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APPROXIMATE SOLUTION OF THE BIHARMONIC PROBLEM IN SMOOTH DOMAINS

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This paper is dedicated to K.E. Atkinson on the occasion of his 65th birthday

ABSTRACT. Using the Goursat representation for the biharmonic function and approximate solutions of a corrected Muskhelishvili equation we construct approximate solutions for biharmonic problems in smooth domains of $\mathbf{R}^2$. It is shown that the sequence of these approximate solutions converges uniformly on each compact subset of the initial domain D. Under additional conditions it converges uniformly on D. We also provide numerical examples.

1. Introduction. Let D be a domain in $\mathbf{R}^2$ bounded by a simple closed regular smooth curve Γ. This means that Γ does not have any intersections with itself and, if $\gamma = (\gamma_1, \gamma_2)$ is a parametrization of Γ, i.e., $\gamma : [a,b] \longrightarrow \Gamma$, where $[a,b] \subset \mathbf{R}$, then γ is continuously differentiable on $[a,b]$ and $(\gamma_1'(s))^2 + (\gamma_2'(s))^2 \neq 0$ for every $s \in [a,b]$. For convenience in the sequel we always assume γ to be a $1-$periodic function on $\mathbf{R}$. Moreover, we also assume that the origin belongs to D.

Let Δ denote the Laplace operator, i.e.,

$$\Delta U(x,y) = \frac{\partial^2 U}{\partial x^2}(x,y) + \frac{\partial^2 U}{\partial y^2}(x,y), \quad (x,y) \in D. \tag{1}$$

It is well known that a vast number of problems in applied sciences can be reduced to the biharmonic equation

$$\Delta^2 U(x,y) = 0, \quad (x,y) \in D \tag{2}$$

with appropriately chosen boundary conditions for the function U. For instance, the problem of bending elastic clamped plates, the equilibrium of elastic bodies, the flow of viscous fluids, are all of this type, [**7–10, 13**].

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2. Reduction to a boundary value problem for analytic functions. In 1934, Muskhelishvili proposed a method for solving problems of the plane theory of elasticity. The method turned out to be very efficient in studying many other problems. Let us explain it on an example.

Lemma 1. *Let U be a solution of the biharmonic problem*

$$\Delta^2 U|_D = 0, \quad \frac{\partial U}{\partial x}\bigg|_\Gamma = G_1, \quad \frac{\partial U}{\partial y}\bigg|_\Gamma = G_2. \tag{3}$$

Then U can be represented in the form

$$\begin{gathered} U\,(x,y) = \operatorname{Re}\,[(x - i\,y)\,\phi\,(x + i\,y) + \chi\,(x + i\,y)]\,, \\ i^2 = -\,1, \quad (x,y) \in D, \end{gathered} \tag{4}$$

where ϕ and χ are analytic functions in the domain D satisfying the boundary condition

$$\phi\,(t) + t\overline{\phi'\,(t)} + \overline{\chi'\,(t)} = f\,(t)\,, \quad t = x + i\,y \in \Gamma, \tag{5}$$

and where $f(t) = G_1(t) + i\,G_2(t)$.

Proof. As usual, we assume that U, together with its partial derivatives, is continuously extendable on Γ. It is well known that any biharmonic function allows the representation (4) with some analytic functions ϕ and χ. Thus the problem is to find ϕ and χ to satisfy boundary conditions (3).

Let $\phi = u + iv$ and $z = x + iy$, with $u = \operatorname{Re}\phi$, $v = \operatorname{Im}\phi$. Then

$$\begin{aligned} U\,(x,y) &= \operatorname{Re}\,[\bar{z}\cdot(u + iv)\,(z) + \chi\,(z)] \\ &= x\cdot u\,(x,y) + y\cdot v\,(x,y) + \operatorname{Re}\,[\chi\,(x,y)]\,, \quad z \in D. \end{aligned}$$

From this, it immediately follows that

$$\begin{aligned} \frac{\partial U}{\partial x} &= u + x\,\frac{\partial u}{\partial x} + y\,\frac{\partial v}{\partial x} + \frac{\partial \operatorname{Re}\,[\chi]}{\partial x}, \\ \frac{\partial U}{\partial y} &= x\,\frac{\partial u}{\partial y} + v + y\,\frac{\partial v}{\partial y} + \frac{\partial \operatorname{Re}\,[\chi]}{\partial y}. \end{aligned}$$

Letting z tend to Γ and taking into account the Cauchy-Riemann equations and boundary conditions (3), we obtain

$$\begin{aligned}\phi\,(t) + t\overline{\phi'\,(t)} + \overline{\chi'\,(t)} &= \phi + x\left(\frac{\partial u}{\partial x} - i\,\frac{\partial v}{\partial x}\right) + i\,y\left(\frac{\partial v}{\partial y} - i\,\frac{\partial v}{\partial x}\right)\\ &\quad + \frac{\partial\,\mathrm{Re}\,[\chi]}{\partial x} - i\,\frac{\partial\,\mathrm{Im}\,[\chi]}{\partial x}\\ &= (u+iv) + x\left(\frac{\partial u}{\partial x} + i\,\frac{\partial u}{\partial y}\right) + y\left(\frac{\partial v}{\partial x} + i\,\frac{\partial v}{\partial y}\right)\\ &\quad + \frac{\partial\,\mathrm{Re}\,[\chi]}{\partial x} + i\,\frac{\partial\,\mathrm{Re}\,[\chi]}{\partial y}\\ &= \frac{\partial U}{\partial x} + i\,\frac{\partial U}{\partial y} = G_1\,(t) + i\,G_2\,(t)\,, \quad t\in\Gamma.\end{aligned}$$

Remark. If, instead of (3), we consider the Dirichlet problem

$$\Delta^2 U|_D = 0, \quad U|_\Gamma = f_1, \quad \left.\frac{\partial U}{\partial \bar n}\right|_\Gamma = f_2, \tag{6}$$

where $\bar n$ denotes the unit normal to Γ, then it reduces to the boundary problem (4) with the right-hand side given by

$$f\,(t) = e^{i\alpha(t)}\left[f_2\,(t) - i\,\frac{\partial f_1}{\partial \bar s}\,(t)\right], \quad t\in\Gamma,$$

where α denotes the angle between $\bar n$ and the real axis $\mathbf{R}$, and $\bar s$ is a unit vector such that the angle between $\bar s$ and $\mathbf{R}$ is $\pi/2-\alpha$.

Now let us assume that we are able to find the boundary value $\phi\,(t)$, $t\in\Gamma$, of the analytic function ϕ satisfying condition (5). Then the same condition allows us to express the boundary value $\chi'\,(t)$, $t\in\Gamma$, for the analytic function χ', viz.,

$$\chi'\,(t) = \overline{f\,(t)} - \overline{\phi\,(t)} - \bar t\,\phi'\,(t)\,. \tag{7}$$

Making the substitution $t=\gamma\,(\sigma)$ in (7) and multiplying the resulting expression by $\gamma'\,(\sigma)$, we obtain

$$\begin{aligned}\chi'\,(\gamma\,(\sigma))\,\gamma'\,(\sigma) &= \overline{f\,(\gamma\,(\sigma))}\,\gamma'\,(\sigma) - \overline{\phi\,(\gamma\,(\sigma))}\,\gamma'\,(\sigma)\\ &\quad - \overline{\gamma\,(\sigma)}\,\phi'\,(\gamma\,(\sigma))\,\gamma'\,(\sigma)\,.\end{aligned} \tag{8}$$

Taking integrals of both sides of (8) and using integration by parts for the last integral in (8), we get

$$
\begin{aligned}
\chi(\gamma(s)) = & \int_0^s \overline{f(\gamma(\sigma))}\,\gamma'(\sigma)\, d\sigma \\
& - \int_0^s \overline{\phi(\gamma(\sigma))}\,\gamma'(\sigma)\, d\sigma \\
& + \int_0^s \overline{\gamma'(\sigma)}\,\phi(\gamma(\sigma))\, d\bar{\sigma} \\
& - \overline{\gamma(s)}\,\phi(\gamma(s)) + \overline{\gamma(0)}\,\phi(\gamma(0)) + c,
\end{aligned} \tag{9}
$$

where $c \in \mathbf{C}$ is an arbitrary constant. If we denote by $t = \gamma(s)$ and by Γ_t the arc of Γ joining the points $t_0 = \gamma(0)$ and $t = \gamma(s)$, then representation (9) takes the form

$$
\begin{aligned}
\chi(t) = & \int_{\Gamma_t} \overline{f(\tau)}\, d\tau - \int_{\Gamma_t} \overline{\phi(\tau)}\, d\tau \\
& + \int_{\Gamma_t} \phi(\tau)\, d\bar{\tau} - \bar{t}\,\phi(t) + \bar{t}_0\,\phi(t_0) + C,
\end{aligned} \tag{10}
$$

where C is a complex constant. Thus, having obtained the boundary representations $\phi(t)$ and $\chi(t)$, $t \in \Gamma$, for the analytic functions ϕ and χ one can retrieve them from

$$
\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\tau)\, d\tau}{\tau - z}, \quad \chi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\chi(\tau)\, d\tau}{\tau - z}, \quad z \in D \tag{11}
$$

and using the Goursat representation (4) one can get a solution of the biharmonic problem (3). However, as a rule, a solution of the boundary problem (5) can be found only in a restricted number of cases. Therefore, the task is to find an approximate solution for (5). The first step in this direction was made by Muskhelishvili himself, cf. [**8, 9**]. Assuming Γ is the unit circle, he proposed looking for a solution in the form of an infinite Fourier series. Finite sections of the series ought to provide approximations to an exact solution. The subsequent research concentrated on this idea using conformal mapping, either exact or approximate, in the case where Γ differs from the unit circle [**2, 3, 6**]. However, the operator corresponding to the boundary problem (5) is not invertible and it is well known that for such operators the finite section method is not stable. Hence, further efforts were needed to make suitable changes of the domain and co-domain to achieve the invertibility of the corresponding operator. In contrast to this approach, we prefer to work in the same space but to use an integral equation for the unknown function ϕ. Though the corresponding integral operator is still not invertible, it can be corrected by adding a compact operator. The newly obtained operator perfectly satisfies all our needs. It is already invertible and the solution of the corresponding integral equation gives us the boundary value for the function ϕ in (5). Therefore, we are able to apply a spline collocation method to the integral equation and to show that this method is stable in the space of continuous functions $C(\Gamma)$. Using this result we construct approximate solutions for the biharmonic problem and give error estimates in the uniform metric norm. Note that in [**4**] the stability of quadrature and Galerkin methods in $L_2(\Gamma)$ for the Muskhelishvili equation on curves with corner points was established. The approximation methods in [**4**] are based on piecewise constant splines and the presence of corner points on the contour requires additional conditions for the corresponding methods to be stable. However, as it is shown later, there is *no stability condition whatsoever* if Γ is a smooth contour. Under a suitable choice of collocation points indeed, the approximation method considered here is *always* stable. In addition, we give examples which illustrate the numerical performance of the method.

3. Reduction to an integral equation. Let us consider the boundary value problem

$$\overline{\phi\,(t)} + \bar{t}\,\phi'\,(t) + \chi'\,(t) = \overline{f\,(t)}. \tag{12}$$

It is well known, [**8**–**10**], that if ϕ is a solution of (12) then it satisfies the following integral equation

$$\begin{aligned} -\,\overline{\phi\,(t)} - \frac{1}{2\pi i}\int_\Gamma \overline{\phi\,(\tau)}\, d\,\log\,\frac{\bar\tau-\bar t}{\tau-t} \\ - \frac{1}{2\pi i}\int_\Gamma \phi\,(\tau)\; d\,\frac{\bar\tau-\bar t}{\tau-t} = f_0\,(t)\,, \quad t\in\Gamma, \end{aligned} \tag{13}$$

where

$$f_0\,(t) = -\,\frac{1}{2}\,\overline{f\,(t)} + \frac{1}{2\pi i}\int_\Gamma \frac{\overline{f\,(\tau)}}{\tau-t}\,d\tau.$$

Unfortunately, the operator corresponding to the left-hand side of (13) is not invertible. Hence, none of the widely used projection approximation methods can be applied to equation (13) directly. However, equation (13) admits a correction which allows us to achieve two goals. First, to obtain an invertible operator, and second, to find an approximate for ϕ.

We say that the curve Γ belongs to the class $\mathcal{C}^{(k)}$, $k \in \mathbf{N}$, if its parametrization γ is a k times continuously differentiable function.

Theorem 1. *Let $\Gamma \in \mathcal{C}^{(2)}$. Then the integral operators*

$$\begin{aligned}
(T_1\phi)(t) &= \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau}\,d\tau, \\
(T_2\phi)(t) &= \frac{1}{2\pi i}\int_\Gamma \left[\frac{\phi(\tau)}{\tau^2} + \frac{\overline{\phi(\tau)}}{\bar{\tau}^2}\right] d\tau, \\
(T_3\phi)(t) &= -\frac{1}{2\pi i}\int_\Gamma \overline{\phi(\tau)}\, d\log\frac{\bar{\tau}-\bar{t}}{\tau - t}, \\
(T_4\phi)(t) &= -\frac{1}{2\pi i}\int_\Gamma \phi(\tau)\; d\,\frac{\bar{\tau}-\bar{t}}{\tau - t},
\end{aligned} \tag{14}$$

are compact on $C(\Gamma)$.

Proof. We consider the integral operators in (14) and show that each of them is compact on $C(\Gamma)$. Indeed, the kernels of the operators $T_1\phi$, $T_2\phi$ are continuous on $\Gamma \times \Gamma$, hence T_1 and T_2 are compact. The kernel $K(t, \tau)$ of the operator T_3

$$(T_3\phi)(t) = -\frac{1}{2\pi i}\int_\Gamma K_1(t,\tau)\,\overline{\phi(\tau)}\,d\tau$$

has the form

$$K_1(t,\tau) = -\frac{1}{2\pi i}\left(\frac{1}{\bar{\tau}-\bar{t}}\frac{d\bar{\tau}}{d\tau} - \frac{1}{\tau - t}\right)$$

and is continuous if $\tau \neq t$. On the other hand, if we set $t = \gamma(s)$, $\tau = \gamma(\sigma)$, then

$$K_1(t,\tau) = -\frac{1}{2\pi}\,\frac{\operatorname{Im}\left(\overline{\gamma''(\sigma)}\gamma'(\sigma)\right) + o(1)}{\left|\gamma'(\sigma)\right|^2}$$

as $\sigma \to s$. Hence under the conditions of Theorem 1, $K_1(t,\tau)$ is continuous for all $t,\tau \in \Gamma$. Therefore, the operator T_3 is compact on $C(\Gamma)$. Compactness of the remaining integral operator T_4 in (14) can be proved analogously. Let $W_2^1(\Gamma)$ refer to the Sobolev space, and let R be defined by

$$R\phi(t) \equiv -\overline{\phi(t)} - \frac{1}{2\pi i}\int_\Gamma \overline{\phi(\tau)}\, d\log\frac{\bar\tau-\bar t}{\tau-t} - \frac{1}{2\pi i}\int_\Gamma \phi(\tau)\, d\,\frac{\bar\tau-\bar t}{\tau-t} + \frac{1}{2\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau}\, d\tau + \frac{1}{t}\,\frac{1}{2\pi i}\int_\Gamma \left(\frac{\phi(\tau)}{\tau^2}\, d\tau + \frac{\overline{\phi(\tau)}}{\bar\tau^2}\, d\bar\tau\right).$$

Theorem 2. *Let $\Gamma \in \mathcal{C}^{(2)}$. Then the operator $R : C(\Gamma) \to C(\Gamma)$ is invertible. Moreover, if $f \in W_2^1(\Gamma)$ and satisfies the condition*

$$\operatorname{Re}\int_\Gamma \overline{f(\tau)}\, d\tau = 0, \tag{15}$$

then the solution of the equation

$$R\phi = f_0 \tag{16}$$

is simultaneously a solution of equation (13) *and of the boundary value problem* (12).

Proof. Since all integral operators appearing in R are compact the operator $R : C(\Gamma) \to C(\Gamma)$ is Fredholm and its index is equal to zero. On the other hand, it follows from [**4**] that R considered on the space $L_2(\Gamma)$ is invertible. Taking into account that the space $C(\Gamma)$ is dense in the space $L_2(\Gamma)$, one obtains that the dimensions of the kernels of R on the spaces $C(\Gamma)$ and $L_2(\Gamma)$ coincide [**5**]. Therefore, $\dim\ker R|_{C(\Gamma)} = 0$, and this implies the invertibility of R on $C(\Gamma)$.

The second assertion of Theorem 2 follows immediately from [**4**]. The above results can be used for the approximate solution of the Muskhelishvili equation and, subsequently, for the approximate solution of the biharmonic problem.

4. Error estimates. Before we start with approximate solution of the Muskhelishvili equation, we would like to give some estimates for the errors appearing in the approximation of the solutions of the biharmonic problem. We will use the scheme described above.

Assume we have available an approximate solution ϕ_n of the Muskhelishvili equation (13), and let

$$\|\phi - \phi_n\|_C \leq \varepsilon_n, \quad n \geq n_0. \tag{17}$$

Using formulae (8) and (9), we can also get an approximation for the function χ, viz.,

$$\begin{aligned}\chi_n(t) = \int_{\Gamma_t} \overline{f(\tau)}\, d\tau - \int_{\Gamma} \overline{\phi_n(\tau)}\, d\tau \\ + \int_{\Gamma} \phi_n(\tau)\; d\bar{\tau} - \bar{t}\phi_n(t) + \bar{t}_0\phi_n(t_0) + C.\end{aligned}$$

Comparing this representation with (17) one can easily find

$$\|\chi_n - \chi\|_C \leq d_1\varepsilon_n \tag{18}$$

where d_1 is a constant independent of n. Then one can set

$$\tilde{\phi}_n(z) \equiv \frac{1}{2\pi i}\int_{\Gamma} \frac{\phi_n(\tau)\, d\tau}{\tau - z}, \quad \widetilde{\chi}_n(z) \equiv \frac{1}{2\pi i}\int_{\Gamma} \frac{\chi_n(\tau)\, d\tau}{\tau - z}, \quad z \in D \tag{19}$$

and subsequently, by the Goursat representation (4),

$$U_n(x,y) \equiv \operatorname{Re}\left[\bar{z}\tilde{\phi}_n(z) + \widetilde{\chi}_n(z)\right], \quad z = x + i\, y \in D. \tag{20}$$

Theorem 3. *Let the approximate solution* ϕ_n, $n \geq n_0$ *of the Muskhelishvili equation* (13) *satisfy inequality* (17). *Then for any compact subset* K *of* D*, the approximate solution* (20) *of the biharmonic problem satisfies the estimate*

$$\sup_{(x,y)\in K} |U_n(x,y) - U(x,y)| \leq d_2\, \varepsilon_n, \quad n \geq n_0 \tag{21}$$

where d_2 *is independent of* $(x,y) \in K$ *and* n *and* U *is given by* (4).

The proof is straightforward. Using the above construction we have

$$\begin{aligned}
&|U_n(x,y) - U(x,y)| \\
&\qquad \le \frac{|z|}{2\pi} \int_\Gamma \frac{|\phi_n(\tau) - \phi(\tau)|}{|\tau - z|}\,|d\tau| + \frac{1}{2\pi} \int_\Gamma \frac{|\chi_n(\tau) - \chi(\tau)|}{|\tau - z|}\,|d\tau| \\
&\qquad \le \frac{|\Gamma|}{2\pi \operatorname{dist}(K,\Gamma)} \Big[\max_{z\in\Gamma} |z| + d_1\Big]\varepsilon_n
\end{aligned}$$

where $|\Gamma|$ stands for the length of Γ and d_1 is defined in (18).

Remark. Estimate (21) contains the constant $[\operatorname{dist}(K,\Gamma)]^{-1}$ which grows if the boundary of K tends to Γ. The estimate can be improved if it is known that the functions $\phi_n(t)$, $\chi_n(t)$, $n \ge n_0$, belong to a subspace $W(\Gamma)$ of $C(\Gamma)$ such that the Cauchy integral operator S

$$Sx(t) \equiv \frac{1}{\pi i} \int_\Gamma \frac{x(\tau)\,d\tau}{\tau - t}, \quad t \in \Gamma$$

is bounded on $W(\Gamma)$, i.e., if there exists a constant d_3 such that

$$\|Sx\|_C \le d_3 \|x\|_C \tag{22}$$

for any $x \in W(\Gamma)$. Then, instead of ϕ_n and χ_n, one can use in (19) the functions

$$\widehat{\phi}_n(t) = (P\phi_n)(t), \quad \widehat{\chi}_n(t) = (P\chi_n)(t) \tag{23}$$

with $P = 1/2\,(I + S)$. Since $\widehat{\phi}_n$ and $\widehat{\chi}$ are boundary values of analytic functions in D, we have

$$\begin{aligned}
\|\phi - \widehat{\phi}_n\|_C &= \|P(\phi - \phi_n)\|_C \le d_3 \|\phi - \phi_n\|_C, \\
\|\chi - \widehat{\chi}_n\|_C &= \|P(\chi - \chi_n)\|_C \le d_3 \|\chi - \chi_n\|_C.
\end{aligned}$$

An approximate solution for the biharmonic problem can now be constructed as follows

$$\widehat{U}_n(x,y) = \operatorname{Re}\Big[\bar{z}\widetilde{\widehat{\phi}_n}(z) + \widetilde{\widehat{\chi}_n}(z)\Big].$$

However, to estimate the error now one can use the maximum principle for analytic functions. This gives us

$$\sup_{(x,y)\in D} \left|\widehat{U}_n\,(x,y) - U\,(x,y)\,\right| \le d_4\,\varepsilon_n. \tag{24}$$

5. Approximate solution of the Muskhelishvili equation. We use equation (16) to find approximations for the boundary function ϕ. Let $\eta = \eta\,(s)$, $s \in \mathbf{R}$, refer to the characteristic function of the interval $[0,1)$, i.e.,

$$\eta\,(s) = \begin{cases} 1 & x \in [0,1), \\ 0 & \text{otherwise.} \end{cases}$$

For any natural number d, consider a function $\eta^{(d)} : \mathbf{R} \to \mathbf{R}$ defined by

$$\eta^{(d)}\,(s) = \int_{\mathbf{R}} \eta\,(s-\sigma)\,\eta^{(d-1)}\,(\sigma)\;d\sigma,$$

where $\eta^{(0)}\,(s) \equiv \eta\,(s)$, $s \in \mathbf{R}$. Then for any fixed $n \in \mathbf{N}$ and for any $d \ge 1$, the set

$$S_n^d\,(\mathbf{R}) \equiv \operatorname{clos}_{C(\mathbf{R})} \operatorname{span} \left\{\widetilde{\eta}_{jn}\,(s) : \widetilde{\eta}_{jn}\,(s) = \eta^d\,(ns-j)\,,\ j \in \mathbf{Z}\right\}$$

forms a spline subspace of $C\,(\mathbf{R})$, [**12**].

We fix $d \ge 1$ and consider a corresponding spline space on Γ. The latter is constructed in the following way. If γ is a $1-$periodic parametrization of Γ then for any t on Γ one puts

$$\eta_{jn}\,(t) = \widetilde{\eta}_{jn}\,(s)\,, \quad t = \gamma\,(s)\,, \quad s \in \mathbf{R}, \quad j \in \mathbf{Z}.$$

The corresponding spline space is denoted by $S_n^d \equiv S_n^d\,(\Gamma)$. It is obvious that the functions η_{jn}, $j = 0, 1, \dots, n-1$, form a basis in S_n^d.

Let δ be a real number $0 \le \delta < 1$. Consider the points

$$t_j^{(n)} \equiv \gamma\left(\frac{j+\delta}{n}\right), \quad j = 0, 1, \dots, n-1.$$

The approximate solution ϕ_n of the equation (16) is sought in the form

$$\phi_n\,(t) = \sum_{k=0}^{n-1} c_k^{(n)} \eta_{kn}\,(t)\,, \quad t \in \Gamma. \tag{25}$$

The coefficients of ϕ_n are found from the system of algebraic equations

$$R\phi_n\big(t_j^{(n)}\big) = f_0\big(t_j^{(n)}\big), \quad j = 0, 1, \dots , n-1. \tag{26}$$

Theorem 4. *Let $\Gamma \in \mathcal{C}^{(2)}$ and $\delta \neq 1/2$ if d is even or $\delta \neq 0$ if d is odd. If $f \in W_2^1(\Gamma)$ and satisfies condition* (15)*, then there exists a number $n_0 \in \mathbf{N}$ such that for all $n \geq n_0$ equations* (26) *are solvable and the approximate solutions* (25) *converge to an exact solution of equation* (13) *in the norm of $C(\Gamma)$.*

Proof. It is well known, cf. [**11**], that under conditions of Theorem 3 there exist interpolation projections $K_n^\delta : C(\Gamma) \to S_n^d(\Gamma)$ with the property

$$K_n^\delta \phi\big(t_j^{(n)}\big) = \phi\big(t_j^{(n)}\big), \quad j = 0, 1, \dots , n-1$$

for any $\phi \in C(\Gamma)$. Therefore, the algebraic equations (26) are equivalent to the operator equations

$$K_n^\delta R\phi_n = K_n^\delta f_0, \quad n \in \mathbf{N}.$$

The latter equations can be written in the form

$$K_n^\delta R\phi_n = -\overline{\phi}_n + K_n^\delta (T_1 + T_2 + T_3 + T_4)\, \phi_n = K_n^\delta f_0.$$

Now, taking into account the strong convergence of K_n^δ to the identity operator on $C(\Gamma)$, Theorems 1 and 2 and the results of [**11**, pp. 25–30], we deduce that the sequence $\{K_n^\delta R\}$ is stable, i.e., there exists an integer $n_0 \in \mathbf{N}$ such that the operators $K_n^\delta R : S_n^d(\Gamma) \to S_n^d(\Gamma)$ are invertible and the norms of their inverses are uniformly bounded. The last fact also implies the convergence of ϕ_n to the exact solution ϕ of (16). By Theorem 2, ϕ is a solution of (13) as well. This completes the proof. $\square$

Remark. From these considerations it follows that the approximate solution ϕ_n obtained by the above collocation method can be used for calculating the approximate solution of the biharmonic problem.

6. Numerical considerations. The numerical scheme proceeds as follows. Given the boundary values G_1 and G_2 of (3), one calculates the right-hand side of (5), i.e., of the boundary value problem (12). The function f_0 can then be evaluated, using for instance the elementary approach described in [**1**], so that the forcing function in (13) is thus known. Replacing now the noninvertible singular integral operator given by the left-hand side of (13) by its invertible counterpart R, see Theorem 2, it is now possible to solve (16) for the function $\phi(t)$, $t \in \Gamma$, using the collocation method described in Section 5, i.e., we solve system (26). The computation of values at inner points of ϕ and χ is achieved again with elementary quadratures via the formulae (11). The final evaluation of the solution at the inner grid nodes can now be obtained using formula (4).

The examples given as well as the related figures empirically illustrate the numerical performance of the algorithm. In all examples the order of the splines used is $d = 2$. Also, the curve Γ is always the unit circle. The collocation points are chosen with different values of δ to demonstrate that the choice of this parameter does not influence the conditioning of the resulting system away from the forbidden value $\delta = 0.5$. The number of basis elements n instead ranges in the first example from 8 to 64 and in the second one from 16 to 128. The rows in the figures correspond respectively to the analytic solution, to the computed solution and to the contour plots of the absolute error. The tables show the excellent conditioning of the algorithm. To study the convergence, in the polar coordinate plane we use a rectangular grid G, with 25 points both in the radial as well as in the angular directions. The convergence is empirically determined from the absolute error $\|U - U_n\|_{G,\infty}$ calculated at the grid points of the grid G. The results of the tables show it to be in line with the theory.

Example 1. Here we solve the problem with the boundary functions $G_1 = 2x$, $G_2 = 2y$, with an analytic solution (up to an arbitrary constant) given by $U = x^2 - y^2 + 1$. The behavior of the numerically evaluated solution is illustrated in Figure 1. Table 1 contains the conditioning and the numerical values of the error.

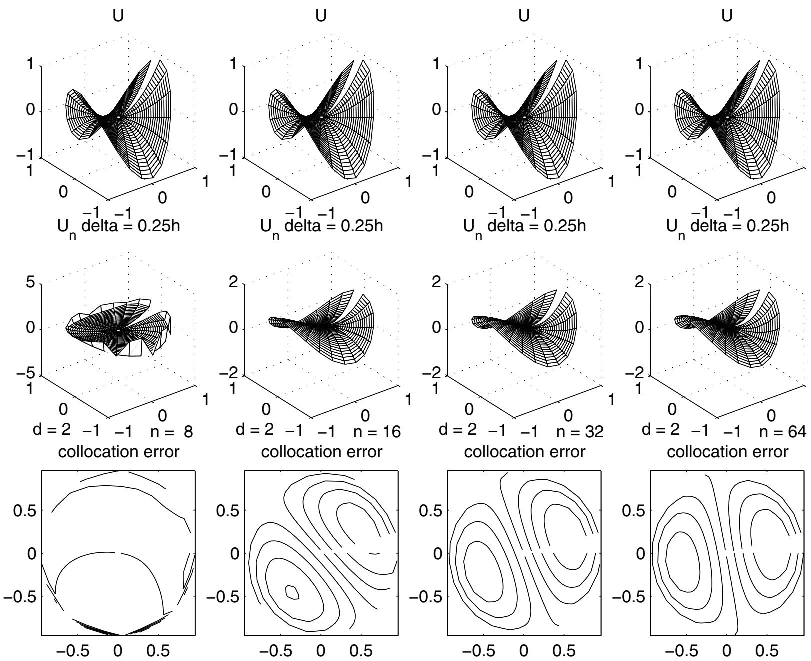

FIGURE 1. Analytic solution (first row) and numerical solution (second row) and contour plot of absolute error (third row) of Example 1.

TABLE 1. Results of Example 1, $\delta = 0.25$.

n	conditioning	$\|U - U_n\|_{G,\infty}$
8	15.85	0.14291
16	15.90	0.04293
32	15.92	0.00898
64	15.92	0.00370

Example 2. The relevant functions in this case are $G_1 = 4x^3 - 12xy^2$, $G_2 = 4y^3 - 12x^2y$, with an analytic solution given by $U =$

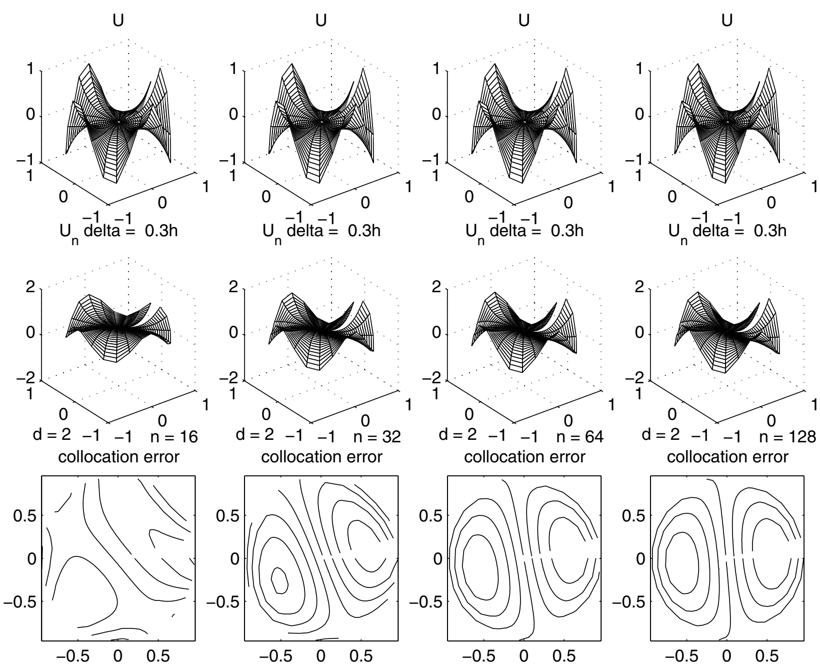

FIGURE 2. Analytic solution (first row) and numerical solution (second row) and contour plot of absolute error (third row) of Example 2.

$x^4 - 6x^2y^2 + y^4$ and results plotted in Figure 2. Conditioning and behavior of the error are found in Table 2.

TABLE 2. Results of Example 2, $\delta = 0.3$.

n	conditioning	$\|U - U_n\|_{G,\infty}$
16	19.88	7.2196 E-2
32	19.89	4.1258 E-2
64	19.90	1.3493 E-2
128	19.90	5.8315 E-3

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