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APPROXIMATION METHODS AND STABILITY OF SINGULAR INTEGRAL EQUATIONS FOR FREUD EXPONENTIAL WEIGHTS ON THE LINE

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ABSTRACT. We investigate approximation methods and the stability of a class of integral equations on the line for Freud exponential weights.

1. Introduction. In this paper, we show that there exist positive, finite numbers μ which allow us to approximate singular integral equations on the line of the form

$$\mu w^2 f - K[f] = g w^{2+\delta}.$$

Here w is a fixed even exponential weight of smooth polynomial decay at $\pm\infty$, $K[\cdot] := H[\cdot w^2]/\pi$ is a weighted Hilbert transform and g is a fixed real valued function in a weighted locally Lipschitz space of order $0 < \lambda < 1$. The exact form of the equations studied is motivated, in part, by concrete applications, see [**1, 2, 18, 19**] and the references cited therein, and so is of current interest and importance.

Our main aim, see Theorems 2a–d below, will be to show that for a large class of weights w (see Definition 1 below), there exist positive, finite numbers μ depending on w and λ so that solutions of the above equation exist, are in the same weighted Lipschitz space as g and may be well approximated. In this sense, our approximation methods are stable. Our results here have been made possible because of recent investigations of the authors dealing with uniform bounds for weighted Hilbert transforms (see [**3**–**7**], Theorems 1a and 1b below) and recent results of the first author and Jung, see [**8**], dealing with pointwise convergence of derivatives of Lagrange interpolation polynomials. Recent results on L_p $(0 < p < \infty)$ bounds for weighted Hilbert transforms can be found in [**9**] and the references cited therein.

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Throughout this paper, C will denote an absolute positive constant which may take on different values from time to time and will be independent of all variables under consideration.

The remainder of this paper is organized as follows. In Section 2, we present the definition of our class of weights and prove that the weighted Hilbert transform, as defined below, is a bounded, locally Lipschitz operator of order $0 < \lambda < 1$. This is contained in Theorem 1a. For compact intervals with $w \equiv 1$, Privalov's classical theorem [**17**, Section 14.1] can be considered an analogue of our result. A variation of this property that includes a more complicated weight function is stated in Theorem 1b. In Section 3, we state and prove our main result, Theorem 2, where we provide an approximation method for the solutions of our integral equation and we study error bounds.

2. The class of weights and the weighted Hilbert transform.

In this section, we present the definition of our class of weights and prove that the weighted Hilbert transform, as defined below, is a bounded, locally Lipschitz operator of order $0 < \lambda < 1$.

2.1 Class of weights. We begin with the definition of a suitable class of weights which is contained in:

Definition 1. Let $w := \exp(-Q)$ where $Q : \mathbf{R} \to \mathbf{R}$ is continuous and even. We shall call such a weight *admissible* if it satisfies the following additional conditions:

(a) Q' is continuous in $(0, \infty)$, $Q(0) = 0$ and

$$\lim_{|x|\to\infty} Q(x) = \infty.$$

(b) Q'' exists and is positive in $(0, \infty)$.

(c) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in $(0, \infty)$, i.e., $T(x) \leq CT(y)$ for $x \leq y$, with

$$\beta \leq T(x) \leq C, \quad x \in (0, \infty)$$

for some $\beta > 1$ which is fixed for the weight w.

(d)

$$\frac{Q''(x)}{|Q'(x)|} \leq C\,\frac{|Q'(x)|}{Q(x)}, \quad x \in \mathbf{R}.$$

Definition 1 defines a general class of weights of smooth polynomial decay at infinity. A typical example of such a weight is given by the example

$$w(x) := w_\alpha(x) := \exp\left(-|x|^\alpha\right), \quad \alpha > 1, \quad x \in \mathbf{R}, \tag{2.1}$$

of which the Hermite weight ($\alpha = 2$) is a special case. Such weights are often called Freud weights in the literature. The conditions (a–b) are weak smoothness assumptions whereas conditions (c–d) are regularity conditions. For example, condition (c) forces Q to grow as a polynomial at infinity. We refer the interested reader to [**3, 4, 6, 7, 9, 21**] and the references cited therein for further perspectives and applications.

Remark. In the approximation procedure that we shall develop below, we will need to use the zeros of the orthogonal polynomials with respect to our weight functions. Gautschi's ORTHPOL package [**15**] provides very good routines for this purpose.

2.2 Function class. Given a fixed admissible weight w and a fixed constant $0 < \lambda < 1$, we are now able to define our function class. This is contained in

Definition 2. Define for fixed $0 < \lambda < 1$ and fixed admissible w

$$\begin{aligned} X := \{f : \mathbf{R} \to \mathbf{R},\ fw \text{ continuous with } \lim_{|x|\to\infty} f(x)w(x) = 0,\\ fw \text{ locally Lipschitz of order } \lambda\}. \end{aligned}$$

Since we will need to approximate in this space, we will need a suitable notion of distance and thus we find it convenient to metrize this space with a natural norm given by

$$\|f\|_X := \|fw\|_\infty + L_\lambda^w(f), \quad f \in X.$$

Here and throughout, $\|.\|_\infty$ denotes the L_∞ norm and $L_\lambda^w(f)$ is the smallest constant $D > 0$ (depending on f, λ and w) such that

$$|f(x)w(x) - f(y)w(y)| \le D|x-y|^\lambda$$

for all x and y sufficiently close in $\mathbf{R}$. Note that such a constant exists for our class of functions because of the local Lipschitz property and the condition $\lim_{|x|\to\infty} f(x)w(x) = 0$ that asserts the boundedness of fw on the real line.

We have:

Lemma 1. *X is a Banach space.*

Proof. Although straightforward, we provide a proof for the reader's convenience. Given $\alpha \in \mathbf{R}$, it is trivial that if $f \in X$,

$$|\alpha fw(x) - \alpha fw(y)| \le |\alpha| L_\lambda^w(f)|x-y|^\lambda$$

for every x and y in $\mathbf{R}$ close enough, thus

$$L_\lambda^w(\alpha f) \le |\alpha| L_\lambda^w(f).$$

Similarly, it follows that

$$L_\lambda^w(f) \le \frac{L_\lambda^w(\alpha f)}{|\alpha|}$$

if $\alpha \ne 0$ and clearly

$$L_\lambda^w(\alpha f) = |\alpha| L_\lambda^w(f)$$

if $\alpha = 0$ so we have

$$L_\lambda^w(\alpha f) = |\alpha| L_\lambda^w(f)$$

and similarly

$$L_\lambda^w(f+g) \le L_\lambda^w(f) + L_\lambda^w(g), \quad f, g \in X.$$

Clearly, if $\|f\|_X = 0$, then $f = 0$ and also as L_λ^w is the smallest constant such that

$$|\alpha fw(x) - \alpha fw(y)| \le |\alpha| L_\lambda^w(f)|x-y|^\lambda$$

for all y and x close enough, we must also have that $f = 0$ implies $L_\lambda^w(f) = 0$. Thus, $\|\cdot\|_X$ is a norm. It remains to show that X is complete. Thus let $(f_k) \in X$ be a Cauchy sequence. Then given $\varepsilon > 0$, there exists N_0 such that for $m, n \geq N_0$,

$$\|(f_n - f_m)w\|_\infty \leq \|f_n - f_m\|_X < \varepsilon. \tag{2.2}$$

This implies that there exists some f with

$$\lim_{|x|\to\infty} f(x)w(x) = 0$$

such that

$$\|(f_n - f)w\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

As $\|\cdot\|_X$ is a norm, we have

$$L_\lambda^w((f_n - f)w) \to L_\lambda^w(0) = 0, \quad n \to \infty$$

pointwise, so it follows that

$$\|f_n - f\|_X \to 0, \quad n \to \infty. \tag{2.3}$$

Finally, we have, for any n and x and y close enough,

$$\begin{aligned}|w(x)f(x)-w(y)f(y)| &\leq |w(x)f(x)-w(x)f_n(x)|+|w(x)f_n(x)-w(y)f_n(y)| \\ &\quad + |w(y)f_n(y)-w(y)f(y)| \\ &\leq \|f_n\|_X \cdot |x-y|^\lambda + 2\|(f-f_n)w\|_\infty.\end{aligned}$$

Thus we conclude that $f \in X$. Thus X is complete. □

Henceforth, when we refer to the space X, we mean that X depends on a fixed and given admissible weight w and a constant $0 < \lambda < 1$.

2.3 Weighted Hilbert transform. We use

Definition 3. Let w be admissible and $0 < \lambda < 1$. We set for $f \in X$ and $x \in \mathbf{R}$,

$$H\left[fw^2\right](x) := \int_{\mathbf{R}} \frac{w^2(t)f(t)}{t-x}\,dt$$

where the integral is understood in the Cauchy principal value sense.

We may now state and prove:

Theorem 1a. *Let* $0 < \lambda < 1$ *and* w *be admissible. Then*

$$H[.w^2] : X \longrightarrow X.$$

Moreover, $H[.w^2]$ *is a bounded operator.*

Remark. Note that for compact intervals and with $w \equiv 1$, Theorem 1a is an analogue of Privalov's classic theorem, see [**17**, Section 14.1]. For $w \sim 1$, a similar result to Theorem 1a is discussed in [**13**, pp. 217–221]. For bounded, continuous functions on $\mathbf{R}$, analogues of Theorem 1a can also be found in [**20**]. We remind the reader that Theorem 1a does not assume, in particular, that f needs to be bounded and hence our result is substantially stronger and requires different methods of proof than the above results in [**20**] and [**13**].

Proof. We first show that

$$H[.w^2] : X \longrightarrow X.$$

To see the Lipschitz property, let $x, y \in \mathbf{R}$ with x close enough to y. Then for such x and y

$$\begin{aligned} &w(x)H\left[fw^2\right](x) - w(y)H\left[fw^2\right](y) \\ &= w(x)\int_{\mathbf{R}} \frac{f(t)w^2(t)}{t-x}\,dt - w(y)\int_{\mathbf{R}} \frac{f(t)w^2(t)}{t-y}\,dt \\ &= w(x)f(x)\int_{\mathbf{R}} \frac{w^2(t)}{t-x}\,dt - w(y)f(y)\int_{\mathbf{R}} \frac{w^2(t)}{t-y}\,dt \\ &\quad + w(x)\int_{\mathbf{R}} \frac{w^2(t)\left[f(t)-f(x)\right]}{t-x}\,dt \\ &\quad - w(y)\int_{\mathbf{R}} \frac{w^2(t)[f(t)-f(y)]}{t-y}\,dt \\ &= I_1 + I_2 \end{aligned} \tag{2.4}$$

where

$$I_1 := w(x)f(x)\int_{\mathbf{R}} \frac{w^2(t)}{t-x}\,dt - w(y)f(y)\int_{\mathbf{R}} \frac{w^2(t)}{t-y}\,dt$$

and

$$I_2 := w(x)\int_{\mathbf{R}} \frac{w^2(t)\,[f(t)-f(x)]}{t-x}\,dt - w(y)\int_{\mathbf{R}} \frac{w^2(t)[f(t)-f(y)]}{t-y}\,dt. \tag{2.5}$$

We proceed to estimate I_1 and I_2. Firstly,

$$\begin{aligned} I_1 &= w(x)f(x)\int_{\mathbf{R}} \frac{w^2(t)}{t-x}\,dt - w(y)f(y)\int_{\mathbf{R}} \frac{w^2(t)}{t-y}\,dt \\ &= w(x)f(x)\left[\int_{\mathbf{R}} \frac{w^2(t)}{t-x}\,dt - \int_{\mathbf{R}} \frac{w^2(t)}{t-y}\,dt\right] \\ &\quad + \int_{\mathbf{R}} \frac{w^2(t)}{t-y}\,[w(x)f(x)-w(y)f(y)]\,dt. \end{aligned}$$

We claim that

$$\begin{aligned} \left|\int_{\mathbf{R}} \frac{w^2(t)}{t-y}\,[w(x)f(x)-w(y)f(y)]\,dt\right| &\le CL_\lambda^w|x-y|^\lambda \\ &\le C\|f\|_X\cdot|x-y|^\lambda. \end{aligned} \tag{2.6}$$

To see this, let $\delta > 0$ be fixed and small enough. Then we write

$$\begin{aligned} \left|\int_{\mathbf{R}} \frac{w^2(t)}{t-y}\,dt\right| &= \left|\int_{|y-t|>\delta} \frac{w^2(t)}{t-y}\,dt + \int_{y-\delta}^{y+\delta} \frac{w^2(t)-w^2(y)}{t-y}\,dt\right| \\ &\le \int_{|y-t|>\delta} \frac{w^2(t)}{|t-y|}\,dt + \int_{y-\delta}^{y+\delta} \left|\frac{w^2(t)-w^2(y)}{t-y}\right|\,dt \\ &\le \frac{1}{\delta}\|w^2\|_1 + 2\delta\|(w^2)'\|_\infty. \end{aligned}$$

As $f \in X$, we have our claim. Next recall that w^2 is differentiable, and hence by [**13**, Corollary to Theorem 2.24], we have for all $\lambda^* \in (\lambda, 1)$ that

$$\left|\int_{\mathbf{R}} \frac{w^2(t)}{t-x}\,dt - \int_{\mathbf{R}} \frac{w^2(t)}{t-y}\,dt\right| \le C|x-y|^{\lambda^*}.$$

So

$$
\begin{aligned}
|I_1| &\le |(w(x)f(x) - w(y)f(y))| \cdot |H[w^2](y)| \\
&\quad + |w(x)f(x)| \cdot \left|H\left[w^2\right](x) - H\left[w^2\right](y)\right| \qquad (2.7)\\
&\le C\|fw\|_\infty (|x-y|^\lambda + |x-y|^{\lambda^*}) \le C\|f\|_X \cdot |x-y|^\lambda.
\end{aligned}
$$

Here we have used $\lambda^* > \lambda$ and (2.6).

Next we bound $|I_2|$: Let us set $\varepsilon := 2|x-y|$ and let us write

$$I_2 := I_{2,1} + I_{2,2}$$

where

$$
\begin{aligned}
I_{2,1} = w(x) &\int_{x-\varepsilon}^{x+\varepsilon} \frac{w^2(t)\,(f(t) - f(x))}{t-x}\,dt \\
&- w(y) \int_{x-\varepsilon}^{x+\varepsilon} \frac{w^2(t)\,(f(t) - f(y))}{t-y}\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
I_{2,2} = w(x) &\int_{|x-t|>\varepsilon} \frac{w^2(t)\,(f(t) - f(x))}{t-x}\,dt \\
&- w(y) \int_{|x-t|>\varepsilon} \frac{w^2(t)\,(f(t) - f(y))}{t-y}\,dt.
\end{aligned}
$$

We proceed to estimate $I_{2,1}$ and $I_{2,2}$. Let us write

$$
\begin{aligned}
&I_{2,1} \\
&= w(x) \int_{x-\varepsilon}^{x+\varepsilon} \frac{w(t)\,(w(t)f(t) - w(x)f(x) - w(t)f(x) + f(x)w(x))}{t-x}\,dt \\
&\quad - w(y) \int_{x-\varepsilon}^{x+\varepsilon} \frac{w(t)\,(f(t)w(t) - w(y)f(y) - w(t)f(y) + w(y)f(y))}{t-y}\,dt.
\end{aligned}
$$

Then

(2.8)

$$\begin{aligned}
|I_{2,1}| &\leq w(x) \int_{x-\varepsilon}^{x+\varepsilon} \frac{w(t)\left(L_\lambda^w |t-x|^\lambda + |f(x)(w(t)-w(x))|\right)}{|t-x|}\, dt \\
&\quad + w(y) \int_{x-\varepsilon}^{x+\varepsilon} \frac{w(t)\left(L_\lambda^w |t-y|^\lambda + |f(y)(w(t)-w(y))|\right)}{|t-y|}\, dt \\
&= w(x)\int_{x-\varepsilon}^{x+\varepsilon} w(t) L_\lambda^w |t-x|^{\lambda-1}\, dt + w(y)\int_{x-\varepsilon}^{x+\varepsilon} w(t) L_\lambda^w |t-y|^{\lambda-1}\, dt \\
&\quad + |w(x)f(x)| \int_{x-\varepsilon}^{x+\varepsilon} w(t)|w'(\eta_{x,t})|\, dt \\
&\quad + |w(y)f(y)| \int_{x-\varepsilon}^{x+\varepsilon} w(t)|w'(\eta_{y,t})|\, dt \\
&\leq C L_\lambda^w \varepsilon^\lambda + C\varepsilon \|fw\|_\infty \leq C\|f\|_X \varepsilon^\lambda.
\end{aligned}$$

Finally we shall estimate $I_{2,2}$: Let us write

$$\begin{aligned}
I_{2,2} &= w(x) \int_{|x-t|>\varepsilon} \frac{w^2(t)\left(f(t)-f(x)\right)}{t-x}\, dt \\
&\quad - w(y) \int_{|x-t|>\varepsilon} \frac{w^2(t)\left(f(t)-f(y)\right)}{t-y}\, dt \\
&= -\int_{|x-t|>\varepsilon} \frac{w^2(t)\left(f(x)w(x)-f(y)w(y)\right)}{t-x}\, dt \\
&\quad + \int_{|x-t|>\varepsilon} w^2(t) \left(\frac{-f(y)w(y)}{t-x} + \frac{f(y)w(y)}{t-y}\right) dt \\
&\quad + \int_{|x-t|>\varepsilon} w^2(t) \left(\frac{f(t)w(x)}{t-x} - \frac{f(t)w(y)}{t-y}\right) dt \\
&= I_{221} + I_{222}.
\end{aligned} \tag{2.9}$$

We must estimate each of the two terms above. Firstly,

$$\begin{aligned}
|I_{221}| &= |w(x)f(x) - w(y)f(y)| \left| \int_{|x-t|>\varepsilon} \frac{w^2(t)}{t-x}\, dt \right| \\
&\leq L_\lambda^w |x-y|^\lambda \left| \int_{|x-t|>\varepsilon} \frac{w^2(t)}{t-x}\, dt \right| \\
&\leq C L_\lambda^w |x-y|^\lambda.
\end{aligned} \tag{2.10}$$

Next let us write

(2.11)
$$\begin{aligned} I_{222} &= \int_{|x-t|>\varepsilon} w^2(t)\, \frac{w(y)f(y)(y-x)+f(t)w(t)(x-y)}{(t-x)(t-y)}\, dt \\ &\quad + \int_{|x-t|>\varepsilon} f(t)w^2(t)\, \frac{w(x)(t-y) - w(y)(t-x) - w(t)(x-y)}{(t-x)(t-y)}\, dt \\ &= I_{2221} + I_{2222}. \end{aligned}$$

Notice that

$$\begin{aligned} |I_{2221}| &\le |x-y| \int_{|x-t|>\varepsilon} w^2(t) L_\lambda^w |t-y|^{\lambda-1} \frac{dt}{|t-x|} \\ &\le \varepsilon |x-y| L_\lambda^w \int_{\mathbf{R}} w^2(t)|t-y|^{\lambda-1}\, dt \\ &\le C\varepsilon |x-y| L_\lambda^w. \end{aligned} \tag{2.12}$$

For I_{2222} we proceed as follows: Assume $y = x + (\varepsilon/2)$ (an analogous argument works for $y = x - (\varepsilon/2)$). Then

$$\begin{aligned} &w(x)(t-y) - w(y)(t-x) - w(t)(x-y) \\ &\qquad = (t-x)\bigl(w(x) - w(x+\varepsilon/2)\bigr) + \varepsilon/2\bigl(w(t) - w(x)\bigr) \end{aligned} \tag{2.13}$$

We now insert the representation of y and (2.13) into the definition of I_{2222} in (2.11). This gives

$$\begin{aligned} I_{2222} &= \int_{|x-t|>\varepsilon} f(t)w^2(t)\, \frac{w(x) - w(x+\varepsilon/2)}{t-x-\varepsilon/2}\, dt \\ &\quad + \frac{\varepsilon}{2} \int_{|x-t|>\varepsilon} f(t)w^2(t)\, \frac{w(t)-w(x)}{(t-x)(t-x-\varepsilon/2)}\, dt, \end{aligned} \tag{2.14}$$

and since

$$\begin{aligned} &\int_{|x-t|>\varepsilon} |f(t)w^2(t)| \left| \frac{w(x) - w(x+\varepsilon/2)}{(t-x-\varepsilon/2)} \right| dt \\ &\qquad \le \frac{\varepsilon}{2} \|w'\|_\infty \|fw\|_\infty \int_{|x-t|>\varepsilon} \frac{w(t)}{|t-x-\varepsilon/2|}\, dt \\ &\qquad \le C\frac{\varepsilon}{2} \log\varepsilon \|w'\|_\infty \|fw\|_\infty \end{aligned} \tag{2.15}$$

and

(2.16)

$$\frac{\varepsilon}{2}\int_{|x-t|>\varepsilon}|f(t)w^2(t)|\left|\frac{w(t)-w(x)}{(t-x)(t-x-\varepsilon/2)}\right|\,dt \le C\frac{\varepsilon}{2}\log\varepsilon\|w'\|_\infty\|fw\|_\infty,$$

equations (2.14), (2.15) and (2.16) easily yield that

$$|I_{2222}| \le C\|f\|_X\cdot|x-y|^\lambda. \tag{2.17}$$

(Recall here that $\lambda<1$.) Equations (2.12) and (2.17) then give

$$|I_{222}| \le C\|f\|_X\cdot|x-y|^\lambda \tag{2.18}$$

and so (2.18) together with (2.10), (2.8) and (2.7) give that indeed

$$H[.w^2]: X \longrightarrow X$$

as required.

Next we need to show that

$$H[.w^2]: X \longrightarrow X$$

is bounded. Let us write

$$\begin{aligned}\|H[.w^2]\|_{X\to X} &:= \sup_{\|f\|_X\le 1}\|H[fw^2]\|_X\\ &= \sup_{\|f\|_X\le 1}\left\{\|wH[fw^2]\|_{L_\infty(\mathbf{R})}+L_\lambda^w(H[fw^2])\right\}\\ &= \sup_{\|f\|_X\le 1}\left\{K_1+K_2\right\}.\end{aligned}$$

We now proceed to bound each of the terms K_j, $j=1,2$. We first deal with K_1. Here we need a crude form of [**3**, Theorem 4], which implies that, for any $\varepsilon>0$ small enough, and $f\in X$,

(2.19)

$$\begin{aligned}&\left|(wH[fw^2])(x)\right|\\ &\quad\le \|fw^2\|_{L_\infty(\mathbf{R})}\\ &\qquad + w(x)\int_0^\varepsilon \frac{|f(x+y/2)w(x+y/2)-f(x-y/2)w(x-y/2)|}{y}\,dy\\ &\quad\le C\left[\|fw^2\|_{L_\infty(\mathbf{R})}+L_\lambda^w(f)w(x)\int_0^\varepsilon y^{\lambda-1}\,dy\right]\\ &\quad\le C\left[\|fw\|_{L_\infty(\mathbf{R})}+L_\lambda^w(f)\right].\end{aligned}$$

We stress here that the constant C is independent of f. Thus

$$K_1 \leq C\|f\|_X.$$

On the other hand, we know from the first part of the proof, that if x and y are close enough, we have

$$\left| w(x)H[fw^2](x) - w(y)H[fw^2](y) \right| \leq C\|f\|_X \cdot |x-y|^\lambda$$

which implies that

$$K_2 \leq C\|f\|_X.$$

Combining the bounds for K_1 and K_2, we see that

$$\|H[.w^2]\|_{X\to X} \leq C.$$

This completes the proof of Theorem 1a. □

In a similar way we can prove

Theorem 1b. *Let $0 < \lambda < 1$, $0 < \gamma < \gamma^*$ and w be admissible. Then*

$$w^{-\gamma}H[.w^{\gamma^*}] : X \longrightarrow X$$

is a bounded operator.

Proof. Using methods similar to those already exploited in the proofs of Theorem 1a, this is a lengthy but straightforward matter. □

3. The integral equation and an approximation method. In this section, we state and prove our main result, Theorem 2. To this end, let us fix for the remainder of this section, w admissible, $0 < \lambda < 1$ and $g \in X$. Let μ be a positive number which will be chosen later and consider the formal integral equation

$$\mu w^2(x)f(x) - K[f](x) = g(x)w^{2+\delta}(x), \quad x \in \mathbf{R}, \tag{3.1}$$

with some $\delta > 0$ where

$$K[.] := \frac{1}{\pi} H[.w^2].$$

As usual we denote by $L(X,X)$ the space of all bounded operators from X to X. We recall that Theorem 1a says that $K \in L(X,X)$.

We are ready to state the first part of our main result. Here and in the following, I denotes the identity operator on X.

Theorem 2a. *We have* $(\mu I - K)^{-1} \in L(X,X)$, *the solution* f *of* (3.1) *satisfies* $f \in X$ *and*

(3.2)

$$w^2(x)f(x) = \frac{\mu}{\mu^2+1}\, g(x)w^{2+\delta}(x) + \frac{1}{\mu^2+1} \cdot \frac{1}{\pi}\, H[w^{2+\delta}g](x), \quad x \in \mathbf{R}.$$

Proof. Apply the operator H to both sides of equation (3.1) and divide by π. Using the identity $H^2 = -\pi^2 I$, which readily follows from [**14**, Table 15.1, Formula (2)], we deduce

$$\frac{\mu}{\pi}\, H[w^2 f](x) + w^2(x)f(x) = \frac{1}{\pi}\, H[w^{2+\delta}g](x).$$

In the case $\mu = 0$ this implies (3.2). Otherwise divide this equation by μ and add the result to (3.1). Then rearranging yields (3.2). Dividing by $w^2(x)$ and applying Theorem 1b then proves that $f \in X$. $\square$

We have thus shown that a unique solution for the analytic problem exists, and this knowledge is essential for the approximation process.

Remark. The explicit solution of our problem is given by equation (3.2) in analytic form. It is of course possible to use this relation in combination with a numerical method for the approximation of the Hilbert transform operator H to construct an approximate solution of our original problem, and indeed this path has been followed elsewhere, see, e.g., [**11, 12**]. However it is evident from the derivation that such an approach is strictly limited to equations with constant coefficients, i.e., with constant μ. We aim to provide and analyze a method that has the potential to be generalized later to the case of non-constant coefficients, where no explicit formula for the exact solution is available.

Given our admissible weight w, we let $p_n(w^2)$ denote the unique nth degree orthonormal polynomial with respect to w^2 defined by

$$\int_{\mathbf{R}} p_n(w^2)(x)p_m(w^2)(x)w^2(x)\,dx = \delta_{mn}, \quad m,n = 0,1,2\dots.$$

Then we introduce a polynomial interpolation operator L_n whose interpolation array consists of the n zeroes $\{x_{j,n}\}$, $1 \le j \le n$ of $p_n(w^2)$ which are contained in $\mathbf{R}$ and may be ordered as

$$x_{n,n} < x_{n-1,n} < \cdots < x_{2,n} < x_{1,n}.$$

It follows that

$$L_n[f] = \sum_{j=1}^{n} l_{j,n}(w^2)$$

where

$$l_{j,n}(w^2)(x) := \frac{p_n(w^2)(x)}{p_n'(w^2)(x_{j,n})(x - x_{j,n})}, \quad 1 \le j \le n, \quad x \in \mathbf{R}.$$

We set

$$\|L_n\|_\infty := \left\| \sum_{k=1}^{n} \left| l_{kn}(\cdot)w^{-1}(x_{kn})w(\cdot) \right| \right\|_{L_\infty(\mathbf{R})}$$

to be the Lebesgue constant for L_n with weight function w and let

$$E_n[f]_{w,\infty} := \inf_{P \in \mathcal{P}_n} \|(f-P)\,w\|_{L_\infty(\mathbf{R})}$$

denote the error of best weighted polynomial approximation to f from the space Π_n of polynomials of degree at most n.

We now define an approximation sequence of functions $\{f_n\}$, $n \ge 1$ by:

$$\mu w^2(x) f_n(x) - K[L_n[f_n]](x) = g(x)w^{2+\delta}(x), \quad x \in \mathbf{R}, \tag{3.3}$$

where, as above, $L_n[h]$ interpolates the function h at the array $\{x_{1,n}, \dots, x_{n,n}\}$ of n interpolation points specified above. In other words, f_n is the approximation to f by a version of the projection method [**16**, Section 4.3.4], and hence it is possible to calculate f_n

explicitly via the solution of a linear system of equations. See also the paper [**6**] where the weighted Hilbert transform was approximated using a quadrature method.

Remark. It is known [**23**] that the Lebesgue constants of this interpolation operator are not optimal; other interpolation operators with smaller Lebesgue constants have been suggested in the literature for related problems [**4, 7, 12, 23**]. However, we are forced to use this approach because for our proofs we need to know some additional properties of the interpolation operator, and such properties are presently known only in this special case. See the methods of [**6**] and Theorem 2(d).

Theorem 2b. *Assume in addition that β is as in Definition 1 with $\beta > 12/5$ and $f'w \in L_\infty(\mathbf{R})$. Then we have $(\mu I - K_n)^{-1} \in L(X,X)$ for each fixed $n \geq 1$ provided μ is not an eigenvalue of K_n. Moreover, if μ is not an eigenvalue of K_n, for all sufficiently large and fixed n, then*

$$fw^2 - f_n w^2 = (\mu I - K_n)^{-1} K[f - L_n[f]]w^2. \tag{3.4}$$

Moreover,

$$\|(f - f_n)w^2\|_{L_\infty(\mathbf{R})} \leq C\|(\mu I - K_n)^{-1}\|_{X\to X} n^{1/6} E_{n-1}[f]_{w,\infty}. \tag{3.5}$$

Proof. Indeed, let us write

$$\begin{aligned}
\|K_n\|_{X\to X} &= \sup_{\|f\|_X \leq 1} \|K_n(f)\|_X \\
&= \sup_{\|f\|_X \leq 1} \left\{\|wK_n[f]\|_{L_\infty(\mathbf{R})} + L_\lambda^w(K_n[f])\right\} \\
&\leq \sup_{\|f\|_X \leq 1} \big[\|wK[f]\|_{L_\infty(\mathbf{R})} + \|w(K - K_n)[f]\|_{L_\infty(\mathbf{R})} \\
&\quad + L_\lambda^w((K - K_n)[f]) + L_\lambda^w(K[f])\big].
\end{aligned}$$

Now we apply [**6**, Theorem 1.3] to deduce that uniformly for large enough n, $\|w(K - K_n)[f]\|_{L_\infty(\mathbf{R})} = o(1)$ and $L_\lambda^w((K - K_n)[f]) = o(1)$.

Then $K_n \in L(X,X)$ follows from (a). Now as the image space of K_n is finite dimensional, K_n is a compact operator. Thus if μ is not an eigenvalue of K_n for the given n, the Riesz-Schauder theorem [**16**, Section 1.3.9] implies that $(\mu I - K_n)^{-1} \in L(X,X)$. Moreover, our method is a projection method, and hence (3.4) follows from a standard result for projection methods [**16**, Lemma 4.1.14]. Equation (3.5) follows from (3.4) and [**6**] using the inequalities

$$\begin{aligned}
&\|(f-f_n)w^2\|_{L_\infty(\mathbf{R})} \\
&\qquad \le \|(\mu I - K_n)^{-1}\|_{X\to X}\|K\|_{X\to X}\|(f-L_n(f))w^2\|_{L_\infty(\mathbf{R})} \\
&\qquad \le C\|(\mu I - K_n)^{-1}\|_{X\to X}\|L_n\|_\infty E_{n-1}[f]_{w,\infty} \\
&\qquad \le C\|(\mu I - K_n)^{-1}\|_{X\to X} n^{1/6} E_{n-1}[f]_{w,\infty}
\end{aligned}$$

since $\|L_n\|_\infty = O(n^{1/6})$, see [**23**]. This completes the proof of Theorem 2b. $\square$

We now proceed to estimate the right-hand side of (3.5). To this end, and in what follows, we let a_n, $n \ge 1$, denote the unique positive solution of the equation

$$n = \frac{2}{\pi}\int_0^1 \frac{a_n x Q'(a_n x)}{\sqrt{1-x^2}}\,dx. \tag{3.6}$$

Then it is well known, see [**21**], that a_n exists, is unique and $a_n = O(n^{1/\beta})$ for large n, where β is given in Definition 1. In particular, if $w = w_\alpha$ as in (2.1), then $T = \beta = \alpha$ and

$$a_n = O(n^{1/\alpha}), \quad n \to \infty.$$

For our weights we have the classical result [**21**] that $E_{n-1}[f]_{w,\infty} = O(a_n/n) = O(n^{1/\beta - 1})$.

Using these results, we obtain an immediate consequence of equation (3.5).

Theorem 2c. *Assume the hypotheses used in the derivation of* (3.5). *Then, for large enough* n *we have*

$$\begin{aligned}
\|(f-f_n)w^2\|_{L_\infty(\mathbf{R})} &\le C\|(\mu I - K_n)^{-1}\|_{X\to X} n^{1/6}\,\frac{a_n}{n} \\
&\le C\|(\mu I - K_n)^{-1}\|_{X\to X} n^{1/\beta - 5/6}
\end{aligned} \tag{3.7}$$

This result implies that we have convergence of the sequence (f_n) of approximate solutions towards the exact solution f in the weighted L_∞ norm with weight w if $\|(\mu I - K_n)^{-1}\|_{X\to X} < C$ as $n \to \infty$ and $\beta > 6/5$. Common methods for the proof of the first inequality, cf. e.g., [**16**], use the compactness of the operator K—a condition that is violated in our case. We therefore use ideas of [**22**, Section 4] to circumvent these difficulties.

It will now be necessary to state more precise conditions on our constants μ. We know from the proof of Theorem 2b that

$$\sup_{n\geq 1} \|K_n\|_{X\to X} < D^*$$

where $D^* > 0$ depends only on w and λ which are fixed from the start. We assume from now on that $|\mu| \geq D^*$. We have:

Theorem 2d. *Assume the hypotheses of Theorem* 2c*, and let* $f''w \in L_\infty(\mathbf{R})$*, and suppose that* $f'w$ *has limit* 0 *at* $\pm\infty$*. Then, for large enough* n*,*

$$\|(f - f_n)w^2\|_{L_\infty(\mathbf{R})} = O(n^{2/\beta - 5/4}\sqrt{\log n}\,). \tag{3.8}$$

Here the O *term depends on* w*,* β *and* λ *which are fixed but is independent of* f*,* f_n *and* n*.*

Notice that the regularity assumptions here are a stronger than in the previous theorems for we need estimates on the error of best weighted approximation which is quite natural.

Remark. Theorem 2d implies that we have convergence for $\beta > 8/5$. In view of similar results for related problems, we believe that this restriction is due to the methods used in the proofs of the results of [**6**] that we required here, and that the result is actually true for a larger range of β whose precise nature is as yet unclear. However it seems that the proof of this conjecture would require substantially different methods. Note though that an inspection of the proof reveals that we can obtain a larger range of permitted values for β if we replace our basic assumption $f^{(2)}w \in L_\infty(\mathbf{R})$ by $f^{(j)}w \in L_\infty(\mathbf{R})$ with some $j \geq 3$.

In this case we can replace the condition $\beta > 8/5$ by $\beta > 1+3/(4j-3)$. The expression on the right-hand side of the last inequality decreases monotonically as j increases.

Proof of Theorem 2d. We recall equation (3.4) that states

$$fw^2 - f_n w^2 = (\mu I - K_n)^{-1} K[f - L_n[f]]w^2.$$

In view of [**6**, Corollary 1.4 and Theorem 1.5], our assumptions on f give

$$K[f - L_n[f]](x) = O(n^{7/12}\sqrt{\log n}\, a_n^2 n^{-2}) = O(n^{2/\beta - 17/12}\sqrt{\log n}\,)$$

uniformly for all $x \in \mathbf{R}$. Applying the method used in the proof of [**6**, Theorem 1.5(a)] (applied there to a formula based on a different interpolation operator) to our operator, we find that

$$\begin{aligned} \|K_n\big[K[f - L_n[f]]\big]\|_X &= O(n^{1/6} n^{2/\beta - 17/12}\sqrt{\log n}\,) \\ &= O(n^{2/\beta - 5/4}\sqrt{\log n}\,). \end{aligned}$$

We may rewrite the right-hand side of (3.4) in the form of a Neumann series,

$$(\mu I - K_n)^{-1} K[f - L_n[f]]w^2 = \frac{1}{\mu}\sum_{\nu=0}^{\infty} \mu^{-\nu} K_n^{\nu}\big[K[f - L_n[f]]\big]w^2,$$

and we see that the series converges with a limit being of the order of $\|K[f - L_n[f]]\|_X$, i.e., we have that

$$\|fw^2 - f_n w^2\|_{L_\infty(\mathbf{R})} = O(n^{2/\beta - 5/4}\sqrt{\log n}\,). \qquad \square$$

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