tag{4.3}$$

where for an $n \times n$ matrix B, $|B|$ denotes the determinant of B. Let $M_{i,j}(\lambda)$ be the determinant of the submatrix of $A(\lambda)$ which remains after the ith row and jth column are deleted from $A(\lambda)$.

Let $A^c(\lambda)$ be the matrix defined by $A^c_{i,j}(\lambda) = (-1)^{i+j}M_{j,i}(\lambda)$, that is, $A^c(\lambda)$ is the transpose of the matrix whose entries are the cofactors $(-1)^{i+j}M_{i,j}$ of the corresponding entries of A. Recall from [**15**, p. 353] that

$$A(\lambda)A^c(\lambda) = |A(\lambda)|I_n,$$

hence, $|A^c(\lambda)| = |A(\lambda)|^{n-1}$, $n \geq 2$, where I_n is the $n \times n$ identity matrix. It follows from the above equation that $A(\lambda)$ is nonsingular if and only if $A^c(\lambda)$ is nonsingular. Since each y_i is an entire function in λ, $\Delta(\lambda)$ is also an entire function in λ. Note that for $\lambda \in \mathbf{C}$

$$U_l(\phi_k(x,\lambda)) = 0 \quad \text{for} \quad l \neq k, \tag{4.4}$$

and

$$U_k(\phi_k(x,\lambda^*)) = 0 \quad \text{for} \quad k = 1, \dots, n, \tag{4.5}$$

if and only if λ^* is an eigenvalue. Fix $m \in \mathbf{Z}$, let $\tilde{\alpha}_1, \dots, \tilde{\alpha}_n \in \mathbf{R}$, and define the map

$$F_m(x) = \tilde{\alpha}_1\phi_1(x,\lambda_m) + \cdots + \tilde{\alpha}_n\phi_n(x,\lambda_m),$$

for some $\tilde{\alpha}_i$, $i = 1, 2, \dots, n$, to be determined. By setting $\alpha_i = (-1)^{i+1}\tilde{\alpha}_i$, $i = 1, 2, \dots, n$, and expanding $\phi_i(x,\lambda_m)$ in terms of the fundamental solutions, we obtain

$$\begin{aligned} F_m(x) &= \sum_{i=1}^n \tilde{\alpha}_i \left[\sum_{j=1}^n (-1)^{j+1} M_{i,j}(\lambda_m) y_j(x,\lambda_m)\right] \\ &= \sum_{j=1}^n \left[\sum_{i=1}^n (-1)^{i+j} \alpha_i M_{i,j}(\lambda_m)\right] y_j(x,\lambda_m) \\ &= \sum_{j=1}^n \left[\sum_{i=1}^n \alpha_i A^c_{j,i}(\lambda_m)\right] y_j(x,\lambda_m). \end{aligned}$$

Suppose that $F_m(x)$ is identically zero, i.e., $F_m(x) \equiv 0$. Then because the $y_i(x)$'s are linearly independent, we have for $j = 1, \dots, n$,

$$\sum_{i=1}^n \alpha_i A^c_{j,i}(\lambda_m) = 0, \tag{4.6}$$

or in a matrix form $A^c(\lambda_m)\boldsymbol{\alpha} = \mathbf{0}$, where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)^T$. Since $A(\lambda_m)$ is singular, so is $A^c(\lambda_m)$, and the rank of $A^c(\lambda_m)$ is k with $1 \leq k \leq n-1$. Hence the solution to equation (4.6) is a vector space of dimension $n-k$, which can be imbedded in a hyperplane S_m of dimension $n-1$ in the form

$$\alpha_{1,m}x_1 + \alpha_{2,m}x_2 + \cdots + \alpha_{n,m}x_n = 0.$$

Here we may identify the vector $(x_1, \dots, x_n)$ with the point $(x_1, \dots, x_n)$.

Let $S = \cup_{m\in\mathbf{Z}} S_m$. First, we show that $\mathbf{R}^n$ is not the union of countably many hyperplanes of dimensions $n-1$ passing through the origin. Since any such hyperplane is of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, we can write it in the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ with $\sum_{i=1}^n a_i^2 = 1$ (just replace a_i by $a_i / \sum_{i=1}^n a_i^2$). Let

$$\{S_m : \alpha_{1,m}x_1 + \alpha_{2,m}x_2 + \cdots + \alpha_{n,m}x_n = 0\}_{m\in\mathbf{Z}}$$

be such a collection.

For each $m \in \mathbf{Z}$, consider the n-tuple $(\alpha_{1,m}, \alpha_{2,m}, \dots, \alpha_{n,m}) = P_m$. Because $\sum_{i=1}^n a_{i,m}^2 = 1$, P_m can be viewed as a point on the unit sphere in $\mathbf{R}^n$, and since the unit sphere is not countable, we can find a point $P = (\alpha_1, \alpha_2, \dots, \alpha_n)$ on the unit sphere such that $P \neq \pm P_m$ for all m. It suffices to find $\alpha_i \neq \pm\alpha_{i,m}$ for all m and $i = 1, 2, \dots, n$. Hence, the hyperplane

$$\alpha_1x_1 + \alpha_2x_2 + \cdots + \alpha_nx_n = 0$$

is a hyperplane of dimension $n-1$ in $\mathbf{R}^n$ that does not belong to S.

Since in $\mathbf{R}^n$ any hyperplane of dimension k with $1 \leq k \leq n-1$ is nowhere dense in $\mathbf{R}^n$, it follows from the Baire category theorem that $\mathbf{R}^n$ cannot be equal to a countable union of hyperplanes of dimensions less than or equal to $n-1$. Similarly, any hyperplane of dimension $n-1$ cannot be a countable union of hyperplanes of dimensions less than $n-1$.

We have shown that to every eigenvalue λ_m, there exists infinitely many $\alpha_{1,m}, \dots, \alpha_{n,m}$ that make

$$\sum_{i=1}^{n} \alpha_{i,m}\, \phi_i(x, \lambda_m) = 0,$$

but all these $\alpha_{i,m}$'s lie on a hyperplane of dimension $n-1$.

Since $\mathbf{R}^n$ is not the union of countably many hyperplanes of dimensions less than or equal to $n-1$ passing through the origin, we can find $(\beta_1, \dots, \beta_n) \in \mathbf{R}^n - S$, so that for all $m \in \mathbf{Z}$,

$$\beta_1 \phi_1(x, \lambda_m) + \cdots + \beta_n \phi_n(x, \lambda_m) \not\equiv 0. \tag{4.7}$$

Now we construct a function $\phi(x, \lambda)$ that satisfies (2.1) and conditions (1)–(3) above. Let

$$\phi(x, \lambda) = \sum_{i=1}^{n} \beta_i\, \phi_i(x, \lambda). \tag{4.8}$$

Then, clearly ϕ is a solution of (2.1), which is also an entire function in λ. By (4.5), for each eigenvalue λ_m

$$U_l(\phi(x, \lambda_m)) = 0 \quad \text{for} \quad l = 1, \dots, n,$$

and hence $\phi(x, \lambda_m)$ is an eigenfunction corresponding to the eigenvalue λ_m. Thus, $\phi(x, \lambda)$ satisfies condition (2).

Next we show that for any noneigenvalue λ, $\phi(x, \lambda)$ does not vanish identically, i.e, it satisfies condition (3).

Let λ^* be a noneigenvalue. We have

$$\phi(x, \lambda^*) = \sum_{i=1}^{n} \beta_i\, \phi_i(x, \lambda^*).$$

By expanding $\phi_i(x, \lambda^*)$ in terms of the fundamental solutions, we obtain

$$\phi(x, \lambda^*) = \sum_{j=1}^{n} C_j\, y_j(x, \lambda^*),$$

where

$$C_j = \sum_{i=1}^{n} \beta_i (-1)^{j+1} M_{i,j}(\lambda^*).$$

Let us set $\beta_i = (-1)^{i+1} w_i$ to obtain

$$\phi(x, \lambda^*) = \sum_{j=1}^{n} \left[\sum_{i=1}^{n} (-1)^{i+j} w_i M_{i,j}(\lambda^*) \right] y_j(x, \lambda^*).$$

Assume that $\phi(x, \lambda^*) \equiv 0$, then we obtain, in view of the linear independence of the fundamental solutions, that

$$\sum_{i=1}^{n} (-1)^{i+j} w_i M_{i,j}(\lambda^*) = 0, \quad j = 1, \dots, n,$$

which can be written in the form

$$A^c(\lambda^*)\mathbf{w} = \mathbf{0},$$

where $\mathbf{w} = (w_1, \dots, w_n)^T$. But since A is nonsingular at λ^*, it follows that $\mathbf{w} = 0$, which implies that $w_i = 0$ for $i = 1, \dots, n$. This in turn implies that $\beta_i = 0$ for $i = 1, 2, \dots, n$, which is a contradiction since $(\beta_1, \dots, \beta_n) \notin S$. Therefore, for every λ which is not an eigenvalue, $\phi(x, \lambda)$ does not vanish identically. $\square$

Definition 5. We say that the BVP (2.1) and (2.2) is one-dimensional if there exists $1 \le k \le n$ such that the rank of the matrix

$$\begin{pmatrix} U_1(y_1) & \cdots & U_1(y_n) \\ \vdots & & \vdots \\ U_{k-1}(y_1) & \cdots & U_{k-1}(y_n) \\ U_{k+1}(y_1) & \cdots & U_{k+1}(y_n) \\ \vdots & & \vdots \\ U_n(y_1) & \cdots & U_n(y_n) \end{pmatrix}$$

is $n-1$ for *all eigenvalues* λ_m.

Clearly, under the assumption that the BVP (2.1) and (2.2) is one-dimensional, we can take

$$\phi(x, \lambda) = \phi_k(x, \lambda)$$

and Theorem 4.1 holds. This is essentially the case considered by Butzer and Schöttler and later by Annaby in his proof of the sampling theorem, see [**2**, equation (3.5), Theorem 3.6].

In the next theorem, we use the kernel function defined by (4.8) to derive the sampling theorem and the Lagrange-type interpolation series associated with problem (2.1) and (2.2). Here it should be emphasized that, under the stringent assumption that the problem is one-dimensional, the Lagrange identity, (2.6), which contains $2n$ terms, reduces to just one term, hence simplifying the proof significantly. The proof of the general case is more complicated and requires a special treatment as can be seen below.

Theorem 4.2. *Consider the regular, self-adjoint eigenvalue problem* (2.1) *and* (2.2). *Assume that all the eigenvalues are simple. Then, there is a kernel function $\phi(x, \lambda)$ such that if f is represented by*

$$f(\lambda) = \int_a^b \phi(x, \lambda) g(x)\, dx, \quad \lambda \in \mathbf{R}, \tag{4.9}$$

for some $g \in L^2(a, b)$, then f is an entire function that admits the sampling representation

$$f(\lambda) = \sum_{\lambda_n \in EV} f(\lambda_n) \frac{\Delta(\lambda)}{\Delta'(\lambda)(\lambda - \lambda_n)}. \tag{4.10}$$

The series converges uniformly on each compact subset of $\mathbf{R}$, *and $\Delta(\lambda)$ is the entire function defined by* (4.3) *whose zeros are exactly the eigenvalues. Without loss of generality, we may take $\Delta(\lambda)$ to be the canonical product of its zeros.*

Proof. Let ϕ be the function defined in Theorem 4.1. Let

$$F(\lambda) = \int_a^b \phi(x, \lambda) g(x)\, dx$$

for some $g \in L^2(a,b)$. Since $\phi(x,\lambda)$ is an entire function in λ, and the integral converges uniformly, F is an entire function. For any eigenvalue λ_m and $\lambda \in \mathbf{R}$, we have

$$\int_a^b [\phi(x,\lambda)L\phi(x,\lambda_m) - \phi(x,\lambda_m)L\phi(x,\lambda)]\,dx$$
$$= (\lambda - \lambda_m)\int_a^b \phi(x,\lambda)\phi(x,\lambda_m)\,dx := G_m(\lambda),$$

so that $G_m(\lambda)$ is an entire function in λ.

Differentiating (4.12) with respect to λ and taking the limit $\lambda \to \lambda_m$, we have

$$G_m'(\lambda_m) = \int_a^b |\phi(x,\lambda_m)|^2\,dx, \quad \text{and} \quad G_m'(\lambda_m) \neq 0. \tag{4.13}$$

Since $\phi(\cdot,\lambda) \in L^2(a,b)$, we can expand it as a series of eigenfunctions [**11**, p. 199]

$$\phi(x,\lambda) = \sum_{\lambda_m \in EV} S_m(\lambda)\phi(x,\lambda_m), \tag{4.14}$$

where the series converges in $L^2(a,b)$ and

$$S_m(\lambda) = \frac{\int_a^b \phi(x,\lambda_m)\phi(x,\lambda)\,dx}{\left[\int_a^b |\phi(x,\lambda_m)|^2\,dx\right]}. \tag{4.15}$$

In view of (4.12), (4.13) and (4.15), we have

$$S_m(\lambda) = \frac{G_m(\lambda)}{G_m'(\lambda_m)(\lambda - \lambda_m)}.$$

By Parseval's equality, it follows that

$$\begin{aligned} f(\lambda) &= \int_a^b \phi(x,\lambda)g(x)\,dx \\ &= \sum_{\lambda_m \in EV} \left\{\int_a^b g(x)\phi(x,\lambda_m)\,dx\right\} \\ &\qquad\qquad \times \left\{\int_a^b \phi(x,\lambda)\phi(x,\lambda_m)\,dx\right\} ||\phi(\cdot,\lambda_m)||_2^{-2} \\ &= \sum_{\lambda_m \in EV} f(\lambda_m)S_m(\lambda) = \sum_{\lambda_m \in EV} f(\lambda_m)\,\frac{G_m(\lambda)}{G_m'(\lambda_m)(\lambda - \lambda_m)}, \end{aligned} \tag{4.16}$$

where $||\phi(\cdot,\lambda_m)||_2 = (\int_a^b |\phi(x,\lambda_m)|^2\,dx)^{1/2}$ and the series converges pointwise.

We next show that $G_m(\lambda^*) = 0$ implies that λ^* is an eigenvalue. From (4.1)–(4.5), it follows that, for $\lambda \in \mathbf{R}$,

$$U_i(\phi(x,\lambda)) = (-1)^{i+1}\beta_i\Delta(\lambda), \quad i = 1,\dots,n, \tag{4.17}$$

and for any eigenvalue λ_l, $G_m(\lambda_\ell) = 0$. Let λ^* be a noneigenvalue and $m \in \mathbf{Z}$. Then

$$G_m(\lambda^*) = (\lambda^* - \lambda_m)\int_a^b \phi(x,\lambda^*)\phi(x,\lambda_m)\,dx. \tag{4.18}$$

In view of (2.6), we also have

$$\begin{aligned} G_m(\lambda^*) &= (\lambda^* - \lambda_m)\int_a^b \phi(x,\lambda^*)\phi(x,\lambda_m)\,dx \\ &= U_1(\phi(x,\lambda^*))V_{2n}(\phi(x,\lambda_m)) + \cdots \\ &\quad + U_n(\phi(x,\lambda^*))V_{n+1}(\phi(x,\lambda_m)) \\ &\quad + U_{n+1}(\phi(x,\lambda^*))V_n(\phi(x,\lambda_m)) + \cdots \\ &\quad + U_{2n}(\phi(x,\lambda^*))V_1(\phi(x,\lambda_m)), \end{aligned}$$

where $V_1(y) = 0,\dots,V_{2n}(y) = 0$ are the adjoint-boundary conditions of (2.2).

Because the boundary-value problem (2.1), (2.2) is self-adjoint

$$V_i(\phi(x,\lambda_m)) = 0, \quad i = 1,\dots,n.$$

Therefore, in view of (4.17),

$$\begin{aligned} G_m(\lambda^*) &= (\lambda^* - \lambda_m)\int_a^b \phi(x,\lambda^*)\phi(x,\lambda_m)\,dx \\ &= U_1(\phi(x,\lambda^*))V_{2n}(\phi(x,\lambda_m)) + \cdots \\ &\quad + U_n(\phi(x,\lambda^*))V_{n+1}(\phi(x,\lambda_m)) \\ &= \Delta(\lambda^*)w_m, \end{aligned}$$

where w_m is a constant independent of λ^*, given by the scalar product $w_m = \boldsymbol{\beta}\cdot\widetilde{V}_m$, of the vectors $\boldsymbol{\beta} = \left[\beta_1, -\beta_2, \dots, (-1)^{n+1}\beta_n\right]$ and $\widetilde{V}_m = [V_{2n}(\phi(x,\lambda_m)),\dots,V_{n+1}(\phi(x,\lambda_m))]$.

Clearly, $w_m \neq 0$, otherwise $G_m(\lambda^*)$ would be identically zero, which would contradict (4.13). Therefore, $G_m(\lambda^*)$ and $\Delta(\lambda^*)$ have exactly the same zeros, and hence the condition $G_m(\lambda^*) = 0$ would imply that λ^* is an eigenvalue.

Because, for any λ, $G_m(\lambda) = \Delta(\lambda) \cdot w_m$, equation (4.16) can be written as

$$f(\lambda) = \sum_{\lambda_m \in EV} f(\lambda_m) \frac{\Delta(\lambda)}{\Delta'(\lambda_m)(\lambda - \lambda_m)}. \tag{4.19}$$

Having proved (4.19), we can now use standard techniques to prove the uniform convergence, cf. [**9**].

5. Example. In this section we give an example to show how to apply Theorem 4.1 to the counterexample given in Section 3, Example 3.1.

Example 5.1. In Example 3.1, let $s = \sqrt{\lambda}$, $\beta_1 = 1$ and $\beta_2 = 2$, so that

$$\begin{aligned}\phi(x, s^2) &= (2U_1(y_2) + U_2(y_2))y_1(x, s^2) - (2U_1(y_1) + U_2(y_1))y_2(x, s^2)\\ &= (3 + \cos \pi s) \cos sx - \left(\frac{4}{\pi} - s \sin \pi s\right) \frac{\sin sx}{s}.\end{aligned}$$

Then for any s, $\phi(x, s^2)$ does not vanish identically and, when s^2 is equal to an eigenvalue, $\phi(x, s^2)$ generates the corresponding eigenfunction. Since $\phi(x, s^2)$ does not vanish identically, it is a kernel function. Note that

$$\Delta(s^2) = \frac{-2s\pi - 2\pi s \cos \pi s + (4 + \pi^2 s^2) \sin \pi s}{\pi^2 s},$$

and

$$\phi(x, 0) = 4\left(1 - \frac{1}{\pi}\, x\right), \quad \text{and} \quad \phi(x, 1) = 2\left(\cos x - \frac{2}{\pi}\, \sin x\right),$$

Let $\{s_n\}_{n \in \mathbf{N}}$ be a sequence of nonnegative zeros of the equation $\Delta(s^2) = 0$. If $f(s^2) = \int_0^\pi \phi(x, s^2) g(x)\, dx$ for some $g \in L^2(0, \pi)$, then f admits the following sampling representation

$$f(s^2) = \sum_{n \in \mathbf{N}} f(s_n) \frac{\Delta(s^2)}{\Delta'(s_n^2)(s^2 - s_n^2)},$$

where $\Delta'(s_n^2) = (d/ds^2)\Delta(s^2)|_{s=s_n}$ and the series converges uniformly on each compact subsets of $\mathbf{R}$.

Acknowledgments. The authors would like to thank the referee and Professor M. Shubov for their helpful suggestions.

REFERENCES

1. M.H. Annaby, *On sampling expansions associated with boundary-value problems*, Proc. 1995 Workshop on Sampling Theory (Jurmala, Latvia, 1995), pp. 137–139.

2. ———, *Interpolation expansions associated with non-self-adjoint boundary-value problems*, Commun. Appl. Anal. **3** (1999), 545–561.

3. M.H. Annaby and G. Freiling, *Sampling expansions associated with Kamke problems*, Math. Z. **234** (2000), 163–189.

4. M.H. Annaby and A.I. Zayed, *On the use of Green's function in sampling theory*, J. Integral Equations Appl. **10** (1998), 117–139.

5. G.D. Birkhoff, *Boundary value and expansion problems of ordinary linear differential equations*, Trans. Amer. Math. Soc. **9** 1908, 373–395.

6. R. Boas, *Entire functions*, Academic Press, New York, 1954.

7. A. Boumenir and A. Zayed, *Sampling with a string*, J. Fourier Anal. Appl. **8** (2002), 211–231.

8. P.L. Butzer and G. Nasri-Roudsari, *Kramer's sampling theorem in signal analysis and its role in mathematics*, in *Image processing*: *Mathematical methods and applications* (J.M. Blackledge, ed.), Clarendon Press, Oxford, 1997.

9. P.L. Butzer and G. Schöttler, *Sampling theorems associated with fourth- and higher-order self-adjoint eigenvalue problems*, J. Comput. Appl. Math. **51** 1994, 159–177.

10. P.L. Butzer and R.L. Stens, *An extension of Kramer's sampling theorem for not necessarily band-limited signals-the aliasing error*, Acta Sci. Math. (Szeged) **60** (1995), 59–69.

11. E.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

12. W.N. Everitt and G. Nasri-Roudsari, *Interpolation and sampling theories, and linear ordinary boundary-value problems*, in *Sampling theory in Fourier and signal analysis*: *Advanced topics* (J.R. Higgins and R.L. Stens, eds.), Clarendon Press, Oxford, 1999.

13. W.N. Everitt, G. Schöttler and P.L. Butzer, *Sturm-Liouville boundary value problems and Lagrange interpolation series*, Rend. Mat. Appl. (7) **14** (1994), 87–126.

14. H.R. Higgins, *Sampling theory in Fourier and signal analysis*: *Foundations*, Oxford Sci. Publications, Oxford, 1996.

15. T. Hungerford, *Algebra*, Springer-Verlag, New York, 1974, pp. 447–448.

16. S.C. Kak, *Sampling theorem in Walsh-Fourier analysis*, Electronics Lett. **6** (1970), 447–448.

17. H.P. Kramer, *A generalized sampling theorem*, J. Math. Phys. **38** 1959, 68–72.

18. B.Ja. Levin, *Distribution of zeros of entire functions*, Amer. Math. Soc. Transl. Math. Monogr., vol. 5, Providence, Rhode Island, 1964.

19. P. McCoy, *On the structure of multivariable sampling series*, Complex Variables Theory Appl. **41** (2000), 1–15.

20. M.A. Naimark, *Linear differential operators*, *Part* I, Frederick Ungar Publ. Co., New York, 1966.

21. G. Nasri-Roudsari, *Der Kramer Abtastsatz im Zusammenhang mit Sturm-Liouville Differentialgleichungen und Quasi-Differentialgleichungen köherer Ordnung*, Ph.D. Thesis, Rheinisch-Westfälische Technische Hochschule Aachen, Aachen, Germany, 1996.

22. A.I. Zayed, *Advances in Shannon's sampling theory*, CRC Press, Boca Raton, FL, 1993.

23. ———, *On Kramer's sampling theorem associated with general Sturm-Liouville boundary value problems and Lagrange interpolation*, SIAM J. Appl. Math. **51** (1991), 575–604.

24. A.I. Zayed, M.A. El-Sayed and M. H. Annaby, *On Lagrange interpolations and Kramer's sampling theorem associated with self-adjoint boundary value problems*, J. Math. Anal. Appl. **158** 1991, 269–284.

25. A. Zayed and A. Garcia, *Kramer's sampling theorem with discontinuous kernels*, Results Math. **34** (1998), 197–206.

26. A.I. Zayed, G. Hinsen and P.L. Butzer, *On Lagrange interpolation and Kramer-type sampling theorems associated with Sturm-Liouville problems*, SIAM J. Appl. Math. **50** (1990), 893–909.

Department of Mathematical Sciences, DePaul University, Chicago, IL, 60614-3250
E-mail address: azayed@condor.depaul.edu

Department of Mathematics, Sogang University, Seoul, 121-742 Korea
E-mail address: shinc@sogang.ac.kr