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NONLINEAR INTEGRAL EQUATIONS WITH INCREASING OPERATORS IN MEASURE SPACES

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ABSTRACT. In this paper we consider a class of integral equations in measure spaces. Remarkable and important special integral equations are contained among them, which have been extensively investigated nowadays. The main results of this paper are existence theorems for the studied integral equations under the condition that the operator defined by the equation is increasing. Moreover, there are some auxiliary results which are interesting in their own rights. We shall see that some of the problems formulated for the classical integral equations can be solved in a very satisfactory way in this essentially more general case, and the results give unified approaches of the problems. Finally, some applications are given.

1. Introduction. In what follows $(X, \mathcal{A}, \mu_i)$, $i = 1, \dots, n$ are measure spaces, S is a function from X into $\mathcal{A}$, and $\mu := \sum_{i=1}^{n} \mu_i$.

In this paper we study integral equations of the form

$$y(x) = f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ y \, d\mu_i, \tag{1}$$

where $f : D_f(\subset X) \to \mathbf{R}$, $g_i : D_{g_i}(\subset X) \to \mathbf{R}$, $i = 1, \dots, n$, and $h_i : I_i$ $(\subset \mathbf{R}) \to \mathbf{R}$ for $i = 1, \dots, n$.

We recall some concepts from measure theory that will be used in the present work. When we consider a measure, we take it as understood that its domain is a σ-algebra. The integrable functions, with respect to a measure, over a measurable set are regarded as almost measurable on this set. The product of finitely many measure spaces is understood as in [**8**].

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Definition 1. We say that a function $s : A \to \mathbf{R}$ is a solution of the integral equation (1) if

(i) A is a nonempty subset of X such that $S(x) \subset A$ for every $x \in A$,

(ii) $h_i \circ s$ is μ_i-integrable over $S(x)$ for every $x \in A$ and $i = 1, \dots, n$,

(iii) $y := s$ satisfies (1) for every $x \in A$.

The function S is assumed to have some of the following properties, see [**5**]:

(C1) $x \notin S(x)$, $x \in X$.

(C2) If $x_1 \in S(x_2)$, then $S(x_1) \subset S(x_2)$, $x_2 \in X$.

(C3) $\{(x_1, x_2) \in X^2 \mid x_2 \in S(x_1)\}$ is $\mu_i \times \mu_j$-measurable for $i,\ j = 1, \dots, n$.

The first reason for the significance of the considered integral equations is seen from the observation that, under the previous assumptions on S, remarkable and important special integral equations are contained among them, which have been extensively investigated nowadays. This is illustrated by the following examples.

Let $X := [0, \infty[$, and let $\mathcal{A}$ be the Lebesgue-measurable subsets of X. Suppose $p : X \to [0, \infty[$ and $q : X \to [0, \infty]$ are measurable functions such that $p \le q$. Define S_1 and S_2 on X by

$$S_1(x) := [p(x), q(x)[\quad \text{and} \quad S_2 := [p(x), q(x)] \cap X.$$

Replacing S by S_1 and S_2 in (C3), respectively, it is not too hard to prove that S_1 and S_2 satisfy (C3) independently of the chosen measures. If p and q are constant functions, then S_1 and S_2 satisfy (C2) too. In particular, when $p(x) := 0$ and $q(x) := \infty$, $x \in X$, we can see that there exist classical Fredholm type integral equations in (1). Finally, if $p(x) := 0$, $0 \le q(x) \le x$, $x \in X$, and q is increasing, then S_1 satisfies (C1)–(C3), and S_2 satisfies (C2) and (C3). These cases show that classical Volterra type integral equations are also contained in (1). It is not difficult to obtain analogues of these examples in higher dimensions.

Similarly, the case $S(x) := X$, $x \in X$ corresponds to Fredholm type integral equations, while the case when S satisfies (C1)–(C3) corresponds to Volterra type integral equations. As for the second part of this statement, we do not go into the details and refer to [**5**, Theorem 2.1]. For further concrete examples, see [**7**].

The second reason for the significance of the considered integral equations comes from the observation that some of the problems formulated for the classical integral equations can be solved in a very satisfactory way in this essentially more general case, and the results give unified approaches of the problems. For example, in the paper [**7**] existence and uniqueness results are given for (1) when S satisfies (C1)–(C3), and h_i is Lipschitzian in I_i for $i = 1, \dots, n$. Especially the linear version of (1) is discussed in [**5**].

In spite of the above-mentioned facts, research on integral equations of the form (1) has proceeded slowly, there are relatively few papers even if the measures μ_i, $i = 1, \dots, n$ are one dimensional Lebesgue-Stieltjes measures, see [**1–3, 9, 10**].

Definition 2. We define two vector spaces.

(a) Let A be a nonempty set from $\mathcal{A}$.

$$\begin{aligned}\mathcal{L}(A) :&= \{z : A \longrightarrow \mathbf{R} \mid z \text{ is } \mu_j\text{-integrable over } A,\ j = 1, \dots, n\} \\ &= \{z : A \longrightarrow \mathbf{R} \mid z \text{ is } \mu\text{-integrable over } A\}.\end{aligned}$$

(b) Let A be a nonempty subset of X such that $S(x) \subset A$ for every $x \in A$.

$$\begin{aligned}\mathcal{L}_{\mathrm{loc}}(A) := \{&z : A \longrightarrow \mathbf{R} \mid z \text{ is } \mu_j\text{-integrable over } S(x) \text{ for every } x \in A \\ &\text{and } j = 1, \dots, n\} \\ = \{&z : A \longrightarrow \mathbf{R} \mid z \text{ is } \mu\text{-integrable over } S(x) \text{ for every } x \in A\}.\end{aligned}$$

If p is a function and A is a subset of the domain of p, we denote by $p|A$ the restriction of p to A. In the sequel we follow the next convention: let F be a set of real-valued functions from $A \subset X$, and let the function T be defined on F; if $p : B \to \mathbf{R}$ with $A \subset B$, then $p \in F$ means $p|A \in F$, and $T(p) := T(p|A)$.

We also make the following assumptions for the functions f, g_i and h_i, $i = 1, \dots, n$:

(H1) A is a nonempty set from $\mathcal{A}$ such that $S(x) \subset A$ for every $x \in A$ and such that f, $g_i \in \mathcal{L}(A)$, $i = 1, \dots, n$.

(H2) A is a nonempty subset of X such that $S(x) \subset A$ for every $x \in A$ and such that f, $g_i \in \mathcal{L}_{\text{loc}}(A)$, $i = 1, \dots, n$.

(H3) I_i is an interval, $i = 1, \dots, n$.

(H4) Either h_i is increasing, and g_i is nonnegative, or h_i is decreasing, and g_i is nonpositive, $i = 1, \dots, n$.

In the present paper existence theorems are given for the integral equation (1). In each of these results we suppose that some of the properties (C1)–(C3) and (H1)–(H4) hold. We study the existence of maximal and minimal solutions, and the convergence to these solutions of the successive approximations determined by some supersolutions and subsolutions. The key to the proof of existence is a fixed point theorem for increasing mappings on ordered sets. Finally, some applications are given to illustrate the results.

2. Some preliminary results. In this section we collect together some results which are needed in the main body of this paper. They do not belong to the main line of development, but they are interesting in their own rights.

An ordered set is an ordered pair $(A, \leq)$, where A is a set endowed with a binary relation, denoted by $\leq$, which is supposed to be reflexive, antisymmetric and transitive. An order interval $[a_1, a_2]$, where a_1, $a_2 \in A$, is the set $\{a \in A \mid a_1 \leq a \leq a_2\}$. If there exists a supremum of a subset B of A, it is denoted by $\sup B$.

In the next section, where we discuss existence theorems for the integral equation (1), our arguments depend on the following fixed point theorem which generalizes the fixed point theorem of Amann, see [**4, 12**].

Theorem 3. *Let $(A, \leq)$ be an ordered set, and let $f : D(\subset A) \to A$ be a function such that*

(i) *f is increasing, that is, if a_1, $a_2 \in D$ and $a_1 \leq a_2$, then $f(a_1) \leq f(a_2)$,*

(ii) *there are a_0, $b_0 \in D$ for which $a_0 \leq b_0$, $a_0 \leq f(a_0)$, $f(b_0) \leq b_0$ and $[a_0, b_0] \subset D$,*

(iii) *every chain of $[a_0, b_0]$ has a supremum.*

Then

(a) f *has a smallest fixed point* $a_{\min}$ *in the set* $I_u := \{a \in D \mid a_0 \leq a\}$,

(b) *every* $b \in I_u$ *with the property* $f(b) \leq b$ *satisfies* $a_{\min} \leq b$.

Proof. Let $I := \{b \in I_u \mid f(b) \leq b\}$, for all $b \in I$ let $H(b) := \{a \in I_u \mid a \leq b \text{ and } a \leq f(a)\}$, and let $H := \cap\{H(b) \mid b \in I\}$. It is obviously enough to show that f has a fixed point in H.

It is easy to see that $H \subset D$ and $a_0 \in H$.

First we prove that $f(H) \subset H$. Let $a \in H$ and $b \in I$. Then by (i), $a_0 \leq a \leq f(a) \leq f(b) \leq b$. In particular, when $b = b_0$, it follows from $[a_0, b_0] \subset D$ that $f(a) \in D$, and this implies by (i) and $a \leq f(a)$, that $f(a) \leq f(f(a))$. The previous statements give $f(a) \in H$.

We show now that $\sup C \in H$ whenever C is a chain in H. Since $H \subset [a_0, b_0]$, (iii) implies that $\sup C$ exists. Suppose $b \in I$. Then by $C \subset H \subset H(b) \subset [a_0, b]$, b is an upper bound of C, and therefore $\sup C \in [a_0, b]$. In the special case $b = b_0$ we have $\sup C \in D$, by $[a_0, b_0] \subset D$. If $a \in C$, then (i) implies that $a \leq f(a) \leq f(\sup C)$, and hence $f(\sup C)$ is an upper bound of C, so that $\sup C \leq f(\sup C)$.

We consider now the ordered set $(H, \leq)$ and the function $f \mid H$. We have already proved that the range of $f \mid H$ is a subset of H and that every chain of H has a supremum under the given order relation $\leq$ on H. It now follows from the Zorn's lemma that there exists a maximal element a_m in H. By the definition of H, $a \leq f(a)$ for every $a \in H$, so that $a_m = f(a_m)$.

The proof is complete. □

Corollary 4. *Let* $(A, \leq)$ *be an ordered set, and let* $f : D(\subset A) \to A$ *be a function such that the hypotheses* (i) *and* (ii) *made above are satisfied, and*

(iii) *every chain of* $[a_0, b_0]$ *has an infimum.*

Then

(a) f *has greatest fixed point* $b_{\max}$ *in the set* $I_d := \{a \in D \mid a \leq b_0\}$,

(b) *every* $a \in I_d$ *with the property* $a \leq f(a)$ *satisfies* $a \leq b_{\max}$.

The next result belongs to integration theory, and it will be used in the proof of a special existence theorem for (1), see Theorem 23.

Theorem 5. *Let $(Y, \mathcal{B}, \nu)$ be a measure space, and let $S : Y \to \mathcal{B}$ satisfy the conditions* (C1)–(C3). *Suppose $\nu(Y) > 0$, and $\nu(B) = 0$ for every measurable subset B of $N := \{x \in Y \mid \nu(S(x)) = 0\}$. If $p : Y \to \mathbf{R}$ is nonnegative and ν-integrable over $S(x)$ for every $x \in Y$, then for each $\varepsilon > 0$ there is $x_\varepsilon \in Y$ such that $\nu(S(x_\varepsilon)) > 0$ and $\int_{S(x_\varepsilon)} p \, d\nu < \varepsilon$.*

We require two lemmas. The n-fold product of a measure space $(Y, \mathcal{B}, \nu)$ is denoted by $(Y^n, \mathcal{B}^n, \nu^n)$, $n \in \mathbf{N}^+$.

The first lemma contains an integral inequality which is a special case of the main result in [**6**].

Lemma 6. *Let $(Y, \mathcal{B}, \nu)$ be a measure space, and let $S : Y \to \mathcal{B}$ satisfy the conditions* (C1)–(C3). *If $p : Y \to \mathbf{R}$ is nonnegative and ν-integrable over Y, then*

$$\int_{H_n(Y)} p \times \cdots \times p \, d\nu^n \le \frac{1}{n!} \left(\int_Y p \, d\nu \right)^n, \quad n \in \mathbf{N}^+,$$

where

$$H_n(Y) := \{(x_1, \dots x_n) \in Y^n \mid x_k \in S(x_{k-1}),\ k = 2, \dots , n\} . \tag{2}$$

In the second lemma we construct a measure space from a given one, and we discuss the connections between them.

Lemma 7. *Let $(Y, \mathcal{B}, \nu)$ be a measure space, and let $H \subset Y$ such that for every measurable set $B \subset Y \setminus H$ we have $\nu(B) = 0$. Let*

$$\mathcal{C} := \{B \cap H \mid B \in \mathcal{B}\} ,$$

and for each $B \in \mathcal{B}$ let $\pi(B \cap H) := \nu(B)$.

(a) *π is well defined. This means that if B_1, $B_2 \in \mathcal{B}$ and $B_1 \cap H = B_2 \cap H$, then $\nu(B_1) = \nu(B_2)$.*

(b) $(H, \mathcal{C}, \pi)$ *is a measure space.*

Suppose $B \in \mathcal{B}$, *and* $p : B \to [-\infty, \infty]$ *is* ν*-almost measurable on* B.

(c) $p|B \cap H$ *is* π*-almost measurable on* $B \cap H$.

(d) *If* p *is* ν*-integrable over* B, *then* p *is* π*-integrable over* $B \cap H$ *and* $\int_B p\, d\nu = \int_{B \cap H} p\, d\pi$.

The product measure ν^2 (π^2) *is defined by an outer measure on the power set of* Y (H), *see* [**8**]. *This outer measure is also denoted by* ν^2 (π^2).

(e) *If* $A \subset H^2$, *then* $\pi^2(A) = \nu^2(A)$.

(f) *If* $A \in \mathcal{B}^2$, *then* $A \cap H^2 \in \mathcal{C}^2$.

Proof. (a) The hypotheses on B_1 and B_2 imply that $B_1 \setminus B_2$ and $B_2 \setminus B_1$ are $\mathcal{B}$-measurable subsets of $Y \setminus H$, whence $\nu(B_1 \setminus B_2) = \nu(B_2 \setminus B_1)$, and therefore

$$\nu(B_1) = \nu(B_1 \setminus B_2) + \nu(B_1 \cap B_2) = \nu(B_2 \setminus B_1) + \nu(B_2 \cap B_1) = \nu(B_2).$$

(b) It is easy to see that $\mathcal{C}$ is a σ-algebra in H. To prove that π is a measure on $\mathcal{C}$, let $B_n \in \mathcal{B}$, $n \in \mathbf{N}$ such that $(B_n \cap H)_{n=0}^{\infty}$ is a sequence of pairwise disjoint sets of $\mathcal{C}$. Then $B_i \cap B_j$, $i, j \in \mathbf{N}$, $i \neq j$ are $\mathcal{B}$-measurable subsets of $Y \setminus H$, so that $\nu(B_i \cap B_j) = 0$, $i, j \in \mathbf{N}$, $i \neq j$. Let $B := \cup_{i,j=0,\ i \neq j}^{\infty} (B_i \cap B_j)$. It follows that B is a $\mathcal{B}$-measurable subset of $Y \setminus H$ and $\nu(B) = 0$. If $\widehat{B}_n := B_n \setminus B$, $n \in \mathbf{N}$, then $\widehat{B}_n \in \mathcal{B}$ and $\widehat{B}_n \cap H = B_n \cap H$, $n \in \mathbf{N}$ as well as

$$\widehat{B}_i \cap \widehat{B}_j \subset (B_i \setminus (B_i \cap B_j)) \cap (B_j \setminus (B_i \cap B_j)) = \varnothing, \quad i, j \in \mathbf{N} \quad i \neq j.$$

Hence

$$\begin{aligned} \pi\left(\bigcup_{n=0}^{\infty} (B_n \cap H)\right) &= \pi\left(\left(\bigcup_{n=0}^{\infty} B_n\right) \cap H\right) = \pi\left(\left(\bigcup_{n=0}^{\infty} \widehat{B}_n\right) \cap H\right) \\ &= \nu\left(\bigcup_{n=0}^{\infty} \widehat{B}_n\right) = \sum_{n=0}^{\infty} \nu(\widehat{B}_n) = \sum_{n=0}^{\infty} \pi\left(B_n \cap H\right). \end{aligned}$$

(c) There exists a $\mathcal{B}$-measurable subset B_1 of B such that $\nu(B_1) = 0$ and p is $\mathcal{B}$-measurable on $B \setminus B_1$. Then $B_1 \cap H \in \mathcal{C}$, $B_1 \cap H \subset B \cap H$

and $\pi(B_1 \cap H) = 0$. Hence it is enough to show that p is $\mathcal{C}$-measurable on $(B \setminus B_1) \cap H$. This follows from

$$\{x \in (B \setminus B_1) \cap H \mid p(x) < c\} = \{x \in B \setminus B_1 \mid p(x) < c\} \cap H \in \mathcal{C}, \quad c \in \mathbf{R}.$$

(d) Suppose that p is a nonnegative $\mathcal{B}$-measurable simple function on B, that is,

$$p = \sum_{i=1}^{n} c_i 1_{B_i}, \quad c_i \in [0, \infty[, \quad B_i \in \mathcal{B}, \quad i = 1, \dots, n,$$

where the B_i's are pairwise disjoint and 1_{B_i} denotes the characteristic function of B_i on B, $i = 1, \dots, n$. Then $p|B \cap H$ is a $\mathcal{C}$-measurable simple function with the $\mathcal{C}$-measurable sets $B_i \cap H$, $i = 1, \dots, n$, and

$$\int_B p \, d\nu = \sum_{i=1}^{n} c_i \nu(B_i) = \sum_{i=1}^{n} c_i \pi(B_i \cap H) = \int_{B \cap H} p \, d\pi.$$

By using this and (c), we obtain the rest of the assertion in the usual way.

(e) Let $(B_1^n \times B_2^n)_{n=0}^{\infty}$ be a sequence of sets from $\mathcal{B} \times \mathcal{B}$ whose union contains A. Since $A \subset H^2$, the members of the sequence $((B_1^n \cap H) \times (B_2^n \cap H))_{n=0}^{\infty}$ belong to $\mathcal{C} \times \mathcal{C}$ and cover A. Further, we have

$$\begin{aligned} \pi^2((B_1^n \cap H) \times (B_2^n \cap H)) &= \pi(B_1^n \cap H)\pi(B_2^n \cap H) \\ &= \nu(B_1^n)\nu(B_2^n) = \nu^2(B_1^n \times B_2^n), \quad n \in \mathbf{N}, \end{aligned}$$

hence $\pi^2(A) \le \nu^2(A)$. The opposite inequality can be proved similarly.

(f) Since $A \in \mathcal{B}^2$, the Carathéodory condition implies that

$$\nu^2(U) \ge \nu^2(U \cap A) + \nu^2(U \setminus A)$$

for every $U \subset Y^2$. Hence, by (e)

$$\begin{aligned} \pi^2(V) &\ge \nu^2(V \cap A) + \nu^2(V \setminus A) = \pi^2(V \cap A) + \pi^2(V \setminus A) \\ &= \pi^2(V \cap (A \cap H^2)) + \pi^2(V \setminus (A \cap H^2)) \end{aligned}$$

for every $V \subset H^2$. This gives $A \cap H^2 \in \mathcal{C}^2$ by the Carathéodory condition. $\square$

The only interesting case of Lemma 7 arises when H is not measurable. If $H \in \mathcal{B}$, then the result is elementary.

Consider now the proof of Theorem 5.

Proof. We begin with the case where $N = \varnothing$. If the result is false, then there exists $\varepsilon_0 > 0$ such that for every $x \in Y$ we have $\int_{S(x)} p\,d\nu \geq \varepsilon_0$. Let $x_0 \in Y$. We prove by induction on n that

$$\int_{H_n(S(x_0))} p \times \cdots \times p\,d\nu^n \geq \varepsilon_0^n, \quad n \in \mathbf{N}^+. \tag{3}$$

Concerning the meaning of the set $H_n(S(x_0))$, see (2). The case $n = 1$ is obvious. Suppose then that $n \in \mathbf{N}^+$ for which (3) holds. Then, by Fubini's theorem, see [**8**],

$$\begin{aligned}
&\int_{H_{n+1}(S(x_0))} p \times \cdots \times p\,d\nu^{n+1} \\
&= \int_{H_n(S(x_0))} \bigg(\int_{S(x_n)} p(x_1)\dots p(x_n)p(x_{n+1})\,d\nu(x_{n+1}) \bigg)\,d\nu^n(x_1, \dots, x_n) \\
&= \int_{H_n(S(x_0))} \bigg(p(x_1)\dots p(x_n) \int_{S(x_0)} p\,d\nu \bigg)\,d\nu^n(x_1, \dots, x_n) \\
&\geq \varepsilon_0 \int_{H_n(S(x_0))} p \times \cdots \times p\,d\nu^n \geq \varepsilon_0^{n+1}.
\end{aligned}$$

Next, Lemma 6 shows that

$$\int_{H_n(S(x_0))} p \times \cdots \times p\,d\nu^n \leq \frac{1}{n!}\bigg(\int_{S(x_0)} p\,d\nu \bigg)^n, \quad n \in \mathbf{N}^+. \tag{4}$$

It now follows from (3) and (4) that

$$1 \le \frac{1}{n!}\left(\left(\int_{S(x_0)} p\,d\nu\right)\Big/\varepsilon_0\right)^n, \quad n \in \mathbf{N}^+,$$

giving a contradiction.

To prove the theorem in the general case, where $N \neq \varnothing$, we construct from $(Y, \mathcal{B}, \nu)$ and N a new measure space $(Y \setminus N, \mathcal{C}, \pi)$, as in Lemma 7. Let $S_N : Y \setminus N \to \mathcal{C}$ be defined by

$$S_N(x) := S(x) \cap (Y \setminus N).$$

Then $\pi(Y \setminus N) = \nu(Y) > 0$, $\pi(S_N(x)) = \nu(S(x)) > 0$ for every $x \in Y \setminus N$, S_N obviously satisfies (C1) and (C2), and by Lemma 7 (f), S_N satisfies (C3) too. By applying the first part of the proof with the measure space $(Y \setminus N, \mathcal{C}, \pi)$ and the function S_N, we obtain that for each $\varepsilon > 0$ there is $x_\varepsilon \in Y \setminus N$ such that $\int_{S_N(x_\varepsilon)} p\,d\pi < \varepsilon$. By Lemma 7 (d), $\int_{S(x_\varepsilon)} p\,d\nu < \varepsilon$. □

3. Main results. We begin this section by introducing some terminology that will be used further.

Definition 8. (a) Let A be a nonempty set from $\mathcal{A}$ such that $S(x) \subset A$ for every $x \in A$. For a given $z \in \mathcal{L}(A)$, the symbol $\|z\|$ is defined by $\|z\| := \int_A |z|\,d\mu$.

(b) Let $\mathcal{N} := \{z \in \mathcal{L}(A) \mid \|z\| = 0\}$, and let $L(A) := \mathcal{L}(A)/\mathcal{N}$. For every $z \in \mathcal{L}(A)$, let $\overline{z} \in L(A)$ be the equivalence class containing z ($\overline{z} = z + \mathcal{N}$), and we set $\|\overline{z}\| := \|z\|$. For $\mathcal{F} \subset \mathcal{L}(A)$ we define $F := \{\overline{z} \in L(A) \mid z \in \mathcal{F}\}$.

(c) We introduce the canonical ordering on $L(A)$: for $\overline{z_1}$, $\overline{z_2} \in L(A)$, $\overline{z_1} \preceq \overline{z_2}$ means that $z_1 \le z_2$, μ almost everywhere on A.

Remark 9. (a) $(\mathcal{L}(A), \|\cdot\|)$ is a complete pseudometric space.

(b) $(L(A), \|\cdot\|, \preceq)$ is an L-normed Banach lattice, briefly, AL-space, see [**11**].

Definition 10. We consider the integral equation (1) under the hypotheses (C3) and (H1).

(a) We define an operator corresponding to (1):

$$\mathcal{D}(A) := \{y \in \mathcal{L}(A) \mid h_i \circ y \text{ is } \mu_i\text{-integrable over } A,\ i = 1, \dots, n\},$$

$$\mathcal{T}_A : \mathcal{D}(A) \longrightarrow \mathcal{L}(A),$$
$$\mathcal{T}_A(y)(x) := f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ y \, d\mu_i, \quad x \in A.$$

(b) The following essential operator is derived from $\mathcal{T}_A$:

$$T_A : D(A) \longrightarrow L(A), \quad T_A(\overline{y}) := \overline{\mathcal{T}_A(y)}.$$

Remark 11. (a) Theorem 2.3 (b) in [**5**] guarantees that the range of $\mathcal{T}_A$ is a subset of $\mathcal{L}(A)$.

(b) T_A is well defined, since $\mathcal{T}_A(y_1) = \mathcal{T}_A(y_2)$ whenever $y_1,\ y_2 \in \overline{y} \in D(A)$.

(c) It is easy to check that $\overline{y} \in D(A)$ implies $\overline{y} \subset \mathcal{D}(A)$.

The following results are technical preliminaries to the main theorems.

Lemma 12. *Assume that the hypotheses* (C3) *and* (H1) *are satisfied.*

(a) *If $s \in \mathcal{D}(A)$ is a solution of* (1), *then $\overline{s}$ is a fixed point of T_A.*

(b) *If $\overline{y} \in D(A)$ is a fixed point of T_A, then there exists exactly one solution $s \in \mathcal{D}(A)$ of* (1) *such that $s \in \overline{y}$.*

(c) *If the condition* (H4) *is also satisfied, and $y_1,\ y_2 \in \mathcal{D}(A)$ for which $y_1 \le y_2$, μ almost everywhere on A, then $\mathcal{T}_A(y_1) \le \mathcal{T}_A(y_2)$.*

Proof. (a) and (c) are obvious.

(b) Let $s := \mathcal{T}_A(y)$. Then $s \in \overline{\mathcal{T}_A(y)} = \overline{y}$, and therefore $s = y$, μ almost everywhere on A. Hence $s = \mathcal{T}_A(y) = \mathcal{T}_A(s)$. If $s_1,\ s_2 \in \overline{y}$ such that $s_i = \mathcal{T}_A(s_i)$, $i = 1,\ 2$, then $s_1 = s_2$, μ almost everywhere on A, whence $s_1 = \mathcal{T}_A(s_1) = \mathcal{T}_A(s_2) = s_2$. $\square$

We are now in a position to state the first existence theorem for (1). We assume the integrability condition (H1) which is more restrictive than (H2), but this has the advantage that only (C3) is required from the conditions concerning the function S.

Theorem 13. *Assume that the hypotheses* (C3), (H1), (H3) *and* (H4) *are satisfied. Suppose further that there exist functions $\varphi, \psi \in \mathcal{D}(A)$ such that $y = \varphi$ satisfies the inequality*

$$y \leq \mathcal{T}_A(y) \quad \mu\text{-a.e. on } A, \tag{5}$$

$y = \psi$ satisfies the inequality

$$\mathcal{T}_A(y) \leq y \quad \mu\text{-a.e. on } A, \tag{6}$$

and $\varphi \leq \psi$, μ almost everywhere on A. Let

$$\mathcal{D}_u(\varphi) := \{y \in \mathcal{D}(A) \mid \varphi \leq y,\ \mu\text{-a.e. on } A\},$$

and

$$\mathcal{D}_d(\psi) := \{y \in \mathcal{D}(A) \mid y \leq \psi,\ \mu\text{-a.e. on } A\}.$$

Then

(a) *T_A has a smallest fixed point $\overline{s}_{\min}$ in $D_u(\varphi)$, and $\overline{s}_{\min} \preceq \overline{y}$ for every $\overline{y} \in D_u(\varphi)$ with $T_A(\overline{y}) \preceq \overline{y}$.*

(b) *There exists a smallest solution $s_{\min}$ of* (1) *in $\mathcal{D}_u(\varphi)$ which means that $s_{\min} \leq s$ whenever $s \in \mathcal{D}_u(\varphi)$ is a solution of* (1).

(c) *If $y \in \mathcal{D}_u(\varphi)$ satisfies* (6), *then $s_{\min}(x) \leq y(x)$ whenever* (6) *holds at $x \in A$.*

(d) *The successive approximations determined by φ*

$$\varphi_0 := \varphi, \quad \varphi_{k+1} := \mathcal{T}_A(\varphi_k), \quad k \in \mathbf{N}$$

are well defined, the sequence $(\varphi_k)_{k=1}^{\infty}$ is increasing, and converge pointwise on A to $s_{\min}$ whenever h_i, $i = 1, \ldots, n$ is left continuous.

(a1) *T_A has a greatest fixed point $\overline{s}_{\max}$ in $D_d(\psi)$, and $\overline{y} \preceq \overline{s}_{\max}$ for every $\overline{y} \in D_d(\psi)$ with $\overline{y} \preceq T_A(\overline{y})$.*

(b1) *There exists a greatest solution $s_{\max}$ of* (1) *in $\mathcal{D}_d(\psi)$ which means that $s \leq s_{\max}$ whenever $s \in \mathcal{D}_d(\psi)$ is a solution of* (1).

(c1) *If $y \in \mathcal{D}_d(\psi)$ satisfies* (5), *then $y(x) \leq s_{\max}(x)$ whenever* (5) *holds at $x \in A$.*

(d1) *The successive approximations determined by ψ*

$$\psi_0 := \psi, \quad \psi_{k+1} := \mathcal{T}_A(\psi_k), \quad k \in \mathbf{N}$$

are well defined, the sequence $(\psi_k)_{k=1}^{\infty}$ is decreasing, and converge pointwise on A to $s_{\max}$ whenever h_i, $i = 1, \dots, n$ is right continuous.

Proof. (a) From the given conditions for φ and ψ it follows that $\overline{\varphi} \preceq \overline{\psi}$, $\overline{\varphi} \preceq T_A(\overline{\varphi})$ and $T_A(\overline{\psi}) \preceq \overline{\psi}$.

We prove first that the order interval $[\overline{\varphi}, \overline{\psi}]$ in $L(A)$ is a subset of $D(A)$. It suffices to show that if $y \in \mathcal{L}(A)$ with $\varphi \leq y \leq \psi$ μ almost everywhere on A, then $y \in \mathcal{D}(A)$. Let i be an integer such that $1 \leq i \leq n$. Since $h_i \circ \varphi$ and $h_i \circ \psi$ are μ_i-integrable over A, there exists $B_i \in \mathcal{A}$, $B_i \subset A$ such that $\mu_i(A \setminus B_i) = 0$ and that $h_i \circ \varphi$ and $h_i \circ \psi$ are measurable on B_i. Further, by $\mu_i \leq \mu$, we can suppose that y is measurable on B_i and $\varphi(x) \leq y(x) \leq \psi(x)$, $x \in B_i$. By applying these statements, the properties of $h_i \circ y$ can be obtained: (H3) implies that B_i is a subset of the domain of $h_i \circ y$; it therefore follows from the monotonicity of h_i (thus h_i is Borel-measurable) that $h_i \circ y$ is measurable on B_i, hence $h_i \circ y$ is μ_i-almost measurable on A; this shows that $h_i \circ y$ is μ_i-integrable over A, since either h_i is increasing, in which case $h_i(\varphi(x)) \leq h_i(y(x)) \leq h_i(\psi(x))$, $x \in B_i$, or h_i is decreasing, that is $h_i(\psi(x)) \leq h_i(y(x)) \leq h_i(\varphi(x))$, $x \in B_i$.

We now prove that T_A is increasing. It is enough to show that if y_1, $y_2 \in \mathcal{D}(A)$ such that $y_1 \leq y_2$ μ-almost everywhere on A, then $\mathcal{T}_A(y_1) \leq \mathcal{T}_A(y_2)$. This comes from Lemma 12 (c).

Remark 9 (b) implies that $(L(A), \|\cdot\|, \preceq)$ is order complete, thus every chain of $[\overline{\varphi}, \overline{\psi}]$ has a supremum.

By what we have already proved, Theorem 3 can be applied to the ordered set $(L(A), \preceq)$ and the operator T_A, and this gives the result.

(b) and (c). By Lemma 12 (b), we can find exactly one solution $s_{\min} \in \mathcal{D}(A)$ of (1) with the property $s_{\min} \in \overline{s}_{\min}$. Then $s_{\min} \in \mathcal{D}_u(\varphi)$ holds. Let $y \in \mathcal{D}_u(\varphi)$ satisfy the inequality (6). Clearly $T_A(\overline{y}) \preceq \overline{y}$, hence, by (a), $\overline{s}_{\min} \preceq \overline{y}$, and therefore $s_{\min} \leq y$ μ-almost everywhere on A. It follows from Lemma 12 (c) that $\mathcal{T}_A(s_{\min}) \leq \mathcal{T}_A(y)$, and hence $s_{\min}(x) \leq y(x)$ whenever (6) holds at $x \in A$.

(d) Let $B \in \mathcal{A}$, $B \subset A$ such that $\mu(A \backslash B) = 0$ and that the inequalities $\varphi(x) \leq \psi(x)$, $\varphi(x) \leq \mathcal{T}_A(\varphi)(x)$ and $\mathcal{T}_A(\psi)(x) \leq \psi(x)$ are satisfied for every $x \in B$.

We prove by induction on k that $\varphi_k \in \mathcal{D}(A)$ and $\varphi(x) \leq \varphi_k(x) \leq \psi(x)$ for every $x \in B$ and $k \in \mathbf{N}$. The case $k = 0$ is obvious. Suppose then that $k \in \mathbf{N}$ for which the result is true. To show that $\varphi_{k+1} \in \mathcal{D}(A)$, we refer to the first part of the proof of (a). By the definition of B and Lemma 12 (c), $\varphi(x) \leq \mathcal{T}_A(\varphi)(x) \leq \mathcal{T}_A(\varphi_k)(x) \leq \mathcal{T}_A(\psi)(x) \leq \psi(x)$, $x \in B$.

To prove that $(\varphi_k)_{k=1}^{\infty}$ is increasing, we also use induction on k. We have seen above that $\varphi_0(x) \leq \varphi_1(x)$, $x \in B$, hence by Lemma 12 (c), $\varphi_1 \leq \mathcal{T}_A(\varphi_0) \leq \mathcal{T}_A(\varphi_1) = \varphi_2$. Now let $k \in \mathbf{N}^+$ such that $\varphi_k \leq \varphi_{k+1}$. This and Lemma 12 (c) imply that $\varphi_{k+1} = \mathcal{T}_A(\varphi_k) \leq \mathcal{T}_A(\varphi_{k+1}) = \varphi_{k+2}$.

Since $\varphi_k(x) \leq \psi(x)$, $x \in B$, it follows from Lemma 12 (c) that $\varphi_k = \mathcal{T}_A(\varphi_{k-1}) \leq \mathcal{T}_A(\psi) =: \psi_1$, $k \in \mathbf{N}^+$. We have established that the sequence $(\varphi_k)_{k=1}^{\infty}$ is increasing and bounded above by ψ_1, hence (φ_k) converges pointwise to a function s on A such that $\varphi(x) \leq s(x) \leq \psi(x)$, $x \in B$. Now the Monotone Convergence Theorem shows that $s \in \mathcal{L}(A)$, thus the previous inequality and the first part of the proof of (a) imply that $s \in \mathcal{D}(A)$.

We observe next that s is a solution of (1). Since $s, \varphi_k \in \mathcal{D}(A)$, $k \in \mathbf{N}$, $(\varphi_k)_{k=1}^{\infty}$ is increasing and h_i is left continuous, $h_i \circ \varphi_k \to h_i \circ s$ μ_i-almost everywhere on A for every $i = 1, \ldots, n$. If h_i is increasing, then $(h_i \circ \varphi_k)$ is increasing μ_i-almost everywhere on A, and if h_i is decreasing, then $(h_i \circ \varphi_k)$ is decreasing μ_i-almost everywhere on A. According to these two statements and the monotone convergence theorem, we have

$$\int_A h_i \circ \varphi_k \, d\mu_i \longrightarrow \int_A h_i \circ s \, d\mu_i, \quad i = 1, \ldots, n.$$

From the definition of (φ_k) and the condition $S(x) \subset A$, $x \in A$, we therefore deduce that s is a solution of (1).

It remains to prove that $s = s_{\min}$, and it is enough to show that if $\hat{s} \in \mathcal{D}_u(\varphi)$ is a solution of (1), then $\varphi_k \leq \hat{s}$, $k \in \mathbf{N}^+$. Since $\varphi \leq \hat{s}$, μ-almost everywhere on A, it follows from Lemma 12 (c) that $\varphi_1 = \mathcal{T}_A(\varphi) \leq \mathcal{T}_A(\hat{s}) = \hat{s}$. We complete the proof by induction on k. If $k \in \mathbf{N}^+$ for which the result holds, then by Lemma 12 (c), $\varphi_{k+1} = \mathcal{T}_A(\varphi_k) \leq \mathcal{T}_A(\hat{s}) = \hat{s}$.

The cases (a1), (b1), (c1) and (d1) follow in a similar manner, we omit the details. □

When there is a function which is not left (right) continuous in (d) ((d1)), the successive approximations φ_k (ψ_k) may fail to converge to $s_{\min}$ ($s_{\max}$), even if (1) has a unique solution $s \in \mathcal{D}(A)$. This can be illustrated by examples.

Remark 15 justifies the following definition.

Definition 14. We consider the integral equation (1) under the hypotheses (C2), (C3) and (H2).

$$\mathcal{D}_{\text{loc}}(A) := \{y : A \to \mathbf{R} \mid \text{for } x \in A \text{ with } S(x) \neq \varnothing, \\ y \mid S(x) \in \mathcal{D}(S(x))\} .$$

Remark 15. We consider the integral equation (1) under the hypotheses (C2), (C3) and (H2). If $s : A \to \mathbf{R}$ is a solution of (1), then Theorem 2.3 (b) in [**5**] shows that $s \in \mathcal{D}_{\text{loc}}(A)$.

Before we turn to another existence theorem for (1), we need a result which will be important in what follows.

Lemma 16. *Assume that the hypotheses* (C2), (C3) *and* (H2) *are satisfied. Let* $L := \{x \in A \mid S(x) \neq \varnothing\}$. *Suppose we are given solutions* $s_x \in \mathcal{D}(S(x))$, $x \in L$ *of* (1) *such that* $s_{x_2}|S(x_1) = s_{x_1}$ *for each* $x_1 \in L$, $x_2 \in A$ *with* $x_1 \in S(x_2)$. *Then there exists exactly one solution* $s : A \to \mathbf{R}$ *of* (1) *for which* $s|S(x) = s_x$, $x \in L$.

Proof. Let

$$s : A \longrightarrow \mathbf{R},$$

$$s(x) := \begin{cases} f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ s_x \, d\mu_i & \text{if } x \in L \\ f(x) & \text{if } x \in A \setminus L. \end{cases}$$

Let $x \in L$ and $u \in S(x)$. The conditions imply that if $u \in L$, then

$$\begin{aligned} s(u) &= f(u) + \sum_{i=1}^{n} g_i(u) \int_{S(u)} h_i \circ s_u \, d\mu_i \\ &= f(u) + \sum_{i=1}^{n} g_i(u) \int_{S(u)} h_i \circ s_x \, d\mu_i = s_x(u), \end{aligned}$$

and if $u \notin L$, then $s(u) = f(u) = s_x(u)$, thus $s|S(x) = s_x$.

Now let $x \in A$. If $x \notin L$, then obviously

$$s(x) = f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ s \, d\mu_i,$$

while if $x \in L$, then by the first part of the proof,

$$s(x) = f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ s_x \, d\mu_i = f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ s \, d\mu_i,$$

hence s is a solution of (1).

The uniqueness of s is obvious. □

We consider next an existence theorem for (1) when the integrability condition (H2) satisfies.

Theorem 17. *Assume that the hypotheses* (C2), (C3), (H2), (H3) *and* (H4) *are satisfied. Suppose further that there exist functions* φ, $\psi \in \mathcal{D}_{\mathrm{loc}}(A)$ *such that* $y = \varphi$ *satisfies the inequality*

$$(7)\quad y(u) \leq f(u) + \sum_{i=1}^{n} g_i(u) \int_{S(u)} h_i \circ y \, d\mu_i \quad \mu\text{-a.e. on } S(x), \quad x \in A,$$

$y = \psi$ satisfies the inequality

$$(8)\quad y(u) \geq f(u) + \sum_{i=1}^{n} g_i(u) \int_{S(u)} h_i \circ y \, d\mu_i \quad \mu\text{-a.e. on } S(x), \quad x \in A,$$

and $\varphi \leq \psi$, μ-almost everywhere on $S(x)$, $x \in A$. Let

$$\mathcal{D}_{\mathrm{loc},u}(\varphi) := \{y \in \mathcal{D}_{\mathrm{loc}}(A) \mid \varphi \leq y,\ \mu\text{-a.e. on } S(x),\ x \in A\},$$

and

$$\mathcal{D}_{\mathrm{loc},d}(\psi) := \{y \in \mathcal{D}_{\mathrm{loc}}(A) \mid y \leq \psi,\ \mu\text{-a.e. on } S(x),\ x \in A\}.$$

Then

(a) *there exists a smallest solution $s_{\min}$ of* (1) *in $\mathcal{D}_{\mathrm{loc},u}(\varphi)$ which means that $s_{\min} \leq s$ whenever $s \in \mathcal{D}_{\mathrm{loc},u}(\varphi)$ is a solution of* (1).

(b) *If $y \in \mathcal{D}_{\mathrm{loc},u}(\varphi)$ satisfies* (8), *then $s_{\min}(x) \leq y(x)$ whenever* (8) *holds at $x \in A$.*

(c) *The successive approximations determined by φ*

$$\varphi_0 := \varphi,$$
$$\varphi_{k+1} : A \longrightarrow \mathbf{R},$$
$$\varphi_{k+1}(x) := f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ \varphi_k \, d\mu_i, \quad k \in \mathbf{N}$$

are well defined, $\varphi_k \in \mathcal{D}_{\mathrm{loc}}(A)$, $k \in \mathbf{N}$, the sequence $(\varphi_k)_{k=1}^{\infty}$ is increasing, and converge pointwise on A to $s_{\min}$ whenever h_i, $i = 1, \dots, n$ is left continuous.

(a1) *There exists a greatest solution $s_{\max}$ of* (1) *in $\mathcal{D}_{\mathrm{loc},d}(\psi)$ which means that $s \leq s_{\max}$ whenever $s \in \mathcal{D}_{\mathrm{loc},d}(\psi)$ is a solution of* (1).

(b1) *If $y \in \mathcal{D}_{\mathrm{loc},d}(\psi)$ satisfies* (7), *then $y(x) \leq s_{\max}(x)$ whenever* (7) *holds at $x \in A$.*

(c1) *The successive approximations determined by ψ*

$$\psi_0 := \psi,$$
$$\psi_{k+1} : A \longrightarrow \mathbf{R},$$
$$\psi_{k+1}(x) := f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ \psi_k \, d\mu_i, \quad k \in \mathbf{N}$$

are well defined, $\psi_k \in \mathcal{D}_{\mathrm{loc}}(A)$, $k \in \mathbf{N}$, the sequence $(\psi_k)_{k=1}^{\infty}$ is decreasing, and converge pointwise on A to $s_{\max}$ whenever h_i, $i = 1, \dots, n$ is right continuous.

Proof. (a) and (b). Let $x \in A$ with $S(x) \neq \varnothing$. By using the hypotheses and the definition of $\mathcal{D}_{\mathrm{loc}}(A)$, we obtain from Theorem 13 (b) that (1) has a smallest solution $s_{\min,x}$ in the set

$$\mathcal{D}_{x,u}(\varphi) := \{y \in \mathcal{D}(S(x)) \mid \varphi \leq y, \ \ \mu\text{-a.e. on } S(x)\}.$$

We show now that if $x_1,\ x_2 \in A$, $S(x_1) \neq \varnothing$ and $x_1 \in S(x_2)$, then $s_{\min,x_2}|S(x_1) = s_{\min,x_1}$. To prove this, let

$$y : S(x_2) \longrightarrow \mathbf{R}, \qquad y(x) := \begin{cases} s_{\min,x_1}(x) & \text{if } x \in S(x_1) \\ s_{\min,x_2}(x) & \text{if } x \in S(x_2) \setminus S(x_1). \end{cases}$$

Since $s_{\min,x_2}|S(x_1)$ is a solution of (1) from $\mathcal{D}_{x_1,u}(\varphi)$, $s_{\min,x_1} \leq s_{\min,x_2}|S(x_1)$. It therefore follows that $y \in \mathcal{D}_{x_2,u}(\varphi)$ satisfies the inequality $\mathcal{T}_{S(x_2)}(y) \leq y$. Hence Theorem 13 (c) implies that $s_{\min,x_2} \leq y$, and this gives that $s_{\min,x_2}|S(x_1) \leq s_{\min,x_1}$.

By Lemma 16 applied to the solution set

$$\{s_{\min,x} \mid x \in A \text{ with } S(x) \neq \varnothing\},$$

we obtain a solution $s_{\min} : A \to \mathbf{R}$ of (1). Since $s_{\min} \in \mathcal{D}_{\mathrm{loc},u}(\varphi)$, it remains to prove (b). Let $y \in \mathcal{D}_{\mathrm{loc},u}(\varphi)$ satisfy (8), and let $x \in A$ such that the inequality in (8) holds at x. If $S(x) = \varnothing$, then

$$y(x) \geq f(x) = s_{\min}(x),$$

and if $S(x) \neq \varnothing$, then Theorem 13 (c) implies that $y|S(x) \geq s_{\min}|S(x)$. It therefore follows from the condition (H4) that

$$\begin{aligned} y(x) &\geq f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ y \, d\mu_i \\ &\geq f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ s_{\min} \, d\mu_i = s_{\min}(x). \end{aligned}$$

(c) If $x \in A$ with $S(x) \neq \varnothing$, then due to condition (C2) and Theorem 13 (d) we have that $\varphi_k | S(x) \in \mathcal{D}(S(x))$, $k \in \mathbf{N}$. Hence $\varphi_k \in \mathcal{D}_{\mathrm{loc}}(A)$ follows provided φ_k, $k \in \mathbf{N}$ is defined on A. This is obvious for φ_0. We can deduce it for φ_{k+1}, $k \in \mathbf{N}$ too: if $x \in A$ with $S(x) \neq \varnothing$, then $h_i \circ \varphi_k$ is μ_i-integrable over $S(x)$, $i = 1, \dots, n$, and if $S(x) = \varnothing$, then $\varphi_{k+1}(x) = f(x)$.

By Theorem 13 (d), the sequence $(\varphi_k | S(x))_{k=1}^{\infty}$ is increasing for every $x \in A$ with $S(x) \neq \varnothing$. Together with the condition (H4), this yields that the sequence $(\varphi_k)_{k=1}^{\infty}$ is increasing.

We have seen in the proof of Theorem 13 (d) that for $x \in A$ with $S(x) \neq \varnothing$

$$\int_{S(x)} h_i \circ \varphi_k \, d\mu_i \longrightarrow \int_{S(x)} h_i \circ s_{\min,x} \, d\mu_i, \quad i = 1, \dots, n,$$

and this is obviously true when $x \in A$ with $S(x) = \varnothing$. The proof of (a) gives that $s_{\min} | S(x) = s_{\min,x}$ for every $x \in A$ with $S(x) \neq \varnothing$, thus

$$\varphi_{k+1}(x) \longrightarrow f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ s_{\min} \, d\mu_i = s_{\min}(x), \quad x \in A.$$

The cases (a1), (b1) and (c1) can be proved similarly. □

4. Applications. The following existence results provide an opportunity to realize the scope and the fruitfulness of the theorems of the preceding section.

Theorem 18. *Assume that the hypotheses* (C3), (H1), (H3) *and* (H4) *are satisfied. Suppose that* $g_i | A$ *and* h_i, $i = 1, \dots, n$ *are nonnegative* (*thus, by* (H4), h_i, $i = 1, \dots, n$ *is increasing*), *and* $f \in \mathcal{D}(A)$. *Suppose further that there exist nonnegative functions* $c_i : A \to \mathbf{R}$, $i = 1, \dots, n$ *such that*

$$\sum_{i=1}^{n} c_i g_i \in \mathcal{L}(A), \quad f + \sum_{i=1}^{n} c_i g_i \in \mathcal{D}(A) \tag{9}$$

and

$$\int_{S(x)} h_i \circ \Big(f + \sum_{j=1}^{n} c_j g_j \Big) \, d\mu_i \leq c_i(x), \quad x \in A, \quad i = 1, \dots, n. \tag{10}$$

Then there exists a smallest solution of (1) *in* $\mathcal{D}(A)$.

Proof. We can apply Theorem 13 (b) to the pair of functions

$$\varphi := f|A \quad \text{and} \quad \psi := f + \sum_{i=1}^{n} c_i g_i.$$

Indeed, the inequalities $\varphi \leq \mathcal{T}_A(\varphi)$ and $\varphi \leq \psi$ are obvious. Furthermore, by (10)

$$\begin{aligned}\mathcal{T}_A(\psi)(x) &= f(x) + \sum_{i=1}^{n} g_i(x) \int_{S(x)} h_i \circ \psi \, d\mu_i \\ &\leq f(x) + \sum_{i=1}^{n} c_i(x) g_i(x) = \psi(x), \quad x \in A.\end{aligned}$$

It remains to show that every solution of (1) in $\mathcal{D}(A)$ belongs to $\mathcal{D}_u(\varphi)$, which follows from the nonnegativity of $g_i|A$ and h_i, $i = 1, \dots, n$. □

The previous theorem is less general than can be achieved; it would be enough to suppose that $g_i|A$, $i = 1, \dots, n$, is nonnegative μ-almost everywhere on A and the inequalities in (9) hold μ-almost everywhere on A.

As an example of an integral equation to which Theorem 18 can be applied, we consider the following result.

Corollary 19. *Assume that the hypotheses* (C3) *and* (H1) *are satisfied. Suppose that* $h_i : [0, \infty[\to \mathbf{R}$, $h_i(t) := t^{\alpha_i}$, *where* $0 < \alpha_i < 1$, $i = 1, \dots, n$. *If* f, $g_i \in \mathcal{D}(A)$, $i = 1, \dots, n$ *and they are nonnegative on* A, *then there exists a smallest solution of* (1) *in* $\mathcal{D}(A)$.

Proof. We show that nonnegative constant functions $c_i : A \to \mathbf{R}$, $i = 1, \dots, n$ can be chosen such that (9) and (10) hold, thus the result can be deduced from Theorem 18.

Let $c_i \geq 0$, $i = 1, \dots, n$. $\mathcal{L}(A)$ is a vector space, hence $\sum_{i=1}^n c_i g_i \in \mathcal{L}(A)$. Since $0 < \alpha_i < 1$, $i = 1, \dots, n$, we have

$$0 \leq \left(f(x) + \sum_{j=1}^n c_j g_j(x)\right)^{\alpha_i} \leq f^{\alpha_i}(x) + \sum_{j=1}^n c_j^{\alpha_i} g_j^{\alpha_i}(x),$$
$$x \in A, \quad i = 1, \dots, n,$$

and this implies that $f + \sum_{i=1}^n c_i g_i \in \mathcal{D}(A)$. Equation (9) is therefore verified.

To prove that (10) can be realized for some $c_i \geq 0$, $i = 1, \dots, n$, it is enough to show that there are numbers c_i, $i = 1, \dots, n$ for which

$$\int_A \left(f^{\alpha_i} + \sum_{j=1}^n c_j^{\alpha_i} g_j^{\alpha_i}\right) d\mu_i \leq c_i, \quad i = 1, \dots, n. \tag{11}$$

To simplify the notations, let

$$a_i := \int_A f^{\alpha_i}\, d\mu_i \quad \text{and} \quad b_{ij} := \int_A g_j^{\alpha_i}\, d\mu_i, \quad i,\ j = 1, \dots, n.$$

Then (11) has the form

$$a_i + \sum_{j=1}^n c_j^{\alpha_i} b_{ij} \leq c_i, \quad i = 1, \dots, n. \tag{12}$$

We observe now that (12) has a solution such that $c_1 = \cdots = c_n > 0$. In fact, under this condition

$$\frac{a_i}{c_1} + c_1^{\alpha_i - 1} \sum_{j=1}^n b_{ij} \leq 1, \quad i = 1, \dots, n$$

which is obviously true for every sufficiently large c_1. □

The next result is contained in Theorem 17 as a special case.

Theorem 20. *Assume that the hypotheses* (C2), (C3), (H2), (H3) *and* (H4) *are satisfied. Suppose that* $g_i|S(x)$, $x \in A$ *and* h_i, $i = 1, \dots, n$ *are nonnegative, and* $f \in \mathcal{D}_{\text{loc}}(A)$. *Suppose further that there exist nonnegative functions* $c_i : A \to \mathbf{R}$, $i = 1, \dots, n$ *such that*

$$\sum_{i=1}^{n} c_i g_i \in \mathcal{L}_{\text{loc}}(A), \quad f + \sum_{i=1}^{n} c_i g_i \in \mathcal{D}_{\text{loc}}(A)$$

and

$$\int_{S(x)} h_i \circ \left(f + \sum_{j=1}^{n} c_j g_j \right) d\mu_i \le c_i(x), \quad x \in A, \quad i = 1, \dots, n.$$

Then there exists a smallest solution of (1) *in* $\mathcal{D}_{\text{loc}}(A)$.

Proof. By using Theorem 17 (a) instead of Theorem 13 (b), we proceed similarly to the proof of Theorem 18. □

To illustrate this theorem, we give the following result. For simplicity, we confine ourselves to the case of (1), where $n = 1$:

$$y(x) = f(x) + g_1(x) \int_{S(x)} h_1 \circ y \, d\mu_1. \tag{13}$$

Corollary 21. *Assume that the hypotheses* (C2), (C3) *and* (H2) *are satisfied. Suppose that* $f|S(x)$ *and* $g_1|S(x)$, $x \in A$ *are nonnegative,* $h_1 : [0, \infty[\, \to [0, \infty[$ *is increasing and* $h_1(t) \le \alpha t$, $t \in [0, \infty[$ *for some* $\alpha > 0$. *If there is a number* $\beta \in \,]0, 1[$ *such that*

$$\int_{S(x)} \left(g_1(u) \int_{S(u)} f \, d\mu_1 \right) d\mu_1(u) \le \beta \int_{S(x)} f \, d\mu_1, \quad x \in A, \tag{14}$$

then there exists a smallest solution of (13) *in* $\mathcal{D}_{\text{loc}}(A)$.

Proof. We can apply Theorem 20 by taking

$$c_1 : A \longrightarrow \mathbf{R}, \quad c_1(x) := \gamma \int_{S(x)} f \, d\mu_1$$

for some $\gamma > 0$. The full argument consists of the verification of the conditions of Theorem 20, the details are as follows. Let $x \in A$ with $S(x) \neq \varnothing$. Then $h_1 \circ f$ is μ_1-almost measurable on $S(x)$ (see the first part of the proof of Theorem 13 (a)) and therefore, by

$$0 \leq h_1(f(u)) \leq \alpha f(u), \quad u \in S(x),$$

$h_1 \circ f$ is μ_1-integrable over $S(x)$. This implies that $f \in \mathcal{D}_{\text{loc}}(A)$. According to Theorem 2.3 (b) in [**5**], we have that $c_1 g_1 \in \mathcal{L}_{\text{loc}}(A)$. Now as we have already proved $f \in \mathcal{D}_{\text{loc}}(A)$, we can show that $f + c_1 g_1 \in \mathcal{D}_{\text{loc}}(A)$. Finally, if $x \in A$ with $S(x) \neq \varnothing$, then the inequality

$$\int_{S(x)} h_1\left(f(u) + g_1(u)\gamma \int_{S(u)} f\, d\mu_1\right) d\mu_1(u)$$
$$\leq \alpha \int_{S(x)} \left(f(u) + g_1(u)\gamma \int_{S(u)} f\, d\mu_1\right) d\mu_1(u) \leq \gamma \int_{S(x)} f\, d\mu_1$$

is satisfied whenever

$$\int_{S(x)} \left(g_1(u) \int_{S(u)} f\, d\mu_1\right) d\mu_1(u) \leq \left(1 - \frac{\alpha}{\gamma}\right) \int_{S(x)} f\, d\mu_1,$$

and this is true when $0 < 1 - (\alpha/\gamma) \leq \beta$. □

Now we apply the preceding result to some particular situations.

Example 22. Let $X := [1, \infty[$, let $\mathcal{A}$ be the Lebesgue measurable subsets of X, and let $S(x) := [px, \infty[$, $x \in X$, where $p \geq 1$. If λ is the Lebesgue measure on $\mathcal{A}$, then S satisfies (C2) and (C3) (see the examples in the Introduction). We consider the integral equation

$$y(x) = \frac{1}{x^a} + \frac{1}{x^b} \int_{px}^{\infty} h \circ y\, d\lambda, \tag{15}$$

where a, $b > 1$, $h : [0, \infty[\to [0, \infty[$ is increasing and $h(t) \leq \alpha t$, $t \in [0, \infty[$ for some $\alpha > 0$.

An easy computation shows that (14) is satisfied for each $x \in [1, \infty[$ with

$$\beta := \frac{1}{(a + b - 2)p^{a+b-2}} \quad \text{if} \quad a + b \geq 3$$

and

$$\beta := \frac{1}{p^{a+b-2}} \quad \text{if} \quad 2 < a + b \leq 3.$$

Hence, by Corollary 21, there exists a smallest solution of (15) in $\mathcal{D}_{\text{loc}}([1, \infty[)$.

We deduce our final result directly from Theorem 18, but it is rather interesting in its own right.

Theorem 23. *Assume that the hypotheses* (C1)–(C3), (H2), (H3) *and* (H4) *are satisfied. Suppose that there is* $x_0 \in A$ *such that* $\mu(S(x_0)) > 0$. *Suppose further that* $f|S(x)$, $g_i|S(x)$, $x \in A$ *and* h_i, $i = 1, \dots, n$, *are nonnegative. If*

$$f \in \mathcal{D}_{\text{loc}}(A), \quad \textit{and} \quad f + \sum_{i=1}^{n} g_i \in \mathcal{D}_{\text{loc}}(A),$$

then there exists a solution $s : B \to \mathbf{R}$ *of* (1) *such that* B *contains a measurable set* C *with* $\mu(C) > 0$.

Proof. Let $N := \{x \in A \mid \mu(S(x)) = 0\}$.

Suppose first that $N \neq \varnothing$ and N contains a measurable set C with $\mu(C) > 0$. N satisfies property (i) in Definition 1, by (C2), hence $s : N \to \mathbf{R}$, $s(x) := f(x)$ is a desired solution.

Suppose now that every measurable subset of N has μ-measure 0. If we show that there is an $x \in S(x_0) \cup \{x_0\}$ for which $\mu(S(x)) > 0$ and

$$\int_{S(x)} h_i \circ \left(f + \sum_{j=1}^{n} g_j \right) d\mu_i \leq 1, \quad i = 1, \dots, n, \tag{16}$$

then it follows from Theorem 18 (by taking $c_i = 1$, $i = 1, \dots, n$) that (1) has a solution on $S(x)$ which is appropriate for us. If (16) is satisfied with $x := x_0$, then the required x is given. Otherwise, let K_1 be the set of those indices from $\{1, \dots, n\}$ for which the inequality in (16) does not hold when $x = x_0$, and let $i_1 \in K_1$. Since $\mu_{i_1}(S(x_0)) > 0$, Theorem 5 implies that there is an $x_1 \in S(x_0)$ such that $\mu_{i_1}(S(x_1)) > 0$ and

$$\int_{S(x_1)} h_{i_1} \circ \left(f + \sum_{j=1}^{n} g_j \right) d\mu_{i_1} \leq 1. \tag{17}$$

If (17) is satisfied for every index from K_1, then $x := x_1$ can be chosen. If not, then we proceed in this manner with the set K_2 of those indices from K_1 for which (17) does not hold. After finite steps we have a suitable x. $\square$

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