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NONCONFORMING FINITE ELEMENT METHODS FOR A CLAMPED PLATE WITH ELASTIC UNILATERAL OBSTACLE

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Dedicated to Professor Kendall Atkinson
It is a privilege being his colleague and collaborator.

ABSTRACT. In this paper we analyze nonconforming finite element methods for solving a fourth order boundary value problem describing the deformation of a clamped elastic thin plate unilaterally constrained by an elastic obstacle. Optimal order error estimates are derived for both continuous and discontinuous nonconforming finite elements.

1. Introduction. Nonconforming methods are popularly employed in solving high order differential equations. For fourth order boundary value problems, conforming finite elements require C^1 continuity. For multi-dimensional spatial domains, it is not easy to construct C^1 elements, and the resulting C^1 elements are usually difficult to use. On the contrary, nonconforming finite elements are easy to construct and easy to use, since the smoothness requirement on finite element functions is weakened to either C^0 continuity or even less than C^0 continuity. Application of nonconforming finite element methods is not limited to fourth order or higher order problems, though; they offer more efficient solution algorithms for numerous other problems, see, e.g., [**3**, p. 208].

It is more delicate to provide convergence and error analysis for nonconforming finite element methods than for conforming finite element methods. An early reference on the mathematical analysis of nonconforming finite element methods for the plate bending problem is [**9**]. A patch test was proposed and is widely used by engineers for

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convergence analysis of nonconforming finite element methods, see [**1, 8**]. However, it is shown in [**15**] that the patch test is neither a necessary nor a sufficient condition for convergence. One finds in [**15**] a rigorous necessary and sufficient condition for convergence of nonconforming finite element solutions to variational equations of some boundary value problems. Some further developments along this line can be found in [**13, 20**] where convergence conditions are studied which are easier to examine. A summary account of nonconforming finite element methods can be found in [**4**], or more recently, [**5**].

In this paper, we study the nonconforming finite element method for solving the problem of determining the deformed configuration of a clamped homogeneous, isotropic thin elastic Kirchhoff plate subject to transversal loads and unilaterally constrained by an elastic obstacle. The model is proposed and studied in [**16**], where a mixed approach is taken for its finite element solution. The model is a boundary value problem for a fourth-order equation with a non-smooth term of the solution, and can also be formulated as a fourth-order elliptic variational inequality. We notice that in the literature, only a few papers can be found on the analysis of nonconforming finite element methods for fourth order variational inequalities, see [**7, 17**–**19**]. Analysis of mixed finite element methods for a fourth order variational inequality in unilaterally supported bent plate problem can be found in [**6**].

The paper is organized as follows. In Section 2, we introduce the plate problem presented in [**16**] and determine some regularity properties for the solution of the problem. Nonconforming finite element methods for the problem are introduced in Section 3. Section 4 is devoted to error estimation of continuous nonconforming finite element methods for the plate contact problem. The error estimates are valid in particular for the Zienkiewicz triangle and Adini's rectangle. Discontinuous nonconforming finite element methods are discussed in Section 5, and we mention Morley's triangle and the Fraeijs de Veubeke triangle as two particular examples. Description of the four sample nonconforming elements can be found in [**4, 5**].

2. The plate contact problem. The physical setting of the contact problem is illustrated in Figure 1. We consider a thin flat plate $\Omega\times(-d/2,d/2)$, where $d>0$ is the thickness of the plate, assumed to be small, and $\Omega\subset\mathbf{R}^2$ is an open, bounded, connected set with a Lipschitz

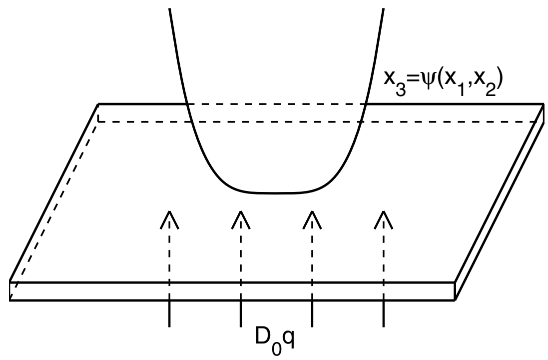

FIGURE 1. A plate subject to a normal force and unilaterally constrained by an obstacle.

continuous boundary $\Gamma = \partial\Omega$. Throughout the paper, we denote a generic point in $\overline{\Omega}$ by $\mathbf{x} = (x_1, x_2)$. Assume the three-dimensional material of the plate is isotropic, linearly elastic with Poisson's ratio $\sigma \in (0, 1/2)$ and Young's modulus $E > 0$. The plate is assumed to be clamped on its lateral boundary $\Gamma \times (-d/2, d/2)$, and is subject to a transversal external force of density $D_0 q(\mathbf{x})$, $q(\mathbf{x}) \geq 0$, where D_0 denotes the stiffness coefficient of the plate:

$$D_0 = \frac{E\, d^3}{12\,(1-\sigma^2)}.$$

The unknown of the problem is the plate vertical deflection $u(\mathbf{x})$. The plate is constrained unilaterally by an obstacle described by the equation $x_3 = \psi(\mathbf{x})$. The obstacle is flexible and offers resistance on regions where contact occurs between the obstacle and the plate. The reaction force of the obstacle acts vertically on the plate and is given by the formula

$$r(\mathbf{x}) = -\,D_0 \kappa(\mathbf{x})\,(u(\mathbf{x}) - \psi(\mathbf{x}))_+.$$

Here κ is a positive-valued function, and $t_+ = \max\{t, 0\}$ is the positive part of t. Observe that $r(\mathbf{x}) = 0$ at a point $\mathbf{x}$ where the plate does not touch the obstacle. The function κ describes the stiffness of the obstacle. The limiting case of a rigid obstacle is recovered as $\kappa(\mathbf{x}) \to \infty$.

For the given data, we make the following assumptions:

$$\begin{aligned} &q \in L^2(\Omega), \quad q \geq 0 \quad \text{a.e. in} \quad \Omega, \quad \psi \in L^2(\Omega), \\ &\kappa \in L^\infty(\Omega), \quad \kappa \geq \kappa_0 > 0 \quad \text{a.e. in} \quad \Omega. \end{aligned} \tag{2.1}$$

Since the plate is clamped, the function space for the deflection u is

$$V = H_0^2(\Omega).$$

We adopt the summation convention over a repeated index. For any $u, v \in V$, introduce the following functionals

$$\begin{aligned} a(u,v) &= \int_\Omega \big[\Delta u\, \Delta v + (1-\sigma) \\ &\qquad \times (2\, \partial_{12} u\, \partial_{12} v - \partial_{11} u\, \partial_{22} v - \partial_{22} u\, \partial_{11} v)\big]\, dx, \\ j(u,v) &= \int_\Omega \kappa\, (u-\psi)_+ v\, dx, \end{aligned}$$

and denote

$$(q,v) = \int_\Omega q\, v\, dx.$$

Later in this paper, we will use the following formula on several occasions: for any $v \in H^3(D)$ and any $w \in H^2(D)$,

$$\begin{aligned} &\int_D (2\, \partial_{12} v\, \partial_{12} w - \partial_{11} v\, \partial_{22} w - \partial_{22} v\, \partial_{11} w)\ dx \\ &\qquad = \int_{\partial D} (-\partial_{\tau\tau} v\, \partial_\nu w + \partial_{\nu\tau} v\, \partial_\tau w)\ ds, \end{aligned} \tag{2.2}$$

where D is a two-dimensional Lipschitz domain, $\boldsymbol{\nu} = (\nu_1, \nu_2)$ is the unit outward normal vector on ∂D, $\boldsymbol{\tau} = (\tau_1, \tau_2) = (-\nu_2, \nu_1)$ is the unit tangential vector along ∂D, and

$$\begin{aligned} \partial_\nu v &= \nu_i \partial_i v, \\ \partial_\tau v &= \tau_i \partial_i v, \\ \partial_{\nu\tau} v &= \nu_i \tau_j \partial_{ij} v, \\ \partial_{\tau\tau} v &= \tau_i \tau_j \partial_{ij} v. \end{aligned}$$

These derivatives and the vectors $\boldsymbol{\nu}$, $\boldsymbol{\tau}$ are defined a.e. on the boundary of the Lipschitz domain.

Note that for $u, v \in V$, we have

$$a(u,v) = \int_\Omega \Delta u \, \Delta v \, dx.$$

Then following [**16**] (with a scaling of the constant D_0 throughout the equation), the weak formulation of the plate problem is

$$u \in V, \quad a(u,v) + j(u,v) = (q,v) \quad \forall\, v \in V. \tag{2.3}$$

It is shown in [**16**] that the problem (2.3) has a unique solution, and the problem is equivalent to the variational inequality of finding $u \in V$ such that

$$a(u, v-u) + \int_\Omega \frac{\kappa}{2} \left[(v-\psi)_+\right]^2 \, dx - \int_\Omega \frac{\kappa}{2} \left[(u-\psi)_+\right]^2 \, dx \geq (q, v-u) \quad \forall\, v \in V,$$

and is also equivalent to the problem of minimizing the energy functional

$$\frac{1}{2}\, a(v,v) + \frac{1}{2} \int_\Omega \kappa \left[(v-\psi)_+\right]^2 \, dx - (q,v)$$

over V. The classical formulation of the problem is

$$\begin{aligned} \Delta^2 u + \kappa\,(u-\psi)_+ = q \quad & \text{in} \quad \Omega, \\ u = \partial_\nu u = 0 \quad & \text{on} \quad \Gamma, \end{aligned}$$

where $\partial_\nu u$ is the outward normal derivative of u on Γ. The boundary condition $\partial_\nu u = 0$ is understood to be valid at the boundary points where the outward normal vector $\boldsymbol{\nu}$ is defined.

We now recall a regularity result for the biharmonic equation [**2**, Theorem 7].

Theorem 2.1. *Let u be the weak solution of the problem*

$$\begin{aligned} \Delta^2 u = f \quad & \text{in} \quad \Omega, \\ u = \partial_\nu u = 0 \quad & \text{on} \quad \Gamma. \end{aligned}$$

Let $\Omega \subset \mathbf{R}^2$ be a bounded polygon and denote its largest internal angle by ω. Then, if $\omega < 180^\circ$ and $f \in H^{-1}(\Omega)$, we have $u \in H^3(\Omega)$; and if $\omega < 126.283696\ldots^\circ$ and $f \in L^2(\Omega)$, we have $u \in H^4(\Omega)$.

For the plate contact problem (2.3), notice that $\kappa\,(u-\psi)_+ \in L^2(\Omega)$. Applying Theorem 2.1, we obtain the following regularity result for the solution.

Theorem 2.2. *Assume* (2.1), *and let $u \in V$ be the weak solution of the problem* (2.3). *Then $u \in H^3(\Omega)$ if $\omega < 180^\circ$, and moreover, $u \in H^4(\Omega)$ if $\omega < 126.283696\ldots^\circ$.*

Assume $u \in H^3(\Omega)$. For later use, we note that, from (2.3),

$$-\int_\Omega \nabla\Delta u \cdot \nabla v\,dx + \int_\Omega \kappa\,(u-\psi)_+ v\,dx = (q,v) \quad \forall\, v \in H_0^2(\Omega).$$

By a density argument, we then have

$$-\int_\Omega \nabla\Delta u \cdot \nabla v\,dx + \int_\Omega \kappa\,(u-\psi)_+ v\,dx = (q,v) \quad \forall\, v \in H_0^1(\Omega). \tag{2.4}$$

For a domain D and an integer $k \ge 0$, we will use $\|\cdot\|_{k,D}$ and $|\cdot|_{k,D}$ for the norm and semi-norm on the Sobolev space $H^k(D)$. When $D = \Omega$, we will simply write $\|\cdot\|_k$ and $|\cdot|_k$.

3. Nonconforming finite element methods. Let $\{\mathcal{T}_h\}_h$ be a family of finite element partitions of the domain $\overline{\Omega}$. Here $h \to 0+$ is a discretization parameter. A typical element in $\mathcal{T}_h$ is denoted by T. Let $\{V_h\}_h$ be a family of corresponding finite element spaces used to approximate the space V. We consider nonconforming approximations. Thus in general, $V_h \not\subset V$. Then the discrete approximation problem is to find $u_h \in V_h$ such that

$$a_h(u_h, v_h) + j(u_h, v_h) = (q, v_h) \quad \forall\, v_h \in V_h, \tag{3.1}$$

where the discrete bilinear form is

$$\begin{aligned} a_h(v,w) = \sum_{T\in\mathcal{T}_h} \int_T \big[&\Delta v\,\Delta w + (1-\sigma) \\ &\times (2\,\partial_{12}v\,\partial_{12}w - \partial_{11}v\,\partial_{22}w - \partial_{22}v\,\partial_{11}w)\big]\,dx. \end{aligned} \tag{3.2}$$

Assume

$$\|v_h\|_h = \Big(\sum_{T \in \mathcal{T}_h} |v_h|^2_{2,T} \Big)^{1/2}, \quad v_h \in V_h \tag{3.3}$$

is a norm on V_h. This assumption is usually easy to verify, and is valid for the nonconforming elements mentioned in later sections. Since

$$a_h(v,w) = \sum_{T \in \mathcal{T}_h} \int_T \big[\sigma\,\Delta v\,\Delta w + (1-\sigma) \\ \times (2\,\partial_{12} v\,\partial_{12} w + \partial_{11} v\,\partial_{11} w + \partial_{22} v\,\partial_{22} w)\big]\,dx,$$

the bilinear form (3.2) is coercive on V_h:

$$a_h(v_h, v_h) \geq \alpha\,\|v_h\|_h^2 \quad \forall\, v_h \in V_h$$

with $\alpha = 1 - \sigma$. Obviously, $a_h(\cdot,\cdot)$ is continuous:

$$|a_h(v,w)| \leq M\,\|v\|_h \|w\|_h \quad \forall\, v, w \in V + V_h.$$

We will need the following elementary inequality

$$(t_+ - s_+)\,(t - s) \geq 0 \quad \forall\, t, s \in \mathbf{R}. \tag{3.4}$$

Proposition 3.1. *The discrete problem* (3.1) *has a unique solution. Moreover,* $\|u_h\|_h$ *and* $\|(u_h - \psi)_+\|_0$ *are uniformly bounded, with a bound that is independent of* h*.*

Proof. It is easy to see that (3.1) is equivalent to the minimization problem

$$J_h(u_h) \leq J_h(v_h) \quad \forall\, v_h \in V_h, \tag{3.5}$$

where the energy functional

$$J_h(v_h) = \frac{1}{2}\,a_h(v_h, v_h) + \frac{1}{2}\int_\Omega \kappa\,[(v_h - \psi)_+]^2\,dx - (q, v_h).$$

Since $q \geq 0$ a.e. in Ω,

$$\begin{aligned}
-(q, v_h) &= -(q, v_h - \psi) - (q, \psi) \\
&\geq -(q, (v_h - \psi)_+) - (q, \psi) \\
&\geq -\frac{1}{4} \int_\Omega \kappa \left[(v_h - \psi)_+\right]^2 dx + c
\end{aligned}$$

with a constant c depending only on q and ψ. Thus,

$$\begin{aligned}
J_h(v_h) &\geq \frac{1}{2} a_h(v_h, v_h) + \frac{1}{4} \int_\Omega \kappa \left[(v_h - \psi)_+\right]^2 dx + c \\
&\geq \frac{\alpha}{2} \|v_h\|_h^2 + \frac{\kappa_0}{4} \int_\Omega \left[(v_h - \psi)_+\right]^2 dx + c,
\end{aligned} \tag{3.6}$$

and so

$$J_h(v_h) \to \infty \quad \text{as } \|v_h\|_h \to \infty.$$

Since V_h is finite dimensional and J_h is continuous, there is a solution to the problem (3.5).

To show the uniqueness, let $u_h^{(1)}$ and $u_h^{(2)}$ be two solutions of (3.1). Then

$$a_h(u_h^{(1)} - u_h^{(2)}, v_h) + j(u_h^{(1)}, v_h) - j(u_h^{(2)}, v_h) = 0 \quad \forall\, v_h \in V_h.$$

Let $v_h = u_h^{(1)} - u_h^{(2)}$ and apply the inequality (3.4),

$$a_h(u_h^{(1)} - u_h^{(2)}, u_h^{(1)} - u_h^{(2)}) \leq 0.$$

Hence, $u_h^{(1)} = u_h^{(2)}$.

Uniform boundedness of $\|u_h\|_h$ and $\|(u_h - \psi)_+\|_0$ follows from the inequality

$$J_h(u_h) \leq J_h(0) = \int_\Omega \frac{\kappa}{2} (-\psi)_+^2\, dx$$

and the lower bound (3.6) for $J_h(u_h)$. $\square$

We note that the discrete problem (3.1) is equivalent to the discrete variational inequality of finding $u_h \in V_h$ such that

$$\begin{aligned}
a_h(u_h, v_h - u_h) + \int_\Omega \frac{\kappa}{2} \left[(v_h - \psi)_+\right]^2 dx - \int_\Omega \frac{\kappa}{2} \left[(u_h - \psi)_+\right]^2 dx \\
\geq (q, v_h - u_h) \quad \forall\, v_h \in V_h.
\end{aligned}$$

We will study continuous family ($V_h \subset C^0$) and discontinuous family ($V_h \not\subset C^0$) of nonconforming finite elements separately. For each family, we consider finite elements satisfying one of the following two assumptions.

(A_1) There exists a constant c such that

$$\Big| \sum_{T \in \mathcal{T}_h} \int_{\partial T} [\Delta w - (1-\sigma)\,\partial_{\tau\tau} w]\,\partial_\nu v_h\, ds \Big| \le c\,h\,\|w\|_3 \|v_h\|_h \quad \forall\, w \in V \cap H^3(\Omega),\ v_h \in V_h, \tag{3.7}$$

$$\|w - \Pi_h w\|_0 + h^2 \|w - \Pi_h w\|_h \le c\,h^3 \|w\|_3 \quad \forall\, w \in V \cap H^3(\Omega), \tag{3.8}$$

where Π_h is the finite element interpolation operator.

(A_2) There exists a constant c such that

$$\Big| \sum_{T \in \mathcal{T}_h} \int_{\partial T} [\Delta w - (1-\sigma)\,\partial_{\tau\tau} w]\,\partial_\nu v_h\, ds \Big| \le c\,h^2 \|w\|_4 \|v_h\|_h \quad \forall\, w \in V \cap H^4(\Omega),\ v_h \in V_h, \tag{3.9}$$

$$\|w - \Pi_h w\|_0 + h^2 \|w - \Pi_h w\|_h \le c\,h^4 \|w\|_4 \quad \forall\, w \in V \cap H^4(\Omega), \tag{3.10}$$

For the concrete nonconforming finite elements, we will comment on the validity of these relations.

4. Continuous nonconforming finite element approximation. We consider some continuous nonconforming plate elements in this section. Assume Ω is a polygonal domain. Let $\{\mathcal{T}_h\}_h$ be a family of *regular* triangulations of $\overline{\Omega}$, and let $\{V_h\}_h \subset C^0(\overline{\Omega})$ be a corresponding family of nonconforming finite element subspaces of V.

The following theorem provides optimal order error estimates.

Theorem 4.1. *Assuming $\omega < 180^\circ$ and* (A_1)*, we have the error bound*

$$\|u - u_h\|_h \le c\,h\,\|u\|_3. \tag{4.1}$$

Assuming $\omega < 126.283696\ldots^\circ$ *and* (A_2), *we have the error bound*

$$\|u-u_h\|_h \le c\, h^2 \|u\|_4. \tag{4.2}$$

Proof. For any $v_h \in V_h$, we have

$$\begin{aligned} \alpha \,\|u_h - v_h\|_h^2 &\le a_h(u_h - v_h, u_h - v_h) \\ &= a_h(u - v_h, u_h - v_h) + a_h(u_h - u, u_h - v_h). \end{aligned} \tag{4.3}$$

The first term in the summation is bounded as follows:

$$\begin{aligned} a_h(u - v_h, u_h - v_h) &\le M\, \|u - v_h\|_h \|u_h - v_h\|_h \\ &\le c\,\|u - v_h\|_h^2 + \frac{\alpha}{2}\,\|u_h - v_h\|_h^2. \end{aligned} \tag{4.4}$$

To bound the second term, we notice that for any $w_h \in V_h$,

$$\begin{aligned} a_h(u, w_h) = &\sum_{T\in\mathcal{T}_h} \int_T [\Delta u\, \Delta w_h + (1-\sigma) \\ &\qquad \times (2\,\partial_{12}u\partial_{12}w_h - \partial_{11}u\partial_{22}w_h - \partial_{22}u\partial_{11}w_h)]\; dx \\ = &\sum_{T\in\mathcal{T}_h} \int_{\partial T} \Delta u\, \partial_\nu w_h\, ds - \sum_{T\in\mathcal{T}_h} \int_T \nabla(\Delta u)\cdot\nabla w_h\, dx \\ &+ (1-\sigma) \sum_{T\in\mathcal{T}_h} \int_{\partial T} (-\,\partial_{\tau\tau}u\partial_\nu w_h + \partial_{\nu\tau}u\partial_\tau w_h)\; ds \\ = &\sum_{T\in\mathcal{T}_h} \int_{\partial T} [\Delta u - (1-\sigma)\,\partial_{\tau\tau}u]\,\partial_\nu w_h\, ds - \int_\Omega \nabla(\Delta u)\cdot\nabla w_h\, dx, \end{aligned}$$

where we have used the relation, see [**4**, pp. 368–369],

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} \partial_{\nu\tau}u\partial_\tau w_h\, ds = 0,$$

valid for continuous nonconforming finite elements. Using the equality (2.4), we get

$$\begin{aligned} a_h(u, w_h) = &\sum_{T\in\mathcal{T}_h} \int_{\partial T} [\Delta u - (1-\sigma)\,\partial_{\tau\tau}u]\,\partial_\nu w_h\, ds \\ &+ (q, w_h) - \int_\Omega \kappa\,(u-\psi)_+ w_h\, dx. \end{aligned}$$

Thus, recalling the defining equation (3.1) for the finite element solution u_h, we have

$$
\begin{aligned}
a_h(u_h - u, w_h) &= a_h(u_h, w_h) - a(u, w_h) \\
&= -\sum_{T \in \mathcal{T}_h} \int_{\partial T} [\Delta u - (1-\sigma)\,\partial_{\tau\tau} u]\,\partial_\nu w_h\,ds \\
&\quad + \int_\Omega \kappa\,[(u-\psi)_+ - (u_h-\psi)_+]\,w_h\,dx.
\end{aligned} \tag{4.5}
$$

The first term on the righthand side will be bounded by (3.7) or (3.9). For the second term with $w_h = u_h - v_h$, we use the elementary inequality (3.4),

$$
\begin{aligned}
&\int_\Omega \kappa\,[(u-\psi)_+ - (u_h-\psi)_+]\,(u_h - v_h)\,dx \\
&\qquad \le \int_\Omega \kappa\,[(u-\psi)_+ - (u_h-\psi)_+]\,(u - v_h)\,dx \\
&\qquad \le c\,[\|(u-\psi)_+\|_0 + \|(u_h-\psi)_+\|_0]\,\|u - v_h\|_0.
\end{aligned}
$$

Since $\|(u_h-\psi)_+\|_0$ is uniformly bounded independently of h, we have

$$
\int_\Omega \kappa\,[(u-\psi)_+ - (u_h-\psi)_+]\,(u_h - v_h)\,dx \le c\,\|u - v_h\|_0. \tag{4.6}
$$

Combining (4.3), (4.4), (4.5) and (4.6), we have

$$
\begin{aligned}
\|u_h - v_h\|_h^2 \le c \Big| &\sum_{T \in \mathcal{T}_h} \int_{\partial T} [\Delta u - (1-\sigma)\,\partial_{\tau\tau} u]\,\partial_\nu (u_h - v_h)\,ds \Big| \\
&+ c\left(\|u - v_h\|_h^2 + \|u - v_h\|_0\right).
\end{aligned}
$$

Under the assumptions $\omega < 180^\circ$ and (A_1), we can then derive the following relation:

$$
\|u_h - v_h\|_h \le c\,h\,\|u\|_3 + c\left(\|u - v_h\|_h + \|u - v_h\|_0^{1/2}\right).
$$

From this, the inequality

$$
\|u - u_h\|_h \le \|u - v_h\|_h + \|u_h - v_h\|_h,
$$

and the arbitrariness of $v_h \in V_h$, we obtain

$$\|u-u_h\|_h \le c\,h\,\|u\|_3 + c \inf_{v_h\in V_h}\left(\|u-v_h\|_h + \|u-v_h\|_0^{1/2}\right).$$

Finally, using the interpolation error estimate (3.8), we get the error bound (4.1).

The error bound (4.2) is derived similarly. □

One example of a continuous nonconforming finite element is the Zienkiewicz triangle. Assume $\overline{\Omega}$ is such that it is possible to split it into triangles with all sides parallel to three fixed directions. This property is valid if Ω is the union of rectangles with sides parallel to two fixed directions and right triangles with two sides parallel to the two fixed directions. Let $\{\mathcal{T}_h\}$ be a regular family of partitions of $\overline{\Omega}$ into such triangles. Then the Zienkiewicz triangle consists of piecewise incomplete polynomials of degree less than or equal to 3. On each triangle, the polynomial is determined by its values and the values of its two first-order derivatives at the three vertices; for details, see [**4**]. For this element, the inequalities (3.7) and (3.8) are valid, cf. [**12**].

Another example is Adini's rectangle. Assume $\overline{\Omega} \subset \mathbf{R}^2$ can be partitioned into rectangles, e.g., if Ω is the union of rectangles with sides parallel to two fixed directions. Let $\{\mathcal{T}_h\}_h$ be a regular family of partitions of $\overline{\Omega}$ into rectangles with sides parallel to the coordinate axes. Then Adini's rectangle is defined as a piecewise polynomial corresponding to the partition $\mathcal{T}_h$ such that on each element, it is a third degree polynomial plus a linear combination of $x_1^3x_2$ and $x_1x_2^3$, with the values of the function and of the two first partial derivatives with respect to x_1 and x_2 at the four vertices of the element as the finite element parameters. The interpolation error estimates (3.8) and (3.10) hold. A proof of the bound (3.7) is found in [**4**] or [**15**]. The bound (3.9) is shown in [**10**], see also [**11**].

Therefore, if $\omega < 180^\circ$, for the Zienkiewicz triangle and Adini's rectangle, we have the optimal order error estimate (4.1), and if $\omega < 126.283696\ldots^\circ$, we have the optimal order error estimate (4.2) for Adini's rectangle.

5. Discontinuous nonconforming finite element approximation. In this section, we consider discontinuous nonconforming finite element approximations of the plate contact problem. Let $\{V_h\}_h \not\subset C^0(\overline{\Omega})$ be a family of nonconforming finite element subspaces of V corresponding to a *regular* family $\{\mathcal{T}_h\}_h$ of triangulations of $\overline{\Omega}$ into triangles such that the finite element functions are continuous at the triangle vertices.

Theorem 5.1. *Assuming $\omega < 180^\circ$ and (A_1), we have the error bound*

$$\|u - u_h\|_h \le c\, h\, \|u\|_3. \tag{5.1}$$

Proof. The proof is similar to that of Theorem 4.1. However, since $V_h \not\subset C(\overline{\Omega})$ implies $V_h \not\subset H_0^1(\Omega)$, we need to make the following modification in bounding the term $a_h(u_h - u, w_h)$ with $w_h = u_h - v_h$.

Let w_h^I be the continuous piecewise linear interpolant of w_h. Since $w_h^I \in C\,(\overline{\Omega})$, $w_h^I \in H^1(\Omega)$. Consider the term

$$\begin{aligned} a_h(u, w_h) = &\sum_{T\in\mathcal{T}_h} \int_T [\Delta u\, \Delta w_h + (1-\sigma) \\ &\qquad \times (2\,\partial_{12}u\partial_{12}w_h - \partial_{11}u\partial_{22}w_h - \partial_{22}u\partial_{11}w_h)]\; dx \\ = &\sum_{T\in\mathcal{T}_h} \int_{\partial T} [\Delta u - (1-\sigma)\,\partial_{\tau\tau}u]\,\partial_\nu w_h\, dx \\ &- \sum_{T\in\mathcal{T}_h} \int_T \nabla(\Delta u)\cdot\nabla w_h\, dx + (1-\sigma) \sum_{T\in\mathcal{T}_h} \int_{\partial T} \partial_{\nu\tau}u\,\partial_\tau w_h\, ds. \end{aligned}$$

Write

$$\begin{aligned} -\sum_{T\in\mathcal{T}_h} \int_T \nabla(\Delta u)\cdot\nabla w_h\, dx = &-\sum_{T\in\mathcal{T}_h} \int_T \nabla(\Delta u)\cdot\nabla w_h^I\, dx \\ &- \sum_{T\in\mathcal{T}_h} \int_T \nabla(\Delta u)\cdot\nabla(w_h - w_h^I)\, dx. \end{aligned}$$

By (2.4), since $w_h^I \in H_0^1(\Omega)$,

$$-\sum_{T\in\mathcal{T}_h} \int_T \nabla(\Delta u)\cdot\nabla w_h^I\, dx = (q, w_h^I) - \int_\Omega \kappa\,(u-\psi)_+ w_h^I\, dx.$$

Hence,

$$\begin{aligned} a_h(u, w_h) = &\sum_{T\in\mathcal{T}_h} \int_{\partial T} [\Delta u - (1-\sigma)\, \partial_{\tau\tau} u]\, \partial_\nu w_h \, dx \\ &+ (q, w_h^I) - \int_\Omega \kappa\, (u-\psi)_+ \, w_h^I \, dx \\ &- L_u + (1-\sigma) \sum_{T\in\mathcal{T}_h} \int_{\partial T} \partial_{\nu\tau} u \, \partial_\tau w_h \, ds, \end{aligned}$$

where

$$L_u = \sum_{T\in\mathcal{T}_h} \int_T \nabla(\Delta u)\cdot \nabla(w_h - w_h^I)\, dx. \tag{5.2}$$

Thus,

$$\begin{aligned} a_h(u_h - u, w_h) &= a_h(u_h, w_h) - a(u, w_h) \\ &= (q, w_h) - \int_\Omega \kappa\, (u_h - \psi)_+ w_h \, dx - a_h(u, w_h) \\ &= (q, w_h - w_h^I) + \int_\Omega \kappa\, (u-\psi)_+ (w_h^I - w_h)\, dx \\ &\quad - \sum_{T\in\mathcal{T}_h} \int_{\partial T} [\Delta u - (1-\sigma)\, \partial_{\tau\tau} u]\, \partial_\nu w_h \, ds \\ &\quad + L_u - (1-\sigma) \sum_{T\in\mathcal{T}_h} \int_{\partial T} \partial_{\nu\tau} u \, \partial_\tau w_h \, ds \\ &\quad + \int_\Omega \kappa\, [(u-\psi)_+ - (u_h - \psi)_+]\, w_h \, dx. \end{aligned}$$

We have

$$\begin{aligned} |(q, w_h - w_h^I)| + \left| \int_\Omega \kappa\, (u-\psi)_+ (w_h^I - w_h)\, dx \right| &\le c\, \|w_h^I - w_h\|_0 \\ &\le c\, h^2 \|w_h\|_h, \end{aligned}$$

where c depends on $\|q\|_0$ and $\|(u-\psi)_+\|_0$. The term

$$\sum_{T\in\mathcal{T}_h} \int_{\partial T} [\Delta u - (1-\sigma)\, \partial_{\tau\tau} u]\, \partial_\nu w_h \, ds$$

is bounded by (3.7). The term

$$\int_\Omega \kappa\,[(u-\psi)_+ - (u_h-\psi)_+]\,w_h\,dx$$

with $w_h = u_h - v_h$ is bounded by (4.6). Finally, write

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\nu\tau}u\,\partial_\tau w_h\,ds = \sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\nu\tau}u\,\partial_\tau w_h^I\,ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\nu\tau}u\,\partial_\tau(w_h - w_h^I)\,ds.$$

For each side γ of the elements, define a piecewise constant projection

$$P_0^\gamma(v) = \frac{1}{|\gamma|}\int_\gamma v\,ds.$$

Since

$$\int_\gamma \partial_\tau(w_h - w_h^I)\,ds = 0,$$

we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}&\int_{\partial T}\partial_{\nu\tau}u\,\partial_\tau(w_h - w_h^I)\,ds\\ &= \sum_{T\in\mathcal{T}_h}\sum_{\gamma\subset\partial T}\int_\gamma \partial_{\nu\tau}u\,\partial_\tau(w_h - w_h^I)\,ds\\ &= \sum_{T\in\mathcal{T}_h}\sum_{\gamma\subset\partial T}\int_\gamma [\partial_{\nu\tau}u - P_0^\gamma(\partial_{\nu\tau}u)]\,\partial_\tau(w_h - w_h^I)\,ds.\end{aligned}$$

By a scaling argument,

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\nu\tau}u\,\partial_\tau(w_h - w_h^I)\,ds \le c\,h\,|u|_3\|w_h\|_h.$$

Since $w_h^I \in C(\overline{\Omega})$ and $u \in V \cap H^3(\Omega)$, we have

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\partial_{\nu\tau}u\,\partial_\tau w_h^I\,ds = \int_{\partial\Omega}\partial_{\nu\tau}u\,\partial_\tau w_h^I\,ds = 0.$$

The term L_u defined in (5.2) is bounded as follows:

$$|L_u| \le \sum_{T\in\mathcal{T}_h} |u|_{3,T} |w_h - w_h^I|_{1,T} \le c\, h\, |u|_3 \|w_h\|_h. \tag{5.3}$$

Thus, by a procedure similar to the proof of Theorem 4.1, we have

$$\begin{aligned}\|u_h - v_h\|_h \le c\, \|u - v_h\|_h &+ c\, (\|(u-\psi)_+\|_0 \\ &+ \|(u_h-\psi)_+\|_0)^{1/2} \|u - v_h\|_0^{1/2} \\ &+ c\, (\|q\|_0 + \|(u-\psi)_+\|_0)\, h^2 + c\, h\, |u|_3.\end{aligned}$$

Finally, we use

$$\|u - u_h\|_h \le \|u - v_h\|_h + \|u_h - v_h\|_h,$$

and take $v_h = \Pi_h u$. Then the error bound (5.1) can be derived. □

As examples of discontinuous nonconforming finite element spaces for the plate contact problem, we mention Morley's triangle and the Fraeijs de Veubeke triangle. Assume Ω is a polygonal domain, and let $\{\mathcal{T}_h\}$ be a regular family of partitions of $\overline{\Omega}$ into triangles. For Morley's triangle, on each element, the finite element function is quadratic and is uniquely determined by the function values at the three vertices and the normal derivative at the three mid-side nodes. For the Fraeijs de Veubeke triangle, on each element, the finite element function is cubic and is uniquely determined by the function values at the three vertices and at the center, and the normal derivative at the Gaussian points of second order on each side. From their constructions, we see that for both Morley's triangle and the Fraeijs de Veubeke triangle, the finite element functions are continuous at the vertices of the corresponding triangulation. For both elements, the bounds (3.7) and (3.8) are valid, see [**12**] for Morley's triangle and [**15**] for the Fraeijs de Veubeke triangle. So if $\omega < 180^\circ$, for both elements, we have the optimal order error estimates (5.1).

For the Fraeijs de Veubeke triangle, we also have (3.9) and (3.10). A proof of (3.9) can be given based on results similar to Lemmas 10.3.7 and 10.3.9 of [**3**]. Thus, assuming $\omega < 126.283696\ldots^\circ$, we have $u \in H^4(\Omega)$ and would expect the following error bound

$$\|u - u_h\|_h \le c\, h^2 \|u\|_4. \tag{5.4}$$

However, the proof of Theorem 5.1 does not work for this, due to the error bound (5.3) for the term L_u defined in (5.2).

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