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QUALOCATION FOR BOUNDARY INTEGRAL EQUATIONS USING SPLINES WITH MULTIPLE KNOTS

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Dedicated to Kendall Atkinson

ABSTRACT. This paper provides an analysis of the qualocation method for periodic pseudodifferential operators, with multiple knot splines used for the trial and test spaces. A recently introduced basis for the multiple knot periodic splines leads to a relatively easy analysis. Convergence is proved with the aid of approximation properties of the qualocation projection and of inverse stability estimates that are characterized by necessary and sufficient algebraic conditions. The analysis of the variable coefficient case uses a local principle and recently established commutator properties.

1. Introduction. In this paper we study the qualocation method for pseudodifferential operators of the form

$$L = L_0 + L_1, \tag{1.1}$$

where

$$L_0 v(x) := \sum_{n=-\infty}^{\infty} \sigma_0(x,n)\, \hat{v}(n) e^{i2\pi n x} \quad \text{for} \quad x \in \mathbf{T}, \tag{1.2}$$

and L_1 is a suitable perturbation (see below). Here $\mathbf{T} := \mathbf{R}\backslash\mathbf{Z}$ is the one-dimensional torus of length 1, and

$$\hat{v}(n) := \int_{\mathbf{T}} v(x) e^{-i2\pi n x}\, dx \quad \text{for} \quad n \in \mathbf{Z}$$

are the complex Fourier coefficients of a 1-periodic distribution $v : \mathbf{T} \to \mathbf{R}$, so that

$$v(x) = \sum_{n=-\infty}^{\infty} \hat{v}(n) e^{i2\pi n x} \quad \text{for} \quad x \in \mathbf{T}.$$

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The 'symbol' σ_0 has the form

$$\sigma_0(x,\xi) := \left\{ \begin{matrix} 1 & \text{if } \xi = 0 \\ a^+(x)|\xi|^\beta + a^-(x)\operatorname{sign}(\xi)|\xi|^\beta & \text{if } 0 \neq \xi \in \mathbf{R} \end{matrix} \right\}, \tag{1.3}$$
$$x \in \mathbf{T},$$

with coefficients a^+ and a^- in $C^\infty(\mathbf{T})$, where $\beta \in \mathbf{R}$ is the 'order' of L_0, in that for $\rho > 0$

$$\sigma_0(x, \rho\xi) = \rho^\beta \sigma_0(x,\xi), \quad x \in \mathbf{T},\ \xi \neq 0.$$

We assume that L is elliptic, i.e., $\sigma_0(x,\xi) \neq 0$ for $x \in \mathbf{T}$ and $|\xi| = 1$, and to have index $\kappa = 0$, where

$$\kappa := \frac{1}{2\pi} \left[\arg \frac{a^+(x) + a^-(x)}{a^+(x) - a^-(x)} \right]_0^1$$

is the winding number of the closed curve $(a^+ + a^-)/(a^+ - a^-)$ in the complex plane. It is known that $L_0 : H^s \to H^{s-\beta}$ is then a Fredholm operator for all $s \in \mathbf{R}$, where $H^s = H^s(\mathbf{T})$ is the usual Sobolev space of 1-periodic distributions f equipped with the norm

(1.4)
$$\|f\|_s := \left(\sum_{n=-\infty}^{\infty} \langle n \rangle^{2s} |\hat{f}(n)|^2 \right)^{1/2}, \quad \text{with} \quad \langle n \rangle := \begin{cases} 1 & \text{if } n = 0, \\ |n| & \text{if } n \neq 0. \end{cases}$$

It is assumed that L_1 maps $H^s \to H^{s-\beta+\delta}$ for some $\delta > 0$ and all $s \in \mathbf{R}$ boundedly, and hence that L is also Fredholm with index 0.

We consider the discretization of (1.1) by qualocation using as test and trial spaces periodic splines with multiple knots on equidistant meshes. Let r, M, N with $1 \leq M \leq r$ be positive integers. We define the set of knots

$$\pi_h := \{x_j = jh,\ j = 0, \dots, N-1\} \quad \text{for} \quad h \in \mathcal{H} := \{1/N,\ N \in \mathbf{N}\},$$

and denote by $S_h^{r,M}$ the space of periodic splines of order r with M-fold knots at the points in π_h. The splines $S_h^{r,M}$ form a subspace of C^{r-M-1} of dimension MN, where $C^k = C^k(\mathbf{T})$ is the space of 1-periodic k times continuously differentiable functions (with C^{-1} meaning piecewise continuous functions with jumps only at the knots in π_h).

Qualocation is based on a composite quadrature rule

$$Q_N f = h \sum_{k=0}^{N-1} \sum_{j=1}^{J} \omega_j f(x_{k,j}), \quad x_{k,j} := x_k + h\xi_j, \tag{1.5}$$

derived by copying onto subintervals of length h the basic quadrature formula

$$Q_1 f = \sum_{j=1}^{J} \omega_j f(\xi_j), \tag{1.6}$$

where the quadrature points $\{\xi_j\}$ and weights $\{\omega_j\}$ satisfy

$$\begin{gathered} 0 \le \xi_1 < \xi_2 < \cdots < \xi_J < 1, \quad J \ge M, \\ \sum_{j=1}^{J} \omega_j = 1, \quad \omega_j > 0. \end{gathered} \tag{1.7}$$

Associated with the quadrature rule we define an inner product

$$(v_h, w_h)_h := Q_N(v_h \overline{w}_h) \tag{1.8}$$

on the linear space W_h of 'grid' functions v_h and w_h, which are functions defined on the grid

$$\pi_h' := \{x_{k,j} = x_k + h\xi_j,\ k = 0, \ldots, N-1,\ j = 1, \ldots, J\}. \tag{1.9}$$

The inner product in (1.8) can be thought of as an approximation to

$$(v, w) := \int_0^1 v(x)\, \overline{w}(x)\, dx \quad \text{for} \quad v, w \in L^2(\mathbf{T}). \tag{1.10}$$

In [**3**] we derived conditions ensuring that $(\cdot,\cdot)_h$ is an inner product on $S_h^{r,M}$.

We choose now splines of order r as trial space and splines of a possibly different order r' as test space, with $r, r' \geq M$. The qualocation method for solving the equation $Lu = f$ approximately is to find $u_h \in S_h^{r,M}$ such that

$$(1.11) \qquad (Lu_h, z_h)_h := (f, z_h)_h \quad \text{for all} \quad z_h \in S_h^{r',M}.$$

This method can be viewed as a discrete version of the Petrov-Galerkin method. Conditions to make (1.11) well-defined are given in Section 2.

The basic stability result for the solution of qualocation equations for operators with constant coefficients is proved in Proposition 4.2. In Theorem 4.4 convergence is derived from stability. An extended stability result is given in Corollary 4.5. The results in [**9, 10**] for smoothest splines and in [**4**] for the collocation method are included as special cases.

A feature of past treatments of the qualocation method (for smoothest splines) is that the freedom in the choice of the quadrature rule Q_1 has been exploited to yield additional orders of convergence beyond the basic convergence results that we prove here. We defer any consideration of additional orders of convergence to a future paper.

The stability analysis for the variable coefficient operators is based on a local principle of Prößdorf [**5**]. Since the qualocation projection lacks some of the boundedness properties needed in the analysis in [**5**], a variant of Prößdorf's principle with weakened assumptions, in the manner of [**2**], is stated in Section 5. As is well known, the analysis of the variable coefficient operators relies on commutator properties. There are two kinds of such properties, called CP I and CP II in [**7**]. These were proved in [**3**] and are restated here as Theorem 2.14. The main stability result for the variable coefficient qualocation method is derived in Corollary 6.2 and the main convergence result in Theorem 6.3.

An important tool in the whole paper is a new spline basis introduced in [**3**]. This basis is also suitable for performing concrete numerical calculations. The new basis is a consistent extension of the one in [**1**] for splines with simple knots, and contrasts with the recursive characterization of multiple knot splines used in [**4**]. Some properties of the spline spaces $S_h^{r,M}$ as proved in [**3**] are collected in Section 2.

2. The multiple knot spline periodic space. For the convenience of the reader we provide here the definition of our spline basis and collect some results for the spline spaces $S_h^{r,M}$ from [**3**]. Also, some consequences needed in the sequel are proved.

For the definition of the spline basis we introduce the functions

$$\tilde{\Delta}_k(\xi, y) := \sum_{\ell \neq 0} \frac{\ell^{k-1}}{(y+\ell)^r} \, \Phi_\ell(\xi) \quad \text{for} \quad |y| \leq \frac{1}{2} \quad \text{and} \quad \xi \in \mathbf{R}, \tag{2.1}$$

$$\Phi_\ell(\xi) := \exp(i2\pi\ell\xi) \quad \text{for} \quad \ell \in \mathbf{Z} \quad \text{and} \quad \xi \in \mathbf{R}, \tag{2.2}$$

$$\Delta_1(\xi, y) := 1 + y^r \tilde{\Delta}_1(\xi, y), \tag{2.3}$$

$$\Delta_k(\xi, y) := \tilde{\Delta}_k(\xi, y) \quad \text{for} \quad k = 2, \dots, M, \tag{2.4}$$

$$\begin{gathered} \psi_{k,\mu}(x) := \Phi_\mu(x) \Delta_k\left(Nx, \frac{\mu}{N}\right) \\ \text{for} \quad k = 1, \dots, M \quad \text{and} \quad \mu \in \Lambda_h, \end{gathered} \tag{2.5}$$

where

$$\Lambda_h := \left(-\frac{N}{2}, \frac{N}{2}\right] \cap \mathbf{Z} \quad \text{for} \quad Nh = 1. \tag{2.6}$$

Proposition 2.1 [**3**, Proposition 2.1]. *The set* $\psi_{k,\mu}$ *for* $k = 1, \dots, M$ *and* $\mu \in \Lambda_h$ *is a basis in* $S_h^{r,M}$.

Lemma 2.2 [**3**, Corollary 7.3]. *For* $\mu, \nu \in \Lambda_h$ *and* $k, \ell = 1, \dots, M$

$$(\psi_{k,\mu}, \psi_{\ell,\nu})_h = \delta_{\mu\nu} \, Q\Big(\Delta_k\Big(\cdot, \frac{\mu}{N}\Big), \Delta_\ell\Big(\cdot, \frac{\nu}{N}\Big)\Big), \tag{2.7}$$

with

$$Q(v, w) := Q_1(v\overline{w}). \tag{2.8}$$

A central role in the analysis is played by the so called qualocation projection $R_h : W_h \to S_h^{r,M}$, defined by

$$(R_h f_h, \psi)_h = (f_h, \psi)_h \quad \text{for all} \quad \psi \in S_h^{r,M}. \tag{2.9}$$

Lemma 2.3 [**3**, Lemma 3.1]. *The positive semi-definite sesquilinear form $(\cdot,\cdot)_h$ is an inner product on $S_h^{r,M}$, and hence R_h is well defined, if and only if for all $\mu \in \Lambda_h$ the functions $\{\Delta_k(\cdot,(\mu/N)),\ k = 1,\dots,M\}$ are linearly independent on the set of quadrature points $\{\xi_j,\ \ j = 1,\dots,J\}$.*

A useful relation for the subsequent analysis is the following norm equivalence.

Proposition 2.4 [**3**, Proposition 3.3]. *Let $1 \le M \le r$ and $s < r - M + 1/2$. On $S_h^{r,M}$ the norm $\|\cdot\|_s$ is equivalent, uniformly for $h \in \mathcal{H}$, to the norm*

$$\|v_h\|_{s,h} := \left(\sum_{\mu \in \Lambda_h} \left[\langle \mu \rangle^{2s} |c_{1,\mu}|^2 + N^{2s} \sum_{k=2}^{M} |c_{k,\mu}|^2 \right] \right)^{1/2}, \tag{2.10}$$

where the coefficients $c_{k,\mu}$ are defined by the unique representation

$$v_h = \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} c_{k,\mu} \psi_{k,\mu}. \tag{2.11}$$

In the special case $s = 0$, Proposition 2.4 yields

Corollary 2.5 [**3**, Corollary 3.4]. *Let $1 \le M \le r$. There exist constants $0 < c < C$ such that for all $h \in \mathcal{H}$ and $v_h \in S_h^{r,M}$ the norm equivalence*

$$c\|v_h\|_0 \le \left(\sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} |c_{k,\mu}|^2 \right)^{1/2} \le C\|v_h\|_0 \tag{2.12}$$

holds, where v_h has the form (2.11).

A further norm equivalence is given in the next result.

Proposition 2.6 [**3**, Proposition 3.5]. *Let $1 \le M \le r$. The norms $\|\cdot\|_0$ and $\|\cdot\|_h$ are equivalent on $S_h^{r,M}$, uniformly for $h \in \mathcal{H}$, if and only if the functions $\{\Delta_k(\cdot, y),\ k = 1, \dots, M\}$ are for all $y \in [-1/2, 1/2]$ linearly independent on the set of quadrature points $\{\xi_j,\ j = 1, \dots, J\}$.*

As test space we will use the spline space $S_h^{r',M}$ of order $r' \ge M$, where r' may be different from r. We denote the corresponding basis of $S^{r',M}$ by $\psi'_{k,\mu}$, and by $\tilde{\Delta}'_k$ and Δ'_k the quantities corresponding to (2.1), (2.3) and (2.4).

Next we cite results on the stability and approximation power of R_h; analogous results hold also for R'_h. We will frequently use the following definition.

Definition 2.7. We say that the condition (R) or (R$'$) is satisfied if the functions $\{\Delta_k(\cdot, y),\ k = 1, \dots, M\}$ or $\{\Delta'_k(\cdot, y),\ k = 1, \dots, M\}$, respectively, are linearly independent on the set of quadrature points $\{\xi_j\}$ for all $|y| \le 1/2$.

Proposition 2.8 [**3**, Proposition 3.7]. *Let condition* (R) *be satisfied, and assume $0 \le s < r - M + 1/2$, $s \le t \le r$ and $1/2 < t$. Then*

$$\|R_h f - f\|_s \le C h^{t-s} \|f\|_t \quad \text{for} \quad f \in H^t. \tag{2.13}$$

With respect to the norm $\|\cdot\|_h$ the qualocation projection R_h has the same approximation power as with respect to the norm $\|\cdot\|_0$. This is one part of the following result.

Proposition 2.9 [**3**, Proposition 3.11]. *Let condition* (R) *be satisfied, and assume $1/2 < t \le r$. Then*

$$\|R_h f - f\|_h + \|P_h f - f\|_h \le C h^t \|f\|_t \quad \text{for} \quad f \in H^t. \tag{2.14}$$

Here $P_h : H^t \to S_h^{r,M}$ is the projection introduced in [**4**, p. 428] through the definition

$$P_h f \in S_h^{r,M}, \quad (P_h f, \Phi)_0 = (f, \Phi)_0 \quad \text{for} \quad \Phi \in S_h^{\infty,M}, \tag{2.15}$$

where

$$S_h^{\infty,M} := \operatorname{span}\{\Phi_{\mu+\ell N},\ \mu \in \Lambda_h,\ \ell \in (-M/2, M/2]\}.$$

It is shown in [**4**, Theorem 3.4] that for $s < r - M + 1/2$ and $s \le t \le r$

$$\|P_h f - f\|_s \le C h^{t-s} \|f\|_t \quad \text{for} \quad f \in H^t. \tag{2.16}$$

It follows that for $s < r - M + 1/2$ and $f \in H^s$

$$P_h f \to f \quad \text{in} \quad H^s \quad \text{for} \quad (h \in \mathcal{H}). \tag{2.17}$$

In the same way as Lemmas 3.12 and 3.13 in [**3**] the following two lemmas can be proved.

Lemma 2.10. *Let condition* (R$'$) *be satisfied, and assume that* $0 \le s < r' - M + 1/2$, $s + \beta < r - M + 1/2$ *and* $\beta + M < r$. *Let* $\{Z_h\}_{\mathcal{H}}$ *be for all* $t \in \mathbf{R}$ *a bounded sequence of mappings in* H^t. *Then the following stability estimate holds*:

$$\|R_h' L Z_h v_h\|_s + \|R_h' Z_h L v_h\|_s \le C \|v_h\|_{s+\beta}$$
$$\textit{for} \quad v_h \in S_h^{r,M} \quad \textit{and} \quad h \in \mathcal{H}.$$

Lemma 2.11. *Let condition* (R$'$) *be satisfied, and assume that* $0 \le s < r' - M + 1/2$, $\beta + s < r - M + 1/2$ *and* $\beta + M < r$. *Let* $g \in C^\infty(\mathbf{T})$. *Then for* $v_h \in S_h^{r,M}$ *the following convergence relation holds*:

$$v_h \longrightarrow v \quad \textit{in} \quad H^{s+\beta} \quad \textit{for} \quad (h \in \mathcal{H})$$
$$\Longrightarrow R_h' L(g v_h) \longrightarrow L(g v) \quad \textit{in} \quad H^s \quad \textit{for} \quad (h \in \mathcal{H}).$$

We shall also need the following variant of Lemma 2.11.

Lemma 2.12. *Let condition* (R′) *and* $\beta + M < r$ *be satisfied. If the sequence* $\{v_h \in S_h^{r,M}\}_{\mathcal{H}}$ *is bounded in* H^β, *then*

$$\text{(2.18)} \quad L_1 v_h \longrightarrow f \quad \text{in} \quad H^0 \quad \text{for} \quad (h \in \mathcal{H}) \\ \Longrightarrow R'_h L_1 v_h \longrightarrow f \quad \text{in} \quad H^0 \quad \text{for} \quad (h \in \mathcal{H}).$$

Proof. Choose $\sigma \in (1/2, r]$ such that $\beta + \sigma < r - M + 1/2$. Recall that L_1 maps for some $\delta > 0$ the space $H^{\sigma+\beta-\delta}$ into H^σ boundedly. We can assume $\delta \le 1/2$. With the aid of Proposition 2.8 (with R_h replaced by R'_h) and the inverse estimate (see [**4**, Theorem 3.4]) we obtain (with I denoting the identity operator)

$$\begin{aligned} \|R'_h L_1 v_h - f\|_0 &\le \|(I - R'_h)\, L_1 v_h\|_0 + \|L_1 v_h - f\|_0 \\ &\le C\,(h^\sigma \|L_1 v_h\|_\sigma + \|L_1 v_h - f\|_0) \\ &\le C\,(h^\sigma \|v_h\|_{\sigma+\beta-\delta} + \|L_1 v_h - f\|_0) \\ &\le C\left(h^\delta \|v_h\|_\beta + \|L_1 v_h - f\|_0\right), \end{aligned}$$

which converges to zero for $(h \in \mathcal{H})$. □

On the space W_h of grid functions on the mesh π'_h the qualocation projection is bounded with respect to the norm $\|\cdot\|_h$.

Proposition 2.13 [**3**, Proposition 3.14]. *Let condition* (R) *be satisfied. Then*

$$\text{(2.19)} \qquad \|R_h f_h\|_0 \le C \|f_h\|_h \quad \text{for} \quad f_h \in W_h \quad \text{and} \quad h \in \mathcal{H}.$$

Finally, we state the superapproximation and commutator properties.

Theorem 2.14 [**3**, Corollary 4.2, Theorem 4.4]. *Let condition* (R) *be satisfied, and let* $g \in C^r(\mathbf{T})$. *If* $M < r$, $0 \le s < r - M + 1/2$ *and* $t \le r - M$, *then*

$$\text{(2.20)} \quad \|(I - R_h) g v_h\|_s \le C h^{1+t-s} \|g'\|_{r-1,\infty} \|v_h\|_t \quad \text{for} \quad v_h \in S_h^{r,M}.$$

If $1/2 < t \le r$ *and* $0 \le s < r - M + 1/2$, *then*

$$\|R_h g(I - R_h) f\|_s \le C h^{1+t-s} \|g'\|_{r-1,\infty} \|f\|_t \quad \textit{for} \quad f \in H^t. \tag{2.21}$$

The same estimates hold with R_h *replaced by the projection* P_h *from* (2.15).

3. Qualocation. Throughout the rest of the paper we assume that the condition (R′) is satisfied and consequently that $(\cdot,\cdot)_h$ is an inner product on $S_h^{r',M}$ and the qualocation projection R'_h is well-defined.

It is easily seen, from the earlier definitions, that

$$L_0 \psi_{k,\mu} = \Phi_\mu(x) \begin{cases} \sigma_0(x,\mu)\, \Omega_1(Nx, (\mu/N); x) & \text{if } k = 1, \\ N^\beta \Omega_k(Nx, (\mu/N); x) & \text{if } k = 2, \dots, M, \end{cases}$$

where the functions $\Omega_k(\xi, y; x)$ are given, for $\xi \in \mathbf{R}$, $|y| \le 1/2$ and $x \in \mathbf{T}$, by

$$\widetilde{\Omega}_k(\xi, y; x) := \sum_{\ell \ne 0} \sigma_0(x, y+\ell)\, \frac{\ell^{k-1}}{(y+\ell)^r}\, \Phi_\ell(\xi), \tag{3.1}$$

$$\Omega_1(\xi, y; x) := 1 + (\sigma_0(x,y))^{-1} y^r\, \widetilde{\Omega}_1(\xi, y; x) \quad \text{for} \quad y \ne 0, \tag{3.2}$$

$$\Omega_1(\xi, 0; x) := 1, \tag{3.3}$$

$$\Omega_k(\xi, y; x) := \tilde{\Omega}_k(\xi, y; x) \quad \text{for} \quad k = 2, \dots, M. \tag{3.4}$$

We omit the variable x in the notation if σ_0 is independent of x. For Ω_k to be well-defined for $k = 1, \dots, M$ we assume that

$$\beta + M < r. \tag{3.5}$$

It follows from (3.5) that $\Omega_1(\xi, y; x)$ is continuous at $y = 0$ with value equal to 1.

An essential role is played by the so-called numerical symbol, defined as the $M \times M$-matrix $D = D(y;x)$ for $|y| \le 1/2$ and $x \in \mathbf{T}$ with elements

$$(3.6) \qquad [D(y;x)]_{k,\ell} := Q\big(\Omega_\ell(\cdot,y;x), \Delta'_k(\cdot,y)\big) \quad \text{for} \quad k,\ell = 1,\dots,M.$$

The numerical symbol D is said to be elliptic if $D(y;x)$ is nonsingular for $x \in \mathbf{T}$ and $|y| \le 1/2$.

Clearly, Ω_k is equal to Δ_k for $\sigma_0 = 1$, i.e., when $L = I$. If additionally, $r' = r$, the numerical symbol is equal to the matrix $B'(y)$ with elements

$$(3.7) \qquad B'_{k,\ell}(y) := Q\big(\Delta'_k(\cdot,y), \Delta'_\ell(\cdot,y)\big) \quad \text{for} \quad k,\ell = 1,\dots,M.$$

The ellipticity is for that case characterized in [**3**, Lemma 3.1]. In the special case of collocation, i.e., $J = M$, we can similarly characterize the ellipticity for a general operator L.

Lemma 3.1. *Assume $J = M$. The numerical symbol D is elliptic if and only if each of the two sets of functions $\{\Delta'_k(\cdot,y),\ k = 1,\dots,M\}$ and $\{\Omega_k(\cdot,y;x),\ k = 1,\dots,M\}$ is for $|y| \le 1/2$ and $x \in \mathbf{T}$ linearly independent on the set of quadrature points $\{\xi_j\}$.*

Proof. From (3.6), if for an $x \in \mathbf{T}$ and a $y \in [-1/2, 1/2]$ one of the two sets is not linearly independent, then D is clearly singular, and hence not elliptic. On the other hand, assume that D is not elliptic, and hence for some $x \in \mathbf{T}$ and $y \in [-1/2, 1/2]$ singular. Either the first set of functions is linearly dependent (and we are done) or it is linearly independent, in which case because D is singular there exists a nontrivial linear combination of the second set that is Q-orthogonal to the first one. Since $J = M$ the set $\{\Delta'_k(\cdot,y)\}$ is a basis in the space of functions on the quadrature points $\{\xi_j\}$ and, hence, that linear combination is equal to zero, i.e., the second set is linearly dependent. $\square$

4. Constant symbol case. In this section we derive the main stability and convergence results for the qualocation method for the case that the main part L_0 of the pseudodifferential operator L has constant coefficients, i.e., the symbol σ_0 is independent of the space variable x.

As a preparation we provide the following lemma that corresponds to Lemma 2.2.

Lemma 4.1. *Let σ_0 be independent of x. For $\mu, \nu \in \Lambda_h$ and $\ell = 1, \dots, M$,*

$$(L_0\psi_{1,\mu}, \psi'_{\ell,\nu})_h = \delta_{\mu\nu}\sigma_0(\mu)\, Q\Big(\Omega_1\Big(\cdot, \frac{\mu}{N}\Big), \Delta'_\ell\Big(\cdot, \frac{\nu}{N}\Big)\Big), \tag{4.1}$$

$$(L_0\psi_{k,\mu}, \psi'_{\ell,\nu})_h = \delta_{\mu\nu} N^\beta Q\Big(\Omega_k\Big(\cdot, \frac{\mu}{N}\Big), \Delta'_\ell\Big(\cdot, \frac{\nu}{N}\Big)\Big) \quad \textit{for} \quad k = 2, \dots, M. \tag{4.2}$$

Proof. With the Fourier series representation of $\psi_{k,\mu}$ from (2.1)–(2.5) (or see [**3**, Lemma 7.5]), we obtain with the definition (1.2) of L_0, for $k > 1$

$$\begin{aligned} L_0\psi_{k,\mu}(x) &= \sum_{\ell \neq 0} \frac{\ell^{k-1}}{(\mu/N + \ell)^r}\, \sigma_0(\mu + \ell N)\Phi_{\mu+\ell N}(x) \\ &= N^\beta \Phi_\mu(x) \sum_{\ell \neq 0} \frac{\ell^{k-1}}{(\mu/N + \ell)^r}\, \sigma_0\Big(\frac{\mu}{N} + \ell\Big)\Phi_\ell(Nx) \\ &= N^\beta \Phi_\mu(x)\, \Omega_k\Big(Nx, \frac{\mu}{N}\Big), \end{aligned}$$

and for $k = 1$,

$$\begin{aligned} L_0\psi_{1,\mu}(x) &= \Phi_\mu(x)\Big[\sigma_0(\mu) + N^\beta \Big(\frac{\mu}{N}\Big)^r \widetilde{\Omega}_1\Big(Nx, \frac{\mu}{N}\Big)\Big] \\ &= \Phi_\mu(x)\sigma_0(\mu)\, \Omega_1\Big(Nx, \frac{\mu}{N}\Big). \end{aligned}$$

Taking [**3**, Lemma 7.2] into account, we obtain for $\ell = 1, \dots, M$

$$\begin{aligned} (L_0\psi_{1,\mu}, \psi'_{\ell,\nu})_h &= \sigma_0(\mu)\, Q_N\Big(\Phi_{\mu-\nu}\Omega_1\Big(N\cdot, \frac{\mu}{N}\Big), \Delta'_\ell\Big(N\cdot, \frac{\nu}{N}\Big)\Big) \\ &= \delta_{\mu\nu}\sigma_0(\mu)\, Q\Big(\Omega_1\Big(\cdot, \frac{\mu}{N}\Big), \Delta'_\ell\Big(\cdot, \frac{\nu}{N}\Big)\Big). \end{aligned}$$

The proof of (4.2) is similar. □

In the present case of an x-independent symbol σ_0, we are now going to set up the linear system relating

$$v_h = \sum_{\mu \in \Lambda_h} \sum_{k=1}^{M} c_{k,\mu} \psi_{k,\mu}$$

and

$$R'_h L_0 v_h = \sum_{\mu \in \Lambda_h} \sum_{k=1}^{M} d_{k,\mu} \psi'_{k,\mu}. \tag{4.3}$$

By forming the inner product $(\cdot,\cdot)_h$ with $\psi'_{\mu,\nu}$ in (4.3) we obtain, taking the definition of R'_h and Lemmas 2.2 and 4.1 into account, the block diagonal system

$$D\left(\frac{\mu}{N}\right)\tilde{c}_\mu = B'\left(\frac{\mu}{N}\right) d_\mu \quad \text{for} \quad \mu \in \Lambda_h, \tag{4.4}$$

where the components of the vectors $\tilde{c}_\mu \in \mathbf{C}^M$ for $\mu \in \Lambda_h$ are given by

$$\tilde{c}_{1,\mu} := \sigma_0(\mu) c_{1,\mu} \quad \text{and} \quad \tilde{c}_{k,\mu} := N^\beta c_{k,\mu} \quad \text{for} \quad k = 2, \dots, M. \tag{4.5}$$

Proposition 4.2. *Let σ_0 be independent of x, let $\beta + M < r$, and let condition* (R′) *be satisfied. Then the inverse stability estimate*

$$\|v_h\|_\beta \le C \|R'_h L_0 v_h\|_0 \quad \textit{for} \quad v_h \in S_h^{r,M} \quad \textit{and} \quad h \in \mathcal{H} \tag{4.6}$$

is satisfied if and only if the numerical symbol D is elliptic.

Proof. Assume first that D is elliptic. Then for each fixed y, $|y| \le 1/2$, the $M \times M$-matrix $D(y)$ is invertible. Since $D(\cdot)$ is continuous the inverse is bounded uniformly in y. Since also $B'(\cdot)$ is continuous and hence bounded it follows from (4.4) that

$$|\tilde{c}_\mu| \le C |d_\mu| \quad \text{for} \quad \mu \in \Lambda_h,$$

where $|\cdot|$ denotes a norm on $\mathbf{C}^M$. Hence, we obtain with Proposition 2.4 and Corollary 2.5, taking also the estimate $|\langle\mu\rangle^\beta| \leq C|\sigma_0(\mu)|$ into account (recall that L_0 is elliptic)

$$\begin{aligned}\|v_h\|_\beta^2 &\leq C \sum_{\mu\in\Lambda_h} \left(\langle\mu\rangle^{2\beta}|c_{1,\mu}|^2 + N^{2\beta}\sum_{k=2}^{M}|c_{k,\mu}|^2\right) \\ &\leq C \sum_{\mu\in\Lambda_h} |\tilde{c}_\mu|^2 \leq C \sum_{\mu\in\Lambda_h} |d_\mu|^2 \leq C\|R'_h L_0 v_h\|_0^2.\end{aligned}$$

For the proof of the converse, choose $\mu \in \Lambda_h$ and apply (4.6) to v_h of the form

$$v_h = \sum_{k=1}^{M} c_{k,\mu}\psi_{k,\mu}$$

yielding with the tools already used before

$$|\tilde{c}_\mu| \leq C\|v_h\|_\beta \leq C\|R'_h L_0 v_h\|_0 \leq C|d_\mu| \tag{4.7}$$

for $\tilde{c}_\mu$ and d_μ standing in the relation (4.4). The matrix $B'(y)$ is a special case of D, obtained for $r = r'$ and $L_0 = I$. Being a Gram matrix for the linearly independent functions $\Delta'_k(\cdot, y)$, $k = 1, \dots, M$, it is nonsingular. Hence, from the first part of the proof we know that it has a bounded inverse uniformly in y. Consequently, it follows from (4.4) and (4.7) that

$$\left| D\left(\frac{\mu}{N}\right)^{-1} \right| \leq C \quad \text{for} \quad \mu \in \Lambda_h \quad \text{and} \quad h \in \mathcal{H}.$$

By continuity the last bound holds for all $|y| \leq 1/2$, proving the ellipticity of D. $\square$

From (4.6) we derive the corresponding inverse stability for the whole operator L. We point out that in the remainder of this section it is not assumed that L_0 has constant coefficients. By $\mathcal{H}_1$ we denote a subset of $\mathcal{H}$ such that $\mathcal{H}\backslash\mathcal{H}_1$ is finite and by $\mathcal{H}'$ a subsequence of $\mathcal{H}$, both not necessarily the same at different occurrences.

Lemma 4.3. *Let $\beta + M < r$, and let condition* (R′) *be satisfied. Then L_0 is injective in H^β and the inverse stability estimate*

$$\|v_h\|_\beta \le C\|R'_h L v_h\|_0 \quad \text{for} \quad v_h \in S_h^{r,M} \quad \text{and} \quad h \in \mathcal{H}_1 \tag{4.8}$$

holds if and only if L is injective in H^β and

$$\|v_h\|_\beta \le C\|R'_h L_0 v_h\|_0 \quad \text{for} \quad v_h \in S_h^{r,M} \quad \text{and} \quad h \in \mathcal{H}_1. \tag{4.9}$$

Proof. For each $h \in \mathcal{H}$ choose $v_h \in S_h^{r,M}$, converging to v in H^β. As a consequence of Lemma 2.11, the estimate (4.8) implies $\|v\|_\beta \le C\|Lv\|_0$ and thus the injectivity of L. Similarly, (4.9) implies the injectivity of L_0. It remains to show that (4.8) follows from (4.9), and vice versa, if L_0 and L are both injective. Assume that (4.9) holds but that (4.8) is not satisfied. Then there exist $\mathcal{H}'_1$ and $v_h \in S_h^{r,M}$ for $h \in \mathcal{H}'_1$ such that (4.10)

$$\|v_h\|_\beta = 1 \quad \text{for} \quad h \in \mathcal{H}'_1 \quad \text{and} \quad \|R'_h L v_h\|_0 \longrightarrow 0 \quad \text{for} \quad (h \in \mathcal{H}'_1).$$

Since L_1 maps $H^\beta \to H^\delta$ for some $\delta > 0$ boundedly, the compactness of the imbedding from $H^\delta \to H^0$ implies the existence of a further subsequence $\mathcal{H}''_1$ and an $f \in H^0$ such that

$$L_1 v_h \longrightarrow f \quad \text{in} \quad H^0 \quad \text{for} \quad (h \in \mathcal{H}''_1).$$

With the aid of Lemma 2.12 and (4.10) we then conclude that

$$R'_h L_0 v_h = R'_h (L - L_1) v_h \longrightarrow -f \quad \text{in} \quad H^0 \quad \text{for} \quad (h \in \mathcal{H}''_1).$$

Let $w \in H^\beta$ be the solution of $L_0 w = -f$. With P_h being the projection from (2.15), we obtain by virtue of (4.9) and Lemma 2.11,

$$\|v_h - P_h w\|_\beta \le C\|R'_h L_0 (v_h - P_h w)\|_0 \longrightarrow 0 \quad \text{for} \quad (h \in \mathcal{H}''_1),$$

i.e., $v_h \to w$ in H^β, for $h \in \mathcal{H}''_1$, and hence $\|w\|_\beta = 1$. Another application of Lemma 2.11, this time to the sequence $\{R'_h L v_h\}_{\mathcal{H}''_1}$, yields with (4.10)

$$\lim_{h \in \mathcal{H}''_1} R'_h \, L v_h = Lw = 0,$$

in contradiction to the injectivity of L. □

A main result of this section is the following convergence theorem.

Theorem 4.4. *Let condition* (R$'$) *be satisfied, and let the inverse stability estimate* (4.8) *hold. Assume $\beta + M < r$, and let s and t be real numbers satisfying*

$$s < r - M + \frac{1}{2}, \quad \beta + \frac{1}{2} < t, \quad \beta \leq s \leq t \leq r.$$

Then the qualocation equations (1.11) *are uniquely solvable if h is sufficiently small. Moreover, if $u \in H^t$*

$$\|u_h - u\|_s \leq C h^{t-s} \|u\|_t \quad \text{for} \quad h \in \mathcal{H}_1. \tag{4.11}$$

Proof. Let P_h be the projection from (2.15). With the aid of the inverse inequality, (4.8) and (2.13) (with R_h replaced by R'_h) we derive

$$\begin{aligned} h^{s-\beta}\|u_h - P_h u\|_s &\leq C\|u_h - P_h u\|_\beta \\ &\leq C\|R'_h L(u_h - P_h u)\|_0 = C\|R'_h L(u - P_h u)\|_0 \\ &\leq C\left(\|(I - R'_h)L(u - P_h u)\|_0 + \|L(u - P_h u)\|_0\right) \\ &\leq C\left(h^\sigma \|L(u - P_h u)\|_\sigma + \|L(u - P_h u)\|_0\right) \\ &\leq C\left(h^\sigma \|u - P_h u\|_{\sigma+\beta} + \|u - P_h u\|_\beta\right), \end{aligned}$$

where $\sigma \in (1/2, 1]$ is chosen such that $\sigma + \beta < \min\{r - M + 1/2, t\}$. Uniqueness then follows by setting $u = 0$, and the assertion is now seen to hold by invoking (2.16). $\square$

For completeness, we prove an extension of the inverse stability result in Proposition 4.2.

Corollary 4.5. *Let condition* (R$'$) *be satisfied, and let the inverse stability estimate* (4.8) *hold. Assume $\beta + M < r$, and let s be a real number satisfying*

$$\beta + s < r - M + \frac{1}{2} \quad \text{and} \quad 0 \leq s < r' - M + \frac{1}{2}.$$

Then the inverse stability estimate

$$\|v_h\|_{\beta+s} \leq C\|R'_h L v_h\|_s \quad \textit{for} \quad v_h \in S_h^{r,M} \tag{4.12}$$

holds true for almost all $h \in \mathcal{H}$*.*

Proof. Let $v_h \in S_h^{r,M}$ be given. We define u as the solution of $Lu = R'_h L v_h$ (note that it follows from (4.8) and Lemma 2.11 that L is injective in H^β and hence $L : H^{t+\beta} \to H^t$ is surjective for $t \geq 0$). In this situation the qualocation approximate solution u_h is equal to v_h. We apply (4.11) with s and t replaced by $s+\beta$ and $t+\beta$, respectively, with t satisfying $t = s$ if $s > 1/2$ and $t \in (1/2, 1)$ if $s \leq 1/2$ and obtain

$$\begin{aligned}
\|v_h\|_{s+\beta} &\leq \|u_h - u\|_{s+\beta} + \|u\|_{s+\beta} \\
&\leq C\left(h^{t-s}\|L^{-1}R'_h L v_h\|_{t+\beta} + \|L^{-1}R'_h L v_h\|_{s+\beta}\right) \\
&\leq C\left(h^{t-s}\|R'_h L v_h\|_t + \|R'_h L v_h\|_s\right) \leq C\|R'_h L v_h\|_s. \qquad \square
\end{aligned}$$

Remark 4.6. Stability estimates and convergence results for projection methods applied to pseudodifferential operators have been proved in a quite general setting, e.g., in [**6**] and [**7**]. In [**7**, p. 425] the expectation is expressed that some of the results in [**7**] carry over to the qualocation method because the 'qualocation projection satisfies the hypotheses of [**7**]'. But it is not clear that this argument really works because the qualocation projection is in a certain range of indices unbounded. The latter is also the case for the collocation method studied in [**4**]. In [**4**, p. 438] the methods of [**7**] are made applicable by the trick of working with a bounded projection that maps on the same space as the collocation projection, but there still remains a gap in the range of indices covered by the proof, see [**4**, p. 439]. Moreover, it is not evident how a similar trick can be applied for the qualocation projection. Note also that the numerical symbol in [**6, 7**] depends on the index s of the $\|\cdot\|_s$-norm in which the stability is studied.

By the way, readers of [**6**] may want to be careful in using the formula [**6**, (12)] for the numerical symbol in the case $k = 0$, where the different homogeneity of the symbol σ for argument zero has to be taken into account in the calculation.

5. A local principle. Our analysis of the variable coefficient case relies on a slightly modified version of Satz 1 in [**5**]. The modification concerns the condition $\|R_n f(I-R_n)\|_X \to 0$ as $n \to \infty$ and $\|Q_n f(I-Q_n)\|_Y \to 0$ as $n \to \infty$, i.e., [**5**, p. 241, condition III.2], which appears to be not easily applicable to the case of unbounded projections Q_n or an unbounded sequence of projections Q_n. A similar observation was already made in [**2**] for the case $X = Y$. In this section we state the local principle in the form we need it.

The following setting is suitable for our purposes. Let X, Y and Z be Banach spaces with $Z \subset Y$, and Z continuously imbedded in Y. Let $\{X_n\}_{n\in\mathbf{N}} \subset X$ and $\{Y_n\}_{n\in\mathbf{N}} \subset Y$ be two sequences of finite dimensional subspaces. For each $n \in \mathbf{N}$ let $P_n, R_n \in \mathcal{L}(X, X_n)$ be linear projections with range X_n, and let Q_n, S_n be linear projections with range Y_n and domains satisfying $Z \subset D(Q_n) = D(S_n) \subset Y$. Here $\mathcal{L}(V, W)$ denotes the space of linear bounded operators between the Banach spaces V and W. Finally, let $\mathcal{S} \subset \mathcal{L}(X, Y)$ be a linear subspace.

The following technical conditions are used in Proposition 5.3 below. By $\mathcal{K}(X, Z)$ we denote the space of linear compact operators from $X \to Z$.

I. $\|P_n x - x\|_X \to 0$ for $x \in X$, and $\|Q_n z - z\|_Y \to 0$ for $z \in Z$ as $n \to \infty$.

II. 1. $\mathcal{K}(X, Z) \subset \mathcal{S}$.

2. $AX_n \subset Z$ for $A \in \mathcal{S}$.

3. $Q_n A P_n \in \mathcal{L}(X, Y)$ and $\|Q_n A P_n x - x\|_Y \to 0$ as $n \to \infty$ for $A \in \mathcal{S}$ and $x \in X$.

III. Let $\mathcal{M}$ be a set of multipliers for $\mathcal{S}$, i.e., $\mathcal{M} \subset \mathcal{L}(X) \cap \mathcal{L}(Y)$ is an algebra such that $Af \in \mathcal{S}$ and $fA \in \mathcal{S}$ for $f \in \mathcal{M}$ and $A \in \mathcal{S}$, such that $\mathcal{M}Y_n \cup \mathcal{M}AX_n \cup A\mathcal{M}X_n \subset D(Q_n)$ for $A \in \mathcal{S}$ and

1. $\sup_n \|R_n f P_n\|_{X\to X} < \infty$ and $\sup_n \|Q_n f S_n\|_{Y\to Y} < \infty$ for $f \in \mathcal{M}$,

2. $\|Q_n f(I-Q_n)AP_n\|_{X\to Y} \to 0$ as $n \to \infty$ for $A \in \mathcal{S}$ and $f \in \mathcal{M}$, $\|Q_n f(I-Q_n)gS_n\|_{Y\to Y} \to 0$ as $n \to \infty$ for $f, g \in \mathcal{M}$,

3. $\|Q_n A(I-R_n)fP_n\|_{X\to Y} \to 0$ as $n \to \infty$ for $A \in \mathcal{S}$ and $f \in \mathcal{M}$, $\|R_n f(I-R_n)gP_n\|_{X\to X} \to 0$ as $n \to \infty$ for $f, g \in \mathcal{M}$.

IV. For an index set K there exists for each $z \in K$ a subset $\mathcal{M}_z \subset \mathcal{M}$ such that

1. $0 \notin \mathcal{M}_z$ and for any two elements $f_z^{[j]} \in \mathcal{M}_z$, $j = 1, 2$, there exists a third element $f_z \in \mathcal{M}_z$ with $f_z^{[j]} f_z = f_z$, for $j = 1, 2$,

2. each set $\{f_z\}_{z \in K}$ of elements $f_z \in \mathcal{M}_z$ contains a finite subset $f_{z_1}, \dots, f_{z_m}$ such that $f_{z_1} + \cdots + f_{z_m}$ is invertible in $\mathcal{M}$.

V. 1. $Af - fA \in \mathcal{K}(X, Z)$ for $A \in \mathcal{S}$, $f \in \mathcal{M}_z$ and $z \in K$,

2. for all $A \in \mathcal{S}$ and $z \in K$ there exist operators $A_z \in \mathcal{S}$, such that for all $\varepsilon > 0$ one can find $T_z \in \mathcal{K}(X, Z)$, $f_z \in \mathcal{M}_z$ and $n_0 \geq 1$ satisfying

$$\|Q_n(A - A_z) f_z P_n - Q_n T_z P_n\|_{X \to Y} < \varepsilon \quad \text{for} \quad n \in \mathbf{N} \quad \text{with} \quad n \geq n_0.$$

The following definitions are adapted from [**5**]. We assume that we have the setting in which conditions I–V are satisfied. Unless stated otherwise, all symbols in these definitions have the same meaning as in those conditions.

Definition 5.1. For an operator $A \in \mathcal{S}$, the sequence $\{Q_n A P_n\}$ is said to be stable if the operators $Q_n A : X_n \to Y_n$ are invertible for $n \geq n_0$ and

$$\sup_{n \geq n_0} \|(Q_n A|_{X_n})^{-1}\|_{Y_n \to X_n} < \infty.$$

Definition 5.2. For an operator $A \in \mathcal{S}$ the sequence $\{Q_n A P_n\}$ is said to be locally stable from the right if for all $z \in K$ and $n \in \mathbf{N}$ with $n \geq n_0$ there exist operators $T_z \in \mathcal{K}(X, Z)$, $D_{z,n} \in \mathcal{L}(Y_n, X_n)$ and an element $f_z \in \mathcal{M}_z$ such that

$$\|Q_n f_z (A_z + T_z) D_{z,n} - Q_n f_z S_n\|_{Y_n \to Y_n} \to 0 \quad \text{as} \quad n \to \infty \tag{5.13}$$

and

$$\sup_{n \geq n_0} \|D_{z,n}\|_{Y_n \to X_n} < \infty. \tag{5.14}$$

We are now in a position to state our version of Prößdorf's Satz 1 from [**5**]. The proof requires only minor modifications of the original one.

Proposition 5.3. *Assume that the above conditions* I–V *hold and that $A^{-1} \in \mathcal{L}(Y, X)$ exists. Then the sequence $\{Q_n A P_n\}$ is stable if and only if it is locally stable from the right.*

6. Variable coefficients. The main aim of this section is to prove inverse stability if the symbol σ_0 is x-dependent.

Theorem 6.1. *Assume condition* (R′) *to be satisfied, and let $\beta + M < r$. The inverse stability estimate*

$$\|v_h\|_\beta \le C \|R'_h L_0 v_h\|_0 \quad \text{for} \quad v_h \in S_h^{r,M} \quad \text{and} \quad h \in \mathcal{H}_1 \tag{6.1}$$

holds true if and only if $L_0 : H^\beta \to H^0$ is injective and for all $z \in \mathbf{T}$ the inverse stability estimate

$$\|v_h\|_\beta \le C \|R'_h L_{0,z} v_h\|_0 \quad \text{for} \quad v_h \in S_h^{r,M} \quad \text{and} \quad h \in \mathcal{H}_1 \tag{6.2}$$

holds true, where $L_{0,z}$ is the pseudodifferential operator with the symbol frozen at z, i.e., with the x-independent symbol $\sigma_0(z, \xi)$.

Once this is established, a combination of Proposition 4.2, Theorem 6.1 and Lemma 4.3 delivers the next corollary.

Corollary 6.2. *Assume condition* (R′) *to be satisfied, and let $\beta + M < r$. The inverse stability estimate*

$$\|v_h\|_\beta \le C \|R'_h L v_h\|_0 \quad \text{for} \quad v_h \in S_h^{r,M} \quad \text{and} \quad h \in \mathcal{H}_1 \tag{6.3}$$

holds true if and only if $L : H^\beta \to H^0$ is injective and the numerical symbol D is elliptic.

Corollary 3.4 also furnishes the stability estimate (4.12) if (6.1) holds. Combining Corollary 6.2 with Theorem 4.4 we can state the following convergence result.

Theorem 6.3. *Assume condition* (R′) *is satisfied. Let the numerical symbol be elliptic, and let $L_0 : H^\beta \to H^0$ be injective. Assume $\beta + M < r$, and let s and t be real numbers satisfying*

$$s < r - M + \frac{1}{2}, \quad \beta + \frac{1}{2} < t, \quad \beta \le s \le t \le r.$$

Then the qualocation equations (1.11) *are uniquely solvable if h is sufficiently small. Moreover, if $u \in H^t$*

$$\|u_h - u\|_s \le C h^{t-s} \|u\|_t \quad \textit{for} \quad h \in \mathcal{H}_1.$$

The proof of Theorem 6.1 relies on Proposition 5.3. We begin with verifying the conditions I–V in Section 5. First we identify the various quantities in the general setting of Section 5. The sequence of quantities is indexed by $n \in \mathbf{N}$ there, but we keep the indices $h \in \mathcal{H}$ when dealing with the qualocation context. We take

$$X = H^\beta, \quad Y = H^0, \quad Z = H^\sigma, \quad X_n = S_h^{r,M}, \quad Y_n = S_h^{r',M}, \tag{6.4}$$

$$P_n = P_h, \quad Q_n = R_h', \quad R_n = P_h, \quad S_n = P_h', \tag{6.5}$$

where $\sigma \in (1/2, r - M - \beta + 1/2)$ and $\sigma \le r'$, and P_h and $P_h^{'}$ are the projections on $S_h^{r,M}$ and $S_h^{r',M}$, respectively, from (2.15). We identify the algebra $\mathcal{M}$ with $C^\infty(\mathbf{T})$ and the subspace $\mathcal{S} \subset \mathcal{L}(X, Y)$ with the linear hull of $\mathcal{K}(X, Z)$ and the space of all operators of the form $L_0(f\cdot)$ with L_0 a variable coefficient pseudodifferential operator with order β and $f \in C^\infty(\mathbf{T})$. For the index set K we take $K = \mathbf{T}$. With this identification the requirements of the general setting in Section 5 are met. Recall from [**8**] that H^σ is continuously embedded in $C(\mathbf{T})$ such that the qualocation projections are well defined there. Note that operators in the form fL_0 are included in the set $\mathcal{S}$ since L_0 has variable coefficients. We now check the conditions I–V.

Condition I is known to hold from (2.17) and Proposition 2.8. Conditions II.1 and II.2 are obvious from the definition of $\mathcal{S}$. Condition II.3 is clear for $A \in \mathcal{K}(X, Z)$ and follows for $A = L_0(f\cdot)$ from Lemma 2.11 and (2.17).

Condition III.1: $\|P_h f P_h v\|_\beta \le \|v\|_\beta$ is satisfied for $v \in H^\beta$ since $\{P_h\}$ is bounded in H^β as is multiplication by a smooth function f. The estimate $\|R_h' f P_h' v\|_0 \le \|v\|_0$ for $v \in H^0$ is contained in Lemma 2.10 by choosing $L = I$ and Z_h to be the operator of multiplication by f.

Condition III.2: With the aid of Theorem 2.14 (with R_h and r replaced by R'_h and r', respectively) and the inverse inequality we conclude, with $v \in H^\beta$,

$$\begin{aligned}\|R'_h g(1-R'_h)L_0 f P'_h v\|_0 &\le Ch^{1+\sigma}\|L_0 f P_h v\|_\sigma \le Ch^{1+\sigma}\|P_h v\|_{\beta+\sigma}\\ &\le Ch\|P_h v\|_\beta \le Ch\|v\|_\beta \to 0 \quad \text{for} \quad (h \in \mathcal{H}).\end{aligned}$$

The first relation in III.2 is thereby verified, since it also holds for $A \in \mathcal{K}(X, Z)$ as is seen from Proposition 2.8. The second part of the condition is included in the first part if we choose A to be multiplication by g.

Condition III.3: With the aid of Lemma 2.10 and Theorem 2.14, we obtain

$$\begin{aligned}\|R'_h L_0(1-P_h) f P_h g\|_0 &\le C\|(1-P_h) f P_h g\|_\beta\\ &\le Ch\|P_h g\|_\beta \le Ch\|g\|_\beta,\end{aligned}$$

showing that the first part of the condition is satisfied since it also holds for $A \in \mathcal{K}(X, Z)$. The second part of the condition is included in the first part if we choose A to be multiplied by f.

Condition IV: This condition is satisfied by choosing

$$\mathcal{M}_z := \Big\{ f_z \in C_0^\infty\Big(z - \frac{1}{4},\, z + \frac{1}{4}\Big),\ f_z = 1 \text{ in a neighborhood of } z,\\ 0 \le f_z(x) \le 1,\ x \in \mathbf{T}\Big\}.$$

Condition V.1: It is proved in [**8**, Proposition 4.3] that the commutator of a pseudodifferential operator L_0 of order β and multiplication by a smooth function f satisfies

$$\|(fL_0 - L_0 f)v\|_{s-\beta+1} \le C\|v\|_s \quad \text{for} \quad v \in H^s \quad \text{and} \quad s \in \mathbf{R}. \tag{6.6}$$

Thus $L_0 f - f L_0$ maps H^β boundedly into H^1, and consequently $L_0(g\cdot)$ satisfies condition V.1. For $A \in \mathcal{K}(X, Z)$ the condition is obvious.

Condition V.2: For $A \in \mathcal{K}(X, Z)$ the condition holds with $A_z = 0$ and $T_z = Af_z$, so we consider the case $A = L_0(g\cdot)$. In view of the already verified Condition V.1 we know that

$$T_z := (L_0 - L_{0,z})gf_z - gf_z(L_0 - L_{0,z}) \in \mathcal{K}(X, Z),$$

and hence it is sufficient to prove for given $\varepsilon > 0$ the existence of $f_z \in \mathcal{M}$ such that

$$\text{(6.7)} \quad \|R_h' g f_z (L_0 - L_{0,z}) P_h v\|_0 \le \varepsilon \|v\|_\beta \quad \text{for} \quad v \in H^\beta \quad \text{and} \quad h \in \mathcal{H}_1.$$

The symbol of L_0 has the form (1.3) with continuous coefficients a^+ and a^-. We denote by L_0^+ and L_0^- the operators with symbol $a^+ = 1$, $a^- = 0$ and $a^+ = 0$, $a^- = 1$, respectively. With the aid of Propositions 2.13, 2.9 and 2.6 (with R_h and r replaced by R_h' and r', respectively) and Lemma 2.10, we obtain for all $\eta > 0$, by choosing the support of f_z small enough,

$$\begin{aligned}
\|R_h' f_z g (L_0 - L_{0,z}) P_h v\|_0 &\le C \|f_z g (L_0 - L_{0,z}) P_h v\|_h \\
&\le C\eta \left(\|L_0^+ P_h v\|_h + \|L_0^- P_h v\|_h \right) \\
&\le C\eta \big(\|(1 - R_h') L_0^+ P_h v\|_h + \|(1 - R_h') L_0^- P_h v\|_h \\
&\qquad + \|R_h' L_0^+ P_h v\|_h + \|R_h' L_0^- P_h v\|_h \big) \\
&\le C\eta \Big[h^\sigma \left(\|L_0^+ P_h v\|_\sigma + \|L_0^- P_h v\|_\sigma \right) \\
&\qquad + \|R_h' L_0^+ P_h v\|_0 + \|R_h' L_0^- P_h v\|_0 \Big] \\
&\le C\eta \left(h^\sigma \|P_h v\|_{\beta+\sigma} + \|P_h v\|_\beta \right) \le C\eta \|v\|_\beta.
\end{aligned}$$

Thus (6.7) and hence Condition V.2 is seen to hold. Having verified all conditions I–V, we are now in a position to use Proposition 5.3.

Proof of Theorem 6.1. Assume (6.2) holds. By choosing

$$\begin{gathered}
Q_n = R_h', \quad P_n = P_h, \quad X_n = S_h^{r,M}, \quad Y_n = S_h^{r',M}, \quad T_z = 0, \\
D_{z,n} = (R_h' L_{0,z}|_{S_h^{r,M}})^{-1} \quad \text{for} \quad h \in \mathcal{H}
\end{gathered}$$

it is seen that the sequence $\{R_h' L_0 P_h\}_{\mathcal{H}}$ is locally stable from the right. Since L_0 is assumed to be injective, it has a bounded inverse $H^0 \to H^\beta$. Then Proposition 5.3 yields (6.1). On the other hand, let (6.1) hold. It then follows as in the proof of Lemma 4.3 that L_0 is injective and by virtue of Proposition 5.3 $\{R_h L_0 P_h\}_{\mathcal{H}}$ is locally stable from the right. Since $(L_{0,z})_z = L_{0,z}$ and $L_{0,z}$ is injective, another application of Proposition 5.3 shows (6.2) holds. $\square$

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