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STABILITY IN LINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS WITH NONLINEAR PERTURBATION

M.N. ISLAM AND Y.N. RAFFOUL

ABSTRACT. A Lyapunov functional is employed to obtain conditions that guarantee stability, uniform stability and uniform asymptotic stability of the zero solution of a scalar linear Volterra integrodifferential equation with nonlinear perturbation.

1. Introduction. In this paper we consider the scalar linear Volterra integrodifferential equation

$$x'(t) = h(t)x(t) + \int_0^t C(at-s)x(s)\,ds \tag{1.1}$$

and its perturbed form

$$x'(t) = h(t)x(t) + \int_0^t C(at-s)x(s)\,ds + g(t,x(t)) \tag{1.2}$$

where a is a constant, $a > 1$. The function $g(t,x(t))$ is continuous in t and x and satisfies $|g(t,x(t))| \le \lambda(t)|x(t)|$, where $\lambda(t)$ is continuous. Moreover, $h(t)$ is continuous for all $t \ge 0$ and $C : \mathbf{R} \to \mathbf{R}$ is continuous. We study the stability properties of the zero solution of either (1.1) or (1.2) and we construct suitable Lyapunov functionals in the analysis.

We point out that if $C \in L^1[0,\infty)$, then the equations (1.1) and (1.2) become fading memory problems. When $a > 1$, the memory term $\int_0^t C(at-s)\,ds = \int_{(a-1)t}^{at} C(u)\,du$ tends to zero as $t \to \infty$, that is, the memory fades away completely. On the other hand, if $0 < a < 1$, the memory term never fades away completely; it tends to a constant as $t \to \infty$. For $a = 1$, equations (1.1) and (1.2) are the well-known convolution equations. Many researchers have studied stability

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properties for both convolution and non-convolution cases of (1.1) and (1.2) by use of the Lyapunov method. For more on the construction of Lyapunov functionals, in the case $a = 1$, we refer the reader to [**1–3, 5, 7, 8, 10, 12, 13**] and the references therein.

As mentioned above, in this article we only address the case $a > 1$. In the case $0 < a < 1$, we could not obtain meaningful stability results. We use a modified version of the Lyapunov functional that was used by Zhang in [**13**] and the methods that were employed by Burton and Somolinos in [**4**] to obtain stability, uniform stability and uniform asymptotic stability of the zero solutions of equations (1.1) and (1.2).

In the case of $C(at - s) = C(t, s)$, requiring $\lambda(t) \in L^1[0, \infty)$, the authors in [**9**] and [**11**] obtained conditions for uniform asymptotic stability of the zero solution of (1.2). In this paper we show that the zero solution of (1.2) is uniformly asymptotically stable without requiring $\lambda(t) \in L^1[0, \infty)$. Also, when $C(at - s) = C(t, s)$, the authors in [**6**] obtained necessary and sufficient conditions for uniform asymptotic stability of (1.1) using the resolvent.

Normally $h(t)$ of (1.1) and (1.2) is expected to be a negative function for stability properties of the zero solution. In this paper we do not require this condition. We use the size of the kernel $C(t)$ to offset the positive effect of the function $h(t)$. At the end of this paper we provide an example (Example 3.3) as an application of Theorem 3.2 showing that the zero solution of (1.1) is uniformly asymptotically stable for positive constant function h.

For any fixed $t \geq 0$, let

$$B(t) = \Big\{\phi : [0, t] \to \mathbf{R}, \\ \phi \text{ is continuous and bounded in the supremum norm}\Big\}.$$

For each $\phi \in B(t_0)$, $t_0 \geq 0$, there is a unique solution $x(t) = x(t, t_0, \phi)$ of equation (1.2) defined on an interval $[t_0, \gamma)$ with $x(s) = \phi$ for $0 \leq s \leq t_0$. For $\phi \in B(t_0)$, the supremum norm of ϕ is given by $||\phi|| = \sup\{|\phi(t)| : 0 \leq t \leq t_0\}$. If the solution remains bounded, then $\gamma = \infty$.

Definition 1.1. The zero solution of (1.2) is said to be stable if for each $\epsilon > 0$ and each $t_0 \geq 0$, there exists a $\delta = \delta\,(\varepsilon, t_0) > 0$ such that $[\phi \in B(t_0),\, \|\phi\| < \delta,\, t \geq t_0]$ imply $|x(t, t_0, \phi)| < \epsilon$. The zero of (1.2) is uniformly stable if it is stable and δ is independent of t_0.

Definition 1.2. The zero solution of (1.2) is said to be uniformly asymptotically stable if it is uniformly stable and if there is a $\gamma_1 > 0$ and for each $\epsilon > 0$ there exists a $T > 0$ such that $[t_0 \geq 0,\, \phi \in B(t_0),\, \|\phi\| < \gamma_1,\ t \geq T + t_0]$ imply $|x(t, t_0, \phi)| < \epsilon$.

2. Lyapunov functionals and stability for equation (1.2). For $\alpha < 0$, let

$$G_\alpha(t) = \int_t^\infty C(u) e^{\alpha u} du\, e^{-\alpha t}. \tag{2.1}$$

Assuming $G_\alpha(t)$ exists and $G_\alpha(t) \in L^1[0, \infty)$, define $V(t)$ by

$$\begin{aligned} V(t) &= \frac{1}{2}\bigg(x(t) + \frac{1}{a}\int_0^t G_\alpha(at - s)x(s)\,ds\bigg)^2 \\ &\quad + p\int_0^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds, \end{aligned} \tag{2.2}$$

where p is a positive constant to be determined in Theorem 2.1. In the next lemma we calculate $V'(t)$ along solutions of (1.2), $V'_{(1.2)}(t)$, and when needed, we can get $V'_{(1.1)}(t)$ simply by letting $g(t, x(t)) = 0$. Let

$$a_1(t) = \frac{h(t)}{a} + \frac{1}{a^2}\, G_\alpha(at - t) \tag{2.3}$$

$$a_2(t) = \lambda(t)\bigg(1 + \frac{\lambda(t)}{2a^2}\bigg). \tag{2.4}$$

Lemma 2.1. *If $V(t)$ is given by (2.2), then for some positive constant L we have*

(2.5)

$$V'_{(1.2)}(t) \le \left[aa_1(t) + a_2(t) + \frac{(a_1(t)-\alpha)^2}{2L^2} + p\int_{t(a-1)}^{\infty} |G_\alpha(u)|\,du\right]x^2(t) + \left[\left(\frac{1}{2} + \frac{L^2}{2} + \frac{|\alpha|}{a}\right)\int_{t(a-1)}^{\infty} |G_\alpha(u)|\,du - ap\right] \cdot \int_0^t |G_\alpha(at-s)|x^2(s)\,ds.$$

Proof. Let $x(t) = x(t, t_0, \phi)$ be a solution of (1.2) and define $V(t)$ by (2.2). Then along solutions of (1.2) we have

$$\begin{aligned} V'_{(1.2)}(t) = {}& \Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,ds\Big)\Big(h(t)x(t) + g(t,x(t)) \\ &+ \frac{1}{a}\,G_\alpha(at-t)x(t) - \alpha\int_0^t G_\alpha(at-s)x(s)\,ds\Big) \\ &+ p\int_{at-t}^{\infty} |G_\alpha(u)|\,du\,x^2(t) - ap\int_0^t |G_\alpha(at-s)|x^2(s)\,ds \\ = {}& h(t)x^2(t) + \frac{1}{a}\,G_\alpha(at-t)x^2(t) + (a_1(t)-\alpha)x(t) \\ &\cdot \int_0^t G_\alpha(at-s)x(s)\,ds - \frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)x(s)\,ds\Big)^2 \\ &+ x(t)g(t,x(t)) + \frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,dsg(t,x(t)) \\ &+ p\int_{at-t}^{\infty} |G_\alpha(u)|\,du\,x^2(t) - ap\int_0^t |G_\alpha(at-s)|x^2(s)\,ds. \end{aligned}$$

For any real numbers y and z and any nonzero constant k, one has $2yz \le (y^2/k^2) + k^2z^2$. Using this inequality along with the Schwarz inequality one can verify that for the positive constant L,

$$\begin{aligned}
&(a_1(t)-\alpha)x(t)\int_0^t G_\alpha(at-s)x(s)\,ds\\
&\quad\le \frac{(a_1(t)-\alpha)^2x^2(t)}{2L^2}+\frac{L^2}{2}\Big(\int_0^t G_\alpha(at-s)x(s)\,ds\Big)^2\\
&\quad\le \frac{(a_1(t)-\alpha)^2x^2(t)}{2L^2}+\frac{L^2}{2}\int_0^t |G_\alpha(at-s)|\,ds\int_0^t |G_\alpha(at-s)|x^2(s)\,ds.
\end{aligned}$$

Also,

$$\text{(I)}\qquad \begin{aligned}
&-\frac{\alpha}{a}\Big(\int_0^t G_\alpha(at-s)x(s)\,ds\Big)^2\\
&\qquad\le \frac{|\alpha|}{a}\int_0^t |G_\alpha(at-s)|\,ds\int_0^t |G_\alpha(at-s)|x^2(s)\,ds,\\
&\qquad x(t)g(t,x(t))\le |x(t)|\,|g(t,x(t))|\le \lambda(t)x^2(t),
\end{aligned}$$

and

$$\text{(II)}\qquad \begin{aligned}
&\frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,ds g(t,x(t))\\
&\quad\le \frac{\lambda(t)|x(t)|}{a}\int_0^t |G_\alpha(at-s)||x(s)|\,ds\\
&\quad\le \frac{\lambda^2(t)x^2(t)}{2a^2}+\frac{1}{2}\Big(\int_0^t |G_\alpha(at-s)||x(s)|\,ds\Big)^2\\
&\quad\le \frac{\lambda^2(t)x^2(t)}{2a^2}+\frac{1}{2}\int_0^t |G_\alpha(at-s)|\,ds\int_0^t |G_\alpha(at-s)|x^2(s)\,ds.
\end{aligned}$$

Employing the above four inequalities in $V'_{(1.2)}(t)$ we easily obtain (2.5). This completes the proof of Lemma 2.1.

Theorem 2.1. *Let* $G_\alpha(u)\in L^1[0,\infty)$ *with* $(1/a)\int_0^\infty |G_\alpha(u)|\,du<1$. *Suppose*

$$\text{(2.6)}\qquad aa_1(t)+a_2(t)+\frac{(a_1(t)-\alpha)^2}{2}+a_3Q\le -\beta,$$

for $\beta \geq 0$, $Q = \left(1/\sqrt{a}\int_0^\infty |G_\alpha(u)|\,du\right)^2$ *and*

$$a_3 = \left(\frac{1}{2} + \frac{L^2}{2} + \frac{|\alpha|}{a}\right). \tag{2.7}$$

Then the zero solution of (1.2) *is stable.*

Proof. First notice that if $V(t)$ is given by (2.2), then

(2.8)

$$\begin{aligned} V(t) &= \frac{1}{2}\Big(x^2(t) + \Big(\frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,ds\Big)^2 + 2\frac{x(t)}{a}\int_0^t G_\alpha(at-s)x(s)\,ds\Big) \\ &\quad + p\int_0^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds \\ &\leq x^2(t) + \frac{1}{a^2}\int_0^t |G_\alpha(at-s)|\,ds \int_0^t |G_\alpha(at-s)|x^2(s)\,ds \\ &\quad + p\int_0^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds \\ &\leq x^2(t) + \frac{1}{a^2}\int_0^\infty |G_\alpha(u)|\,du \int_0^t |G_\alpha(at-s)|x^2(s)\,ds \\ &\quad + p\int_0^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds \\ &\leq x^2(t) + \frac{1}{a}\int_0^t |G_\alpha(at-s)|x^2(s)\,ds + p\int_0^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds. \end{aligned}$$

Now for $L = 1$ take $p = a_3/a\int_0^\infty |G_\alpha(u)|\,du$. Then, from (2.5) and (2.6) we get

(2.9)

$$\begin{aligned} V'_{(1.2)}(t) &\leq \left[aa_1(t) + a_2(t) + \frac{(a_1(t)-\alpha)^2}{2} + p\int_0^\infty |G_\alpha(u)|\,du\right]x^2(t) \\ &\leq -\beta x^2(t). \end{aligned}$$

Let $J = (1/a)\int_0^\infty |G_\alpha(u)|\,du$. Given an $\epsilon > 0$ and a fixed $t_0 \geq 0$, we choose $\delta > 0$ with $0 < \delta < \epsilon$ such that

$$\sqrt{2}\,(1 + J + J\,ap\,t_0)^{1/2}\,\delta < \epsilon\,(1 - J). \tag{2.10}$$

Let $x(t) = x(t, t_0, \phi)$ be a solution of (1.2) with $||\phi|| < \delta$. Then for $t \geq t_0$, using (2.8) and (2.9) we have

$$
\begin{aligned}
&\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,ds\Big)^2 \\
&\qquad \leq V(t) \leq V(t_0) \\
&\qquad \leq \Big(1 + \frac{1}{a}\int_0^\infty |G_\alpha(u)|\,du + p\,t_0\int_0^\infty |G_\alpha(u)|\,du\Big)\delta^2 \\
&\qquad \leq (1 + J + J\,ap\,t_0)\delta^2.
\end{aligned}
\tag{2.11}
$$

We claim that $|x(t)| < \epsilon$ for all $t \geq t_0$. Note that $|x(u)| < \delta < \epsilon$ for all $0 \leq u \leq t_0$. If the claim is not true, let $t = t_*$ be the first t such that $|x(t_*)| = \epsilon$ and $|x(s)| < \epsilon$ for $t_0 \leq s < t_*$. Then, using (2.11) we obtain

$$
\begin{aligned}
\epsilon\,(1-J) &= \epsilon\,\Big(1 - \frac{1}{a}\int_0^\infty |G_\alpha(u)|\,du\Big) \\
&\leq \Big|x(t_*) + \frac{1}{a}\int_0^{t_*} G_\alpha(at_* - s)x(s)\,ds\Big| \\
&\leq \sqrt{2}\,(1 + J + J\,ap\,t_0)^{1/2}\,\delta,
\end{aligned}
$$

which contradicts (2.10) and completes the proof. □

Theorem 2.2. *Suppose the hypotheses of Theorem* 2.1 *hold and there is a positive constant* R *such that*

$$
\int_{(a-1)t}^{at}\int_v^\infty |G_\alpha(u)|\,du\,dv < R \tag{2.12}
$$

for all $t > 0$. *Then the zero solution of* (1.2) *is uniformly stable.*

Proof. For any $t_0 \geq 0$ we have

$$
\int_0^{t_0}\int_{at_0-s}^\infty |G_\alpha(u)|\,du\,ds = \int_{(a-1)t_0}^{at_0}\int_v^\infty |G_\alpha(u)|\,du\,dv < R.
$$

Given an $\epsilon > 0$ we choose $\delta > 0$ with $0 < \delta < \epsilon$ such that

$$
\sqrt{2}\,(1 + J + pR)^{1/2}\,\delta < \epsilon\,(1-J). \tag{2.13}
$$

Let $x(t) = x(t, t_0, \phi)$ be a solution of (1.2) with $||\phi|| < \delta$. Then for $t \geq t_0$, using (2.8) and (2.9) we have

$$\begin{aligned}\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,ds\Big)^2 &\\ &\leq V(t) \leq V(t_0)\\ &\leq \Big(1 + \frac{1}{a}\int_0^\infty |G_\alpha(u)|\,du + p\int_0^{t_0}\int_{at_0-s}^\infty |G_\alpha(u)|\,du\,ds\Big)\,\delta^2\\ &< (1 + J + pR)\delta^2.\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.1. This completes the proof of Theorem 2.2.

Theorem 2.3. *Suppose the hypotheses of Theorem* 2.2 *hold with* $\beta > 0$, *where* β *satisfies* (2.6). *If* $\int_v^\infty |G_\alpha(u)|\,du \in L^1[0,\infty)$, *then the zero solution of* (1.2) *is uniformly asymptotically stable.*

Proof. By Theorem 2.2 the zero solution is uniformly stable. So, for $\epsilon = 1$, find the δ of uniform stability. Let $\gamma_1(t) > 0$ be given. We will find $T > 0$ such that $[t_0 \geq 0, ||\phi|| < \delta, t \geq t_0 + T]$ implies $|x(t, t_0, \phi)| < \gamma_1(t)$. Since $V'(t) \leq 0$, if we find a t_f such that $V(t_f) < \gamma^2$ for a given $\gamma > 0$, then

$$\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,ds\Big)^2 \leq V(t) \leq V(t_f) < \gamma^2$$

for all $t \geq t_f$. Then we use the lower bound on $V(t)$ to show that $|x(t, t_0, \phi)| < \gamma_1(t)$, ($\gamma_1$ is a function of t). We now find a T so that for any such solution there will be a $t_f \in [t_0, t_0 + T]$.

Since $G_\alpha(u) \in L^1[0,\infty)$, there is a T_* such that

$$\int_{(a-1)T_*}^\infty |G_\alpha(u)|\,du \leq \frac{a\gamma^2}{4}. \tag{2.14}$$

Also from the hypotheses, there is a T_1 such that for all $T > T_1$,

$$\int_{(a-1)T}^\infty \int_v^\infty |G_\alpha(u)|\,du\,dv \leq \frac{\gamma^2}{4p}.$$

Thus for $t \geq T$ we have

$$\begin{aligned}\int_T^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,ds &= \int_{at-t}^{at-T} \int_v^\infty |G_\alpha(u)|\,du\,dv \\ &\leq \int_{(a-1)t}^\infty \int_v^\infty |G_\alpha(u)|\,du\,dv \leq \frac{\gamma^2}{4p}.\end{aligned} \tag{2.15}$$

Fix a $T_2 > T_1$. For all $t > T_2$, we have

$$\begin{aligned}\int_0^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds &= \int_0^{T_2} \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds \\ &\quad + \int_{T_2}^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)ds\,.\end{aligned} \tag{2.16}$$

Since $|x(t)| < 1$, the first integral on the right side of (2.16) satisfies

$$\begin{aligned}\int_0^{T_2} \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds &\leq \int_0^{T_2} \int_{at-T_2}^\infty |G_\alpha(u)|\,du\,ds \\ &\leq T_2 \int_{at-T_2}^\infty |G_\alpha(u)|\,du.\end{aligned}$$

Also, since $G_\alpha(u) \in L^1$, we can pick a $T_3 > T_2$, such that for $t > T_3$

$$\int_{at-T_2}^\infty |G_\alpha(v)|\,dv < \frac{\gamma^2}{4pT_2}. \tag{2.17}$$

Using (2.17), the first integral on the right of (2.16) satisfies

$$\int_0^{T_2} \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds \leq T_2 \int_{at-T_2}^\infty |G_\alpha(u)|\,du < \frac{\gamma^2}{4p}.$$

Also, using (2.15), the second integral on the right side of (2.16) becomes

$$\int_{T_2}^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,x^2(s)\,ds \leq \int_{T_2}^t \int_{at-s}^\infty |G_\alpha(u)|\,du\,ds \leq \frac{\gamma^2}{4p},$$

for $t > T_3$.

Thus, from (2.16) we get

$$\int_0^t \int_{at-s}^{\infty} |G_\alpha(u)|\, du\, x^2(s)\, ds \le \frac{\gamma^2}{2p}, \tag{2.18}$$

for $t > T_3$. Next, we claim that $x^2(t) \in L^1$. To see this, let $t \ge t_0 \ge 0$. Then by integrating (2.9) from t_0 to t we have

$$V(t) - V(t_0) \le -\int_{t_0}^t \beta x^2(s)\, ds,$$

from which we get

$$\int_{t_0}^t \beta x^2(s)\, ds \le V(t_0) - V(t) \le V(t_0) < (1 + J + pR).$$

Let $T_4 = (1 + J + pR)/\beta(\gamma/2)^2$. Now we claim that every interval of length T_4 contains a τ such that $|x(\tau)| < \gamma/2$. If the claim is not true, then $|x(t)| \ge \gamma/2$ for $t \in [t_1, t_1 + T_4]$ for some $t_1 \ge t_0$. By (2.9), we have

$$\begin{aligned} V(t_1 + T_4) &\le V(t_1) - \int_{t_1}^{t_1+T_4} \beta x^2(s)\, ds \le V(t_0) - \beta(\gamma/2)^2 T_4 \\ &= V(t_0) - (1 + J + pR) < 0, \end{aligned}$$

which contradicts $V(t) \ge 0$ for all $t \ge 0$. For $t > t_0 + T_* + T_3$, observe that both (2.14) and (2.18) hold. Moreover, there is a $t_f \in [t_0 + T_* + T_3,\, t_0 + T_* + T_3 + T_4]$ such that $x^2(t_f) < \gamma^2/4$ since this interval has length T_4. Consequently, by (2.2) and (2.8), for $t \ge t_f$ we have

$$\begin{aligned} &\frac{1}{2}\Big(x(t) + \frac{1}{a}\int_0^t G_\alpha(at - s)x(s)\, ds\Big)^2 \\ &\quad \le V(t) \le V(t_f) \\ &\quad \le x^2(t_f) + \frac{1}{a}\int_0^{t_f} |G_\alpha(at_f - s)| x^2(s)\, ds \\ &\qquad + p\int_0^{t_f}\int_{at_f - s}^{\infty} |G_\alpha(u)|\, du\, x^2(s)\, ds \end{aligned}$$

$$\begin{aligned}
&\le x^2(t_f) + \frac{1}{a}\int_0^{t_f} |G_\alpha(at_f - s)|\,ds \\
&\quad + p\int_0^{t_f}\int_{at_f-s}^{\infty} |G_\alpha(u)|\,du\,ds \\
&\le x^2(t_f) + \frac{1}{a}\int_{(a-1)t_f}^{\infty} |G_\alpha(u)|\,du \\
&\quad + p\int_0^{t_f}\int_{at_f-s}^{\infty} |G_\alpha(u)|\,du\,ds \\
&< \gamma^2/4 + \gamma^2/4 + \gamma^2/2 = \gamma^2.
\end{aligned}$$

We then have, for $t \ge t_f$

$$\left| x(t) + \frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,ds \right| < \sqrt{2}\,\gamma.$$

It follows from the above inequality that

$$|x(t)| - \left| \frac{1}{a}\int_0^t G_\alpha(at-s)x(s)\,ds \right| < \sqrt{2}\,\gamma. \tag{2.19}$$

Since $t_f \ge T_*$, it follows from (2.14) and (2.19) that

$$\begin{aligned}
|x(t)| &< \frac{1}{a}\int_0^t |G_\alpha(at-s)||x(s)|\,ds + \sqrt{2}\,\gamma \\
&\le \frac{1}{a}\int_0^t |G_\alpha(at-s)|\,ds + \sqrt{2}\,\gamma \\
&\le \frac{1}{a}\int_{(a-1)t}^{\infty} |G_\alpha(u)|\,du + \sqrt{2}\,\gamma \\
&\le \frac{\gamma^2}{4} + \sqrt{2}\,\gamma := \gamma_1(t),
\end{aligned}$$

for $t \ge t_f$. This completes the proof of Theorem 2.3.

In the next section we give conditions that guarantee the stability of the zero solution of (1.1). To do so, we will use the same Lyapunov functional defined by (2.2) and set $g(t,x(t)) = 0$. Note that if $g(t,x(t)) = 0$, then the terms (I) and (II) drop out and a_3 becomes,

$$a_3 = \left(\frac{1}{2} + \frac{|\alpha|}{a}\right), \quad \text{when} \quad L = 1.$$

3. Stability for equation (1.1). In this section we generalize the results of [**13**] to equation (1.1) by displaying different types of Lyapunov functionals.

Theorem 3.1. *Let* $a_3^* = ((1/2)+(|\alpha|/a))$, $Q = ((1/\sqrt{a})\int_0^\infty |G_\alpha(u)|\,du)^2$ *with*

$$\frac{1}{a}\int_0^\infty |G_\alpha(u)|\,du < 1. \tag{3.1}$$

Suppose

$$aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2} + a_3^* Q \leq -\beta$$

for $\beta \geq 0$.

(i) *If* $G_\alpha(u) \in L^1[0,\infty)$, *then the zero solution of* (1.1) *is stable.*

(ii) *If* $G_\alpha(u) \in L^1[0,\infty)$ *and* (2.12) *holds, then the zero solution of* (1.1) *is uniformly stable.*

(iii) *If* $G_\alpha(u) \in L^1[0,\infty)$, $\int_v^\infty |G_\alpha(u)|\,du \in L^1[0,\infty)$ *and* $\beta > 0$, *then the zero solution of* (1.1) *is uniformly asymptotically stable.*

The proofs of (i), (ii) and (iii) follow directly from Theorems 2.1, 2.2 and 2.3, respectively.

Theorem 3.2. *Let* Q *be as of Theorem* 3.1 *with* (3.1) *holding. Suppose*

$$\frac{h(t)}{a} + \frac{1}{a^2}\, G_\alpha(at-t) \; + |\alpha| < 0. \tag{3.2}$$

(i) *If* $G_\alpha(u) \in L^1[0,\infty)$, *then the zero solution of* (1.1) *is stable.*

(ii) *If* $G_\alpha(u) \in L^1[0,\infty)$ *and* (2.12) *holds, then the zero solution of* (1.1) *is uniformly stable.*

(iii) *If* $G_\alpha(u) \in L^1[0,\infty)$ *and* $\int_v^\infty |G_\alpha(u)|\,du \in L^1[0,\infty)$, *then the zero solution of* (1.1) *is uniformly asymptotically stable.*

Proof. Let $V(t)$ and $a_1(t)$ be defined by (2.2) and (2.3), respectively. Then by setting $g(t,x(t))=0$ we obtain from (2.5) that

$$V'_{(1.1)}(t) \le \Big[aa_1(t) + \frac{(a_1(t)-\alpha)^2}{2L^2} + p\int_0^\infty |G_\alpha(u)|\,du\Big]x^2(t) + \Big[\Big(\frac{L^2}{2}+\frac{|\alpha|}{a}\Big)\int_0^\infty |G_\alpha(u)|\,du - ap\Big]\int_0^t |G_\alpha(at-s)|x^2(s)\,ds.$$

Let $L=2\sqrt{\delta}$. We will choose the best δ and p so that

$$aa_1(t) + \frac{(a_1(t)-\alpha)^2}{4\delta} + p\int_0^\infty |G_\alpha(u)|\,du < 0 \tag{3.4}$$

and

$$\Big(\delta + \frac{|\alpha|}{a}\Big)\int_0^\infty |G_\alpha(u)|\,du - ap < 0. \tag{3.5}$$

Since $\int_0^\infty |G_\alpha(u)|\,du > 0$, from (3.4) and (3.5) we obtain

$$\Big(\delta+\frac{|\alpha|}{a}\Big)\frac{1}{a}\Big(\int_0^\infty |G_\alpha(u)|\,du\Big)^2 < p\int_0^\infty |G_\alpha(u)|\,du < a\,|a_1(t)| - \frac{(a_1(t)-\alpha)^2}{4\,\delta}$$

from which we arrive at

$$4a\,Q\delta^2 + 4\big(|\alpha|Q - a^2|a_1(t)|\big)\,\delta + a\,\big(a_1(t)-\alpha\big)^2 < 0. \tag{3.6}$$

The quadratic inequality in (3.6) has its minimum at

$$\delta_* = \frac{a^2|a_1(t)| - |\alpha|Q}{2aQ}. \tag{3.7}$$

We claim that $\delta_* > 0$. Because of (3.1) we get $Q < a$ and hence $Q < a^2$. Also, since $a > 1$ and $a_1(t) < 0$, we have

$$\begin{aligned} a^2|a_1(t)| - |\alpha|\, Q &> a\,|a_1(t)| - |\alpha| a \\ &= -\,a(|\alpha| - |a_1(t)|) \\ &\geq -\,a(|\alpha| + a_1(t)) \\ &= -\,a\Big(\frac{h(t)}{a} + \frac{1}{a^2}\, G_\alpha(at-t) + |\alpha|\Big) > 0. \end{aligned}$$

Next we show that δ_* satisfies inequality (3.6). Substituting δ_* for δ into (3.6) we obtain

$$\begin{aligned} &4aQ\,\delta_*^2 + 4\left(|\alpha|Q - a^2|a_1(t)|\right)\,\delta_* + a(a_1(t)-\alpha)^2 \\ &= -\frac{\left(|\alpha|Q - a^2|a_1(t)|\right)^2}{aQ} + a(a_1(t)-\alpha)^2 \\ &= \frac{\left(a^2 - Q\right)}{aQ}\,\left(|\alpha|^2 Q - a^2|a_1(t)|^2\right) \\ &< a\,\frac{\left(a^2 - Q\right)}{Q}\,\left(|\alpha|^2 - |a_1(t)|^2\right) \\ &= a\,\frac{\left(a^2 - Q\right)}{Q}\,(|\alpha| + |a_1(t)|)\,(|\alpha| - |a_1(t)|) \\ &= a\,\frac{\left(a^2 - Q\right)}{Q}\,(|\alpha| + |a_1(t)|)\,\Big(\frac{h(t)}{a} + \frac{1}{a^2}\,G_\alpha(at-t) + |\alpha|\Big) < 0. \end{aligned}$$

Let

$$\xi_1 := \Big(\delta_* + \frac{|\alpha|}{a}\Big)\,Q = \frac{(a^2|a_1(t)| + |\alpha|Q)}{2a}$$

and

$$\begin{aligned} \xi_2 &:= a\,|a_1(t)| - \frac{(a_1(t)-\alpha)^2}{4\delta_*} \\ &= \frac{a}{2(a^2|a_1(t)| - |\alpha|Q)}\,\left[(a\,|a_1(t)|)^2 - |\alpha|^2 Q + (a^2 - Q)|a_1(t)|^2\right]. \end{aligned}$$

One can verify that $\xi_1 > 0$ and $\xi_2 > 0$ by (3.2) and the fact that $Q < a^2$. Define p by

$$(a\,Q)^{1/2}\,p = \frac{\xi_1 + \xi_2}{2}.$$

Clearly p is positive. Moreover, after some calculation we have

$$(a\,Q)^{1/2}\,p = \frac{a(a^2-Q)|a_1(t)|^2 + 2a(a|a_1(t)|)^2 - Q(a+Q/a)|\alpha|^2}{4(a^2|a_1(t)| - |\alpha|Q)}.$$

Substituting δ_* for δ in (3.4), and by noting that

$$(a\,Q)^{1/2} = \int_0^\infty |G_\alpha(u)|\,du,$$

we obtain after tedious calculations from (3.4) that

$$aa_1(t) + \frac{(a_1(t)-\alpha)^2}{4\delta_*} + (aQ)^{1/2}p = \frac{(Q-a^2)(a^2|a_1(t)|^2 - |\alpha|^2 Q)}{4a(a^2|a_1(t)| - |\alpha|Q)} < 0. \tag{3.8}$$

In a similar manner, we have

$$\Big(\delta_* + \frac{|\alpha|}{a}\Big)Q - (aQ)^{1/2}p = \frac{(Q-a^2)(a^2|a_1(t)|^2 - |\alpha|^2 Q)}{4a(a^2|a_1(t)| - |\alpha|Q)} < 0. \tag{3.9}$$

Next we note that

$$\left(\delta_* + \frac{|\alpha|}{a}\right)\int_0^\infty |G_\alpha(u)|\,du - ap = (a/Q)^{1/2}\left[\left(\delta_* + \frac{|\alpha|}{a}\right)Q - (aQ)^{1/2}p\right].$$

Now we define $\beta_1(t)$ by

$$\beta_1(t) = \frac{(a^2-Q)(a^2|a_1(t)|^2 - |\alpha|^2 Q)}{4a(a^2|a_1(t)| - |\alpha|Q)}\,\min\{1, (a/Q)^{1/2}\}. \tag{3.10}$$

Thus using (3.8), (3.9), (3.10) and (3.3) we obtain

$$\begin{aligned} V'_{(1.1)}(t) &\le -\beta_1(t)x^2(t) - \beta_1(t)\int_0^t |G_\alpha(at-s)|x^2(s)\,ds \\ &< -\beta_1(t)x^2(t). \end{aligned}$$

Thus, we have shown that, along solutions of (1.1), $V'(t) < 0$ and therefore the rest of the proofs of (i), (ii) and (iii) follow along the lines of the proofs of Theorems 2.1, 2.2 and 2.3, respectively.

Next, we give an example as an application of Theorem 3.2, in which $h(t) = h > 0$ where h is a constant.

Example 3.3. Consider the scalar Volterra integral equation

$$x'(t) = hx(t) - \int_0^t e^{-k(at-s)}\, x(s)\, ds \tag{3.12}$$

where $a > 1, k$ and h are positive constants to be chosen later in the example. We need to show that (3.1) and (3.2) hold. For $\alpha \leq 0$, we have

$$G_\alpha(t) = -\int_t^\infty e^{-ku}\, e^{\alpha u}\, du\, e^{-\alpha t} < \infty.$$

Thus,

$$G_\alpha(at-t) = -\int_{at-t}^\infty e^{u(\alpha-k)}\, du\, e^{-\alpha(at-t)} = \frac{1}{\alpha - k}\, e^{-k(at-t)}.$$

Next,

$$G_\alpha(u) = -\int_u^\infty e^{s(\alpha-k)}\, ds\, e^{-\alpha u} = \frac{1}{\alpha-k}\, e^{-ku}.$$

This implies that

$$\frac{1}{a}\int_0^\infty |G_\alpha(u)|\, du = \frac{1}{ak(k-\alpha)}.$$

Thus, in order for (3.1) to be satisfied, we must have

$$\frac{1}{k(k-\alpha)} < a. \tag{3.13}$$

On the other hand, to satisfy (3.2) we ask that

$$h + \frac{1}{a(\alpha-k)}\, e^{-k(at-t)} + a\,|\alpha| < 0,$$

which is satisfied for

$$h < \frac{1}{a(k-\alpha)} - a\,|\alpha|. \tag{3.14}$$

If we take $k = 1$, $a = 1.2$ and $\alpha = -0.1$, then (3.13) and (3.14) are satisfied for $h < 0.8$. One can easily check that (iii) of Theorem 3.2 is satisfied and hence, by Theorem 3.2, the zero solution of

$$x'(t) = hx(t) - \int_0^t e^{-(1.2t-s)}\, x(s)\, ds$$

is uniformly asymptotically stable for $h < 0.8$.

We end this paper with the following open problems:

1) What can be said about the uniform asymptotic stability of the zero solution of equation (1.2) when $0 < a < 1$?

2) Can Theorems 3.1 and 3.2 be generalized for equation (1.2)?

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Department of Mathematics, University of Dayton, Dayton, OH 45469-2316
E-mail address: muhammad.islam@notes.udayton.edu

Department of Mathematics, University of Dayton, Dayton, OH 45469-2316
E-mail address: youssef.raffoul@notes.udayton.edu