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POSITIVE SOLUTIONS OF A HAMMERSTEIN INTEGRAL EQUATION WITH A SINGULAR NONLINEAR TERM, II

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Dedicated to Professor Enrico Magenes for his 80th birthday

ABSTRACT. This paper concerns the existence of a positive locally summable solution of a Hammerstein equation with a singular nonlinear term at the origin.

1. Introduction. In this paper we establish some new existence principles for the following Hammerstein equation:

$$u(x) = \int_\Omega K(x,y)g(y,u(y))dy, \quad x \in \Omega, \tag{1.1}$$

where $\Omega \subset \mathbf{R}^N$, $1 \le N$, $K(x,y) \ge 0$; $g(y,s) \ge 0$; $x, y \in \Omega$, $0 < s$ and $g(y,s)$ that can be nonsmooth when $s \to 0^+$.

The literature on the Hammerstein equations with the integrand depending on the reciprocal of the solution is rather limited, nevertheless it arises, more or less directly, in a variety of settings: semi-linear boundary value problems with a nonlinear term depending on the reciprocal of the solution, see [**1, 5–7, 10, 12, 13, 15, 16**], mathematical models of signal theory, see [**21, 22**], ecological models, see [**28**, pp. 103–104], continuous extension of the results on the double stochastic matrix proposed by Hartfiel, see [**23, 27**], Boussinesq's equation in filtration theory, see [**18**].

Karlin and Nirenberg in [**19**], at first, proved an existence principle for (1.1), considering $K(x,x) > 0$, $0 \le x \le 1$; however, they proved also

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that this assumption cannot be completely discarded. In [**19**], as in the pioneering paper of Nowosad [**22**], they assumed $K \in C([0,1] \times [0,1])$.

With the exception of the regularity, the Green's functions of some boundary value problems do not satisfy the previous assumption. For example, the Green's function

$$G(x,y) = \begin{cases} x(1-y) & 0 \le x \le y, \\ y(1-x) & y \le x \le 1, \end{cases}$$

of the boundary value problem:

$$-u'' = f(x), \quad u(0) = u(1) = 0,$$

does not satisfy the Karlin and Nirenberg assumption because it is equal to 0 on the boundary of the square $[0,1] \times [0,1]$. In this paper we prove an existence principle valid also for $G(x,y)$. Define $a(x) = x(1-x)$; we find

$$a(x)a(y) \le G(x,y); \quad \int_0^1 2G(x,y)\,dx \le a(y).$$

These hypotheses, different from the ones of [**19, 22, 27**, p. 1172], are sufficient to guarantee the existence of at least one solution. Also the Green's function of the Laplacian in a bounded open set of $\mathbf{R}^N$, $2 \le N$, with zero Dirichlet condition satisfies similar assumptions. Other examples are listed in the next section.

We dwell upon the structure of $K(x,y)$ in a neighborhood of the diagonal set of $\Omega \times \Omega$. Another existence result has been proved by the author in [**8, 9**] via assumptions formulated in terms of integral means of the kernel on the sets of a cover of the diagonal set of $\Omega \times \Omega$.

In this article there are considered kernels which can be discontinuous on the diagonal set of $\Omega \times \Omega$. Precisely we consider kernels greater than a strictly positive number in at least "half" of $\Omega \times \Omega$, and this region with its symmetric, with respect to the diagonal set of $\Omega \times \Omega$, cover $\Omega \times \Omega$. For example, the Green's function:

$$G(x,y) = \begin{cases} 1 & 0 \le x \le y, \\ e^{y-x} & y \le x, \end{cases}$$

of the boundary value problem

$$-(u'' + u') = f(x), \quad u'(0) = \lim_{x \to +\infty} u(x) = 0,$$

satisfies this assumption.

There is no hypothesis on the behavior of $g(y, s)$, when $s \to 0^+$, therefore the following possibilities are not excluded

$$\liminf_{s \to 0^+} g(y, s) = 0; \qquad \limsup_{s \to 0^+} g(y, s) = +\infty.$$

Consequently, the main results of this paper (Theorems 2 and 4) follow from the behavior of the "approximate solution" $u_\varepsilon \in L^1(\Omega)$, of the problem

$$u_\varepsilon(x) = \int_\Omega K(x, y) g(y, \varepsilon + u_\varepsilon(y))\, dy, \quad x \in \Omega.$$

when $\varepsilon \to 0^+$. The existence of these "approximate solutions" is proved in Appendix 2 of [**9**]. The first step of such analysis is based on an assumption of "local compactness," as in [**4**], see the assumption $(\mathcal{H}_1)$. This implies the existence of a sequence $(u_{\varepsilon_k})_{k \in \mathbf{N}}$ which converges almost everywhere and in every $L^1(\Omega_n)$, toward a measurable function u_0. This belongs to every $L^1(\Omega_n)$ and satisfies the identity

$$u_0(x) = \int_{0 < u_0} K(x, y) g(y, u_0(y))\, dy,$$

where $(0 < u_0) = \{x \in \Omega \,|\, 0 < u_0(x)\}$. Next, to prove that u_0 is nontrivial, we have to prescribe the behavior of $g(y, s)$ as $s \to 0^+$. Some advances with respect to the literature on this problem, see [**1, 2, 12, 13, 18, 19, 22, 23, 26, 27**], have been achieved by the author in [**8, 9**], under the assumption

$$\lim_{s \to 0} \frac{g(y, s)}{s} = +\infty, \tag{1.2}$$

uniformly in Ω. The result of Karlin and Nirenberg is covered by (1.2), because they assumed: $0 < c_0 \le g(y, s) \le c_1 s^{-\beta}$, $0 < s < 2$, $y \in \overline{\Omega}$

with c_0, c_1, $\beta > 0$. Here (1.2) is replaced by the assumption that there exists μ such that

$$\liminf_{s\to 0} \frac{g(y,s)}{s} \geq \mu > \lambda(E). \tag{1.3}$$

uniformly in some measurable set $E \subset \Omega$, $|E| > 0$. $\lambda(E)$ is a positive number, defined in the next section, which coincides with the minimum characteristic value of the operator

$$\varphi \longmapsto \int_E K(\cdot, y)\varphi(y)\, dy, \quad \varphi \in L^1_+(E),$$

when it exists, see [**11**, Vol. II, proof of Frobenius' theorem, p. 51]; [**29**, Theorem IV.3.1]. In particular, if $K(x,y)$ is the Green's function of

$$-u'' = f(x), \quad u(0) = u(1) = 0,$$

$\lambda([0,1])^{-1}$ is the first eigenvalue of the one-dimensional Dirichlet problem in $[0,1]$, see [**16**].

Assumption (1.3) cannot be completely discarded. In fact, the function

$$g(s) = \left|\frac{1}{s}\, \sin\frac{1}{s}\right|, \quad s > 0,$$

does not satisfy (1.3). Let

$$\varepsilon_k = \frac{1}{k\pi}, \quad k \in \mathbf{N}^*,$$

be one solution of

$$u(x) = \int_0^1 K(x,y) \left|\frac{1}{\varepsilon + u(y)}\, \sin\frac{1}{\varepsilon + u(y)}\right| dy, \tag{1.4}$$

when $\varepsilon = \varepsilon_k$, is $u_{\varepsilon_k} = 0$. Therefore, there exists a sequence $(u_\varepsilon)_{\varepsilon>0}$, of approximate solutions of (1.4) which admits one subsequence converging to 0. The solvability of (1.4) with $\varepsilon = 0$ is still an open problem.

In the next section we present the assumptions and the results. The other sections are devoted to the proofs.

2. Notations, assumptions and results. Let us list the notations mostly used in this paper.

$\mathbf{R}_+ := [0, +\infty[$; $\mathbf{R}_+^* :=]0, +\infty[$. Let $E \subset \mathbf{R}^N$, $N \geq 1$, be a measurable set, $|E|$ or meas (E) is the measure of E, $|\cdot|_{1,E}$ the norm of $L^1(E)$ and $L^1_+(E)$ the cone of the $\varphi \in L^1(E)$, $\varphi > 0$ almost everywhere in E.

Let $S \subset \mathbf{R}^N \times \mathbf{R}^N$, S^T denotes the symmetric set of S with respect the diagonal set of $\mathbf{R}^N \times \mathbf{R}^N$. Finally for two fixed functions u, v, $u \leq v$ is the set of the x such that $u(x) \leq v(x)$. The same holds for $u < v$, $u \geq v$, $u > v$.

Let $\Omega \subset \mathbf{R}^N$ be a measurable set and $g : \Omega \times \mathbf{R}_+^* \to \mathbf{R}$ be a non-negative almost everywhere Carathéodory function, i.e., $g(\cdot, s)$ is measurable in Ω, for all $s > 0$ and $g(y, \cdot)$ is continuous in $]0, +\infty[$, for almost every $y \in \Omega$ such that

$$g^*(y, s) := \sup_{s \leq t} g(y, t) \in L^1(\Omega), \quad s > 0.$$

$g^*(y, s)$ is also a Carathéodory nonincreasing function with respect to s and $g(y, s) \leq g^*(y, s)$.

Let $K : \Omega \times \Omega \to \mathbf{R}$ be a measurable non-negative almost every kernel such that

$$\int_\Omega K(\cdot, y)\varphi(y)\, dy \in L^1(\Omega); \qquad \int_\Omega K(x, \cdot)\varphi(x)\, dx \in L^1(\Omega), \tag{2.1}$$

for each $\varphi \in L^1(\Omega)$. It is well known that, for every $\varepsilon > 0$ there exists $u_\varepsilon \in L^1(\Omega)$, $u_\varepsilon > 0$ almost everywhere, such that

$$u_\varepsilon(x) = \int_\Omega K(x, y) g(y, \varepsilon + u_\varepsilon(y))\, dy,$$

see [**9**, Appendix 2]. The results of this paper follow from a suitable analysis of u_ε as $\varepsilon \to 0$.

To guarantee the existence of at least one convergent subsequence we assume that the following hypothesis of "local compactness" holds.

($\mathcal{H}_1$) *There exists an increasing sequence* $(\Omega_n)_{n\in\mathbf{N}}$, $\Omega_n \subset \Omega$, *of measurable sets which covers* Ω *such that the operator*

$$\varphi \mapsto K_n(\varphi) := \int_{\Omega_n} K(\cdot, y)\varphi(y)\, dy$$

is compact from $L^1(\Omega_n)$ *into itself, for all* $n \in \mathbf{N}$.

In [**14, 17, 20, 24, 25, 29**] are listed some assumptions which imply ($\mathcal{H}_1$).

Using the diagonal method, as in the proof of Lemma 4 of [**9**], we are able to prove

Lemma 0. *Assume* ($\mathcal{H}_1$), *and let* $(g(\cdot, \varepsilon + u_\varepsilon))_{\varepsilon>0}$ *be bounded in every* $L^1(\Omega_n)$. *Then there exists* $(\varepsilon_k)_{k\in\mathbf{N}}$, $\varepsilon_k \to 0$, *such that*

$$\left(\int_{\Omega_n} K(\cdot, y) g(y, \varepsilon_k + u_{\varepsilon_k}(y))\, dy\right)_{k\in\mathbf{N}}$$

converges in $L^1(\Omega_n)$, *for all* $n \in \mathbf{N}$.

For simplicity of notations we write

$$g_\varepsilon := g(\cdot, \varepsilon + u_\varepsilon); \qquad u'_{k,n} := \int_{\Omega_n} K(\cdot, y) g_{\varepsilon_k}(y)\, dy;$$

$$u''_{k,n} := u_k - u'_{k,n} = \int_{\Omega\setminus\Omega_n} K(\cdot, y) g_{\varepsilon_k}(y)\, dy.$$

According to the previous lemma there exists an increasing sequence $(v_n)_{n\in\mathbf{N}}$, $v_n \in L^1(\Omega)$, such that

$$\lim_k \left|u'_{k,n} - v_n\right|_{1,\Omega_n} = 0; \quad v_n = 0 \quad \text{in} \quad \Omega \setminus \Omega_n.$$

Then

$$v_n(x) := \begin{cases} \lim_k u'_{k,n}(x), & x \in \Omega_n \text{ a.e.} \\ 0, & x \in \Omega \setminus \Omega_n. \end{cases}$$

Therefore, there exists a non-negative measurable function $u_0 : \Omega \to \mathbf{R}$ such that

$$u_0 = \lim_n v_n = \operatorname{ess\,sup}_n v_n, \quad \text{a.e. in } \Omega.$$

Since $g(y, \cdot)$ not is defined at 0 we need that $u_0 \neq 0$ or, as we will see, $u_{\varepsilon_k} \not\to 0$ almost everywhere in Ω. Therefore, we assume that

($\mathcal{H}_2$) *There exist μ and a measurable set $E \subset \Omega$, $0 < |E| < +\infty$, such that*

$$\liminf_{s \to 0} \frac{g(y, s)}{s} \geq \mu > 0,$$

uniformly with respect to y in E.

We also need the following definition. Let $\varphi \in L^1_+(E)$ and we write

$$E^*(\varphi) := \left\{ y \in E \,\middle|\, \int_E K(x, y)\varphi(x)\, dx \neq 0 \right\}.$$

Define

$$\lambda(E) := \inf \left\{ \lambda(E, \varphi) \,\middle|\, \varphi \in L^1_+(\Omega) \right\}$$

where

$$\lambda(E, \varphi) := \sup_{y \in E^*(\varphi)} \frac{\varphi(y)}{\int_E K(x, y)\varphi(x)\, dx}.$$

If the operator

$$\varphi \mapsto \int_E K(x, \cdot)\varphi(x)\, dx, \quad \varphi \in L^1(E), \tag{2.2}$$

has characteristic values, its minimum positive characteristic value coincides with $\lambda(E)$, see [**29**, Theorem IV.3.1]. Also when it is an operator of $\mathbf{R}^N$ in itself, associated with an irreducible matrix with positive terms, $\lambda(E)$ is the minimum positive characteristic value of (2.2), [**11**, Vol. II, proof of the Frobenius theorem, p. 51].

Theorem 1. *Assume*

($\mathcal{H}_3$) *for all $\alpha > 0$ and $\varphi \in L^1_+(\Omega)$ there exists $\delta > 0$ such that*

$$|E \setminus F| < \delta \quad \Longrightarrow \quad \int_{E\setminus F} K(x,y)\varphi(x)\,dx < \alpha \int_E K(x,y)\varphi(x)\,dx, \quad y \in F.$$

Then $\lambda(\cdot)$ is left continuous in E, namely, for all $\alpha > 0$, there exists $\delta > 0$ such that, for all measurable sets, $F \subset E$, we have

$$|E \setminus F| < \delta \quad \Longrightarrow \quad \lambda(E) \le \lambda(F) \le \lambda(E) + \alpha.$$

The hypothesis ($\mathcal{H}_3$) is fulfilled, for example, in the following three cases.

i) E is compact, $K \in C(E \times E)$ and, for each $\varphi \in L^1_+(E)$, there exists $m(\varphi) > 0$ such that

$$0 < m(\varphi) \le \int_E K(x,y)\varphi(x)\,dx, \quad y \in E. \tag{2.3}$$

ii) $\varphi \mapsto \int_E K(x,\cdot)\varphi(x)\,dx$ is continuous from $L^1(E)$ in $L^\infty(E)$ and, for all $\varphi \in L^1_+(E)$, (2.3) holds.

iii) There exist some positive functions α_i, β_i, $1 \le i \le k$, with $\alpha_i \in L^\infty(E)$, $\beta_i \in L^1(E)$ such that

$$K(x,y) = \sum_1^k \alpha_i(x)\beta_i(y).$$

The behavior of u_ε, as $\varepsilon \to 0$, has been analyzed for two distinct sets of hypotheses on $K(x,y)$ and/or $g(y,s)$. The first of them is the following one.

($\mathcal{K}$) a) *There exists $\eta \in L^1_+(\Omega)$ such that*

$$\int_\Omega K(x,\cdot)\,\eta(x)\,dx \in L^\infty(\Omega).$$

b) *There exists* $c_1 > 0$ *such that*

$$\text{meas}\left[\Omega^2 \setminus \left[(c_1 < K) \cup (c_1 < K)^T\right]\right] = 0.$$

Remark 1. The last assumption can be rewritten as follows

$$\exists\, c_1 > 0: \ \text{meas}\left[(K \le c_1) \cap (K \le c_1)^T\right] = 0.$$

Moreover, $(\mathcal{K})$ can be replaced by the following

$(\mathcal{K}_{b'})$ *There exist a finite cover* $(E_i)_{1\le i\le N}$ *of* Ω *and* $c_1 > 0$ *such that*

$$\text{meas}\left[\bigcup_{i=1}^{N} E_i^2 \cap (K \le c_1) \cap (K \le c_1)^T\right] = 0.$$

Clearly this is sharper than $(\mathcal{K})$, but it adds some more technicalities in the proofs.

Examples. The assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_3)$, $(\mathcal{K})$ are satisfied by the Green's function

$$G(x,y) = \begin{cases} 1 & \text{if } 0 \le x \le y \\ e^{y-x} & \text{if } y \le x, \end{cases}$$

of the following boundary value problem

$$-y'' - y' = f(x); \qquad y'(0) = \lim_{x\to+\infty} y(x) = 0.$$

They are also satisfied by the Green's function

$$G(x,y) = \begin{cases} e^{-y}\cosh x & \text{if } 0 \le x \le y, \\ e^{-x}\cosh y & \text{if } y \le x \le 2, \end{cases}$$

of the boundary value problem

$$-y'' + y = f(x); \qquad y'(0) = y'(2) + y(2) = 0.$$

Theorem 2. *Assume* $(\mathcal{H}_1)$, $(\mathcal{K})$. *Then*

i) *there exists a measurable non-negative almost everywhere function* $u_0 : \Omega \to \mathbf{R}$ *with* $\eta\, u_0 \in L^1(\Omega)$, *such that*

$$u_0(x) = \int\limits_{0<u_0} K(x,y) g(y, u_0(y))\, dy.$$

ii) *If also* $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ *hold and*

$$\mu > \lambda(E),$$

then u_0 *is nontrivial.*

Corollary 3. *Assume* $(\mathcal{H}_1), (\mathcal{H}_2), (\mathcal{H}_3), (\mathcal{K})$. *For all* $\lambda > \lambda(E)/\mu$, *there exists a nontrivial map* $u_{0,\lambda} : \Omega \to \mathbf{R}$, $\eta\, u_{0,\lambda} \in L^1(\Omega)$, *such that*

$$u_{0,\lambda}(x) = \lambda \int\limits_{0<u_{0,\lambda}} K(x,y) g(y, u_{0,\lambda}(y))\, dy.$$

Next, instead of $(\mathcal{K})$, let us consider the following hypotheses.

$(\widetilde{\mathcal{G}})$ *There exist a measurable function* φ_0, $\varphi_0 \geq 0$ *almost everywhere in* Ω, *and* $p \geq 1$ *such that*

$$g(y,s) \leq \frac{\varphi_0(y)}{s^p}, \quad y \in \Omega, \quad 0 < s \leq 1.$$

$(\widetilde{\mathcal{K}})$ *There exist two measurable non-negative almost everywhere functions* a, η *such that*

a) $a(x)a(y) \leq K(x,y)$; $\int_\Omega K(x,y)\, \eta(x)\, dx \leq a(y)$.

b) *For all* $n \in \mathbf{N}$: $\quad 0 < \operatorname{ess\,inf}_{y \in \Omega_n} a(y)$.

c) $\int_\Omega (\varphi_0(y))/(a(y)^{p-1})\, dy < +\infty$.

Remark 2. In light of (2.1) and $(\widetilde{\mathcal{K}}_a)$, we find

$$\int\limits_\Omega a(y)\varphi(y)\, dy < +\infty, \quad \varphi \in L^1_+(\Omega).$$

Examples. The assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_3)$, $(\widetilde{\mathcal{K}})$ on $K(x,y)$ are satisfied by the following kernels.

i) The Green's function

$$G(x,y) = \begin{cases} x & \text{if } 0 \le x \le y, \\ y & \text{if } y \le x, \end{cases}$$

of the following boundary value problem

$$-y'' = f(x); \qquad y(0) = 0,\ \limsup_{x\to+\infty} |y(x)| < +\infty,$$

with $a(x) = x/(1+x)$ and $\eta(x) = 1/(1+x)^3$.

ii) The Green's function of the Laplacian in a smooth bounded domain $\Omega \subset \mathbf{R}^N$, $2 \le N$, with zero Dirichlet condition. In this case we consider $a(x) = c_0 \operatorname{dist}(x, \partial\Omega)$, $x \in \Omega$, and $\eta(x) = c_1\varphi(x)$, where c_0, c_1 are positive constants and $\varphi(x)$ is a positive eigenfunction of the Dirichlet problem corresponding to the first eigenvalue, see [**3**, Lemma 3.2], [**5**, Theorem 9], [**30**, Theorem 1].

iii) The Green's function

$$G(x,y) = \begin{cases} e^y(1-e^{-x}) & \text{if } 0 \le x \le y, \\ e^y - 1 & \text{if } y \le x \le 1, \end{cases}$$

of the boundary value problem

$$-y'' - y' = f(x); \qquad y(0) = y'(1) = 0,$$

with $a(x) = 1 - e^{-x}$, $\eta(x) = e^{-1}$.

Theorem 4. *Assume* $(\mathcal{H}_1)$, $(\widetilde{\mathcal{G}})$, $(\widetilde{\mathcal{K}})$. *Then*

i) *there exists a measurable function* $u_0 : \Omega \to \mathbf{R}$, $0 \le u_0$ *almost everywhere, with* $\eta\, u_0 \in L^1(\Omega)$, *such that*

$$u_0(x) = \int_{0<u_0} K(x,y) g(y, u_0(y))\, dy.$$

Moreover

ii) $u_0 = 0$ *or* $u_0 > 0$ *almost everywhere in* Ω.

iii) *If also* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$ *hold and*

$$\mu > \lambda(E),$$

then $u_0 > 0$ *almost everywhere in* Ω.

Corollary 5. *Assume* $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_3)$, $(\widetilde{\mathcal{G}})$, $(\widetilde{\mathcal{K}})$. *Then, for all*

$$\lambda > \frac{\lambda(E)}{\mu},$$

there exists a positive function $u_{0,\lambda} : \Omega \to \mathbf{R}$, $\eta\, u_{0,\lambda} \in L^1(\Omega)$, *such that*

$$u_{0,\lambda}(x) = \lambda \int_\Omega K(x,y) g(y, u_{0,\lambda}(y))\, dy.$$

3. Proof of Theorem 1.

Lemma 3.1. *Let* E, $F \subset \Omega$ *be measurable sets. Then*

$$F \subset E \quad \Longrightarrow \quad \lambda(E) \le \lambda(F).$$

Proof. Let $\varphi \in L^1_+(E)$ and χ_F be the characteristic function of F. As $F \cap E^*(\varphi\chi_F) = F^*(\varphi\chi_F)$, we deduce

$$\begin{aligned} \lambda(F, \varphi\chi_F) &= \sup_{y \in F^*(\varphi\chi_F)} \frac{(\varphi\chi_F)(y)}{\int_F K(x,y)(\varphi\chi_F)(x)\, dx} \\ &= \sup_{y \in F^*(\varphi\chi_F)} \frac{(\varphi\chi_F)(y)}{\int_E K(x,y)(\varphi\chi_F)(x)\, dx} \\ &= \sup_{y \in E^*(\varphi\chi_F)} \frac{(\varphi\chi_F)(y)}{\int_E K(x,y)(\varphi\chi_F)(x)\, dx} \\ &= \lambda(E, \varphi\chi_F). \end{aligned}$$

Next in view of the definition of $\lambda(\cdot)$, we find

$$\lambda(F) = \inf_{\varphi\in L^1_+(F)} \lambda(F,\varphi) = \inf_{\varphi\in L^1_+(E)} \lambda(F,\varphi\chi_F).$$

According to the previous identity we conclude that

$$\lambda(F) = \inf_{\varphi\in L^1_+(E)} \lambda(E,\varphi\chi_F) \geq \inf_{\varphi\in L^1_+(E)} \lambda(E,\varphi) = \lambda(E). \qquad \square$$

Proof of Theorem 1. If $\lambda(E) = +\infty$ the theorem is true. Then consider the case $\lambda(E) < +\infty$. Let $\sigma > 0$ and write

$$\alpha = \frac{\sigma}{1+\lambda(E)+\sigma}. \tag{3.1}$$

According to the definition of $\lambda(E)$, there exists $\overline{\varphi} \in L^1_+(E)$ such that

$$\lambda(E,\overline{\varphi}) < \lambda(E) + \alpha. \tag{3.2}$$

In light of $(\mathcal{H}_3)$, there exists $\delta > 0$ such that for all $F \subset E$ we have

$$|E\setminus F| < \delta \quad \Longrightarrow \quad \int\limits_{E\setminus F} K(x,y)\overline{\varphi}(x)\,dx < \alpha \int\limits_E K(x,y)\overline{\varphi}(x)\,dx, \quad y\in F.$$

Therefore,

$$\begin{aligned}\lambda(F) &= \inf_{\varphi\in L^1_+(E)} \lambda(F,\varphi\chi_F) \leq \lambda(F,\overline{\varphi}\chi_F)\\ &= \sup_{y\in F^*(\overline{\varphi})} \left(\frac{(\overline{\varphi}\chi_F)(y)}{\int_E K(x,y)\overline{\varphi}(x)\,dx} \, \frac{\int_E K(x,y)\overline{\varphi}(x)\,dx}{\int_F K(x,y)\overline{\varphi}(x)\,dx} \right).\end{aligned}$$

According to the definition of $\lambda(E,\overline{\varphi})$ and (3.2), we compute

$$\begin{aligned}\lambda(F) &\leq \lambda(E,\overline{\varphi}) \sup_{y\in F^*(\overline{\varphi})} \frac{\int_E K(x,y)\overline{\varphi}(x)\,dx}{\int_F K(x,y)\overline{\varphi}(x)\,dx}\\ &< (\lambda(E)+\alpha) \sup_{y\in F^*(\overline{\varphi})} \frac{1}{1-\int_{E\setminus F} K(x,y)\overline{\varphi}(x)\,dx / \int_E K(x,y)\overline{\varphi}(x)\,dx}\\ &= \frac{\lambda(E)+\alpha}{1-\sup_{y\in F^*(\overline{\varphi})} \left(\int_{E\setminus F} K(x,y)\overline{\varphi}(x)\,dx / \int_E K(x,y)\overline{\varphi}(x)\,dx\right)}\\ &< \frac{\lambda(E)+\alpha}{1-\alpha}.\end{aligned}$$

In view of (3.1) we thus deduce for all $F \subset E$:

$$|E \setminus F| < \delta \quad \Longrightarrow \quad \lambda(F) < \lambda(E) + \sigma.$$

Finally, recalling Lemma 3.1, the proof is concluded. $\square$

4. Proof of Theorem 2.

Lemma 4.1 (coercivity). *Let $\psi \in L^1(\Omega)$ be almost everywhere non-negative. Then*

$$\frac{c_1}{2(1+c_1)}\,|\psi|_{1,\Omega}^2 \leq \int\limits_{\Omega\times\Omega} \frac{K(x,y)}{1+K(x,y)}\,\psi(x)\psi(y)\,dx\,dy.$$

Proof. Since

$$|\psi|_{1,\Omega}^2 = \int\limits_{\Omega\times\Omega} \psi\otimes\psi\,dx\,dy \leq \Bigg(\int\limits_{c_1<K} + \int\limits_{(c_1<K)^T}\Bigg)\psi\otimes\psi\,dx\,dy,$$

changing the variables in the second integral, according to $(\mathcal{K}_b)$, we discover

$$\begin{aligned}|\psi|_{1,\Omega}^2 &\leq 2\int\limits_{c_1<K} \psi\otimes\psi\,dx\,dy\\ &\leq \frac{2(1+c_1)}{c_1}\int\limits_{c_1<K}\frac{K(x,y)}{1+K(x,y)}\,\psi(x)\psi(y)\,dx\,dy,\end{aligned}$$

as desired. $\square$

Lemma 4.2 (bounds on $(g_\varepsilon)_{\varepsilon>0}$). *Let $\lambda > 0$ and $E \subset \Omega$ be a measurable set. Then we have the estimate*

$$|g_\varepsilon|_{1,E} \leq 2\big(1+c_1^{-1}\big)\big(|g^*(\cdot,\lambda)|_{1,E} + \lambda\big).$$

Proof. By Tonelli's theorem

$$\begin{aligned}\int_{(u_\varepsilon \le \lambda)\cap E} u_\varepsilon g_\varepsilon \, dx &= \int\limits_{[(u_\varepsilon \le \lambda)\cap E]\times\Omega} K(x,y) g_\varepsilon(x) g_\varepsilon(y)\, dx\, dy \\ &\ge \int\limits_{[(u_\varepsilon \le \lambda)\cap E]\times E} \frac{K(x,y)}{1+K(x,y)}\, g_\varepsilon(x) g_\varepsilon(y)\, dx\, dy \\ &= \Bigg(\int\limits_{E^2} - \int\limits_{[(u_\varepsilon > \lambda)\cap E]\times E}\Bigg) \frac{K(x,y)}{1+K(x,y)}\, g_\varepsilon(x) g_\varepsilon(y)\, dx\, dy.\end{aligned}$$

Now since $g^*(x,\cdot)$ is decreasing we discover

$$\lambda |g_\varepsilon|_{1,E} + \int\limits_{E^2} g^*(x,\lambda) g_\varepsilon(y)\, dx\, dy \ge \int\limits_{E^2} \frac{K(x,y)}{1+K(x,y)}\, g_\varepsilon(x) g_\varepsilon(y)\, dx\, dy.$$

According to the previous lemma, we deduce

$$\big(|g^*(\cdot,\lambda)|_{1,E} + \lambda\big)|g_\varepsilon|_{1,E} \ge \frac{c_1}{2(1+c_1)}\, |g_\varepsilon|_{1,E}^2.$$

Then the stated estimate follows. □

The absolute continuity of the indefinite integral of $g^*(\cdot,\lambda)$ implies the following

Corollary 4.3. *Letting $\lambda > 0$, there exists $\delta > 0$ such that for each measurable set $E \subset \Omega$ we have*

$$|E| < \delta \quad \Longrightarrow \quad |g_\varepsilon|_{1,E} < 4\big(1 + c_1^{-1}\big)\lambda.$$

Corollary 4.4 ("local" convergence). *Assume $(\mathcal{H}_1)$, $(\mathcal{K})$. Then Lemma* 0 *holds.*

Lemma 4.5 (pointwise convergence). *$u_k \to u_0$ almost everywhere in Ω.*

Proof. Since

$$\int_{\Omega} u''_{k,n}\eta\,dx = \int_{\Omega\setminus\Omega_n} g_{\varepsilon_k}(y)\,dy \int_{\Omega} K(x,y)\,\eta(x)\,dx,$$

according to the Lemma 4.2 and $(\mathcal{K}_a)$, we find

$$\int_{\Omega} u''_{k,n}\eta\,dx \le \Big(\sup_{y\in\Omega}\int_{\Omega} K(x,y)\,\eta(x)\,dx\Big) 2\big(1+c_1^{-1}\big)\big(|g^*(\cdot,\lambda)|_{1,\Omega\setminus\Omega_n}+\lambda\big).$$

Since λ was arbitrary and the indefinite integral of $g^*(\cdot,\lambda)$ is absolutely continuous we have

$$\lim_n \int_{\Omega} u''_{k,n}\eta\,dx = 0,$$

uniformly with respect to k. Consequently, fixing any $\sigma > 0$, there exists $M_0 \in \mathbf{N}$ such that

$$\int_{\Omega} u''_{k,n}\eta\,dx < \sigma, \quad n > M_0, \quad k \in \mathbf{N}. \tag{4.1}$$

For each $M > M_0$ we deduce

$$\begin{aligned}\int_{\Omega M} \frac{\eta}{1+\eta}\,\frac{|u_k-u_0|}{1+u_0}\,dx &= \int_{\Omega_M} \frac{\eta}{1+\eta}\,\frac{|u'_{k,n}+u''_{k,n}-v_n+v_n-u_0|}{1+u_0}\,dx \\ &\le \int_{\Omega_M} |u'_{k,n}-v_n|\,dx + \int_{\Omega} u''_{k,n}\eta\,dx + \int_{\Omega} \frac{u_0-v_n}{1+u_0}\,dx.\end{aligned}$$

Since $\lim_k |u'_{k,n}-v_n|_{1,\Omega_n} = 0$ holds for each $n > M$, according to (4.1) we find

$$\limsup_k \int_{\Omega_M} \frac{\eta}{1+\eta}\,\frac{|u_k-u_0|}{1+u_0}\,dx \le \sigma + \int_{\Omega} \frac{u_0-v_n}{1+u_0}\,dx.$$

Applying Beppo Levi's theorem we thus deduce

$$\limsup_k \int_{\Omega_M} \frac{\eta}{1+\eta}\,\frac{|u_k-u_0|}{1+u_0}\,dx = 0, \quad M > M_0.$$

Then $u_k \to u_0$ almost everywhere in Ω_M, for all $M > M_0$, as required. □

Next we claim a technical lemma of "partial" convergence which depends essentially on Corollary 4.3 and the definition of v_n. In the next section we prove Lemma 5.5 which is slightly more general than the following.

Lemma 4.6. *Assume* $\varphi \in C(\mathbf{R}_+) \cap L^\infty(\mathbf{R}_+)$. *Then*

$$\lim_k \int_{\Omega_n} |\varphi(u'_{k,n}) - \varphi(v_n)| g_{\varepsilon_k}\, dx = 0$$

and

$$\limsup_k \int_{\Omega_n} \varphi(u'_{k,n}) g_{\varepsilon_k}\, dx = \limsup_k \int_{\Omega_n} \varphi(v_n) g_{\varepsilon_k}\, dx,$$

for each $n \in \mathbf{N}$.

Proof. Let $\lambda > 0$. Since the integral of $g^*(\cdot, \lambda)$ is absolutely continuous, there exists $\delta > 0$ such that, for all measurable sets $E \subset \Omega$, we have

$$|E| < \delta \quad \Longrightarrow \quad |g^*(\cdot, \lambda)|_{1,E} < \lambda.$$

Since $\lim_k u'_{k,n} = v_n$ almost everywhere in Ω_n, it follows that $\lim_k \varphi(u'_{k,n}) = \varphi(v_n)$ almost everywhere in Ω_n and then in measure. Thus, set

$$\Omega_{k,n} = \big\{x \in \Omega_n \;\big|\; |\varphi(u'_{k,n})(x) - \varphi(v_n)(x)| > \lambda\big\},$$

there exists $\bar{k}$ such that

$$k > \bar{k} \implies |\Omega_{k,n}| < \delta.$$

Next, observe that

$$
\begin{aligned}
k > \bar{k} \quad \Longrightarrow \quad I_{k,n} &:= \int\limits_{\Omega_n} |\varphi(u'_{k,n}) - \varphi(v_n)| g_{\varepsilon_k}\, dx \\
&= \Big(\int\limits_{\Omega_{k,n}} + \int\limits_{\Omega_n \setminus \Omega_{k,n}} \Big) |\varphi(u'_{k,n}) - \varphi(v_n)| g_{\varepsilon_k}\, dx \\
&\le 2|\varphi|_{\infty,\mathbf{R}_+} \int\limits_{\Omega_{k,n}} g_{\varepsilon_k}\, dx + \lambda \int\limits_{\Omega_n \setminus \Omega_{k,n}} g_{\varepsilon_k}\, dx.
\end{aligned}
$$

Recalling Corollary 4.3 and Lemma 4.2, we have

$$
\begin{aligned}
k > \bar{k} \quad \Longrightarrow \quad I_{k,n} &< 2|\varphi|_{\infty,\mathbf{R}_+} 4\big(1+c_1^{-1}\big)\lambda + |g_{\varepsilon_k}|_{\Omega_n}\lambda \\
&\le 8|\varphi|_{\infty,\mathbf{R}_+}\big(1+c_1^{-1}\big)\lambda + 2\big(1+c_1^{-1}\big)(|g^*(\cdot,1)|_{1,\Omega_n}+1)\lambda.
\end{aligned}
$$

This holds true for each $\lambda > 0$, therefore $\lim_k I_{k,n} = 0$. This concludes the proof. $\square$

Proof of Theorem 2_i. If $0 < \operatorname{ess\,inf}\, u_0$, the claim follows by Lemma 4.5 and Lemma 3 of [**9**]. If $0 = \operatorname{ess\,inf}\, u_0$, there exists a decreasing sequence $(X_l)_{l\in\mathbf{N}}$, $|X_l| > 0$, such that

$$
\forall\, x \in X_l :\ u_0(x) \le \frac{1}{1+l}; \qquad \forall\, x \notin X_l :\ \frac{1}{1+l} < u_0(x). \tag{4.2}
$$

Since $u_0 = \operatorname{ess\,sup}\, v_n$, we see:

$$
\forall\, x \in \Omega_n \cap X_l :\ v_n(x) \le \frac{1}{1+l}.
$$

In view of the definition of $u'_{k,n}$, we deduce:

$$
\begin{aligned}
&\int\limits_{\Omega_n \cap X_l} \frac{u'_{k,n}(x)}{1+u'_{k,n}(x)}\, g_{\varepsilon_k}(x)\, dx \\
&\qquad = \int\limits_{(\Omega_n \cap X_l)\times\Omega} \frac{K(x,y)}{1+u'_{k,n}(x)}\, g_{\varepsilon_k}(x) g_{\varepsilon_k}(y)\, dx\, dy \\
&\qquad \ge \int\limits_{(\Omega_n \cap X_l)^2} \frac{K(x,y)}{1+K(x,y)}\, \frac{1}{1+u'_{k,n}(x)}\, g_{\varepsilon_k}(x) g_{\varepsilon_k}(y)\, dx\, dy.
\end{aligned}
$$

Now

$$\Delta_{k,n} := \int\limits_{(\Omega_n \cap X_l)^2} \frac{K(x,y)}{1+K(x,y)} \left| \frac{1}{1+u'_{k,n}(x)} - \frac{1}{1+v_n(x)} \right| g_{\varepsilon_k}(x) g_{\varepsilon_k}(y)\, dx\, dy$$

$$\leq |g_{\varepsilon_k}|_{1,\Omega_n} \int\limits_{\Omega_n} \left| \frac{1}{1+u'_{k,n}(x)} - \frac{1}{1+v_n(x)} \right| g_{\varepsilon_k}(x)\, dx,$$

according to Lemma 4.2 we have:

$$\Delta_{k,n} \leq 2\big(1 + c_1^{-1}\big)\big(|g^*(\cdot,1)|_{1,\Omega_n} + 1\big) \times \int\limits_{\Omega_n} \left| \frac{1}{1+u'_{k,n}(x)} - \frac{1}{1+v_n(x)} \right| g_{\varepsilon_k}(x)\, dx.$$

Next, employing Lemma 4.6 we deduce

$$\lim_k \Delta_{k,n} = 0. \tag{4.4}$$

Therefore, applying Lemma 4.6 and (4.4) at (4.3), letting $k \to +\infty$, we compute

$$\limsup_k \int\limits_{(\Omega_n \cap X_l)} \frac{v_n(x)}{1+v_n(x)}\, g_{\varepsilon_k}(x)\, dx \geq \limsup_k \int\limits_{(\Omega_n \cap X_l)^2} \frac{K(x,y)}{1+K(x,y)} \frac{1}{1+v_n(x)}\, g_{\varepsilon_k}(x) g_{\varepsilon_k}(y)\, dx\, dy.$$

Recalling Lemma 4.1, we discover

$$\frac{1}{l+2} \limsup_k \int\limits_{(\Omega_n \cap X_l)} g_{\varepsilon_k}(x)\, dx \geq \frac{l+1}{l+2} \limsup_k \int\limits_{(\Omega_n \cap X_l)^2} \frac{K(x,y)}{1+K(x,y)}\, g_{\varepsilon_k}(x) g_{\varepsilon_k}(y)\, dx\, dy$$

$$\geq \frac{l+1}{l+2} \limsup_k \left(\frac{c_1}{2(1+c_1)} \left| g_{\varepsilon_k} \right|^2_{1,\Omega_n \cap X_l} \right).$$

Then, since $(g_\varepsilon)_\varepsilon > 0$ is bounded, see Lemma 4.2, we deduce that

$$\frac{1}{l+1}\,2\big(1+c_1^{-1}\big) \geq \limsup_k |g_{\varepsilon_k}|_{1,\Omega_n\cap X_l}. \tag{4.5}$$

Remember the identity

$$u'_{k,n}(x) = \Bigg(\int\limits_{\Omega_n\setminus X_l} + \int\limits_{\Omega_n\cap X_l}\Bigg) K(x,y) g_{\varepsilon_k}(y)\,dy.$$

Employing the second part of (4.2) and Lemma 3 of [**9**], sending $k \to +\infty$, we discover

$$\begin{aligned} v_n(x) = &\int\limits_{\Omega_n\setminus X_l} K(x,y) g(y,u_0(y))\,dy \\ &\qquad + \lim_k \int\limits_{\Omega_n\cap X_l} K(x,y) g_{\varepsilon_k}(y)\,dy, \quad x\in\Omega_n \quad \text{a.e.} \end{aligned}$$

Then

$$\begin{aligned} \Gamma_{k,n} &:= \int\limits_{\Omega_n} \eta(x) \Bigg| v_n(x) - \int\limits_{\Omega_n\setminus X_l} K(x,y) g(y,u_0(y))\,dy \Bigg|\,dx \\ &= \int\limits_{\Omega_n} \eta(x)\,dx\, \lim_k \int\limits_{\Omega_n\cap X_l} K(x,y) g_{\varepsilon_k}(y)\,dy \\ &\leq \Bigg(\operatorname*{ess\,sup}_{y\in\Omega} \int\limits_{\Omega} K(x,y)\,\eta(x)\,dx\Bigg) \liminf_k\; |g_{\varepsilon_k}|_{1,\Omega_n\cap X_l}. \end{aligned}$$

Furthermore, by (4.5),

$$\Gamma_{k,n} \leq \Bigg(\operatorname*{ess\,sup}_{y\in\Omega} \int\limits_{\Omega} K(x,y)\,\eta(x)\,dx\Bigg) \frac{1}{l+1}\,2\big(1+c_1^{-1}\big).$$

Thus,

$$\forall\, n\in\mathbf{N}:\ \lim_l \Gamma_{k,n} = 0.$$

Consequently, for all $n \in \mathbf{N}$, we have

$$\begin{aligned} v_n &= \lim_l \int\limits_{\Omega_n \setminus X_l} K(x,y) g(y, u_0(y))\, dy \\ &= \int\limits_{\Omega_n \cap (0<u_0)} K(x,y) g(y, u_0(y))\, dy, \quad y \in \Omega_n \text{ a.e.} \end{aligned}$$

Passing to the limit, as $n \to +\infty$, we find:

$$u_0(x) = \int\limits_{0<u_0} K(x,y) g(y, u_0(y))\, dy.$$

Next, we conclude the proof proving that $\eta\, u_0 \in L^1(\Omega)$. Remembering the definition of u_{ε_k}, using Tonelli's theorem, we discover:

$$|\eta\, u_{\varepsilon_k}|_{1,\Omega} = \int\limits_{\Omega} g_{\varepsilon_k}\, dy \int\limits_{\Omega} K(x,y)\, \eta(x)\, dx.$$

In light of Lemma 4.2 and the assumption $(\mathcal{K}_a)$, the sequence $(|\eta\, u_{\varepsilon_k}|_{1,\Omega})_{k \in \mathbf{N}}$ is bounded. Moreover, since $|\eta\, u_0|_{1,\Omega} \leq \liminf_k |\eta\, u_{\varepsilon_k}|_{1,\Omega}$, we can conclude the proof. $\square$

Proof of Theorem 2_{ii}. Let $0 < \rho < \mu$ and $\sigma_0 > 0$ be such that

$$y \in E, \quad 0 < s < \sigma_0 \quad \Longrightarrow \quad \frac{g(y,s)}{s} > \mu - \rho. \tag{4.6}$$

Assume that $(u_{\varepsilon_k})_{k \in \mathbf{N}}$ converges to u_0. Suppose, on the contrary, that $u_0 = 0$ in Ω. Then $u_{\varepsilon_k} \to 0$ almost everywhere in Ω and thus in E. Since the measure of E is finite, according to Egorov-Severini's theorem, we have $u_{\varepsilon_k} \to 0$ almost uniformly in E. Then for each $\sigma > 0$, there exist $E_\sigma \subset E$ and k_0 such that

$$|E_\sigma| < \sigma \quad \text{and} \quad (y \in E \setminus E_\sigma, \quad k > k_0 \quad \Longrightarrow \quad \varepsilon_k + u_{\varepsilon_k} < \sigma_0).$$

Therefore (4.6) implies

$$y \in E \setminus E_\sigma, \quad k > k_0 \quad \Longrightarrow \quad g_{\varepsilon_k}(y) > (\mu - \rho)(\varepsilon_k + u_{\varepsilon_k}(y)).$$

Let $\varphi \in L^1_+(E)$. In view of the definition of u_{ε_k}, (1.1) and Tonelli's theorem we compute, for $k > k_0$,

$$\begin{aligned}\int_{E\setminus E_\sigma} u_{\varepsilon_k}(x)\varphi(x)\,dx &= \int_\Omega g_{\varepsilon_k}(y)\,dy \int_{E\setminus E_\sigma} K(x,y)\varphi(x)\,dx\\ &\geq \int_{E\setminus E_\sigma} g_{\varepsilon_k}(y)\,dy \int_{E\setminus E_\sigma} K(x,y)\varphi(x)\,dx\\ &\geq (\mu-\rho)\int_{E\setminus E_\sigma} (\varepsilon_k + u_{\varepsilon_k}(y))\,dy \int_{E\setminus E_\sigma} K(x,y)\varphi(x)\,dx\\ &\geq (\mu-\rho)\int_{E\setminus E_\sigma} (\varepsilon_k + u_{\varepsilon_k}(y))\,\frac{\varphi(y)}{\lambda(E\setminus E_\sigma,\varphi)}\,dy.\end{aligned}$$

Then

$$\int_{E\setminus E_\sigma} (\varepsilon_k + u_{\varepsilon_k}(x))\varphi(x)\,dx \geq \frac{\mu-\rho}{\lambda(E\setminus E_\sigma,\varphi)} \int_{E\setminus E_\sigma} (\varepsilon_k + u_{\varepsilon_k}(y))\varphi(y)\,dy$$

and

$$\lambda(E\setminus E_\sigma,\varphi) \geq \mu-\rho.$$

Passing to the infimum with respect to $\varphi \in L^1_+(E)$, we find

$$\lambda(E\setminus E_\sigma) \geq \mu-\rho.$$

Since σ and ρ are arbitrary, according to Theorem 1 we deduce

$$\lambda(E) \geq \mu,$$

but this contradicts the assumption of the Theorem. Then the proof is complete. □

5. Proof of Theorem 4.

Lemma 5.1 (bounds on $(ag_\varepsilon)_{\varepsilon>0}$). *Assuming $\lambda > 0$ and $E \subset \Omega$ a measurable set, if $\varepsilon + \lambda \leq 1$ we have the estimate*

$$|ag_\varepsilon|_{1,E} \leq (A_E + B_E)^{p/(1+p)} + (A_E + B_E),$$

where $A_E = \left|ag^*(\cdot,\lambda)\right|_{1,E}$, $B_E = \left|\varphi_0 a^{1-p}\right|_{1,E}^{1/p}$.

Proof. Employing Tonelli's theorem and the assumption $(\widetilde{\mathcal{K}}_a)$, we deduce

$$\int\limits_{(u_\varepsilon\le\lambda)\cap E} u_\varepsilon(x)g_\varepsilon(x)\,dx = \int\limits_{[(u_\varepsilon\le\lambda)\cap E]\times\Omega} K(x,y)g_\varepsilon(x)g_\varepsilon(y)\,dx\,dy$$
$$\ge \int\limits_{[(u_\varepsilon\le\lambda)\cap E]\times E} a(x)g_\varepsilon(x)a(y)g_\varepsilon(y)\,dx\,dy.$$

Then

$$\int\limits_{(u_\varepsilon\le\lambda)\cap E} u_\varepsilon(x)g_\varepsilon(x)\,dx + \int\limits_{(u_\varepsilon>\lambda)\cap E} a(x)g_\varepsilon(x)\,dx\cdot\left|ag_\varepsilon\right|_{1,E} \ge \left|ag_\varepsilon\right|_{1,E}^2.$$

Making use of the definition of $g^*(\cdot,\lambda)$ and Remark 2, we discover

$$\int\limits_{(u_\varepsilon\le\lambda)\cap E} u_\varepsilon(x)g_\varepsilon(x)\,dx + \left|ag^*(\cdot,\lambda)\right|_{1,E}\cdot\left|ag_\varepsilon\right|_{1,E} \ge \left|ag_\varepsilon\right|_{1,E}^2. \tag{5.1}$$

In light of the assumptions $(\widetilde{\mathcal{G}})$, $(\widetilde{\mathcal{K}}_c)$, since $\varepsilon+\lambda<1$, we compute

$$\int\limits_{(u_\varepsilon\le\lambda)\cap E} u_\varepsilon(x)g_\varepsilon(x)\,dx \le \int\limits_{(u_\varepsilon\le\lambda)\cap E} (u_\varepsilon(x)+\varepsilon)g_\varepsilon(x)^{1/p}g_\varepsilon(x)^{1/p'}\,dx$$
$$\le \int\limits_{(u_\varepsilon\le\lambda)\cap E} (u_\varepsilon(x)+\varepsilon)\left(\frac{\varphi_0(x)}{(u_\varepsilon(x)+\varepsilon)^p}\right)^{1/p}$$
$$\cdot\, a(x)^{-1/p'}(a(x)g_\varepsilon(x))^{1/p'}\,dx$$
$$= \int\limits_E \left(\frac{\varphi_0(x)}{a(x)^{p-1}}\right)^{1/p}(a(x)g_\varepsilon(x))^{1/p'}\,dx$$
$$\le \left|\varphi_0 a^{1-p}\right|_{1,E}^{1/p}\cdot\left|ag_\varepsilon\right|_{1,E}^{1/p'}.$$

Substituting this estimate in (5.1) we obtain:

$$(5.2) \qquad \big|\varphi_0 a^{1-p}\big|_{1,E}^{1/p} \cdot \big|ag_\varepsilon\big|_{1,E}^{1/p'} + \big|ag^*(\cdot,\lambda)\big|_{1,E} \cdot \big|ag_\varepsilon|_{1,E} \ge \big|ag_\varepsilon\big|_{1,E}^2.$$

For short, we define $\theta = |ag_\varepsilon|_{1,E}$, $A = A_E$, $B = B_E$, so that (5.2) becomes:

$$\theta^{1+1/p} \le \theta^{1/p} A + B.$$

Then

$$\theta \le 1 \quad \Longrightarrow \quad \theta^{1+(1/p)} \le A + B \implies \theta \le (A+B)^{p/(p+1)},$$

$$\theta > 1 \quad \Longrightarrow \quad \theta^{1+(1/p)} \le \theta^{1/p}(A+B) \implies \theta \le (A+B).$$

Thus

$$\theta \le (A+B)^{p/(p+1)} + (A+B),$$

as required. □

The absolute continuity of the indefinite integral of $ag^*(\cdot,\lambda)$ and of $\varphi_0 a^{1-p}$ implies that

Corollary 5.2. *Assuming* $0 < \lambda < 1$*, then there exists* $\delta > 0$ *such that, for every measurable set* $E \subset \Omega$ *and* $\varepsilon + \lambda \le 1$*,*

$$|E| < \delta \quad \Longrightarrow \quad \big|ag_\varepsilon\big|_{1,E} < \lambda.$$

Lemma 5.3 ("local" convergence). *Assume* $\mathcal{H}_1$*,* $(\widetilde{\mathcal{G}})$*,* $(\widetilde{\mathcal{K}})$*. Then Lemma* 0 *holds.*

Proof. By the previous lemma $(ag_\varepsilon)_{\varepsilon>0}$ is bounded in $L^1(\Omega)$ and consequently in all $L^1(\Omega_n)$. In view of the assumption $(\widetilde{\mathcal{K}}_b)$, also $(g_\varepsilon)_{\varepsilon>0}$ is bounded in every $L^1(\Omega_n)$. Thus, the hypotheses of Lemma 0 are satisfied. □

Lemma 5.4 (pointwise convergence). $u_k \to u_0$ *almost everywhere in* Ω*.*

Proof. According to Tonelli's and Fubini's theorems and the assumption $(\widetilde{\mathcal{K}}_a)$, we discover

$$\int\limits_{\Omega} u''_{k,n}(x)\,\eta(x)\,dx = \int\limits_{\Omega\setminus\Omega_n} g_{\varepsilon_k}(y)\,dy \int\limits_{\Omega} K(x,y)\,\eta(x)\,dx$$
$$\leq \int\limits_{\Omega\setminus\Omega_n} g_{\varepsilon_k}(y)a(y)\,dy$$
$$= \int\limits_{\Omega\setminus\Omega_n} ag_{\varepsilon_k}\,dy.$$

Using Corollary 5.2 we obtain

$$\lim_n \int\limits_{\Omega} u''_{k,n}\eta\,dx = 0,$$

uniformly with respect to k.

The rest of the proof runs as that of the Lemma 4.5. □

We next claim a lemma of "partial" convergence slightly more general than Lemma 4.6.

Lemma 5.5. *Assume $\varphi \in C(\mathbf{R}_+) \cap L^\infty(\mathbf{R}_+)$, $\beta \in L^1(\Omega)$, $\beta \geq 0$, almost everywhere. Then*

$$\lim_k \int\limits_{\Omega_n} \left|\varphi(u'_{k,n}) - \varphi(v_n)\right| \beta^{1/p}(ag_{\varepsilon_k})^{1/p'}\,dx = 0$$

and

$$\limsup_k \int\limits_{\Omega_n} \varphi(u'_{k,n})\beta^{1/p}(ag_{\varepsilon_k})^{1/p'}\,dx = \limsup_k \int\limits_{\Omega_n} \varphi(v_n)\beta^{1/p}(ag_{\varepsilon_k})^{1/p'}\,dx,$$

for each $n \in \mathbf{N}$.

Proof. Let $0 < \lambda \leq 1/2$ and $k_0 \in \mathbf{N}$ such that $\varepsilon_k \leq 1/2$, $k_0 < k$. According to Corollary 5.2, there exists $\delta > 0$ such that

$$|E| < \delta \quad\Longrightarrow\quad \left|ag_{\varepsilon_k}\right|_{1,E} < \lambda,$$

for every measurable set $E \subset \Omega$ and $k_0 < k$. Since $\lim_k u'_{k,n} = v_n$ almost everywhere in Ω_n, then $\lim_k \varphi(u'_{k,n}) = \varphi(v_n)$ almost everywhere in Ω_n and also in measure. Precisely, if we define

$$\Omega_{k,n} = \big\{ x \in \Omega_n \,\big|\, |\varphi(u'_{k,n})(x) - \varphi(v_n)(x)| > \lambda \big\},$$

there exists $\bar{k} \geq k_0$ such that

$$k > \bar{k} \quad \Longrightarrow \quad |\Omega_{k,n}| < \delta \quad \Longrightarrow \quad |a g_{\varepsilon_k}|_{1,\Omega_{k,n}} < \lambda. \tag{5.3}$$

Then we compute

$$\begin{aligned} k > \bar{k} \quad \Longrightarrow \quad I_{k,n} &:= \int\limits_{\Omega_n} |\varphi(u'_{k,n}) - \varphi(v_n)|\, \beta^{1/p} (a g_{\varepsilon_k})^{1/p'}\, dx \\ &\leq \big|\beta\big|_{1,\Omega_n}^{1/p} \Big\{ 2\big|\varphi\big|_{\infty,\mathbf{R}_+} \big|a g_{\varepsilon_k}\big|_{1,\Omega_{k,n}}^{1/p'} + \lambda \big|a g_{\varepsilon_k}\big|_{1,\Omega_n}^{1/p'} \Big\}. \end{aligned}$$

Next, utilizing (5.3) and Lemma 5.1, we deduce that

$$\begin{aligned} k > \bar{k} \quad \Longrightarrow \quad & I_{k,n} \\ & \leq \big|\beta\big|_{1,\Omega_n}^{1/p} \Big\{ 2|\varphi|_{\infty,\mathbf{R}_+} \lambda^{1/p'} + \lambda \big[T(\Omega_n)^{p/(p+1)} + T(\Omega_n) \big]^{1/p'} \Big\}. \end{aligned}$$

Since λ was arbitrary we conclude the proof. $\square$

Proof of Theorem 4_i. If $0 < \operatorname{ess\,inf} u_0$, applying Lemma 5.4 and Lemma 3 of [**9**], we deduce the claim. If $0 = \operatorname{ess\,inf} u_0$, there exists a decreasing sequence $(X_l)_{l \in \mathbf{N}}$ of measurable sets such that $|X_l| > 0$, and

$$\forall\, x \in X_l : \; u_0(x) \leq \frac{1}{1+l}, \qquad \forall\, x \notin X_l : \; \frac{1}{1+l} < u_0(x). \tag{5.4}$$

Since $u_0 = \sup_n v_n$, it follows also that

$$\forall\, x \in X_l : \; v_n(x) \leq \frac{1}{1+l}. \tag{5.5}$$

Remembering the definition of $u'_{k,n}$ and the assumption $(\widetilde{\mathcal{K}}_a)$, we discover

$$u'_{k,n}(x) \geq a(x) \int\limits_{\Omega_n} a(y) g_{\varepsilon_k}(y)\, dy, \quad x \in \Omega.$$

Hence, if we multiply both sides for

$$\frac{g_{\varepsilon_k}}{1+u'_{k,n}}$$

and integrate on $\Omega_n \cap X_l$, it follows that

$$\begin{aligned}
&|ag_{\varepsilon_k}|_{1,\Omega_n} \int\limits_{\Omega_n\cap X_l} \frac{ag_{\varepsilon_k}}{1+u'_{k,n}}\,dx \\
&\le \int\limits_{\Omega_n\cap X_l} \frac{u'_{k,n}g_{\varepsilon_k}}{1+u'_{k,n}}\,dx \\
&= \Bigg(\int\limits_{\Omega_n\cap X_l\cap(\varepsilon_k+u_{\varepsilon_k}\le 1)} + \int\limits_{\Omega_n\cap X_l\cap(\varepsilon_k+u_{\varepsilon_k}>1)} \Bigg) \frac{u'_{k,n}}{1+u'_{k,n}}\, g_{\varepsilon_{k,n}}^{1/(2p)} g_{\varepsilon_{k,n}}^{1/(2p)'}\,dx \\
&\le \int\limits_{\Omega_n\cap X_l\cap(\varepsilon_k+u_{\varepsilon_k}\le 1)} \frac{u'_{k,n}}{1+u'_{k,n}} \left(\frac{\varphi_0}{(\varepsilon_k+u_{\varepsilon_k})^p}\right)^{1/(2p)} a^{-1/(2p)'}(ag_{\varepsilon_k})^{1/(2p)'}\,dx \\
&\quad + \int\limits_{\Omega\cap X_l\cap(\varepsilon_k+u_{\varepsilon_k}>1)} \frac{u'_{k,n}}{1+u'_{k,n}}\, g^*(\cdot,1)\,dx \\
&\le \int\limits_{\Omega_n\cap X_l} \frac{\sqrt{u'_{k,n}}}{1+u'_{k,n}} \left(\frac{\varphi_0}{a^{2p-1}}\right)^{1/(2p)} (ag_{\varepsilon_k})^{1/(2p)'}\,dx + \int\limits_{\Omega_n\cap X_l} \frac{u'_{k,n}}{1+u'_{k,n}}\, g^*(\cdot,1)\,dx.
\end{aligned}$$

Since $(|ag_{\varepsilon_k}|_{1,\Omega_n})_{k\in\mathbf{N}}$ is bounded, see Lemma 5.1, according to Lemma 5.5 we discover

$$\lim_k \Bigg(|ag_{\varepsilon_k}|_{1,\Omega_n} \int\limits_{\Omega_n\cap X_l} \left| \frac{1}{1+u'_{k,n}} - \frac{1}{1+v_n} \right| ag_{\varepsilon_k}\,dx \Bigg) = 0.$$

Therefore, from the latter estimate, in view of Lemma 5.5 and (5.5), we deduce

$$\begin{aligned}
&\frac{1+l}{2+l}\Big(\limsup_k |ag_{\varepsilon_k}|_{1,\Omega_n\cap X_l}\Big)^2 \\
&\qquad \le \limsup_k \left(|ag_{\varepsilon_k}|_{1,\Omega_n} \int\limits_{\Omega_n\cap X_l} \frac{ag_{\varepsilon_k}}{1+u'_{k,n}}\,dx \right) \\
&\qquad \le \limsup_k \int\limits_{\Omega_n\cap X_l} \frac{\sqrt{v_n}}{1+v_n}\left(\frac{\varphi_0}{a^{2p-1}}\right)^{1/(2p)} (ag_{\varepsilon_k})^{1/(2p)'}\,dx \\
&\qquad\quad + \int\limits_{\Omega_n\cap X_l} \frac{v_n}{1+v_n}\, g^*(\cdot,1)\,dx \\
&\qquad \le \frac{\sqrt{1+l}}{2+l}\,\frac{|\varphi_0 a^{1-p}|^{1/(2p)}_{1,\Omega_n\cap X_l}}{\sqrt{\operatorname*{ess\,inf}_{\Omega_n} a}}\, \limsup_k |ag_{\varepsilon_k}|^{1/(2p)'}_{1,\Omega_n} \\
&\qquad\quad + \frac{1}{2+l}\,|g^*(\cdot,1)|_{1,\Omega_n},
\end{aligned}$$

and consequently

$$\begin{aligned}
&\big(\limsup_k |ag_{\varepsilon_k}|_{1,\Omega_n\cap X_l}\big)^2 \\
&\qquad \le \frac{1}{\sqrt{1+l}}\,\frac{|\varphi_0 a^{1-p}|^{1/(2p)}_{1,\Omega_n}}{\sqrt{\operatorname*{ess\,inf}_{\Omega_n} a}}\,\big(\limsup_k |ag_{\varepsilon_k}|_{1,\Omega_n\cap X_l}\big)^{1/(2p)'} \\
&\qquad\quad + \frac{1}{1+l}\,|g^*(\cdot,1)|_{1,\Omega_n}.
\end{aligned}$$

Define

$$S_n = \frac{|\varphi_0 a^{1-p}|^{1/(2p)}_{1,\Omega_n}}{\sqrt{\operatorname*{ess\,inf}_{\Omega_n} a}} + |g^*(\cdot,1)|_{1,\Omega_n},$$

exactly as in the proof of Lemma 5.1, and we have

$$\limsup_k |ag_{\varepsilon_k}|_{1,\Omega_n\cap X_l} \le \Big(\frac{S_n}{\sqrt{1+l}}\Big)^{1/2} + \Big(\frac{S_n}{\sqrt{1+l}}\Big)^{(2p)/(1+2p)}. \tag{5.6}$$

Since

$$u'_{k,n}(x) = \Bigg(\int\limits_{\Omega_n\setminus X_l} + \int\limits_{\Omega_n\cap X_l}\Bigg) K(x,y) g_{\varepsilon_k}(y)\,dy,$$

we see from the second of (5.4), Lemma 3 of [**9**] and Lemma 5.4, upon passing to the limit, as $k \to +\infty$, that

$$v_n(x) = \int\limits_{\Omega_n \setminus X_l} K(x,y)g(y,u_0(y))\,dy + \lim_k \int\limits_{\Omega_n \cap X_l} K(x,y)g_{\varepsilon_k}(y)\,dy.$$

Multiplying the above identity by $\eta(x)$ and integrating over Ω_n, we deduce

$$\begin{aligned}\Delta_n &:= \int\limits_{\Omega_n} \eta(x)\left| v_n(x) - \int\limits_{\Omega_n \setminus X_l} K(x,y)g(y,u_0(y))\,dy \right| dx \\ &\le \liminf_k \int\limits_{\Omega_n \cap X_l} g_{\varepsilon_k}(y)\,dy \int\limits_{\Omega_n} K(x,y)\,\eta(x)\,dx \\ &\le \liminf_k \int\limits_{\Omega_n \cap X_l} a(y)g_{\varepsilon_k}(y)\,dy.\end{aligned}$$

Employing (5.6) in the last inequality we find

$$\Delta_n \le \left(\frac{S_n}{\sqrt{1+l}}\right)^{1/2} + \left(\frac{S_n}{\sqrt{1+l}}\right)^{(2p)/(1+2p)}.$$

Passing to the limit, as $l \to +\infty$, we deduce for all $n \in \mathbf{N}$:

$$\lim_l \int\limits_{\Omega_n} \eta(x) \left| v_n(x) - \int\limits_{\Omega_n \cap (1/(1+l) < u_0)} (x,y)g(y,u_0(y))\,dy \right| dx = 0.$$

Thus, for $n \in \mathbf{N}$,

$$\begin{aligned}v_n(x) &= \lim_l \int\limits_{\Omega_n \cap ((1/1+l) < u_0)} K(x,y)g(y,u_0(y))\,dy \\ &= \int\limits_{\Omega_n \cap (0 < u_0)} K(x,y)g(y,u_0(y))\,dy, \quad x \in \Omega_n \quad \text{a.e.}\end{aligned}$$

Letting $n \to +\infty$, we deduce the proof. $\square$

Proof of Theorem 4_{ii}. Let $\mathcal{N} := (u_0 = 0)$. If $|\mathcal{N}| = 0$, the claim is true, if $|\mathcal{N}| > 0$ we observe, remembering the condition $(\widetilde{\mathcal{K}}_a)$, that

$$\forall\, x \in \mathcal{N}: \ a(x) \int\limits_{\Omega_n \setminus \mathcal{N}} a(y) g(y, u_0(y))\, dy \leq 0.$$

Then

$$\forall\, n \in \mathbf{N}: \int\limits_{\Omega_n \setminus \mathcal{N}} a(y) g(y, u_0(y))\, dy = 0.$$

Now, according to the assumption $(\widetilde{\mathcal{K}}_b)$, we deduce

$$\forall\, n \in \mathbf{N}: \ g(\cdot, u_0) = 0 \quad \text{a.e. in} \quad \Omega_n \setminus \mathcal{N},$$

and thus

$$g(\cdot, u_0) = 0 \quad \text{a.e. in} \quad \Omega \setminus \mathcal{N}.$$

The latter equality implies

$$u_0(x) = \int\limits_{\Omega \setminus \mathcal{N}} K(x,y) g(y, u_0(y))\, dy = 0, \quad x \in \Omega \quad \text{a.e.},$$

then $\Omega = \mathcal{N}$. $\square$

Proof of Theorem 4_{iii}. The proof is like that of Theorem 2_{ii}. $\square$

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