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# NUMERICAL APPROXIMATIONS FOR A CLASS OF VOLTERRA EQUATIONS WITH REALIZABLE KERNELS

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This paper is dedicated to Kendall Atkinson in honor of his many contributions to the field of numerical analysis.

ABSTRACT. In this paper we consider numerical approximations for systems governed by Volterra integro-differential equations with realizable kernels. We investigate and compare numerical methods based on direct integration of the Volterra equations with methods based on internal state realizations. Internal state methods depend on constructing a specific realization and since these realizations are not unique the selection of an internal state model could impact the resulting numerical algorithm. We illustrate this idea by focusing on Volterra equations which can be realized by delay systems and present numerical examples to illustrate the ideas.

**1. Introduction.** In this paper we provide a comparison of numerical algorithms for a class of Volterra integro-differential equations of the form

$$\dot{x}(t) = A_0 x(t) + \int_0^t K(t-s)x(s)\,ds, \quad t > 0, \quad x(0) = x_0 \in \mathbf{R}^N, \tag{1.1}$$

where $A_0$ is an $n \times n$ constant matrix and the kernel $K(\xi)$ is the transfer function of a well-posed linear control system. We assume that there exist a Hilbert space $H$, linear operators $\mathcal{A} : D(\mathcal{A}) \subseteq H \to H$, $\mathcal{B} : \mathbf{R}^N \to H$ and $\mathcal{C} : D(\mathcal{C}) \subseteq H \to \mathbf{R}^N$ such that $\mathcal{A}$ generates a $C_0$-semigroup $S(t)$ on $H$ and for all $x \in \mathbf{R}^N$, $S(t)\mathcal{B}x \in D(\mathcal{C})$ and

$$K(t)x = \mathcal{C}S(t)\mathcal{B}x, \quad t > 0. \tag{1.2}$$

Under these conditions we have a well-defined function

$$K(t) : \mathbf{R}^N \longrightarrow \mathbf{R}^N,$$

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defined by (1.2) which we assume belongs to $L^2(0,T)$ for all finite $T > 0$. The triple of linear operators $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is called a realization of the kernel $K(t)$. Note that neither $\mathcal{C}$ nor $\mathcal{B}$ are required to be bounded and the triple $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is not unique. In particular, there are many possible realizations of the same kernel $K(t)$.

If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a realization of $K(t)$, then one can view the realization as a control system

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u(t), \quad z(0) = z_0 \in H$$

with output

$$y(t) = \mathcal{C}z(t)$$

and input $u(\cdot) \in L^2(0,T)$. Therefore, if $x(\cdot) \in L^2(0,T)$, then the mild solution to

$$\dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}x(t), \quad z(0) = 0 \in H \tag{1.3}$$

is given by the variation of parameters formula

$$z(t) = \int_0^t S(t-s)\mathcal{B}x(s)\,ds$$

and, under suitable assumptions on $\mathcal{C}$, it follows that

$$\begin{aligned} \mathcal{C}z(t) &= \mathcal{C}\int_0^t S(t-s)\mathcal{B}x(s)\,ds \\ &= \int_0^t \mathcal{C}S(t-s)\mathcal{B}x(s)\,ds \\ &= \int_0^t K(t-s)x(s)\,ds. \end{aligned}$$

Hence, the Volterra equation (1.1) can be written in internal state form as the system

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + \mathcal{C}z(t), \quad x(0) = x_0 \in \mathbf{R}^N, \\ \dot{z}(t) &= \mathcal{A}z(t) + \mathcal{B}x(t), \quad z(0) = 0 \in H. \end{aligned} \tag{1.4}$$

If in addition we define

$$y(t) = \mathcal{C}z(t) = \int_0^t K(t-s)x(s)\,ds, \tag{1.5}$$

then the Volterra equation (1.1) and the internal state system (1.4) are equivalent to

$$\dot{x}(t) = A_0 x(t) + y(t), \quad x(0) = x_0 \in \mathbf{R}^N, \tag{1.6}$$

where $y(t)$ is given in (1.5).

Here, the "state" $z(t)$ is called the internal state and the Volterra integral equation (1.1) comes about by using the variation of parameters formula to eliminate this internal state.

These observations lead to (at least) two approaches to the development of numerical methods for approximating the Volterra integro-differential equation (1.1). The direct method is based on constructing numerical schemes specifically for Volterra equations. For example, one could apply a Runge-Kutta type scheme directly to the Volterra equation such as in [**1**]. There is a rich and fruitful literature devoted to this approach and the books by Brunner and van der Houwen (see [**9, 10**]) provide excellent examples of this approach. On the other hand, one could develop numerical schemes based on the coupled system (1.4) which we call an internal state method. Obviously both approaches have benefits and drawbacks. Banks, Buksas and Lin [**2**] note that the internal state method often works much better in certain applications in materials science. Also, Cliff, Herdman and Nguyen have observed similar benefits for other Volterra equations, see [**15, 16**].

The internal state methods are based on a specific realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$, and it is not always clear how to select a "good" realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$. On the other hand, certain problems begin with a "natural" internal state model. Therefore, it is important to understand how the choice of a realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ impacts the resulting numerical method. The goal of this paper is to illustrate how numerical schemes of the same order can produce different results depending on the particular realization. In order to keep the paper within reasonable bounds, we focus on various numerical schemes designed explicitly for delay integro-differential systems, see [**1, 6, 9, 10**]. We provide numerical examples that compare the performance of standard methods when these methods are applied to different internal models of the same problem.

In order to provide a specific example, consider the case where $K(t)$ is the inverse Laplace transform of the scalar function $\widehat{K}(s) = (e^{-s}/s)$

and $A_0 = a_0$ is a scalar. Let $H = \mathbf{R} \times L_2(-1, 0)$, and define $\mathcal{A}$ on

$$D(\mathcal{A}) = \left\{ \begin{bmatrix} \eta \\ \varphi(\xi) \end{bmatrix} : \varphi(\xi) \in H^1(-1, 0), \eta = \varphi(0) \right\} \tag{1.7}$$

by

$$\mathcal{A} \begin{bmatrix} \eta \\ \varphi(\xi) \end{bmatrix} = \begin{bmatrix} 0 \\ -\varphi'(\xi) \end{bmatrix}. \tag{1.8}$$

The operators $\mathcal{B} : \mathbf{R} \to H$ and $\mathcal{C} : H \to \mathbf{R}$ are defined by

$$\mathcal{B}u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = \begin{bmatrix} u \\ 0 \end{bmatrix} \tag{1.9}$$

and

$$\mathcal{C} \begin{bmatrix} \eta \\ \varphi(\xi) \end{bmatrix} = \varphi(-1), \tag{1.10}$$

where $D(\mathcal{C}) = D(\mathcal{A})$. It is well known, see [**3, 12**], that $\mathcal{A}$ generates a $C_0$-semigroup $S(t)$ on $H = \mathbf{R} \times L_2(-1, 0)$, $\mathcal{B}$ is bounded but $\mathcal{C}$ is not bounded. In Section 3 below we show that $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a well-posed realization of $K(t)$. The resulting internal state system is defined by the trivial delay system

$$\begin{aligned} \dot{x}(t) &= a_0 x(t) + y(t), \\ \dot{z}(t) &= x(t) = 0z(t) + 0z(t-1) + x(t), \\ y(t) &= z(t-1). \end{aligned} \tag{1.11}$$

Although it is somewhat more complex to describe, see [**12**] for details, it is possible to construct a second realization $(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1)$ with $\mathcal{A}_1 = \mathcal{A}$, $\mathcal{B}_1$ unbounded and $\mathcal{C}_1$ bounded. This realization corresponds to a system with delays in the control and leads to the delay system

$$\begin{aligned} \dot{x}(t) &= a_0 x(t) + y(t), \\ \dot{z}(t) &= 0z(t) + 0z(t-1) + x(t-1), \\ y(t) &= z(t), \end{aligned} \tag{1.12}$$

where $x(t-1)$ is set to zero when $t-1<0$. As we see below, two such realizations $(\mathcal{A},\mathcal{B},\mathcal{C})$ and $(\mathcal{A}_1,\mathcal{B}_1,\mathcal{C}_1)$ can lead to different numerical problems. Observe that the two previous systems may be written as the delay equations

$$\begin{aligned} &\dot{x}(t) = a_0 x(t) + z(t-1), \quad x(0) = x_0, \\ &\dot{z}(t) = x(t), \quad z(0) = 0, \\ &x(s) = z(s) = 0, \quad -1 < s < 0, \end{aligned} \tag{1.13}$$

and

$$\begin{aligned} &\dot{x}(t) = a_0 x(t) + z(t), \quad x(0) = x_0, \\ &\dot{z}(t) = x(t-1), \quad z(0) = 0, \\ &x(s) = z(s) = 0, \quad -1 < s < 0, \end{aligned} \tag{1.14}$$

respectively. In Section 3 we use these realizations to construct approximations and compare the resulting algorithms.

We note that it is not always clear when a kernel has a well-posed realization. Although this is not the focus of this paper, we provide some basic results that are sufficient for our model problems. If $(\mathcal{A},\mathcal{B},\mathcal{C})$ is a realization of the kernel $K(\xi)$ with both $\mathcal{B}$ and $\mathcal{C}$ bounded, i.e., $\mathcal{B}\in\mathcal{L}(\mathbf{R}^N,H)$ and $\mathcal{C}\in\mathcal{L}(H,\mathbf{R}^N)$, then we say that $(\mathcal{A},\mathcal{B},\mathcal{C})$ is a bounded realization of $K(t)$ and that $K(t)$ is bounded realizable. In order to address unbounded realizations and to provide a precise meaning to the statement that a kernel has a well-posed realization, we use the following definitions (see [**5, 13**]). The definitions are stated in terms of the unilateral Laplace transform defined by

$$\widehat{K}(s) = \int_{0^+}^{\infty} K(t)e^{-st}\,dt$$

for all values of $s$ for which the integral exists (converges). Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace and inverse Laplace operators, respectively. In addition, $L(H_1,H_2)$ is the space of all bounded linear operators from $H_1$ to $H_2$ with the norm

$$||A|| = \max_{u\in H_1} \frac{||Au||_{H_2}}{||u||_{H_1}}.$$

Here $H_1$ and $H_2$ are Hilbert spaces. In addition, we let $\Pi_\rho^+$ denote the right half plane where Re $s > \rho$ and $H^2(\Pi_\rho^+)$ denotes the space of functions which are analytic in $\Pi_\rho^+$ and square integrable along vertical lines in $\Pi_\rho^+$ such that

$$\sup_{u>\rho} \int_{-\infty}^{\infty} |f(u+iw)|^2 \, dw \leq M < \infty.$$

**Definition 1.1.** The kernel $K(t)$ is said to be bounded realizable if its Laplace transform $\widehat{K}(s) \in H^2(\Pi_\rho^+)$ and $(s\widehat{K}(s) - K(0)) \in H^2(\Pi_\rho^+)$ for some $\rho > 0$. A necessary condition for $K(t)$ to be bounded realizable is that its Laplace transform $\widehat{K}(s)$ belongs to $H^2(\Pi_\rho^+) \cap H^\infty(\Pi_\rho^+)$ for some $p \geq 0$.

**Definition 1.2.** If the Laplace transform of $K(t)$, $\widehat{K}(s)$, is a uniformly bounded $L(U, Y)$-valued analytic function on the right half plane with a limit $\widehat{K}(\infty)$ in the positive direction, then we say $K(t)$ is realizable.

We turn now to specific model problems and use these problems to illustrate the basic results.

**2. A Volterra integro-differential equation with bounded-realizable kernel.** In this section, we provide an example of a Volterra integro-differential equation where the kernel is bounded realizable. We apply the internal state method and compare our numerical results to those obtained by the direct numerical method. The numerical approximations from both the internal state method and the direct numerical method are also compared against the exact solution. We report the computing time for each method.

Consider the scalar Volterra integro-differential equation

$$\dot{x}(t) = -x(t) + \int_0^t K(t-s)x(s)\, ds, \quad t > 0, \quad x(0) = 1, \tag{2.1}$$

where

$$K(t) = \begin{cases} 0 & t < 1, \\ t-1 & t \geq 1. \end{cases} \tag{2.2}$$

To apply the internal state method to this problem, we first show that the kernel $K(t)$ is realized by a bounded realization. Taking the Laplace transform of $K(t)$, we find that

$$\widehat{K}(s) = \frac{e^{-s}}{s^2},$$

In [**5**] Baras and Brocket stated without proof that $\widehat{K}(s) = e^{-s}/s^2$ is bounded realizable. For completeness and to set the stage for the examples below, we verify that $\widehat{K}(s)$ satisfies the sufficient conditions for $K(t)$ to be bounded realizable.

**Theorem 2.1.** *The kernel $K(t)$ given by* (2.2) *is bounded realizable.*

*Proof.* First we establish that

$$\widehat{K}(s) = \frac{e^{-s}}{s^2} \in H^2(\Pi_\rho^+)$$

with

$$\sup_{x>\rho} \int_{-\infty}^{\infty} \left| \frac{e^{-(x+iy)}}{(x+iy)^2} \right|^2 dy < \infty.$$

We fix $\rho > 0$; then we have

$$\begin{aligned}
\sup_{x>\rho} \int_{-\infty}^{\infty} \left| \frac{e^{-(x+iy)}}{(x+iy)^2} \right|^2 dy &\le \sup_{x>\rho} \int_{-\infty}^{\infty} \frac{|e^{-2x}|\,|e^{-2iy}|}{|x^2+2ixy-y^2|^2}\, dy \\
&= \sup_{x>\rho} |e^{-2x}| \int_{-\infty}^{\infty} \frac{|e^{-2iy}|}{x^4+2x^2y^2+y^4}\, dy \\
&\le \sup_{x>\rho} |e^{-2x}| \int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^2}\, dy,
\end{aligned}$$

knowing $|e^{-2iy}| = 1$. We note that the function $(x^2+y^2)^{-2}$ is dominated by $1/x^4$ for $y \in [-1,1]$ and by $1/y^4$ otherwise. Thus, it is obvious that the right side of equation (2.3) is bounded by

$$\sup_{x>\rho} |e^{-2x}| \left[ \int_{-1}^{1} \frac{1}{x^4}\, dy + \int_{-\infty}^{-1} \frac{1}{y^4}\, dy + \int_{1}^{\infty} \frac{1}{y^4}\, dy \right].$$

Completing these integrals together with the estimate (2.3) gives

$$\begin{aligned}\sup_{x>\rho}\int_{-\infty}^{\infty}\left|\frac{e^{-(x+iy)}}{(x+iy)^2}\right|^2 dy &\le \sup_{x>\rho}|e^{-2x}|\left[\frac{2}{x^4}-\frac{1}{3y^3}\Big|_{-\infty}^{-1}-\frac{1}{3y^3}\Big|_{1}^{\infty}\right]\\ &= \sup_{x>\rho}|e^{-2x}|\left[\frac{2}{x^4}+\frac{2}{3}\right]\\ &< |e^{-2\rho}|\left[\frac{2}{\rho^4}+\frac{2}{3}\right].\end{aligned}$$

Now we let $s\in\mathcal{C}$. We know $e^{-s}$ and $s^2$ are differentiable for $\operatorname{Re}s>0$. Thus, for $\operatorname{Re}s>0$, $\widehat{K}(s)$ is analytic and

$$\widehat{K}'(s)=\frac{e^{-(s)}}{(s)^2}=\frac{-s^2e^{-s}-2e^{-s}}{s^3}.$$

Next we show that $\widehat{K}(s)$ defined above satisfies

$$s\widehat{K}(s)-K(0)\in H^2(\Pi_\rho^+).$$

We first note that

$$\sup_{x>\rho}\int_{-\infty}^{\infty}\left|\frac{e^{-(x+iy)}}{(x+iy)}\right|^2 dy<\infty,$$

since

$$s\widehat{K}(s)=s\frac{e^{-s}}{s^2}=\frac{e^{-s}}{s}$$

and $K(0)=0$. So when we fix $\rho>0$, we have the following estimates

$$\begin{aligned}\sup_{x>\rho}\int_{-\infty}^{\infty}\left|\frac{e^{-(x+iy)}}{(x+iy)}\right|^2 dy &\le \sup_{x>\rho}\int_{-\infty}^{\infty}\frac{|e^{-2x}|\,|e^{-2iy}|}{x^2+y^2}\,dy\\ &= \sup_{x>\rho}|e^{-2x}|\int_{-\infty}^{\infty}\frac{|e^{-2iy}|}{x^2+y^2}\,dy\\ &\le \sup_{x>\rho}|e^{-2x}|\int_{-\infty}^{\infty}\frac{1}{x^2+y^2}\,dy,\end{aligned}\tag{2.4}$$

where we use $|e^{-2iy}| = 1$. The expression $1/(x^2+y^2)$ is bounded by $1/x^2$ for $y \in [-1,1]$ and $1/y^2$ for $y \notin [-1,1]$. Therefore, the right side of equation (2.4) is bounded by

$$\begin{aligned}\sup_{x>\rho} |e^{-2x}| \left[\int_{-1}^{1} \frac{1}{x^2}\,dy + \int_{-\infty}^{-1} \frac{1}{y^2}\,dy + \int_{1}^{\infty} \frac{1}{y^2}\,dy\right] & \\ = \sup_{x>\rho} |e^{-2x}| \left[\frac{2}{x^2} - \frac{1}{y}\Big|_{-\infty}^{-1} - \frac{1}{y}\Big|_{1}^{\infty}\right] & \\ = \sup_{x>\rho} |e^{-2x}| \left[\frac{2}{x^2} + 2\right] & \\ < |e^{-2\rho}| \left[\frac{2}{\rho^2} + 2\right]. &\end{aligned}$$

We let $h(s) = s$ on $\mathcal{C}$. Obviously, $h$ is differentiable for $\operatorname{Re} s > 0$ and we know $\widehat{K}(s)$ is differentiable for $\operatorname{Re} s > 0$. It follows that $s\widehat{K}(s)$ is analytic for $\operatorname{Re} s > 0$,

$$[s\widehat{K}(s)]' = \frac{-e^{-s}}{s^2} - \frac{e^{-s}}{s},$$

and the proof is complete. □

Following the procedures in [**11**], one can easily show that the delay control system

$$\begin{aligned} \dot{z_1}(t) &= z_2(t-1), \\ \dot{z_2}(t) &= u(t), \end{aligned} \tag{2.5}$$

with output

$$y(t) = z_1(t),$$

provides a bounded realization of the kernel in (2.2) above. In particular, let $H = \mathbf{R}^2 \times L_2(-1,0;\mathbf{R}^2)$ and define $\mathcal{A}$ on

$$D(\mathcal{A}) = \left\{ \begin{bmatrix} \eta \\ \varphi(\xi) \end{bmatrix} : \varphi(\cdot) \in H^1(-1,0;\mathbf{R}^2),\ \eta = \varphi(0) \right\}$$

by

$$\mathcal{A} \begin{bmatrix} \eta \\ \varphi(\xi) \end{bmatrix} = \begin{bmatrix} E_0\eta + E_1\varphi(-1) \\ -\varphi'(\xi) \end{bmatrix},$$

where $E_0$ and $E_1$ are $2 \times 2$ matrices given by

$$E_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

If the operators $\mathcal{B} : \mathbf{R} \to H$ and $\mathcal{C} : H \to \mathbf{R}$ are defined by

$$\mathcal{B}u = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ u \\ 0 \\ 0 \end{bmatrix}$$

and

$$\mathcal{C} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \varphi_1(\cdot) \\ \varphi_2(\cdot) \end{bmatrix} = \eta_1,$$

then $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ provide a bounded realization of $K(t)$ given by (2.2).

It follows that the internal state representation for the integro-differential equation (2.1) can be written as the delay differential equation (DDE)

$$\begin{aligned} \dot{x}(t) &= -x(t) + z_1(t), \\ \dot{z}_1(t) &= z_2(t-1), \\ \dot{z}_2(t) &= x(t), \end{aligned} \tag{2.6}$$

with initial conditions

$$x(0) = 1, \quad z_1(0) = 0, \quad z_2(0) = 0, \tag{2.7}$$

and initial histories

$$x(\tau) = 0, \quad z_1(\tau) = 0, \quad z_2(\tau) = 0, \quad \tau \in (-1, 0). \tag{2.8}$$

It is straightforward to verify that the above delay system (2.6) is an internal state representation of the integro-differential equation (2.1). In particular, use (2.5) to note that

$$\mathcal{L} \begin{pmatrix} \dot{z_1}(t) \\ \dot{z_2}(t) \end{pmatrix} = \mathcal{L} \begin{pmatrix} z_2(t-1) \\ x(t) \end{pmatrix},$$

which implies

$$s\widehat{Z_1}(s) = e^{-s}\widehat{Z_2}(s)$$
$$s\widehat{Z_2}(s) = \widehat{X}(s).$$

Here $\widehat{Z_1}(s)$, $\widehat{Z_2}(s)$, and $\widehat{X}(s)$ represent the Laplace transforms of $z_1(t)$, $z_2(t)$, and $x(t)$, respectively. Solving for $\widehat{Z_1}$, we get

$$\widehat{Z_1}(s) = \frac{e^{-s}}{s^2}\,\widehat{X(s)},$$

and by the convolution theorem,

$$z_1(t) = \int_0^t \mathcal{L}^{-1}\left(\frac{e^{-(t-\tau)}}{(t-\tau)^2}\right)x(\tau)\,d\tau$$

which yields

$$z_1(t) = \int_0^t K(t-\tau)x(\tau)\,d\tau,$$

where $K(t)$ is defined in (2.2).

In order to compare numerical schemes we approximate the internal state model defined by the delay system (2.6) with initial data given by (2.7)–(2.8) by the method of steps using a fourth order Runge-Kutta method, see [**6**]. We can solve for the exact solution of Volterra equation (2.1) using a change of variables and the method of steps. We let $w = t - s$, then equation (2.1) becomes

$$\dot{x}(t) = -\,x(t) + \int_0^t K(w)x(t-w)\,dw, \quad t > 0, \quad x(0) = 1.$$

On the interval $[0, 1)$, equation (2.1) is simply

$$\dot{x}(t) = -\,x(t)$$

since $K(t) = 0$ for $t \in [0, 1)$. With the initial condition, $x(0) = 1$, it follows that $x(t) = e^{-t}$ is the solution of equation (2.1) on $[0, 1)$. For $t \geq 1$, we split the integral in equation (2.1) into two integrals as follows

$$\dot{x}(t) = -\,x(t) + \int_0^1 K(s)x(t-s)\,ds + \int_1^t K(s)x(t-s)\,ds. \tag{2.9}$$

We then substitute for $K(t)$ and with a change of variables, we are left with

$$\dot{x}(t) = -x(t) + \int_0^{t-1} (t-s-1)x(s)\,ds,$$

since $K(t) = 0$ for $t \in [0,1)$ we have $\int_0^1 K(s)x(t-s)\,ds = 0$. Now on the interval $[1,2]$, $x(s)$ is known as $e^{-s}$ for $s \in [0,1]$; thus, equation (2.1) becomes

$$\dot{x}(t) = -x(t) + \int_0^{t-1} (t-s-1)e^{-s}\,ds, \quad x(1) = e^{-1}.$$

After we integrate $\int_0^{t-1}(t-s-1)e^{-s}\,ds$, the integro-differential equation (2.1) is simply the nonhomogenous ODE,

$$\dot{x}(t) = -x(t) + t + e^{(1-t)} - 2, \quad x(1) = e^{-1},$$

for $t \in [1,2)$. It follows that the solution of this ODE is given by

$$x(t) = t + te^{(1-t)} + (1+e)e^{-t} - 3.$$

Similarly, the ideas above can be used to solve for $x(t)$ for $t \in [n, n+1)$, where $n = 2, 3, \ldots, N$. In order to obtain the solution $x$ on the interval $[2,3)$, we solve the integro-differential equation

$$\begin{aligned}\dot{x}(t) = &-x(t) + \int_0^1 (t-s-1)e^{-s}\,ds \\ &+ \int_1^{t-1} (t-s-1)\big(s + se^{(1-s)} + (1+e)e^{-s} - 3\big)\,ds\end{aligned}$$

with the initial condition

$$x(2) = e^{-2} + 3e^{-1} - 1.$$

Therefore, it follows that for $t \in [2,3)$ the solution is given by

$$\begin{aligned}x(t) = &(4e^2 + e + 1)e^{-t} + te^{(1-t)} \\ &+ \left(\frac{t^2}{2} + 2t\right)e^{(2-t)} + \frac{1}{6}\,t^3 - \frac{5}{2}\,t^2 + 15t - \left(15 + \frac{104}{6}\right).\end{aligned}$$

We construct a direct numerical method for the Volterra equation (2.9). In particular, we define $g(t, x)$ by

$$g(t, x) = -x(t) + \int_0^t -1(t - s - 1)x(s)\, ds$$

and then approximate the solution $x(t)$ for $t \in [0, T]$ using the explicit fourth order Runge-Kutta formulas

$$\begin{aligned} x_{n+1} &= x_n + \frac{h}{6}(K_1 + 2K_2 + 2K_3 + K_4) \\ K_1 &= g(t_n,\, x_n) \\ K_2 &= g\Big(t_n + \frac{h}{2},\, x_n + \frac{h}{2} K_1\Big) \\ K_3 &= g\Big(t_n + \frac{h}{2},\, x_n + \frac{h}{2} K_2\Big) \\ K_4 &= g(t_n + h,\, x_n + hK_3), \end{aligned}$$

where $x_n = x(t_n)$. Simpson's rule is employed to calculate the integral. Using the exact solution given above, we can verify the performances of these two numerical schemes.

In Table 1, we provide the numerical values of the exact solution and the solutions from the two methods for the step sizes $h = 0.1$ and $h = 0.2$. We recognize from Table 1 that the two numerical approximations are very close to the exact solution. We then let $h = 0.01$ and plot the two numerical solutions against the exact solution on the interval $[0, 3]$ in Figure 1. We then observe the computing time for each method in Table 2. We note that, although we don't see a big difference in the computing time of these two numerical methods, the internal state method takes half the computing time compare to the direct numerical method. This observation agrees with results from more complex systems ([**2, 16**]).

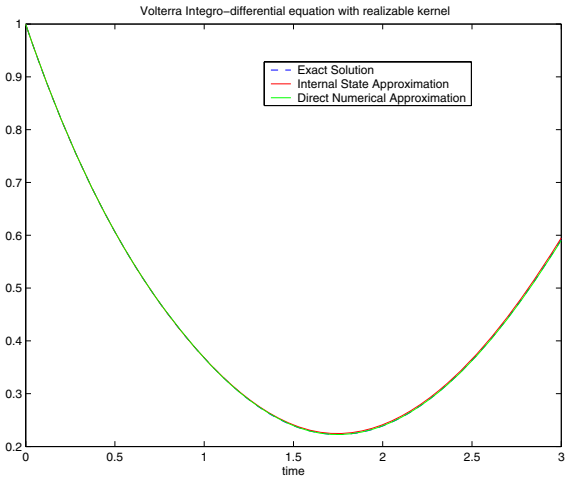


FIGURE 1. Exact solution and approximated solutions from internal state and direct numerical method ($h = 0.01$).

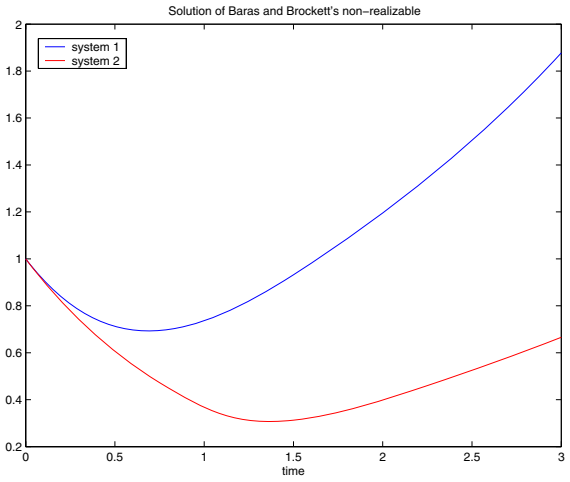


FIGURE 2. A comparison of the solutions based on the two internal state models: $x(\tau) = 1$, $\tau \in (-1, 0)$.

TABLE 1. Numerical values of the solutions with different time steps, $h = 0.1$ and $h = 0.2$.

| $t$ | Internal state solution | | Direct method solution | | Exact solution | |
|---|---|---|---|---|---|---|
| | $h=0.2$ | $h=0.1$ | $h=0.2$ | $h=0.1$ | $h=0.2$ | $h=0.1$ |
| 0 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| 0.1000 | | 0.9048 | | 0.9048 | | 0.9048 |
| 0.2000 | 0.8187 | 0.8187 | 0.8187 | 0.8187 | 0.8187 | 0.8187 |
| 0.3000 | | 0.7408 | | 0.7408 | | 0.7408 |
| 0.4000 | 0.6703 | 0.6703 | 0.6703 | 0.6703 | 0.6703 | 0.6703 |
| 0.5000 | | 0.6065 | | 0.6065 | | 0.6065 |
| 0.6000 | 0.5488 | 0.5488 | 0.5488 | 0.5488 | 0.5488 | 0.5488 |
| 0.7000 | | 0.4966 | | 0.4966 | | 0.4966 |
| 0.8000 | 0.4493 | 0.4493 | 0.4493 | 0.4493 | 0.4493 | 0.4493 |
| 0.9000 | | 0.4066 | | 0.4066 | | 0.4066 |
| 1.0000 | 0.3679 | 0.3679 | 0.3679 | 0.3679 | 0.3679 | 0.3679 |
| 1.1000 | | 0.3330 | | 0.3331 | | 0.3330 |
| 1.2000 | 0.3023 | 0.3022 | 0.3031 | 0.3026 | 0.3024 | 0.3024 |
| 1.3000 | | 0.2760 | | 0.2767 | | 0.2764 |
| 1.4000 | 0.2541 | 0.2545 | 0.2570 | 0.2558 | 0.2554 | 0.2554 |
| 1.5000 | | 0.2381 | | 0.2400 | | 0.2395 |
| 1.6000 | 0.2255 | 0.2270 | 0.2313 | 0.2294 | 0.2288 | 0.2288 |
| 1.7000 | | 0.2210 | | 0.2242 | | 0.2235 |
| 1.8000 | 0.2177 | 0.2204 | 0.2267 | 0.2242 | 0.2234 | 0.2234 |
| 1.9000 | | 0.2250 | | 0.2295 | | 0.2286 |
| 2.0000 | 0.2306 | 0.2347 | 0.2429 | 0.2400 | 0.2390 | 0.2390 |

TABLE 2. Computing time of the two numerical methods in seconds ($h=0.01$).

| Numerical methods | Computing time |
|---|---|
| Internal state method | 0.1500 |
| Direct numerical method | 0.2500 |

**4. Volterra integro-differential equation with unbounded realizable kernel.** In this section we focus on a Volterra integro-differential equation defined by a kernel which is realizable by a well-posed internal model but it is not bounded realizable. In addition, we compare numerical schemes based on two different realizations. Consider the Volterra integro-differential equation

$$\dot{x}(t) = -x(t) + \int_0^t K(t-s)x(s)\,ds \quad t > 0, \quad x(0) = 1, \tag{3.1}$$

where

$$K(t) = \begin{cases} 0 & t < 1, \\ 1 & t \geq 1. \end{cases} \tag{3.2}$$

As for the previous model problem above, we can construct the exact solution to (3.1) by the method of steps. After a change of variables, system (3.1) becomes

$$\dot{x}(t) = -x(t) + \int_0^t K(s)x(t-s)\,ds, \quad t > 0, \quad x(0) = 1. \tag{3.3}$$

Then on the interval $[0, 1)$, we have

$$\dot{x}(t) = -x(t), \quad x(0) = 1,$$

which yields $x(t) = e^{-t}$ as the solution and $x(1) = e^{-1}$.

For $t \in [1, 2)$, we split the integral limits in (3.1) from 0 to 1 and from 1 to $t$, substitute for $K(t)$, and do another change of variables. We then are left with an ODE initial value problem

$$\dot{x}(t) = -x(t) + \int_0^{t-1} e^{-s}\,ds, \quad x(1) = e^{-1}. \tag{3.4}$$

It follows that the solution for the ODE (3.4) is given by

$$x(t) = -te^{(1-t)} + e^{-t} + 1,$$

for $t \in [1, 2)$ with

$$x(2) = 2e^{-1} + e^{-2} + 1.$$

Following the procedure given above, the solution $x(t)$ for $t \in [2,3)$ is

$$x(t) = -3 - te^{(1-t)} + t - 1 + \frac{t^2}{2}\, e^{(2-t)} + (1+e^2)e^{-t}.$$

We can obtain the solution $x(t)$ for $t \in [n, n+1)$, where $n = 2, 3, \ldots, N$ by the method of steps employing the same procedure.

Clearly $K(t)$ is not bounded realizable since $K(t)$ is not continuous at $t = 1$, which is a necessary condition for $K(t)$ to be bounded realizable (see [**5**]). However, $K(t)$ is realizable by a well-posed system.

**Theorem 3.1.** *The kernel $K(t)$ defined in* (3.2) *is realizable.*

*Proof.* Consider the Laplace transform of $K(t)$ defined in (3.2), that is,

$$\widehat{K}(s) = \frac{e^{-s}}{s}.$$

From the Volterra equation (3.1), for $s > 0$, $\mathbf{R}$ is both the input space and the output space. Therefore, for $u_1, u_2 \in \mathbf{R}$, we want to show $\widehat{K}(s)$ is in $L(\mathbf{R}, \mathbf{R})$. It follows that

$$\begin{aligned}
\widehat{K}(s)(u_1 + u_2) &= \frac{e^{-s}}{s}\,(u_1 + u_2) \\
&= \frac{e^{-s}}{s}\, u_1 + \frac{e^{-s}}{s}\, u_2 \\
&= \widehat{K}(s)u_1 + \widehat{K}(s)u_2.
\end{aligned}$$

Also, for $u \in \mathbf{R}$, and $\alpha \in \mathbf{R}$ we have

$$\begin{aligned}
\widehat{K}(s)\alpha u &= \frac{e^{-s}}{s}\, \alpha u \\
&= \alpha\, \frac{e^{-s}}{s}\, u \\
&= \alpha \widehat{K}(s)u.
\end{aligned}$$

Hence, $\widehat{K}(s)$ is in $L(\mathbf{R},\mathbf{R})$. As for $\widehat{K}(s)$ being uniformly bounded on $L(\mathbf{R},\mathbf{R})$, we fix $s=\beta>0$ and then we have

$$\begin{aligned}
\|\widehat{K}(s)\| &= \max_{u\in\mathbf{R}\setminus\{0\}} \frac{\|\widehat{K}(s)u\|R}{\|u\|R} \\
&= \max_{u\in\mathbf{R}\setminus\{0\}} \frac{\|(e^{-s}/s)u\|R}{\|u\|R} \\
&\le \max_{u\in\mathbf{R}\setminus\{0\}} \frac{\|e^{-s}/s\|R\|u\|R}{\|u\|R} \\
&= \left\|\frac{e^{-s}}{s}\right\|R \\
&< \left|\frac{e^{-\beta}}{\beta}\right|.
\end{aligned}$$

Following the analysis of Section 2, it is obvious that $\widehat{K}(s)$ is analytic on $\{s\in\mathcal{C}\mid \operatorname{Re} s>0\}$ and

$$\widehat{K}'(s) = -\frac{e^{-s}}{s^2} - \frac{e^{-s}}{s}.$$

Next we show that the limit of $\widehat{K}$ exists in the positive direction, i.e.,

$$\lim_{x\to\infty} \max\, y\|\widehat{K}(x+iy)\| = \widehat{K}(\infty) < \infty.$$

We have the estimate

$$\begin{aligned}
\lim_{x\to\infty} \max\, y\,\|\widehat{K}(x+iy)\| &= \lim_{x\to\infty} \max\, y\left|\frac{e^{-(x+iy)}}{x+iy}\right| \\
&= \lim_{x\to\infty} \max\, y\,\frac{|e^{-x}e^{-iy}|}{|x+iy|} \\
&= \lim_{x\to\infty} \max\, y\,\frac{|e^{-x}e^{-iy}|}{\sqrt{x^2+y^2}} \\
&\le \lim_{x\to\infty} y\,\frac{|e^{-x}|}{x} \\
&= y\lim_{x\to\infty} \frac{|e^{-x}|}{x} \\
&= 0,
\end{aligned} \tag{3.5}$$

which completes the proof. □

As noted in the introduction, we may construct two realizations $(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1)$ and $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ of $K(t)$ which lead to two internal state models defined by delay systems

$$\begin{aligned} \dot{x}(t) &= -x(t) + z(t), \quad x(0) = 1, \\ \dot{z}(t) &= x(t-1), \quad z(0) = 0, \\ x(s) &= z(s) = 0, \quad -1 < s < 0, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} \dot{x}(t) &= -x(t) + z(t-1), \quad x(0) = 1, \\ \dot{z}(t) &= x(t), \quad z(0) = 0, \\ x(s) &= z(s) = 0, \quad -1 < s < 0, \end{aligned} \tag{3.7}$$

respectively.

The realization $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ that produces the internal state model (3.7) is defined by (1.7)–(1.10) in the introduction and has a bounded input operator $\mathcal{B}$ and unbounded output operator $\mathcal{C}$. The realization $(\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1)$ that produces the internal state model (3.6) comes from a control system with delays in the control and results in the case where the input operator $\mathcal{B}_1$ is unbounded and the output operator $\mathcal{C}_1$ is bounded (see [**12**] for details). Although both delay systems (3.6) and (3.7) are well-posed realizations, the internal state model (3.7) is in some sense more natural. It is a minimal realization of the kernel (3.2) (see [**5, 11**]). On the other hand, although the original Volterra equation (3.1) does not require any initial data for $-1 < s < 0$, the internal state model (3.6) contains the past history $x(t-1)$ of the "state" $x(t)$. In the realization given by (3.7) there is no need to specify $x(s)$ for $-1 < s < 0$ so we set the initial data to $x(s) = 0$ when $-1 < s < 0$. However, since $x(0) = 1$ one might think that in the internal model (3.6) it is "natural" to set $x(s) \equiv 1$ for $-1 < s < 0$. We shall return to this issue when we present the numerical results below.

In order to verify that both (3.6) and (3.7) provide internal state models for the kernel $K(t)$ given by (3.2), one simply applies the Laplace transform to both systems. The details are similar to the example with bounded realization given in the previous section. We turn now to numerical results.

We apply the method of steps together with the fourth order Runge-Kutta method above to both systems (3.6) and (3.7). Unless stated otherwise, we denote system 1 and system 2 in the plots to be the DDE (3.6) and DDE (3.7), respectively. The first issue we address concerns the choice of initial data for the internal state model (3.6). Since the choice of initial function $x(\tau)$ for $\tau \in (-1, 0)$ in the system (3.6) is not naturally defined by the Volterra equation (3.1) nor the construction of the state representation, we consider two choices for the initial history for the state $x(t)$. In particular, we will use $x(\tau) \equiv 1$ and $x(\tau) \equiv 0$, $\tau \in (-1, 0)$. When $x(\tau) \equiv 1$ for $\tau \in (-1, 0)$, we expect the solutions to be different for $t > 0$, and this is confirmed in Figure 2. Moreover, since we know that the past history of the state $x(t)$ does not effect the solution of the internal model (3.7) nor the original Volterra equation (3.1), we expect the second internal model to produce the correct solution regardless of the choice of the initial history $x(\tau)$ for $\tau \in (-1, 0)$. This is clearly illustrated in Figure 3 which shows the numerical solution of the internal state model (3.7) is essentially the exact solution of Volterra equation (3.1) regardless of the fact that we have used $x(\tau) \equiv 1$ for $\tau \in (-1, 0)$. Therefore, it is clear that $x(\tau) \equiv 0$, $\tau \in (-1, 0)$ is the "correct" initial data for both internal state models.

Even if we use the initial history $x(\tau) \equiv 0$, $\tau \in (-1, 0)$ and the fourth order method above is applied to both delay systems (3.6) and (3.7), we see a difference in the numerical solutions. This is illustrated in Figure 4 where we compare the numerical solutions produced by the two models (3.6) and (3.7) with a mesh size $h = 1/64$. There is a clear difference between these solutions when $1 < t < 3$.

Figure 5 provides a comparison of the numerical solutions of the Volterra integro-differential equation (3.1) based on using the internal state model (3.7), the direct numerical method, and the exact solution. At this level, both schemes are highly accurate but the direct numerical method takes approximately twice the amount of computing time (0.2200 seconds) compared to the internal state method (0.0910 seconds). This simple example is typical of all the runs we made on similar problems and illustrates the importance of selecting an appropriate internal state realization.

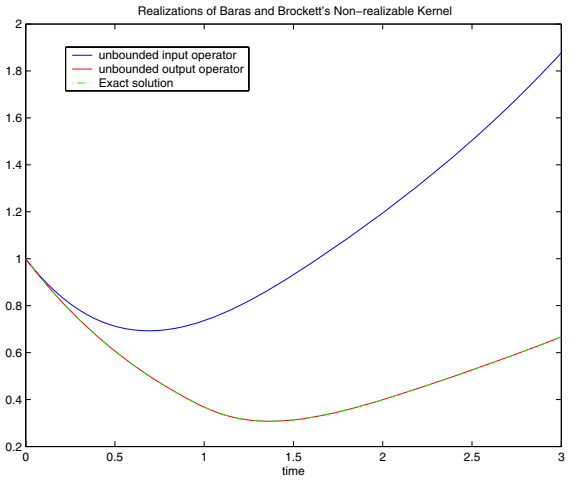


FIGURE 3. A comparison of the solutions based on the two internal state models with the exact solution of the Volterra equation: $x(\tau) = 1$, $\tau \in (-1, 0)$.

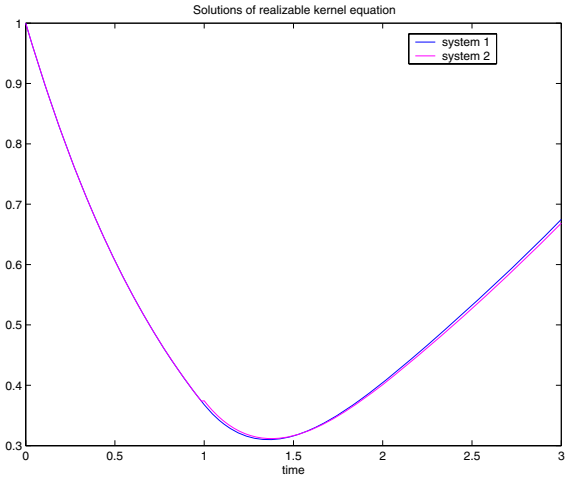


FIGURE 4. Solutions of the two internal state models: $x(\tau) = 0$, $\tau \in (-1, 0)$.

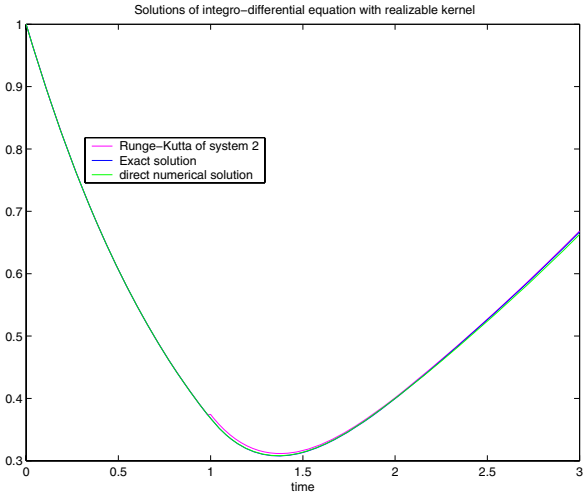


FIGURE 5. A comparison of the numerical solutions to the exact solution.

**4. Conclusions.** The results above illustrate that numerical methods based on internal state models and direct numerical integration of the Volterra equations can be comparable in terms of accuracy and performance. Moreover, we have shown that the choice of the internal state model can impact accuracy even if one uses the same numerical method on both internal models. In most cases, we also observed that the internal state method can reduce computing time when compared to a direct numerical method of the same order. These results are consistent with what others have found in electromagnetic and HIV systems ([**2, 16**]).

The choice of realization clearly impacts accuracy so it is worthwhile to investigate how one might select the "best" internal state model. We conjecture that minimal realizations may provide insight into this issue (see [**5, 11, 13**]). We plan to investigate this issue in future papers.

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