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VARIABLE COEFFICIENT TRANSMISSION PROBLEMS AND SINGULAR INTEGRAL OPERATORS ON NON-SMOOTH MANIFOLDS

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Dedicated to Professor K. Atkinson in recognition of
his many contributions to the field of integral equations

1. Introduction. In this paper we discuss a new approach and an extension of the results in [**11**] regarding transmission boundary value problems and spectral theory for singular integral operators on Lipschitz domains. The main novelty here is the consideration of *variable coefficient* operators and systems which, in turn, requires a change in the strategy employed in [**11**]. In that paper, an approach based on the Serrin-Weinberger asymptotic theory, akin to the influential work of Dahlberg and Kenig [**9**], has been used. By further building on the work in [**11, 20, 34, 44**], here we develop an alternative approach, based on the regularity of the Neumann function, which is capable of handling variable coefficient operators of Schrödinger type on Lipschitz subdomains of Riemannian manifolds. One key feature of this approach is that it avoids the discussion of the asymptotic behavior at infinity for solutions of elliptic PDE's with bounded, measurable coefficients. In order to be more specific we shall now introduce some notation, starting with the geometric setting we have in mind.

Assume that $\mathcal{M}$ is a compact Riemannian manifold, of real dimension $n := \dim \mathcal{M} \geq 2$, equipped with a Lipschitz metric tensor $\mathfrak{g} := \sum g_{jk} dx_j \otimes dx_k$. Throughout the paper we let $d\mathcal{V} := g^{1/2} dx_1 \dots dx_n$, where $g := \det g_{jk}$, be the volume element on $\mathcal{M}$, and denote by

$$(1.1) \qquad \Delta\, u := g^{-1/2} \sum_{j,k} \partial_j \big(g^{jk} g^{1/2}\, \partial_k u \big), \qquad (g^{jk})_{jk} := (g_{jk})^{-1}_{jk},$$

the Laplace-Beltrami operator on $\mathcal{M}$.

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Recall that a domain $\Omega \subset \mathcal{M}$ is called *Lipschitz* provided its boundary is given in local systems of coordinates by graphs of real-valued Lipschitz functions. If Ω is a Lipschitz domain in $\mathcal{M}$, we denote by $d\sigma$ the surface measure on $\partial\Omega$, and by ν the outward unit conormal defined almost everywhere (with respect to $d\sigma$) on $\partial\Omega$. Also, we set $\Omega_+ := \Omega$, $\Omega_- := \mathcal{M} \setminus \overline{\Omega}$, and pick some $V \in L^\infty(\mathcal{M})$ which satisfies $V \geq 0$ on $\mathcal{M}$ and $V > 0$ on some set of positive measure in each connected component of Ω_+ and Ω_-. In particular, with $L^p_s(\mathcal{M})$, $1 < p < \infty$, $s \in [-1, 1]$, denoting the usual Sobolev scale on $\mathcal{M}$, the operator

$$\Delta - V : L^2_1(\mathcal{M}) \longrightarrow L^2_{-1}(\mathcal{M}) \tag{1.2}$$

is bounded and, in fact, invertible provided $V \neq 0$. When $V = 0$, the inverse Δ^{-1} should be understood as

$$\Delta^{-1} : \{u \in L^2_{-1}(\mathcal{M}) : \langle u, 1\rangle = 0\} \longrightarrow L^2_1(\mathcal{M})/\mathbf{R}. \tag{1.3}$$

In either case, we denote by $E_V(x, y)$ the Schwartz kernel of $(\Delta - V)^{-1}$. It follows that $E_V(y, x) = E_V(x, y)$. Next, corresponding to the Lipschitz domain $\Omega \subset \mathcal{M}$, we introduce the single and double layer potential operators, by

$$\mathcal{S}_V f(x) := \int_{\partial\Omega} E_V(x, y)\, f(y)\, d\sigma_y, \qquad x \notin \partial\Omega, \tag{1.4}$$

$$\mathcal{D}_V f(x) := \int_{\partial\Omega} \partial_{\nu_y} E_V(x, y)\, f(y)\, d\sigma_y, \quad x \notin \partial\Omega. \tag{1.5}$$

Here, ∂_ν is the conormal derivative associated with the metric g. The boundary versions of these operators are

$$S_V f(x) := \int_{\partial\Omega} E_V(x, y)\, f(y)\, d\sigma_y, \qquad x \in \partial\Omega, \tag{1.6}$$

$$K_V f(x) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_y} E_V(x, y)\, f(y)\, d\sigma_y, \quad x \in \partial\Omega, \tag{1.7}$$

where p.v. indicates that the integral is taken in the principal value sense, i.e., removing small geodesic balls and passing to the limit. These operators have been studied in some detail in [**34**], where it has been proved that they satisfy many of the most important properties of their 'flat-space' Laplacian counterparts. What enables us to make this extension is a key result proved in [**34**] to the effect that, in local coordinates in which the metric tensor $\mathfrak{g}$ is given by $\sum g_{jk}dx_j \otimes dx_k$, the following asymptotic expansion holds:

$$E_V(x,y) = [\det(g_{jk}(y))]^{-1/2}[e_0(x-y,y) + e_1(x,y)], \tag{1.8}$$

where the main term is given by

$$e_0(z,y) := C_n\Big(\sum g_{jk}(y)z_j z_k\Big)^{-(n-2)/2}, \tag{1.9}$$

and the remainder satisfies

$$|\nabla_x^j \nabla_y^k e_1(x,y)| \le C|x-y|^{-j-k}, \quad 0 \le j,\, k \le 1. \tag{1.10}$$

As a result, the classical Calderón-Zygmund theory applies and yields:

$$\begin{aligned} \partial_\nu \mathcal{S}_V\big|_{\partial\Omega_\pm} &= \mp\tfrac{1}{2}I + K_V^*, & \nabla_{\tan}\mathcal{S}_V\big|_{\partial\Omega_+} &= \nabla_{\tan}\mathcal{S}_V\big|_{\partial\Omega_-}, \\ \mathcal{D}_V\big|_{\partial\Omega_\pm} &= \pm\tfrac{1}{2}\,I + K_V, & \mathcal{S}_V\big|_{\partial\Omega_+} &= \mathcal{S}_V\big|_{\partial\Omega_-} =: S_V, \end{aligned} \tag{1.11}$$

where I denotes the identity operator and K_V^* is the formal adjoint of K_V. Here and elsewhere, $\nabla_{\tan} := \nabla - \nu\partial_\nu$ stands for the tangential gradient on $\partial\Omega$.

Fix some $\kappa = \kappa(\partial\Omega) > 1$ sufficiently large and define the nontangential maximal operator M acting on an arbitrary $u : \Omega_\pm \to \mathbf{R}$ by

$$M(u)(x) := \sup\{|u(y)| : \, y \in \Gamma^\pm(x)\}, \quad x \in \partial\Omega, \tag{1.12}$$

where

$$\Gamma^\pm(x) := \{y \in \Omega_\pm : \, \operatorname{dist}(x,y) < \kappa \operatorname{dist}(y,\partial\Omega)\}, \quad x \in \partial\Omega, \tag{1.13}$$

are nontangential approach regions (lying in Ω_+ and Ω_-, respectively). These cone-like regions also play a role in defining nontangential restrictions to the boundary, i.e.,

$$u\big|_{\partial\Omega}(x) := \lim_{y\in\Gamma^\pm(x)} u(y), \quad \text{for a.e.} \quad x \in \partial\Omega, \tag{1.14}$$

the choice of the sign depending on whether the function u is defined in Ω_+ or Ω_-.

For $1 < p < \infty$ we denote by $L^p(\partial\Omega)$ the Lebesgue space of the measurable, pth power integrable functions on $\partial\Omega$, with respect to the surface measure $d\sigma$. The Sobolev space of order one is then defined as

$$L^p_1(\partial\Omega) := \{f \in L^p(\partial\Omega) : |\nabla_{\tan} f| \in L^p(\partial\Omega)\}. \tag{1.15}$$

We equip it with the natural norm, i.e., $\|f\|_{L^p_1(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \|\nabla_{\tan} f\|_{L^p(\partial\Omega)}$.

The theorem below, dealing with the well-posedness of the transmission problem for the Laplace-Beltrami operator across Lipschitz interfaces, is indicative of the type of results we wish to establish in this paper. See also Theorem 5.6 in the body of the paper for a related version.

Theorem 1.1. *Assume that the manifold $\mathcal{M}$, the metric $\mathfrak{g}$, the Lipschitz domain $\Omega \subset \mathcal{M}$ and the potential V are as above and fix a parameter $\mu \in (0,1)$. Then there exists $\varepsilon = \varepsilon(\mathfrak{g}, \partial\Omega, V, \mu) > 0$ such that the transmission boundary value problem*

$$\begin{cases} (\Delta - V)u^+ = 0 & \text{in } \Omega_+, \\ (\Delta - V)u^- = 0 & \text{in } \Omega_-, \\ M(\nabla u^\pm) \in L^p(\partial\Omega), \\ u^+\big|_{\partial\Omega} - u^-\big|_{\partial\Omega} = f \in L^p_1(\partial\Omega), \\ \partial_\nu u^+ - \mu\, \partial_\nu u^- = g \in L^p(\partial\Omega), \end{cases} \tag{1.16}$$

has a unique solution provided that $1 < p < 2 + \varepsilon$. In addition, this solution satisfies

$$\|M(\nabla u^\pm)\|_{L^p(\partial\Omega)} \le C\left(\|f\|_{L^p_1(\partial\Omega)} + \|g\|_{L^p(\partial\Omega)}\right) \tag{1.17}$$

and has an integral representation formula in terms of the operators (1.4)–(1.7).

The proof of this theorem occupies Sections 2–4 of the paper. At the heart of the matter is the regularity of the Neumann functions naturally associated with the problem (1.16), which is a topic addressed in Section 2.

In Section 5, the discussion is centered around the issue of Fredholmness and invertibility for operators defined by singular integrals of layer potential type. Our results in this regard cover a wide range of spaces, including Lebesgue, Sobolev, Hardy and Besov scales defined on boundaries of Lipschitz domains. This is done via interpolation and a general functional analytic scheme based on stability and extrapolation. In turn, these Fredholmness/invertibility results are used to treat inhomogeneous Laplace transmission problems in Lipschitz domains.

Extending the L^p-theory of transmission problems from single equations to systems of equations presents a whole new set of challenges, as many of the basic ingredients (most notably, the local Hölder regularity of weak solutions) cease to function in this context. Our second round of results in this paper deal with the case of *three-dimensional* electro-magnetic inverse scattering phenomena. In Section 6 we extend the scope of our earlier analysis by including systems of differential operators. The starting point is the study of L^p transmission problems for the Maxwell system. A key ingredient is the so-called magnetostatic operator, cf. (6.19), and we rely on certain operator theoretical identities linking this vector-valued object to the scalar harmonic layer potentials (treated in previous sections). Having dealt with the L^p-theory we also treat inhomogeneous problems for the Maxwell system with data in Sobolev-Besov spaces.

Most of the transmission problems considered in the literature fall under several categories, depending on the nature of the domain and solution. First, there is the class of problems in domains with sufficiently smooth boundaries (so that they can be flattened and/or pseudodifferential operator techniques, with a limited amount of smoothness, can be used). See, e.g., [**21**–**23**] for scalar equations and [**2, 3, 6, 24, 36, 38, 47**], for Maxwell's equations. Second, there is the class of problems in domains with isolated singularities (in which scenario, Mellin transforms are applicable); cf. [**37, 39**]. Weak (variational) solutions for transmission problems in Lipschitz domains are discussed in [**1, 41**]. Finally, strong solutions in Dahlberg's sense [**7, 8**] for transmission problems in Lipschitz domains are treated in [**10, 12, 29, 42**] for single equations and [**12, 29**] for systems (such as Lamé and Maxwell).

Compared to previous work on transmission problems, our results are the first to establish well-posedness and estimates for *optimal* ranges of indices (of Lebesgue, Sobolev and Besov spaces) in arbitrary Lipschitz domains and for *variable coefficient* operators.

2. Transmission Neumann functions. We consider here the case when $n = \dim \mathcal{M} \geq 3$; the situation when $n = 2$ is analogous, requiring only minor alterations, of technical character. In each case, we shall nonetheless specify how our main results should read when $n = 2$.

Fix a Lipschitz subdomain Ω of the manifold $\mathcal{M}$ and consider two pairs of functions,

$$(N^{+,+}(x,y), N^{-,+}(x,y))$$

and

$$(N^{+,-}(x,y), N^{-,-}(x,y)),$$

defined as follows. First, for each fixed $y \in \Omega_+$, the pair $(N^{+,+}(x,y), N^{-,+}(x,y))$ is defined as the (unique) solution of

$$\begin{cases} (\Delta_x - V(x))N^{+,+}(x,y) = \delta_x(y) & x \in \Omega_+, \\ (\mu\Delta_x - \mu V(x))N^{-,+}(x,y) = 0 & x \in \Omega_-, \\ N^{+,+}(\cdot,y)\big|_{\partial\Omega} = N^{-,+}(\cdot,y)\big|_{\partial\Omega}, & \\ \partial_\nu N^{+,+}(\cdot,y) = \mu\,\partial_\nu N^{-,+}(\cdot,y) & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

(here δ stands for the Dirac delta function), whereas for each fixed $y \in \Omega_-$, the pair $(N^{+,-}(x,y), N^{-,-}(x,y))$ is defined as the solution of

$$\begin{cases} (\Delta_x - V(x))N^{+,-}(x,y) = 0 & x \in \Omega_+, \\ (\mu\Delta_x - \mu V(x))N^{-,-}(x,y) = \delta_x(y) & x \in \Omega_-, \\ N^{+,-}(\cdot,y)\big|_{\partial\Omega} = N^{-,-}(\cdot,y)\big|_{\partial\Omega}, & \\ \partial_\nu N^{+,-}(\cdot,y) = \mu\,\partial_\nu N^{-,-}(\cdot,y) & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

The existence of such pairs is a consequence of the L^p theory with p near 2 from [**32**]. Then the usual integration by parts argument continues to work in this setting and yields the symmetry conditions

$$\begin{cases} N^{+,+}(y,x) = N^{+,+}(x,y) & \text{for each } x,y \in \Omega_+, \\ N^{-,-}(y,x) = N^{-,-}(x,y) & \text{for each } x,y \in \Omega_-, \\ N^{+,-}(y,x) = N^{-,+}(x,y), & \text{for each } x \in \Omega_-,\ y \in \Omega_+. \end{cases} \tag{2.3}$$

For $x \in \mathcal{M}$ and $y \in \mathcal{M} \setminus \partial\Omega$, we then set

$$N(x,y) := \begin{cases} N^{+,+}(x,y) & \text{if } x \in \Omega_+,\ y \in \Omega_+, \\ N^{+,-}(x,y) & \text{if } x \in \Omega_+,\ y \in \Omega_-, \\ N^{-,+}(x,y) & \text{if } x \in \Omega_-,\ y \in \Omega_+, \\ N^{-,-}(x,y) & \text{if } x \in \Omega_-,\ y \in \Omega_-, \end{cases} \tag{2.4}$$

i.e., with χ_A denoting the characteristic function of A,

$$N(x,y) = \sum_{j,k\in\{\pm\}} N^{j,k}(x,y)\chi_{\Omega_j}(x)\chi_{\Omega_k}(y), \tag{2.5}$$

where the summation is performed over all possible choices of the signs $j,k \in \{\pm\}$. From (2.4) it follows that

$$N(y,x) = N(x,y), \quad \text{for all} \quad x,y \in \mathcal{M} \setminus \partial\Omega. \tag{2.6}$$

To highlight the importance of this Neumann function in the context of the transmission problem (1.16) we note that successive integrations by parts, along with the symmetry formulas (2.4), show that if $u^\pm$ solve

$$\begin{cases} (\Delta - V)u^+ = 0 & \text{in } \Omega_+, \\ (\Delta - V)u^- = 0 & \text{in } \Omega_-, \\ M(\nabla u^\pm) \in L^2(\partial\Omega), \\ u^+\big|_{\partial\Omega} = u^-\big|_{\partial\Omega}, \\ \partial_\nu u^+ - \mu\,\partial_\nu u^- = f \in L^2(\partial\Omega), \end{cases} \tag{2.7}$$

then the following integral representation formulas hold:

$$u^+(x) = -\int_{\partial\Omega} N^{+,+}(x,y)\,f(y)\,d\sigma_y, \quad x \in \Omega_+, \tag{2.8}$$

$$u^-(x) = -\int_{\partial\Omega} N^{-,-}(x,y)\,f(y)\,d\sigma_y, \quad x \in \Omega_-. \tag{2.9}$$

Another way of introducing the Neumann kernel is as follows. Consider

$$L := \operatorname{div}(A\nabla\,), \qquad A := \chi_{\Omega_+} I + \mu\,\chi_{\Omega_-} I \tag{2.10}$$

where the gradient and the divergence are those associated with the Riemannian metric $\mathfrak{g}$ on $\mathcal{M}$. It is then not difficult to check that if

$$\widetilde{V} := \chi_{\Omega_+} V + \mu\, \chi_{\Omega_-} V, \tag{2.11}$$

then

$$L - \widetilde{V} : L^2_1(\mathcal{M}) \longrightarrow L^2_{-1}(\mathcal{M}) \tag{2.12}$$

is invertible. A direct calculation then shows that $N(x,y)$ introduced in (2.4)–(2.5) is the Schwartz kernel of $(L-\widetilde{V})^{-1}$. In particular,

$$\begin{aligned}(L-\widetilde{V})w = F &\Longleftrightarrow w = (L-\widetilde{V})^{-1}F\\ &\Longleftrightarrow w(x) = \int_{\mathcal{M}} N(x,y)F(y)\, d\mathcal{V}_y.\end{aligned} \tag{2.13}$$

On the other hand, $w \in L^2_1(\mathcal{M})$ is the unique solution of $(L-\widetilde{V})w = F$, for a given $F \in L^2_{-1}(\mathcal{M})$, if and only if the pair $w^{\pm} := w|_{\Omega_\pm}$ solves

(2.14)

$$\text{(TBVP-inhomogeneous)}\qquad \begin{cases} (\Delta - V)w^+ = F^+ & \text{in } \Omega_+,\\ (\mu\Delta - \mu V)w^- = F^- & \text{in } \Omega_-,\\ \operatorname{Tr} w^+ = \operatorname{Tr} w^- & \text{in } L^2_{1/2}(\partial\Omega),\\ \partial_\nu w^+ = \mu\, \partial_\nu u^- & \text{in } L^2_{-1/2}(\partial\Omega),\end{cases}$$

where $F^{\pm} := F|_{\Omega_\pm}$, Tr denotes the trace operator (in the sense of Sobolev spaces), and $L^2_{\pm 1/2}(\partial\Omega)$ are L^2-based Sobolev spaces of order of smoothness $\pm 1/2$ on $\partial\Omega$. Consequently, the solution $w^{\pm}$ of (2.14) can be represented in the form

(2.15)

$$\begin{aligned} w^+(x) &= \int_{\mathcal{M}} N(x,y)F(y)\, d\mathcal{V}_y,\\ &= \int_{\Omega_+} N^{+,+}(x,y)F^+(y)\, d\mathcal{V}_y + \int_{\Omega_-} N^{+,-}(x,y)F^-(y)\, d\mathcal{V}_y, \quad x \in \Omega_+,\end{aligned}$$

(2.16)

$$\begin{aligned} w^-(x) &= \int_{\mathcal{M}} N(x,y)F(y)\, d\mathcal{V}_y,\\ &= \int_{\Omega_+} N^{-,+}(x,y)F^+(y)\, d\mathcal{V}_y + \int_{\Omega_-} N^{-,-}(x,y)F^-(y)\, d\mathcal{V}_y, \quad x \in \Omega_-.\end{aligned}$$

We now analyze the behavior of the Neumann kernel near the diagonal. The starting point is the following simple but useful estimate on solutions of the inhomogeneous problem. Specifically, given $F \in L^2(\mathcal{M})$, the solution $w^\pm \in L^2_1(\Omega_\pm)$ of $(L - \widetilde{V})w = F$ in $\mathcal{M}$ (recall that $w^\pm := w|_{\Omega_\pm}$), satisfies

$$\|w^+\|_{L^{2n/(n-2)}(\Omega_+)} + \|w^-\|_{L^{2n/(n-2)}(\Omega_-)} \le C\,\|F\|_{L^{2n/(n+2)}(\mathcal{M})}. \tag{2.17}$$

Indeed, Sobolev's and Poincaré's inequalities give

$$\|w^\pm\|^2_{L^{2n/(n-2)}(\Omega_\pm)} \le C\,\|w^\pm\|^2_{L^2_1(\Omega_\pm)} \le C \int\limits_{\Omega_\pm} \left\{|\nabla w^\pm|^2 + V|w^\pm|^2\right\} d\mathcal{V}. \tag{2.18}$$

Thus,

$$\begin{aligned} &\|w^+\|^2_{L^{2n/(n-2)}(\Omega_+)} + \|w^-\|^2_{L^{2n/(n-2)}(\Omega_-)} \\ &\le C\bigg(\int_{\Omega_+} \left\{|\nabla w^+|^2 + V|w^+|^2\right\} d\mathcal{V} + \mu \int_{\Omega_-} \left\{|\nabla w^-|^2 + V|w^-|^2\right\} d\mathcal{V}\bigg). \end{aligned} \tag{2.19}$$

The variational characterization of w as a solution of $(L - \widetilde{V})w = F$ gives that the right-hand side of (2.19) is equal to $C\int_{\mathcal{M}} wF\,d\mathcal{V}$ so that

$$\|w\|^2_{L^{2n/(n-2)}(\mathcal{M})} \le C\bigg|\int_{\mathcal{M}} wF\,d\mathcal{V}\bigg|. \tag{2.20}$$

From this, (2.17) follows, by Hölder's inequality.

Next we seek a pointwise estimate on solutions of $(L - V)w = F$. To this end, suppose $K \subset \mathcal{M}$ is a compact set and assume that $F \in L^2(\mathcal{M})$ satisfies

$$\operatorname{supp} F \subset K. \tag{2.21}$$

Let $w := (L - \widetilde{V})^{-1}F \in L^2_1(\mathcal{M})$ so that $(L - \widetilde{V})w = 0$ in $\mathcal{M} \setminus K$. Then, so we claim,

$$|w(x)| \le C\,\|F\|_{L^{2n/(n+2)}(K)}\ \operatorname{dist}(x, K)^{-(n-2)/2} \quad \text{for} \quad x \in \mathcal{M} \setminus K. \tag{2.22}$$

To see this, we recall the estimate

$$\|w\|_{L^{2n/(n-2)}(\mathcal{M})} \leq C\,\|F\|_{L^{2n/(n+2)}(\mathcal{M})} \tag{2.23}$$

established in the previous discussion. Fix $x \in \mathcal{M} \setminus K$, and set $r_0 := \operatorname{dist}(x, K)$. In particular, (with $B_r(x)$ denoting the ball of center x and radius r),

$$(L - \widetilde{V})w = 0 \quad \text{in} \quad B_{r_0}(x), \tag{2.24}$$

so (2.22) follows from the L^∞ estimate of Moser in which we take $p = 2n/(n-2)$. Recall that Moser's L^∞ estimate, i.e., the sub-mean inequality for nonnegative sub-solutions of L, asserts that

$$\sup\{w(x) : x \in B_{R/2}\} \leq C_{\mu,n,p}\bigg(R^{-n}\int_{B_R} w^p\,dx\bigg)^{1/p}, \quad 0 < p < \infty, \tag{2.25}$$

uniformly for any sub-solution $w \geq 0$ of L in B_R. This is proved in Theorem 2 [**35**, pp. 581–582] when $1 < p < \infty$. The extension to $p \leq 1$ uses an argument of Dahlberg and Kenig which may be found in [**13**, pp. 1004–1005]. See also [**19**, Lemma 1.1.8].

We can now use (2.22) to estimate $N(x,y)$. Specifically, we aim to show that

$$|N(x,y)| \leq C\operatorname{dist}(x,y)^{-(n-2)}, \quad \text{uniformly for} \quad (x,y) \in \mathcal{M}\times\mathcal{M} \setminus \text{diag}. \tag{2.26}$$

In order to justify this estimate, fix $x, y \in \mathcal{M}$, $x \neq y$ otherwise arbitrary, set $r := \operatorname{dist}(x,y) > 0$ and take $K := \overline{B}_r(x)$. Applying (2.22) to F supported in K, we have—by relying on the integral representation formulas (2.15)–(2.17) and Riesz duality—the following sequence of estimates

$$\begin{aligned} &\bigg\{\int_{B_r(x)} |N(x,y)|^{2n/(n-2)}\,d\mathcal{V}_y\bigg\}^{(n-2)/2n} \\ &= \sup\bigg\{\Big|\int_{\mathcal{M}} N(x,y)F(y)\,d\mathcal{V}_y\Big| : \|F\|_{L^{2n/(n+2)}(\mathcal{M})} = 1,\ \operatorname{supp} F \subseteq \overline{B}_r(x)\bigg\} \\ &\leq C\,r^{-(n-2)/2}. \end{aligned} \tag{2.27}$$

Now $N(x,y) = N(y,x)$, so $N(x,y)$ also solves a uniformly elliptic, divergence-form PDE with L^∞ coefficients as a function of y, for $y \in B_r(x)$. Hence (2.27) plus another application of Moser's L^∞ estimate (2.25) readily gives (2.26).

The De Giorgi-Nash-Moser theory, in concert with (2.27), also gives Hölder estimates. Specifically, there exists $\alpha \in (0,1)$ depending only on the dimension and the ellipticity constant of the operator such that $|u(p) - u(q)| \leq C|(p-q)/R|^\alpha \|u\|_{L^\infty(B_{2R})}$, for $p, q \in B_R$, for any u null-solution in B_{2R}, cf. also [**14**, Theorem 8.22]. In our case, given $x, y \in \mathcal{M}$, $x \neq y$, we apply this result to the ball centered at y with radius $R := \operatorname{dist}(x,y)/2$, and to the function $u := N(\cdot, y)$. This and (2.26) then yield

$$
\begin{aligned}
&|N(x,y) - N(x',y)| \leq C \frac{\operatorname{dist}(x,x')^\alpha}{\operatorname{dist}(x,y)^{n-2+\alpha}}, \\
&\qquad \text{if} \quad \operatorname{dist}(x,x') \leq (1/2)\operatorname{dist}(x,y).
\end{aligned}
\tag{2.28}
$$

Furthermore, from (2.28) and the symmetry property (2.6), we can also deduce

$$
\begin{aligned}
&|N(x,y) - N(x,y')| \leq C \frac{\operatorname{dist}(y,y')^\alpha}{\operatorname{dist}(x,y)^{n-2+\alpha}}, \\
&\qquad \text{if} \quad \operatorname{dist}(y,y') \leq (1/2)\operatorname{dist}(x,y).
\end{aligned}
\tag{2.29}
$$

3. Hardy and Sobolev-Besov spaces. For the reader's convenience, here we recall some well-known facts and definitions about Hardy and Sobolev-Besov spaces. For the latter scale, see [**5**] for the setting of homogeneous spaces, as well as [**45**] for an excellent up-to-date account.

Let Ω be a Lipschitz domain and $(n-1)/n < p \leq 1$. A surface ball $S_r(x)$ is any set of the form $B_r(x) \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \infty$. Call a function $a : \partial\Omega \to \mathbf{R}$ an atom for the Hardy space $H^p_{at}(\partial\Omega)$ (p-atom for short), if either

(i) there exists an S_r-surface ball:

$$
\operatorname{supp} a \subseteq S_r, \quad \|a\|_{L^\infty(\partial\Omega)} \leq r^{-(n-1)/p}, \quad \text{and} \quad \int_{\partial\Omega} a \, d\sigma = 0,
\tag{3.1}
$$

or

(ii) there exists an S_r-surface ball, with $r \geq 1$, such that

$$\operatorname{supp} a \subseteq S_r, \quad \|a\|_{L^\infty(\partial\Omega)} \leq r^{-(n-1)/p}. \tag{3.2}$$

We then set

$$H^p_{at}(\partial\Omega) := \Big\{ \sum_j \lambda_j a_j : \{\lambda_j\}_j \in \ell^p,\ a_j \text{ satisfies either (3.1) or (3.2)} \Big\}, \tag{3.3}$$

and endow it with the natural infimum norm. The series is convergent in $(C^\alpha(\partial\Omega))^*$ with $\alpha := (n-1)(1/p-1) \in (0,1)$ and in $L^1(\partial\Omega)$ if $p = 1$. The Hardy space (3.3) is local in the sense that $H^p_{at}(\partial\Omega)$ is a module over the Hölder space $C^\alpha(\partial\Omega)$ with $\alpha > (n-1)(p^{-1}-1)$.

We shall also work with $H^{1,p}_{at}(\partial\Omega)$, $(n-1)/n < p \leq 1$, the ℓ^p-span of 'regular' atoms on $\partial\Omega$. More specifically, for $1/q := 1/p - 1/(n-1)$ define

$$\begin{aligned} &H^{1,p}_{at}(\partial\Omega) \\ &\quad := \Big\{ \sum_j \lambda_j a_j \text{ convergent in } L^q(\partial\Omega) : (\lambda_j)_j \in \ell^p,\ a_j \text{ as in (3.5)} \Big\}, \end{aligned} \tag{3.4}$$

and set $\|f\|_{H^{1,p}_{at}(\partial\Omega)} := \inf\,[\sum |\lambda_j|^p]^{1/p}$, where the infimum is taken over all possible representations. Here, for $(n-1)/n < p \leq 1$ and a fixed $\max\{1,p\} < p_0 < \infty$, a function $a : \partial\Omega \to \mathbf{R}$ is called a *regular atom* if there exists a surface ball S_r so that

$$\operatorname{supp} a \subseteq S_r, \quad \|\nabla_{\tan} a\|_{L^{p_0}(\partial\Omega)} \leq r^{(n-1)((1/p_0)-(1/p))} \tag{3.5}$$

where $\nabla_{\tan}$ denotes the tangential gradient on $\partial\Omega$. Different choices of the parameter p_0 above yield the same topology on $H^{1,p}_{at}(\partial\Omega)$. This regular Hardy space is then a module over $\mathrm{Lip}_{\mathrm{comp}}(\partial\Omega)$, the class of Lipschitz, compactly supported functions on $\partial\Omega$.

Besov spaces, $B^{p,q}_s(\partial\Omega)$ with $(n-1)/n < p$, $q \leq \infty$ and $0 < |s| < 1$ on the boundary of a domain Ω can be introduced via localization involving a smooth partition of unity and pull-back. As such, it suffices to consider the case when Ω is an Euclidean domain, lying above the graph of a Lipschitz function $\varphi : \mathbf{R}^{n-1} \to \mathbf{R}$. In this scenario, we set

$$f \in B_s^{p,q}(\partial\Omega) \Longleftrightarrow x' \longmapsto f(x', \varphi(x')) \in B_s^{p,q}(\mathbf{R}^{n-1}), \tag{3.6}$$

whenever $(n-1)/n < p,\ q \leq \infty,\ (n-1)(1/p-1)_+ < s < 1$, where $(a)_+ := \max\{a, 0\}$. Also, we set

$$f \in B_{s-1}^{p,q}(\partial\Omega) \Longleftrightarrow f(\cdot, \varphi(\cdot))\sqrt{1+|\nabla\varphi(\cdot)|^2} \in B_{s-1}^{p,q}(\mathbf{R}^{n-1}), \tag{3.7}$$

whenever $(n-1)/n < p,\ q \leq \infty$ and $(n-1)(1/p-1)_+ < s < 1$. In particular, for $-1 < s < 0$ and $1 < p, q < \infty$, there holds

$$B_s^{p,q}(\partial\Omega) = (B_{-s}^{p',q'}(\partial\Omega))^*, \quad 1/p + 1/p' = 1, \quad 1/q + 1/q' = 1, \tag{3.8}$$

where the duality pairing between $f \in B_s^{p,q}(\partial\Omega)$ and $g \in B_{-s}^{p',q'}(\partial\Omega)$ is (a natural extension of) $\int_{\partial\Omega} fg\, d\sigma$.

The Hölder class $C^s(\partial\Omega)$ corresponds to the limiting case $p = q = \infty$ of the Besov scale. Besov spaces with $p > 1$ and $s \in (0,1)$ can also be obtained from Sobolev spaces via real interpolation, i.e.,

$$B_s^{p,q}(\partial\Omega) = \big(L^p(\partial\Omega),\, L_1^p(\partial\Omega)\big)_{s,q}, \quad 1 < p, q < \infty, \quad 0 < s < 1. \tag{3.9}$$

The standard Sobolev scale $L_s^p(\mathcal{M})$, $1 < p < \infty$, $s \geq 0$, is obtained by lifting $L_s^p(\mathbf{R}^n) := \{(I-\Delta)^{s/2} f : f \in L^p(\mathbf{R}^n)\}$ to $\mathcal{M}$ via a smooth partition of unity and pull-back. Let $L_s^p(\Omega)$ denote the restriction of elements in $L_s^p(\mathcal{M})$ to the Lipschitz domain Ω, and $L_{s,0}^p(\Omega)$ stand for the subspace consisting of restrictions to Ω of elements from $L_s^p(\mathcal{M})$ with support contained in $\overline{\Omega}$. Finally, for $s > 0$ and $1 < p, q < \infty$ with $1/p + 1/q = 1$, we set $L_{-s}^p(\Omega) := (L_{s,0}^q(\Omega))^*$.

For a more detailed exposition, the interested reader is referred to [**40, 46**].

4. Proof of Theorem 1.1. We proceed in a series of steps, starting with

Step I. Here we prove that there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that (1.16) is well-posed whenever $p \in (2-\varepsilon, 2+\varepsilon)$. The key ingredient in this regard, a proof of which can be found in [**32**], is the fact that

(4.1)
$$\begin{cases} \lambda I + K_V^* : L^p(\partial\Omega) \xrightarrow{\sim} L^p(\partial\Omega) & \text{and} \\ S_V : L^p(\partial\Omega) \xrightarrow{\sim} L_1^p(\partial\Omega) & \text{isomorphically for each} \quad \lambda \in \mathbf{R}, \\ |\lambda| \geq \frac{1}{2}, \quad 2-\varepsilon < p < 2+\varepsilon. & \end{cases}$$

Granted (4.1), the issue of existence for (1.16) when $2-\varepsilon < p < 2+\varepsilon$ can be handled as follows. First, pick $\psi \in L^p(\partial\Omega)$ such that $S\psi = f$ and then take $u^\pm := \mathcal{S}h^\pm$ in $\Omega_\pm$ where, with $\lambda := -1/2(1+\mu)/(1-\mu)$, the functions $h^\pm \in L^p(\partial\Omega)$ are given by

(4.2)
$$h^+ := \frac{1}{1-\mu}\left(\lambda I + K_V^*\right)^{-1}\left[g - \mu((1/2)I + K_V^*)\psi\right], \quad \text{and} \quad h^- := h^+ - \psi.$$

Finally, uniqueness for (1.16) also follows from (4.1) since, as proved in [**32**], any functions $u^\pm$ such that $(\Delta - V)u^\pm = 0$ in $\Omega_\pm$ and $M(\nabla u^\pm) \in L^p(\partial\Omega)$, $|p-2| < \varepsilon$ can be represented in terms of a single layer in Ω_+ and Ω_-, respectively.

Step II. Here we prove certain Rellich estimates for the transmission problem (1.16). The departure point is a Rellich-type identity proved in [**32**]. Specifically, let Ω be a Lipschitz subdomain of $\mathcal{M}$, and suppose that $u \in C^1_{\mathrm{loc}}(\Omega)$ is well behaved near $\partial\Omega$ and has $\Delta u \in L^2(\Omega)$. Also, fix a smooth vector field θ on the manifold $\mathcal{M}$, and denote by $\theta_{\tan}$ the tangential component of θ on $\partial\Omega$. Then,

$$\begin{aligned} &\int_{\partial\Omega} \langle \nu, \theta\rangle \left\{|\nabla_{\tan} u|^2 - (\partial_\nu u)^2\right\} d\sigma \\ &\quad = 2\int_{\partial\Omega} (\nabla_{\theta_{\tan}} u)(\partial_\nu u)\, d\sigma - 2\int_\Omega (\nabla_\theta u)\Delta u \, d\mathcal{V} \\ &\qquad + \int_\Omega \left\{(\operatorname{div}\theta)|\nabla u|^2 - 2(\mathcal{L}_\theta \mathfrak{g})(\nabla u, \nabla u)\right\} d\mathcal{V} \end{aligned} \tag{4.3}$$

where, in the last integral in (4.3), $\mathcal{L}_\theta \mathfrak{g}$ denotes the Lie derivative of the metric tensor $\mathfrak{g}$ with respect to θ.

Next, fix $x^* \in \partial\Omega$, $0 < R < \operatorname{diam}\Omega$, and assume that $\langle \theta, \nu \rangle \geq 0$ on $\partial\Omega$, $\langle \theta, \nu \rangle \geq \kappa > 0$ on $B_{R/4}(x^*) \cap \partial\Omega$, $\operatorname{supp}(\theta) \subset B_{R/2}(x^*) \cap \partial\Omega$, and $|\nabla\theta| \leq C/R$. Also, fix a potential V as before. It is convenient to assume that its support is sufficiently small so that θ vanishes on it. Finally, assume that the functions $u^\pm$, defined in Ω_+ and Ω_-, respectively, are well behaved near $\partial\Omega$ and satisfy $(\Delta - V)u^\pm = 0$ in $B_R(x^*) \cap \Omega_\pm$ as well as $\nabla_{\tan} u^+|_{B_R(x^*)\cap\partial\Omega} = \nabla_{\tan} u^-|_{B_R(x^*)\cap\partial\Omega}$ and $\partial_\nu u^+|_{B_R(x^*)\cap\partial\Omega} = \mu\, \partial_\nu u^-|_{B_R(x^*)\cap\partial\Omega}$.

The Rellich identity (4.3) written for $u^\pm$ in Ω_+ and Ω_-, respectively, then gives

$$
\begin{aligned}
(4.4)\quad & \int_{B_R(x^*)\cap\partial\Omega} \langle \theta, \nu \rangle \left[|\nabla_{\tan} u^\pm|^2 - |\partial_\nu u^\pm|^2 \right] d\sigma \\
&= 2\int_{B_R(x^*)\cap\partial\Omega} (\nabla_{\theta_{\tan}} u^\pm)(\partial_\nu u^\pm)\, d\sigma + \int_{B_R(x^*)} \mathcal{O}(|\nabla u^\pm|^2 |\nabla\theta|)\, d\mathcal{V} \\
&\quad + \int_{\partial B_R(x^*)\cap\Omega_\pm} \mathcal{O}(|\nabla u^\pm|)^2\, d\sigma.
\end{aligned}
$$

Next, multiplying with μ formula (4.4), written for the choice '$-$' and subtracting it from formula (4.4), written for the choice '$+$' yields

$$
\begin{aligned}
(4.5)\quad & \int_{B_{R/4}(x^*)\cap\partial\Omega} \left[(1-\mu)|\nabla_{\tan} u^+|^2 + ((1/\mu) - 1)|\partial_\nu u^+|^2 \right] d\sigma \\
&\leq C\Big\{ \int_{\partial B_R(x^*)} |\nabla u|^2\, d\sigma + R^{-1} \int_{B_R(x^*)} |\nabla u|^2\, d\mathcal{V} \Big\},
\end{aligned}
$$

where $u := u^+$ in Ω_+ and $u := u^-$ in Ω_-. By suitably averaging both sides of this estimate in the parameter R, e.g., replacing R by τR and integrating in $\tau \in [1/2, 2]$, finally gives

$$
\begin{aligned}
(4.6)\quad & \int_{B_{R/8}(x^*)\cap\partial\Omega} \left[(1-\mu)|\nabla_{\tan} u^+|^2 + ((1/\mu) - 1)|\partial_\nu u^+|^2 \right] d\sigma \\
&\leq \frac{C}{R} \int_{B_{2R}(x^*)} |\nabla u|^2\, d\mathcal{V}.
\end{aligned}
$$

Step III. Here we begin the process of analyzing (1.16) with atomic data. For the time being, the goal is to establish a key estimate, to the effect that if $a \in L^\infty(\partial\Omega)$ is a 1-atom, then the solution $u = (u^+, u^-)$ of (1.16) with $f = 0$ and $g = a$ satisfies

$$\|M(\nabla u^+)\|_{L^1(\partial\Omega)} + \|M(\nabla u^-)\|_{L^1(\partial\Omega)} \leq C. \tag{4.7}$$

In the proof of this result, the following pointwise decay estimate plays a crucial role. If $u^\pm$ are as above and $\alpha \in (0,1)$ is as in (2.28)–(2.29), then, so we claim,

$$|u^\pm(x)| \leq C \, \frac{r^\alpha}{\mathrm{dist}\,(x, x_0)^{n-2+\alpha}}, \tag{4.8}$$

for $x \in \Omega_\pm$, $\mathrm{dist}\,(x, x_0) \geq 4r$. Here r and x_0 are, respectively, the radius and the center of the minimal surface ball containing the support of the atom a. The case $r \geq 1$ is elementary and below we assume that $r < 1$ so that the atom satisfies a vanishing moment condition. In this latter scenario, as pointed out before, cf. (2.8)–(2.9), the solution (u^+, u^-) to the transmission problem (1.16) can be written in the form

$$u^+(x) = -\int_{\partial\Omega} \big\{N^{+,+}(x,y) - N^{+,+}(x,x_o)\big\} a(y)\, d\sigma_y, \quad x \in \Omega_+, \tag{4.9}$$

$$u^-(x) = -\int_{\partial\Omega} \big\{N^{-,-}(x,y) - N^{-,-}(x,x_o)\big\} a(y)\, d\sigma_y, \quad x \in \Omega_-, \tag{4.10}$$

by using $\int_{\partial\Omega} a\, d\sigma = 0$. Then the estimate (4.8) follows from this and (2.29), given the hypotheses on the support, size and oscillations of the atom a.

After this preamble, we turn our attention to the proof of (4.7). Let $S_1 := B_{4r}(x_0) \cap \partial\Omega$, and for $\ell \geq 2$ (and $2^\ell r \leq \mathrm{diam}\,\Omega$), set

$$B_\ell := B_{2^{\ell+1}r}(x_0) \setminus B_{2^\ell r}(x_0), \quad S_\ell := B_\ell \cap \partial\Omega. \tag{4.11}$$

We will estimate $M(\nabla u^\pm)$ on each set S_ℓ. First, the contribution coming from S_1 can be readily controlled using Hölder's inequality and the L^2-theory from Step I.

Handling the contribution from S_ℓ for $\ell \geq 2$ will involve several ingredients, including (4.8), Caccioppoli's inequality and the local Rellich estimate from Step II. To proceed, for each point $x \in \partial\Omega$ and $R > 0$, we introduce some 'truncated' maximal operators, i.e.,

$$M_{j,R}(u)(x) := \sup\big\{|u(y)| : \, y \in \Gamma^{\pm}_{j,R}(x)\big\}, \quad j = 1, 2, \tag{4.12}$$

where

$$\Gamma^{\pm}_{1,R}(x) := \{y \in \Gamma^{\pm}(x) : \, |x - y| > R\}, \tag{4.13}$$

$$\Gamma^{\pm}_{2,R}(x) := \{y \in \Gamma^{\pm}(x) : \, |x - y| \leq R\}. \tag{4.14}$$

Accordingly, we then set

$$I_{j,\ell} := \int_{S_\ell} [M_{j,r}(\nabla u^+) + M_{j,r}(\nabla u^-)]\, d\sigma, \quad j = 1, 2, \tag{4.15}$$

for $\ell \geq 2$, if $2^\ell r$ is not large; say $2^\ell r \leq A$. Also, pick A so that there is a set Q of positive measure in $\mathcal{M}$, disjoint from all the sets B_ℓ with $2^\ell r \leq A$, such that $V > 0$ on Q. The expressions $I_{1,\ell}$, $I_{2,\ell}$ will now be analyzed separately.

Step IV. Here we deal with estimates away from the boundary. As regards the contribution from $M_{1,r}(\nabla u^{\pm})$, if $\Delta_j := B(x_0, 2^j r) \cap \partial\Omega$ and if the points $x \in \Delta_{j+1} \backslash \Delta_j$ and $z \in \Gamma^{\pm}(x)$ are such that $\mathrm{dist}\,(z, x) \geq 2^j r$, then interior estimates give

$$\delta(z)|\nabla u^{\pm}(z)| \leq C\, r^{-n/2} \bigg(\int_{B_{\delta(z)/2}(z)} |u^{\pm}(y)|^2 \, d\mathcal{V}_y \bigg)^{1/2}, \tag{4.16}$$

where δ is the distance function to $\partial\Omega$. Note that $y \in B_{\delta(z)/2}(z)$ forces $|u^{\pm}(y)| \leq C r^\alpha \mathrm{dist}\,(x, x_0)^{-n+2-\alpha}$, by the decay estimate (4.8). Therefore, keeping also in mind that $\delta(z) \geq C 2^j r \geq C \mathrm{dist}\,(x, x_0)$, we have

$$|\nabla u^{\pm}(z)| \leq C \, \frac{r^\alpha}{\mathrm{dist}(x, x_0)^{n-1+\alpha}}. \tag{4.17}$$

This, in turn, allows us to write

$$M_{1,r}(\nabla u^{\pm})(x) \leq C \, \frac{r^\alpha}{\mathrm{dist}(x, x_0)^{n-1+\alpha}}, \quad \text{for} \quad x \in \Delta_{j+1} \setminus \Delta_j, \tag{4.18}$$

so that, ultimately,

$$\int_{\partial\Omega\setminus\Delta_1}[M_{1,r}(\nabla u^+)+M_{1,r}(\nabla u^-)]\,d\sigma\leq C, \tag{4.19}$$

as desired. The missing piece, i.e., $\|M_{1,r}(\nabla u^\pm)\|_{L^1(\Delta_1)}$, is easily estimated using the L^2-theory from Step I, and this finishes the proof of (4.7) with $M_{1,r}$ in place of M.

Step V. Here we deal with estimates near the boundary. Much as in [**11**], the contribution from $M_{2,r}$ can be estimated as follows:

(4.20)
$$\begin{aligned}I_{2,\ell}&\leq C(2^\ell r)^{(n-1)/2}\Big\{\int_{S_\ell}\Big[|M_{2,r}(\nabla u^+)|^2+|M_{2,r}(\nabla u^-)|^2\Big]\,d\sigma\Big\}^{1/2}\\&\leq C(2^\ell r)^{(n-1)/2}(2^\ell r)^{-3/2}\Big\{\int_{c_1 2^\ell r\leq \mathrm{dist}\,(x,x_0)\leq c_2 2^\ell r}|u(x)|^2\,d\mathcal{V}_x\Big\}^{1/2}\end{aligned}$$

where the first inequality is just Hölder's, and the second relies on the well-posedness of the L^2 Neumann problem from [**32**], the Rellich estimate (4.6) and Caccioppoli's inequality, cf., e.g., [**19**, p. 2].

At this stage, we invoke the estimate (4.8) on u to deduce

$$I_{2,\ell}\leq C(2^\ell r)^{(n-4)/2}(2^\ell r)^{n/2}\,\frac{r^\alpha}{(2^\ell r)^{n-2+\alpha}}=C\,2^{-\alpha\ell}, \tag{4.21}$$

for some $\alpha\in(0,1)$; note that all the r's cancel above. From this we conclude that

$$\sum_{\ell=1}^{N}\Big[\|M_{2,r}(\nabla u^+)\|_{L^1(S_\ell)}+\|M_{2,r}(\nabla u^-)\|_{L^1(S_\ell)}\Big]\leq C, \tag{4.22}$$

where N is chosen so that $2^N r\approx A$. The estimate of $M_{2,r}(\nabla u^\pm)$ on the remainder of $\partial\Omega$ follows from the same analysis as that for $M(\nabla u^\pm)$ on S_N just done. Thus, the estimate (4.7) is proven.

Step VI. Here we finish the proof of the well-posedness of (1.16) with atomic data. Concretely, consider the existence issue for the transmission problem (1.16) when $f \in H^{1,1}_{at}(\partial\Omega)$ and $g \in H^1_{at}(\partial\Omega)$. Since, as proved in [**34**], $S_V : H^1_{at}(\partial\Omega) \xrightarrow{\sim} H^{1,1}_{at}(\partial\Omega)$ isomorphically, matters can easily be reduced to the case when $f = 0$, by subtracting from u^+ a suitable single layer potential. On the other hand, when $f = 0$ and $g = \sum_j \lambda_j a_j \in H^1_{at}(\partial\Omega)$, existence and L^1-estimates for (1.16) follow from Steps I–IV by treating one atom at a time.

To prove uniqueness, let $u^\pm \in C^1_{loc}(\Omega_\pm)$ solve (1.16) with $p = 1$ and $f = g = 0$. The claim is that $u^\pm \equiv 0$ in $\Omega_\pm$ (recall that we are assuming that $V > 0$ on a set of positive measure in each component of Ω_+ and Ω_-). Fix an arbitrary point $x_0 \in \Omega_+$. By the De Giorgi-Nash-Moser theory and our assumptions,

$$
\begin{gathered}
u^\pm \in C^\alpha(\bar{\Omega}_\pm), \quad M(\nabla u^\pm) \in L^1(\partial\Omega), \\
M(\nabla N^{\pm,+}(\cdot, x_0)) \in L^2(\partial\Omega), \\
N^{\pm,+}(\cdot, x_0) \in L^\infty \quad \text{near} \quad \partial\Omega.
\end{gathered}
\tag{4.23}
$$

These suffice to justify (via a limiting process which involves two suitable sequences of smooth subdomains $\Omega_j^\pm \nearrow \Omega_\pm$) the integral representations

$$
\begin{aligned}
u^+(x_0) = & \int_{\partial\Omega} \partial_{\nu_y} N^{+,+}(y, x_0) u^+(y)\, d\sigma_y \\
& - \int_{\partial\Omega} N^{+,+}(y, x_0) \partial_\nu u^+(y)\, d\sigma_y
\end{aligned}
\tag{4.24}
$$

$$
\begin{aligned}
0 = & -\int_{\partial\Omega} \partial_{\nu_y} N^{-,+}(y, x_0) u^-(y)\, d\sigma_y \\
& + \int_{\partial\Omega} N^{-,+}(y, x_0) \partial_\nu u^-(y)\, d\sigma_y.
\end{aligned}
\tag{4.25}
$$

In concert with the transmission boundary conditions satisfied by the pairs (u^+, u^-) and $(N^{+,+}(\cdot, x_0), N^{-,+}(\cdot, x_0))$, the above identities yield $u^+(x_0) = 0$, after some minor algebra. Since $x_0 \in \Omega_+$ was arbitrary, it follows that $u^+ \equiv 0$ in Ω_+; similarly, $u^- \equiv 0$ in Ω_-. This finishes the proof of the fact that (1.16) is well-posed when $p = 1$ (with the understanding that, in this case, the boundary data (f, g) are selected from $H^{1,1}_{at}(\partial\Omega) \oplus H^1_{at}(\partial\Omega)$).

Step VII. Here we prove the well-posedness of (1.16) for $1 < p < 2$. Existence follows by interpolating between the results in Step I and Step VI, while uniqueness is proved much as in Step VI.

This finishes the proof of Theorem 1.1, modulo the claim about the integral representation of the solution. This, in turn, is obtained *a posteriori*, from the L^p theory for the Neumann problem from [**34**] and the invertibility results established in the next section.

5. The invertibility of singular integral operators. Based on the fact that (1.16) is well-posed the same type of argument as in [**11**] proves the following.

Corollary 5.1. *Under the assumptions of Theorem* 1.1*, there exists* $\varepsilon > 0$ *such that*

$$\lambda I + K_V^* : L^p(\partial\Omega) \longrightarrow L^p(\partial\Omega), \tag{5.1}$$

$$\lambda I + K_V : L^p_1(\partial\Omega) \longrightarrow L^p_1(\partial\Omega), \tag{5.2}$$

$$\lambda I + K_V^* : H^1_{at}(\partial\Omega) \longrightarrow H^1_{at}(\partial\Omega), \tag{5.3}$$

$$\lambda I + K_V : H^{1,1}_{at}(\partial\Omega) \longrightarrow H^{1,1}_{at}(\partial\Omega), \tag{5.4}$$

are Fredholm with index zero for any potential V *and any* $\lambda \in \mathbf{R}$ *with* $|\lambda| \geq 1/2$*, provided* $1 < p < 2 + \varepsilon$*. In fact, these operators are genuine isomorphisms when the potential* V *is a positive constant, say* $V \equiv \omega$*, for some* $\omega \in \mathbf{R}$*,* $\omega > 0$*.*

Our goal is to extend the above Fredholmness/invertibility results to other function spaces of interest via stability and extrapolation. To set the stage, we recall an abstract result from [**17**].

Proposition 5.2. *Let* $\{X_p\}_{p \in I}$*,* I *open interval, be quasi-Banach spaces forming a complex interpolation scale, i.e., for any* $p_0, p_1 \in I$

$$[X_{p_0}, X_{p_1}]_\theta = X_p, \quad \textit{if} \quad \theta \in (0,1) \quad \textit{and} \quad 1/p = (1-\theta)/p_0 + \theta/p_1, \tag{5.5}$$

and assume that

$$T : X_p \longrightarrow X_p \tag{5.6}$$

is a linear operator, bounded for each $p \in I$. Call a certain property of (5.6) stable *if the collection of p's for which it holds is open.*

Then being Fredholm and being an isomorphism are stable states.

Furthermore, if $J \subset I$ is an open interval such that T is an isomorphism of X_p for each $p \in J$, then $T_{p_0}^{-1}$ agrees with $T_{p_1}^{-1}$ on $X_{p_0} \cap X_{p_1}$ for any $p_0, p_1 \in J$ (where T_p^{-1} stands for the inverse of T on X_p).

Two such scales of quasi-Banach spaces are going to be of importance for us. First,

$$H^p(\partial\Omega) := \begin{cases} L^p(\partial\Omega) & \text{if } p > 1, \\ H^p_{at}(\partial\Omega) & \text{if } (n-1)/n < p \leq 1, \end{cases} \tag{5.7}$$

is known to be a complex interpolation scale, see [**4, 15, 26**]. Second,

$$H^{1,p}(\partial\Omega) := \begin{cases} L^p_1(\partial\Omega) & \text{if } p > 1, \\ H^{1,p}_{at}(\partial\Omega) & \text{if } (n-1)/n < p \leq 1, \end{cases} \tag{5.8}$$

has also been shown to be a complex interpolation scale in [**26**].

We continue to review the functional analytic tools we are going to rely upon in our analysis of the operators $\lambda I + K_V$ and $\lambda I + K_V^*$. This time, fix X a complete, metrizable, locally bounded, linear space. We say that X^* separates the points in X if $x = 0 \Leftrightarrow f(x) = 0$, for all $f \in X^*$. All quasi-Banach spaces considered in this paper are assumed to have duals which separate their points, i.e., dual rich.

Let $0 < p \leq 1$. A set $S \subseteq X$ is called absolutely p-convex if S coincides with its absolutely p-convex hull, defined as

$$\Big\{ \sum_{\text{finite}} \lambda_j a_j \; : \; a_j \in S, \; \sum |\lambda_j|^p \leq 1 \Big\}. \tag{5.9}$$

For each $0 < p \leq 1$, let $W_{X,p}$ be the absolutely p-convex hull of the unit ball in X and set

$$|||x|||_p := \inf\big\{ \lambda > 0 : \; x/\lambda \in W_{X,p} \big\}. \tag{5.10}$$

The above formula defines a p-norm, that is, $|||x|||_p = 0$ if and only if $x = 0$, for every $\lambda \in \mathbf{C}$ it holds that $|||\lambda x|||_p = |\lambda|\, |||x|||_p$, and

$$|||x+y|||_p^p \le |||x|||_p^p + |||y|||_p^p, \quad \text{for all} \quad x, y \in X. \tag{5.11}$$

The $|||\cdot|||_p$ "norm" generates a locally p-convex topology, weaker than the original topology on X.

Recall that, according to the classical Aoki-Rolewicz theorem, cf., e.g., [**18**], any locally bounded linear space is p-convex, for some $0 < p \le 1$; i.e., its topology comes from a suitable p-norm. Call X a p-Banach space if its topology is given by a p-norm, with respect to which X is complete.

For each X as above, we denote by $\mathcal{E}_p(X)$ the p-envelope of X, i.e., the completion of X in the quasi-norm $|||\cdot|||_p$. It follows that $\mathcal{E}_p(X)$ is a p-Banach space, which should be thought of as the "smallest" locally p-convex topological space containing X. In fact, if X is locally bounded, then $\mathcal{E}_p(X)$ is the "smallest" p-Banach space containing X. In particular, if X is a p-Banach space to begin with, then $\mathcal{E}_p(X) = X$. When $p = 1$, $\mathcal{E}_p(X)$ corresponds to the so-called Banach envelope of X, i.e., the "smallest" Banach space containing X. See [**26**].

Next we record a useful abstract extrapolation result from [**25**].

Proposition 5.3. *Let X be as above and fix $0 < p \le 1$. Any isomorphism of X extends uniquely to an isomorphism of $\mathcal{E}_p(X)$. Furthermore, any endomorphism onto X extends to an endomorphism onto $\mathcal{E}_p(X)$.*

Some specific calculations of p-envelopes are as follows; cf. [**25**].

Proposition 5.4. *If Ω is a Lipschitz domain and $(n-1)/n < q < p \le 1$, then*

$$\mathcal{E}_p(H^{1,q}_{at}(\partial\Omega)) = B^{p,p}_{1-(n-1)(1/q-1/p)}(\partial\Omega), \tag{5.12}$$

$$\mathcal{E}_p(H^{q}_{at}(\partial\Omega)) = B^{p,p}_{-(n-1)(1/q-1/p)}(\partial\Omega). \tag{5.13}$$

We are now ready to discuss the main result of this section. Fix a Lipschitz domain Ω in $\mathcal{M}$ and assume that $(n-1)/n < p \leq \infty$, $(n-1)((1/p)-1)_+ < s < 1$. Then, for each $\varepsilon \in (0,1]$ consider the following four conditions

(5.14)

$$
\begin{aligned}
&(\mathrm{I}): \quad \frac{n-1}{n-1+\varepsilon} < p \leq 1 \quad \text{and} \quad (n-1)\left(\frac{1}{p}-1\right)+1-\varepsilon < s < 1;\\
&(\mathrm{II}): \quad 1 \leq p \leq \frac{2}{1+\varepsilon} \quad \text{and} \quad \frac{2}{p}-1-\varepsilon < s < 1;\\
&(\mathrm{III}): \quad \frac{2}{1+\varepsilon} \leq p \leq \frac{2}{1-\varepsilon} \quad \text{and} \quad 0 < s < 1;\\
&(\mathrm{IV}): \quad \frac{2}{1-\varepsilon} \leq p \leq \infty \quad \text{and} \quad 0 < s < \frac{2}{p}+\varepsilon,
\end{aligned}
$$

if $n \geq 3$, and the following three conditions

$$
\begin{aligned}
&(\mathrm{I}'): \quad \frac{2}{1+\varepsilon} \leq p \leq \frac{2}{1-\varepsilon} \quad \text{and} \quad 0 < s < 1;\\
&(\mathrm{II}'): \quad \frac{2}{3+\varepsilon} < p < \frac{2}{1+\varepsilon} \quad \text{and} \quad \frac{1}{p}-\frac{1+\varepsilon}{2} < s < 1;\\
&(\mathrm{III}'): \quad \frac{2}{1-\varepsilon} < p \leq \infty \quad \text{and} \quad 0 < s < \frac{1}{p}+\frac{1+\varepsilon}{2},
\end{aligned}
\tag{5.15}
$$

if $n = 2$. It is illuminating to point out that the conditions (5.14) amount to the membership of the point with coordinates $(s, 1/p)$ to the two-dimensional region depicted in Figure 1. An appropriate interpretation applies to the set of conditions (5.15).

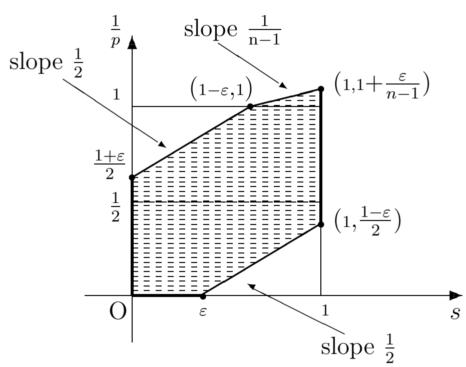

FIGURE 1.

Theorem 5.5. *Retain the same geometrical assumptions as in Theorem* 1.1 *and recall the layer potential operators* (1.4)–(1.7)*. For any* Ω *Lipschitz subdomain of the manifold* $\mathcal{M}$*,* $n \geq 2$*, there exists* $\varepsilon = \varepsilon(\partial\Omega) > 0$ *so that whenever* $\lambda \in \mathbf{R}$ *has* $|\lambda| > 1/2$*, the operators*

$$\lambda I + K_V : B_s^{p,p}(\partial\Omega) \xrightarrow{\sim} B_s^{p,p}(\partial\Omega) \tag{5.16}$$

$$\lambda I + K_V^* : B_{-s}^{q,q}(\partial\Omega) \xrightarrow{\sim} B_{-s}^{q,q}(\partial\Omega) \tag{5.17}$$

are isomorphisms provided $(s, 1/p)$ *and* $(1-s, 1/q)$ *satisfy* (5.15) *if* $n = 2$ *and* (5.14) *if* $n \geq 3$.

Proof. For simplicity, assume that $n \geq 3$, and fix $\lambda \in \mathbf{R}$ with $|\lambda| > 1/2$. The idea is to start with the fact that $\lambda I + K_V$ is an isomorphism of $H_{at}^{1,1}(\partial\Omega)$ if V is a positive constant and then conclude that $\lambda I + K_V$ is an isomorphism of $H_{at}^{1,p}(\partial\Omega)$ for $1 - \varepsilon < p \leq 1$, for some $\varepsilon = \varepsilon(\partial\Omega) > 0$, by virtue of Proposition 5.2. With this in hand, Propositions 5.3–5.4 then prove (5.16) if $0 \leq (n-1)(1/p-1) < 1-s < \varepsilon$.

A similar approach, starting with the fact that $\lambda I + K_V^*$ is an isomorphism of $H_{at}^1(\partial\Omega)$ if V is a positive constant, yields that $\lambda I + K_V^*$ is an isomorphism of the space $B_{-s}^{p',p'}(\partial\Omega)$ whenever $0 \leq (n-1)\cdot(1/p-1) < s < \varepsilon$. Since $(B_{-s}^{1,1}(\partial\Omega))^* = C^s(\partial\Omega)$, this also gives that

$$\lambda I + K_V \quad \text{is an isomorphism of} \quad C^s(\partial\Omega) \quad \text{if} \quad 0 < s < \varepsilon, \quad |\lambda| > \frac{1}{2}. \tag{5.18}$$

The final result, when $n \geq 3$, is then obtained by interpolation. The case $n = 2$ is similar, except that we start perturbing the case $p = 2/3$ (rather than $p = 1$). Passing from constant potentials to the general case when $V \in L^\infty(\mathcal{M})$ is then done by noticing that, for any $V_1, V_2 \in L^\infty(\mathcal{M})$, the differences $K_{V_1} - K_{V_2}$ and $K_{V_1}^* - K_{V_2}^*$ are compact operators on the corresponding Besov spaces. This finishes the proof of the theorem. □

Incidentally, the above reasoning also proves that

$$\lambda I + K_V^* : H_{at}^p(\partial\Omega) \longrightarrow H_{at}^p(\partial\Omega), \tag{5.19}$$

$$\lambda I + K_V : H_{at}^{1,p}(\partial\Omega) \longrightarrow H_{at}^{1,p}(\partial\Omega), \tag{5.20}$$

are Fredholm with index zero for any potential V and any number $\lambda \in \mathbf{R}$ with $|\lambda| \geq 1/2$, granted that $1-\varepsilon < p \leq 1$ when $n \geq 3$ and $2/3-\varepsilon < p \leq 1$ when $n = 2$. As before, these operators are genuine isomorphisms when the potential V is a positive constant.

The conclusions in Theorem 5.5 are particularly relevant in the context of transmission boundary problems with boundary data in Besov spaces. Here is such an example, dealing with the inhomogeneous version of (1.16).

Theorem 5.6. *Let $\mathcal{M}$ be a compact Riemannian manifold, of real dimension ≥ 3, with a Lipschitz metric tensor. Denote by Δ the associated Laplace-Beltrami operator and fix Ω a connected Lipschitz domain in $\mathcal{M}$. Finally, pick some $V \in L^\infty(\mathcal{M})$ such that $V \geq 0$ on $\mathcal{M}$ and $V > 0$ on some set of positive measure in each connected component of Ω_+ and Ω_-.*

Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that the inhomogeneous transmission problem

(5.21)

$$\text{(TBVP-}\textit{inhomogeneous}\text{)} \quad \begin{cases} (\Delta - V)u^+ = f^+ \in L^p_{s+1/p-2,0}(\Omega_+), \\ (\Delta - V)u^- = f^- \in L^p_{s+1/p-2,0}(\Omega_-), \\ u^\pm \in L^p_{s+1/p}(\Omega_\pm), \\ \operatorname{Tr} u^+ - \operatorname{Tr} u^- = g \in B^{p,p}_s(\partial\Omega), \\ \partial_\nu u^+ - \mu\, \partial_\nu u^- = h \in B^{p,p}_{s-1}(\partial\Omega), \end{cases}$$

is well-posed whenever the pair $(s, 1/p)$ satisfies either of the conditions in (5.14) *if $n \geq 3$ and* (5.15) *if $n = 2$.*

Proof. Subtracting appropriate volume potentials from u^+ and u^-, cf. [**33**] for details in similar circumstances, matters are readily reduced to the case when $f^+ = 0$ and $f^- = 0$. In this situation, the result follows from Theorem 5.5 and the mapping properties of single and double layer potentials (1.4)–(1.5) on Besov scales from [**25, 28, 33**]. □

6. The magnetostatic integral operator. Throughout this section, $\mathcal{M}$ is assumed to be a smooth, oriented, compact, boundaryless, Riemannian manifold of real dimension *three*. We debut by discussing the surface divergence operator. Consider $\Omega\subset\mathcal{M}$ an arbitrary Lipschitz domain. First, at the L^p-level with $1<p<\infty$, we set

$$L^p_{\tan}(\partial\Omega):=\{f\in L^p(\partial\Omega,T\mathcal{M}):\langle\nu,f\rangle=0\text{ a.e. on }\partial\Omega\},\tag{6.1}$$

where $T\mathcal{M}$ stands for the tangent bundle to $\mathcal{M}$ (whose sections are vector fields), and introduce

$$\mathrm{Div}:L^p_{\tan}(\partial\Omega)\to L^p_{-1}(\partial\Omega),\tag{6.2}$$

by requiring

$$\int_{\partial\Omega}g\,\mathrm{Div}\,f\,d\sigma=-\int_{\partial\Omega}\langle f,\nabla_{\tan}g\rangle\,d\sigma,\tag{6.3}$$

for each $f\in L^p_{\tan}(\partial\Omega)$, and $g\in L^{p'}_1(\partial\Omega)=(L^p_{-1}(\partial\Omega))^*$, $1/p+1/p'=1$. A space which is going to be important for us in the sequel is

$$L^{p,\mathrm{Div}}_{\tan}(\partial\Omega):=\big\{f\in L^p_{\tan}(\partial\Omega):\mathrm{Div}\,f\in L^p(\partial\Omega)\big\},\tag{6.4}$$

$1<p<\infty$, which we equip with the natural norm

$$\|f\|_{L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)}:=\|f\|_{L^p(\partial\Omega,T\mathcal{M})}+\|\mathrm{Div}\,f\|_{L^p(\partial\Omega)}.\tag{6.5}$$

A closed subspace of (6.5) is

$$L^{p,0}_{\tan}(\partial\Omega):=\big\{f\in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega):\mathrm{Div}\,f=0\big\}.\tag{6.6}$$

Following [**28**], we introduce the class of *tangential Besov spaces*

(6.7)

$$\begin{aligned}TH^p_s(\partial\Omega):=\ &\text{the completion of }\nu\times C^\infty(\mathcal{M},T\mathcal{M})|_{\partial\Omega}\text{ in the norm}\\ &f\longmapsto\|f\|_{B^{p,p}_{(s-1)/p}(\partial\Omega,T\mathcal{M})}+\|\mathrm{Div}\,f\|_{B^{p,p}_{(s-1)/p}(\partial\Omega)},\end{aligned}$$

assuming that $1<p<\infty$, $-1+1/p<s<1/p$. In particular,

$$\begin{aligned}&\mathrm{Div}:TH^p_s(\partial\Omega)\longrightarrow B^{p,p}_{s-(1/p)}(\partial\Omega),\\ &1<p<\infty,\quad -1+1/p<s<1/p,\end{aligned}\tag{6.8}$$

is well-defined and bounded. A convenient way to extend the definition of $\nu \times \nabla_{\tan}$ is via (6.2) and the identity

$$\langle (\nu \times \nabla_{\tan})g, \nu \times f \rangle = \langle g, \operatorname{Div} f \rangle \tag{6.9}$$

if $g \in B^{p,p}_{1-(1/p)+s}(\partial\Omega)$ and $f \in TH^p_s(\partial\Omega)$. Then

$$\begin{gathered} \nu \times \nabla_{\tan} : B^{p,p}_{1-(1/p)+s}(\partial\Omega) \longrightarrow TH^p_s(\partial\Omega), \\ 1 < p < \infty, \quad -1 + 1/p < s < 1/p, \end{gathered} \tag{6.10}$$

is well-defined and bounded. The kernel of the operator (6.9)–(6.10) is the space

$$TH^{p,0}_s(\partial\Omega) := \big\{ f \in TH^p_s(\partial\Omega) : \operatorname{Div} f = 0 \big\}. \tag{6.11}$$

Next we observe that, when considered between appropriate spaces, Div and $\nu \times \nabla_{\tan}$ are Fredholm operators (with indices depending exclusively on the topology of Ω and its boundary). This is made precise in the following proposition proved in [**28**].

Proposition 6.1. *Assume that Ω is an arbitrary Lipschitz domain in $\mathcal{M}$, and suppose that $1 < p < \infty$, $-1 + 1/p < s < 1/p$. Then*

$$\operatorname{Div} : \frac{TH^p_s(\partial\Omega)}{TH^{p,0}_s(\partial\Omega)} \longrightarrow B^{p,p}_{s-(1/p)}(\partial\Omega), \tag{6.12}$$

$$\nu \times \nabla_{\tan} : B^{p,p}_{1+s-(1/p)}(\partial\Omega) \longrightarrow TH^{p,0}_s(\partial\Omega), \tag{6.13}$$

$$\operatorname{Div} : \frac{L^{p,\operatorname{Div}}_{\tan}(\partial\Omega)}{L^{p,0}_{\tan}(\partial\Omega)} \longrightarrow L^p(\partial\Omega), \tag{6.14}$$

$$\nu \times \nabla_{\tan} : L^p_1(\partial\Omega) \longrightarrow L^{p,0}_{\tan}(\partial\Omega) \tag{6.15}$$

are Fredholm operators.

Denote by Δ the Hodge-Laplacian on 1-forms, and let $V \geq 0$ be a bounded, scalar-valued function. Under the current assumptions,

$$\Delta - V : L^2_1(\mathcal{M}, T\mathcal{M}) \longrightarrow L^2_{-1}(\mathcal{M}, T\mathcal{M}) \tag{6.16}$$

is a bounded, negative, formally self-adjoint operator, which is invertible whenever V is not identically zero. In fact, the same is true for $V \equiv 0$ if and only if the first Betti number of $\mathcal{M}$ vanishes, i.e.,

$$H^1_{\mathrm{sing}}(\mathcal{M} : \mathbf{R}) = 0. \tag{6.17}$$

From now on, unless specifically mentioned otherwise, we shall assume that $V \neq 0$. In particular, $\Delta - V$ in (6.16) has an inverse, $(\Delta - V)^{-1}$, whose Schwartz kernel, $\Gamma_V(x,y)$, is a symmetric double form of bidegree $(1,1)$. In local coordinates $\Gamma_V(x,y)$ satisfies, cf. [**31**],

$$\begin{aligned}\Gamma_V(x,y) = {} & \frac{-1}{4\pi\sqrt{\det(g_{jk}(y))}}\Big(\sum_{j,k} g_{jk}(y)(x_j - y_j)(x_k - y_k)\Big)^{-1/2} \\ & \times \sum_{\alpha,\beta} g_{\alpha\beta}(y)\, dx_\alpha \otimes dy_\beta + \text{ a less singular term.}\end{aligned} \tag{6.18}$$

We shall now concern ourselves with the principal value, singular (magnetostatic) integral operator

$$M_V f(x) := \text{ p.v.}\int_{\partial\Omega} \langle \nu(x) \times \mathrm{curl}_x \Gamma_V(x,y), f(y)\rangle\, d\sigma_y, \quad x \in \partial\Omega. \tag{6.19}$$

From [**27, 28, 30**], we know that

$$M_V \text{ is bounded on } \quad L^p_{\tan}(\partial\Omega),\ L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \quad \text{and} \quad TH^p_s(\partial\Omega), \tag{6.20}$$

for each $1 < p < \infty$, $-1 + 1/p < s < 1/p$. This is most transparent in the case when the potential V is constant, say $V \equiv \omega$, for some non-negative $\omega \in \mathbf{R}$. In this case, it has been shown in the aforementioned papers that

$$\mathrm{Div}\, M_\omega f = -\,\omega\,\langle \nu, S_\omega f\rangle - K^*_\omega(\mathrm{Div}\, f) \tag{6.21}$$

for each $f \in TH^p_s(\partial\Omega)$, and

$$(\nu \times \nabla_{\tan}) K_\omega f = -\,\omega\,\nu \times S_\omega(\nu f) + M_\omega(\nu \times \nabla_{\tan} f) \tag{6.22}$$

for each $f \in B^{p,p}_{s+1-1/p}(\partial\Omega)$. Here we are assuming that $1 < p < \infty$, $-1 + 1/p < s < 1/p$, and S_ω, K_ω, K^*_ω are the scalar integral operators from (1.4)–(1.7) associated with the constant potential $V \equiv \omega$.

To state our first theorem, for each fixed $\varepsilon > 0$ consider the following three conditions:

$$
\begin{aligned}
&\frac{2}{1+\varepsilon} < p < \frac{2}{1-\varepsilon} \quad \text{and} \quad -1+\frac{1}{p} < s < \frac{1}{p};\\
&1 < p < \frac{2}{1+\varepsilon} \quad \text{and} \quad \frac{3}{p} - 2 - \varepsilon < s < \frac{1}{p};\\
&\frac{2}{1-\varepsilon} < p < \infty \quad \text{and} \quad -1+\frac{1}{p} < s < \frac{3}{p} - 1 + \varepsilon.
\end{aligned} \tag{6.23}
$$

Theorem 6.2. *Fix a nonnegative potential $V \in L^\infty(\mathcal{M})$ and a Lipschitz domain $\Omega \subseteq \mathcal{M}$. Then there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ such that for each $\lambda \in \mathbf{R}$, $|\lambda| \geq 1/2$, the operator*

$$\lambda I + M_V : TH^p_s(\partial\Omega) \longrightarrow TH^p_s(\partial\Omega) \tag{6.24}$$

is Fredholm with index zero for all s, p satisfying any of the three conditions in (6.23).

Furthermore, for each $\lambda \in \mathbf{R}$, $|\lambda| \geq 1/2$, the operator

$$\lambda I + M_V : L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \longrightarrow L^{p,\mathrm{Div}}_{\tan}(\partial\Omega) \tag{6.25}$$

is also Fredholm with index zero for each $1 < p < 2 + \varepsilon$.

In each case, for a constant, positive potential V, $\lambda I + M_V$ is an isomorphism.

Proof. We shall first assume that $\mathcal{M}$ is such that (6.17) holds. Dispensing with this extra topological hypothesis can then be achieved as in [**28**]. We proceed in a series of steps starting with

Step I. The topological assumption (6.17) guarantees the absence of global monogenic 1-forms on $\mathcal{M}$. Consequently, the unperturbed Hodge-Laplacian Δ has a global fundamental solution, i.e., (6.16) remains invertible when $V \equiv 0$. In particular, (6.21)–(6.22) become genuine intertwining identities when $\omega = 0$, i.e.,

$$\mathrm{Div}\, M_0 = -\, K_0^* \,\mathrm{Div} \quad \text{on} \quad TH^p_s(\partial\Omega), \tag{6.26}$$

and

$$(\nu \times \nabla_{\tan}) K_0 = M_0 (\nu \times \nabla_{\tan}) \quad \text{on} \quad B^{p,p}_{s+1-1/p}(\partial\Omega). \tag{6.27}$$

In turn, (6.26)–(6.27) imply that the diagrams

(6.28)
$$\begin{array}{ccc} B^{p,p}_{s+1-1/p}(\partial\Omega) & \xrightarrow{\lambda I+K_0} & B^{p,p}_{s+1-1/p}(\partial\Omega) \\ \Big\downarrow{\scriptstyle \nu\times\nabla_{\tan}} & & \Big\downarrow{\scriptstyle \nu\times\nabla_{\tan}} \\ TH^{p,0}_s(\partial\Omega) & \xrightarrow{\lambda I+M_0} & TH^{p,0}_s(\partial\Omega) \end{array}$$

and

(6.29)
$$\begin{array}{ccc} TH^{p}_s(\partial\Omega)\big/TH^{p,0}_s(\partial\Omega) & \xrightarrow{\lambda I+M_0} & TH^{p}_s(\partial\Omega)\big/TH^{p,0}_s(\partial\Omega) \\ \Big\downarrow{\scriptstyle \mathrm{Div}} & & \Big\downarrow{\scriptstyle \mathrm{Div}} \\ B^{p,p}_{s-1/p}(\partial\Omega) & \xrightarrow{\lambda I-K_0^*} & B^{p,p}_{s-1/p}(\partial\Omega) \end{array}$$

are commutative. From these and the fact that the operators (5.16)–(5.17) are Fredholm with index zero on the spaces at hand, it follows that

(6.30)
$$\lambda I+M_0 \quad \text{is Fredholm with index zero on the spaces}$$
$$TH^{p,0}_s(\partial\Omega) \quad \text{and} \quad TH^{p}_s(\partial\Omega)\big/TH^{p,0}_s(\partial\Omega), \quad \text{for each } s,p \text{ as in (6.23).}$$

In order to continue, we need the following general fact from the theory of Fredholm operators: *Let A, B, C be Banach spaces and consider the commutative diagram*

(6.31)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

where the two horizontal sequences are exact. Then, if two vertical arrows are Fredholm operators then so is the third one. Furthermore, the index of the middle arrow is the sum of the indexes of the other two vertical arrows.

To implement this result we take

(6.32)
$$A := TH^{p,0}_s(\partial\Omega), \quad B := TH^{p}_s(\partial\Omega),$$
$$C := TH^{p}_s(\partial\Omega)\big/TH^{p,0}_s(\partial\Omega).$$

Also, take the first two horizontal arrows to be inclusions and the next two to be projections (in each short sequence), while all vertical arrows are taken to be natural manifestations of the operator $\lambda I + M_0$ on the spaces listed above.

Thus, at this stage, we have proved the claim made about (6.24) for the choice $V \equiv 0$. In fact the same conclusion remains valid when $V \not\equiv 0$ as well since, thanks to (6.18), the difference $M_{V_1} - M_{V_2}$ is compact on $TH^p_s(\partial\Omega)$ for each $1 < p < \infty$, $0 \le s \le 1$.

Step II. The claim made about the operator (6.25) is proved by following a program similar in spirit to our approach in Step I. This time, the important intertwining identities are

$$\operatorname{Div} M_0 = -K_0^* \operatorname{Div} \quad \text{on} \quad L^p_{\tan}(\partial\Omega), \tag{6.33}$$

and

$$(\nu \times \nabla_{\tan}) K_0 = M_0 (\nu \times \nabla_{\tan}) \quad \text{on} \quad L^p_1(\partial\Omega), \tag{6.34}$$

whereas the relevant commutative diagrams read (with $1 < q < \infty$):

$$\begin{array}{ccc} L^q_1(\partial\Omega) & \xrightarrow{\lambda I + K_0} & L^q_1(\partial\Omega) \\ {\scriptstyle \nu\times\nabla_{\tan}}\downarrow & & \downarrow{\scriptstyle \nu\times\nabla_{\tan}} \\ L^{q,0}_{\tan}(\partial\Omega) & \xrightarrow{\lambda I + M_0} & L^{q,0}_{\tan}(\partial\Omega) \end{array} \tag{6.35}$$

and

$$\begin{array}{ccc} L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)\big/L^{q,0}_{\tan}(\partial\Omega) & \xrightarrow{\lambda I + M_0} & L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)\big/L^{q,0}_{\tan}(\partial\Omega) \\ {\scriptstyle \mathrm{Div}}\downarrow & & \downarrow{\scriptstyle \mathrm{Div}} \\ L^q(\partial\Omega) & \xrightarrow{\lambda I - K_0^*} & L^q(\partial\Omega). \end{array} \tag{6.36}$$

Also, this time, we use the fact that the operators (5.1)–(5.2) are Fredholm. The conclusion is that the operator $\lambda I + M_0$ is Fredholm with index zero on $L^{q,\mathrm{Div}}_{\tan}(\partial\Omega)$ for each $1 < q < 2 + \varepsilon$. Passing to nonzero potentials is then done as before.

Step III. Assume that $\lambda \in \mathbf{R}$, $|\lambda| \ge 1/2$, and that V is a constant, positive potential. Then the operator (6.25) is an isomorphism for each $1 < p < 2 + \varepsilon$.

When $2-\varepsilon < p < 2+\varepsilon$, this can be established as in [**30**], where the Euclidean case was considered. That the same conclusion holds for the full range $1 < p < 2+\varepsilon$ then follows easily from this and Step II.

Step IV. For each $\lambda \in \mathbf{R}$, $|\lambda| \geq 1/2$, the operator (6.24) is in fact an isomorphism provided s, p are as in (6.23) and V is a constant, positive potential.

In order to prove that the operator in (6.24) is in fact invertible when V is a positive constant, say $V \equiv \omega$, it suffices to show that this operator has a dense range for each s, p as in (6.23). This, in turn, will be a consequence of Step III in concert with the observation, proved in [**28**], that

(6.37)
$$\bigcap_{1<q<2+\varepsilon} L^{q,\mathrm{Div}}_{\tan}(\partial\Omega) \hookrightarrow TH^p_s(\partial\Omega) \quad \text{densely for each } s, p \text{ as in (6.23).}$$

Clearly, with these at hand, the desired conclusion follows. This finishes the proof of the theorem. □

Let $k \in \mathbf{R}_+$ and $\mu \in (0,1)$ be two positive constants, and fix a bounded Lipschitz domain $\Omega \subset \mathbf{R}^3$. Then the L^p-transmission problem for the Maxwell equations consists of finding four vector fields, $E_i, H_i : \Omega_+ \to \mathbf{R}^3$ and $E_e, H_e : \Omega_- \to \mathbf{R}^3$, satisfying the following boundary value problem:

(6.38) (TBVP-Maxwell)
$$\begin{cases} \operatorname{curl} E_i - ikH_i = 0 & \text{in } \Omega_+, \\ \operatorname{curl} H_i + ikE_i = 0 & \text{in } \Omega_+, \\ \operatorname{curl} E_e - ikH_e = 0 & \text{in } \Omega_-, \\ \operatorname{curl} H_e + ikE_e = 0 & \text{in } \Omega_-, \\ M(E_i),\, M(H_i),\, M(E_e),\, M(H_e) \in L^p(\partial\Omega), & \\ \nu \times E_e\big|_{\partial\Omega} - \nu \times E_i\big|_{\partial\Omega} = \vec{f} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega), & \\ \nu \times H_e\big|_{\partial\Omega} - \mu\, \nu \times H_i\big|_{\partial\Omega} = \vec{g} \in L^{p,\mathrm{Div}}_{\tan}(\partial\Omega), & \\ x/|x| \times H_e + E_e = o\big(1/|x|\big) & \text{as } |x| \to \infty. \end{cases}$$

The above problem models the scattering of electromagnetic waves by a penetrable bounded obstacle Ω in which case k is the wave number and μ is the transmission parameter. See, e.g., [**2, 24, 29, 36**].

Our main result in this regard is the following.

Theorem 6.3. *For any bounded Lipschitz domain $\Omega \subset \mathbf{R}^3$ there exists $\varepsilon = \varepsilon(\partial\Omega, k, \mu) > 0$ such that the boundary value problem* (6.38) *has a unique solution for any $1 < p < 2 + \varepsilon$. Also, integral representation formulas for the solution in terms of vector layer potentials, as well as natural accompanying estimates hold.*

Proof. The case when $2 - \varepsilon < p < 2 + \varepsilon$ has been proved in [**29**], via layer potential techniques. The same approach works in the more general case discussed here, thanks to Theorem 6.2. □

As a further application, for $1 < p < \infty$, $-1 + 1/p < s < 1/p$, and $\Omega \subset \mathcal{M}$ Lipschitz, we can now consider the *inhomogeneous* transmission boundary value problem for the Maxwell system for a Lipschitz interface on the Riemannian manifold $\mathcal{M}$:

(6.39) (TBVP Maxwell-inhomogeneous)

$$\begin{cases} E_i, H_i \in L^p_s(\Omega_+, T\mathcal{M}), \\ E_e, H_e \in L^p_s(\Omega_-, T\mathcal{M}), \\ \operatorname{curl} E_i - ikH_i = K_i \in L^p_s(\Omega_+, T\mathcal{M}), \\ \operatorname{curl} H_i + ikE_i = J_i \in L^p_s(\Omega_+, T\mathcal{M}), \\ \operatorname{curl} E_e - ikH_e = K_e \in L^p_s(\Omega_-, T\mathcal{M}), \\ \operatorname{curl} H_e + ikE_e = J_e \in L^p_s(\Omega_-, T\mathcal{M}), \\ \nu \times E_e - \nu \times E_i = \vec{f} \in TH^p_s(\partial\Omega), \\ \nu \times H_e - \mu\nu \times H_i = \vec{g} \in TH^p_s(\partial\Omega), \end{cases}$$

where, as before, $k \in \mathbf{R}_+$ and $\mu \in (0,1)$ are fixed constants.

Theorem 6.4. *For any bounded Lipschitz domain $\Omega \subset \mathcal{M}$ there exists a positive constant $\varepsilon = \varepsilon(\partial\Omega, k, \mu)$ such that the boundary value* *problem* (6.39) *has a unique solution for any* s, p *as in* (6.23)*. Also, a naturally accompanying estimate is valid.*

Proof. Subtracting suitable volume potentials, as in [**28**], matters can be reduced to the case when $K_i = J_i = 0$ in Ω_+ and $K_e = J_e = 0$ in Ω_-. At this stage, we may proceed in a fashion similar to [**29**] where the Euclidean case of a similar problem has been dealt with. □

We conclude with an application to the spectral theory of the magnetostatic operator introduced in (6.19). First, if X is a Banach space and $T : X \to X$ is linear and bounded, we denote by $\sigma(T;X)$ the spectrum of T and by $r(T;X)$, the spectral radius of T on X, i.e. the radius of the smallest disk (centered at the origin) containing $\sigma(T;X)$. Alternatively,

$$r(T;X) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}. \tag{6.40}$$

Corollary 6.5. *For any convex bounded domain* $\Omega \subset \mathbf{R}^3$*, there exists* $\varepsilon = \varepsilon(\partial\Omega) > 0$ *with the following significance. For any constant potential* $V \equiv \omega \in \mathbf{R}_+$*,*

$$r\big(M_\omega;\, TH^p_s(\partial\Omega)\big) < \frac{1}{2} \tag{6.41}$$

if $0 < s < 1$, $1 < p < \infty$*, satisfy*

$$\left(\frac{1-\varepsilon}{2}\right)s < \frac{1}{p} < \left(\frac{1-\varepsilon}{2}\right)s + \left(\frac{1+\varepsilon}{2}\right). \tag{6.42}$$

Moreover,

$$r\left(M_\omega;\, L^{p,\mathrm{Div}}_{\tan}(\partial\Omega)\right) < \frac{1}{2} \tag{6.43}$$

if $1 < p < 2 + \varepsilon$.

Proof. This follows from the corresponding results for the (scalar) harmonic layer potentials from [**11**] and (the proof of) Theorem 6.2. See [**11**] for more details. □

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REFERENCES

1. M.S. Agranovich, *Elliptic boundary problems*, Encyclopaedia Math. Sci. **79**, Partial differential equations, IX, Springer, Berlin, 1997, pp. 1–144, 275–281.

2. T. Angell and A. Kirsch, *The conductive boundary condition for Maxwell's equations*, SIAM J. Appl. Math. **52** (1992), 1597–1610.

3. R.B. Barrar and C.L. Dolph, *On a three dimensional transmission problem of electromagnetic theory*, J. Rational Mech. Anal. **3** (1954), 725–743.

4. A.P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution*. II, Adv. Math. **24** (1977), 101–171.

5. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

6. D. Colton and R. Kress, *Integral equation methods in scattering theory*, Wiley, New York, 1983.

7. B.E.J. Dahlberg, *On estimates of harmonic measure*, Arch. Rational Mech. Anal. **65** (1977), 272–288.

8. ———, *On the Poisson integral for Lipschitz and C^1 domains*, Studia Math. **66** (1979), 13–24.

9. B. Dahlberg and C. Kenig, *Hardy spaces and the L^p-Neumann problem for Laplace's equation in a Lipschitz domain*, Ann. of Math. **125** (1987), 437–465.

10. L. Escauriaza, E. Fabes and G. Verchota, *On a regularity theorem for weak solutions to transmission problems with internal Lipschitz boundaries*, Proc. Amer. Math. Soc. **115** (1992), 1069–1076.

11. L. Escauriaza and M. Mitrea, *Transmission problems and spectral theory for singular integral operators on Lipschitz domains*, J. Funct. Anal. **216** (2004), 141–171.

12. L. Escauriaza and J.K. Seo, *Regularity properties of solutions to transmission problems*, Trans. Amer. Math. Soc. **338** (1993), 405–430.

13. E.B. Fabes and D.W. Stroock, *The L_p-integrability of Green's functions and fundamental solutions for elliptic and parabolic equations*, Duke Math. J. **51** (1984), 997–1016.

14. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, reprint of the 1998 ed., *Classics in mathematics*, Springer-Verlag, Berlin, 2001.

15. S. Janson and P.W. Jones, *Interpolation between H_p spaces: the complex method*, J. Funct. Anal. **48** (1982), 58–80.

16. B. Jawerth and M. Mitrea, *Higher dimensional scattering theory on C^1 and Lipschitz domains*, Amer. J. Math. **117** (1995), 929–963.

17. N. Kalton and M. Mitrea, *Stability of Fredholm properties on interpolation scales of quasi-Banach spaces and applications*, Trans. Amer. Math. Soc. **350** (1998), 3837–3901.

18. N.J. Kalton, N.T. Peck and J.W. Roberts, *An F-space sampler*, London Math. Soc. Lecture Note Ser., vol. 89, Cambridge Univ. Press, Cambridge, 1984.

19. C.E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conf. Ser. Math. **83**, Amer. Math. Soc., Providence, RI, 1994.

20. C.E. Kenig and J. Pipher, *The Neumann problem for elliptic equations with nonsmooth coefficients*, Invent. Math. **113** (1993), 447–509.

21. A. Kirsch and L. Päivärinta, *On recovering obstacles inside inhomogeneities*, Math. Meth. Appl. Sci. **21** (1998), 619–651.

22. R.E. Kleinman and P.A. Martin, *On single integral equations for the transmission problem of acoustics*, SIAM J. Appl. Math. **48** (1988), 307–325.

23. O.A. Ladyženskaja, V.J. Rivkind and N.N. Ural'ceva, *Classical solvability of diffraction problems for equations of elliptic and parabolic types*, Dokl. Akad. Nauk SSSR **158** (1964), 513–515.

24. P.A. Martin and P. Ola, *Boundary integral equations for the scattering of electromagnetic waves by a homogeneous dielectric obstacle*, Proc. Roy. Soc. Edinburgh Sect. A **123** (1993), 185–208.

25. S. Mayboroda and M. Mitrea, *The Poisson problem for the Laplacian in Lipschitz domains with data in quasi-Banach spaces*, preprint (2005).

26. O. Mendez and M. Mitrea, *The Banach envelope of Besov and Triebel-Lizorkin spaces and applications*, J. Fourier Anal. Appl. **6** (2000), 503–531.

27. M. Mitrea, *The method of layer potentials in electro-magnetic scattering theory on non-smooth domains*, Duke Math. J. **77** (1995), 111–133.

28. ———, *Sharp Hodge decompositions, Maxwell's equations, and vector Poisson problems on non-smooth three-dimensional Riemannian manifolds*, Duke Math. J. **125** (2004), 467–574.

29. D. Mitrea and M. Mitrea, *Uniqueness for inverse conductivity and transmission problems in the class of Lipschitz domains*, Comm. Partial Differential Equations **23** (1998), 1419–1448.

30. D. Mitrea, M. Mitrea and J. Pipher, *Vector potential theory on non-smooth domains in $\mathbf{R}^3$ and applications to electromagnetic scattering*, J. Fourier Anal. Appl. **3** (1997), 131–192.

31. D. Mitrea, M. Mitrea and M. Taylor, *Layer potentials, the Hodge Laplacian, and global boundary problems in nonsmooth Riemannian manifolds*, Mem. Amer. Math. Soc. **150**, 2001.

32. M. Mitrea and M. Taylor, *Boundary layer methods for Lipschitz domains in Riemannian manifolds*, J. Funct. Anal. **163** (1999), 181–251.

33. ———, *Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev-Besov space results and the Poisson problem*, J. Funct. Anal. **176** (2000), 1–79.

34. ———, *Potential theory on Lipschitz domains in Riemannian manifolds: L^p, Hardy, and Hölder space results*, Comm. Anal. Geom. **9** (2001), 369–421.

35. J. Moser, *On Harnack's theorem for elliptic differential operators*, Comm. Pure Appl. Math. **14** (1961), 577–591.

36. C. Müller, *Foundations of the mathematical theory of electromagnetic waves*, Springer-Verlag, Berlin, 1969.

37. S. Nicaise and A.-M. Sändig, *Transmission problems for the Laplace and elasticity operators: regularity and boundary integral formulation*, Math. Models Methods Appl. Sci. **9** (1999), 855–898.

38. M. Reissel, *On a transmission boundary value problem for the time-harmonic Maxwell equations without displacement currents*, SIAM J. Math. Anal. **24** (1993), 1440–1457.

39. S. Rempel, *Corner singularity for transmission problems in three dimensions*, Integral Equations Operator Theory **12** (1989), 835–854.

40. T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential operators*, de Gruyter, Berlin, 1996.

41. G. Savaré, *Regularity results for elliptic equations in Lipschitz domains*, J. Funct. Anal. **152** (1998), 176–201.

42. J.K. Seo, *Regularity for solutions of transmission problems across internal non-smooth boundary*, Proc. Miniconf. of Partial Differential Equations and Applications, (Seoul, 1995), Lecture Notes Ser. **38**, Seoul National Univ., Seoul, 1997, pp. 189–199,.

43. J. Serrin and H.F. Weinberger, *Isolated singularities of solutions of linear elliptic equations*, Amer. J. Math. **88** (1966), 258–272.

44. Z. Shen, *On the Neumann problem for Schrödinger operators in Lipschitz domains*, Indiana Univ. Math. J. **43** (1994), 143–176.

45. E.M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton Math. Ser. **43**, Monographs in Harmonic Anal., III, Princeton Univ. Press, Princeton, NJ, 1993.

46. H. Triebel, *Function spaces on Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers*, Rev. Mat. Complut. **15** (2002), 475–524.

47. P. Wilde, *Transmission problems for the vector Helmholtz equation*, Proc. Roy. Soc. Edinburgh Sect. A **105** (1987), 61–76.

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