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CONVERGENCE THEOREMS AND MEASURES OF NONCOMPACTNESS FOR NONCOMPACT URYSOHN OPERATORS IN IDEAL SPACES

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ABSTRACT. A result about the uniform convergence of sequences of Urysohn operators in ideal spaces is proved when the limit operator is too singular to be compact. An estimate about the measure of noncompactness of such (weakly) singular Urysohn operators is obtained.

1. Introduction. Let S and T be σ-finite measure spaces, M a metric space, and V a Banach space. Given some function $f\colon T \times S \times M \to V$, we are interested in the corresponding Urysohn operator

$$A(f)x(t) := \int_S f(t, s, x(s))\, ds, \quad t \in T,$$

where the integral is understood in the Lebesgue-Bochner sense. If $M = V = \mathbf{R}$ and f is a so-called Carathéodory function, it is known that under some growth assumptions on f the operator $A(f)$ is compact in L_p-spaces or, more generally, in ideal spaces. These are classical results of Krasnosel'skiĭ [**4**] and Zabreĭko, see e.g., [**5, 14**]. It is also possible to weaken the growth conditions slightly [**6, 9**].

However, there are situations where f does not satisfy these growth assumptions but where one nevertheless would like to say something about the compactness of $A(f)$; if $A(f)$ is not compact, one would at least like to find good estimates for the measure of noncompactness of its image. If such a measure is sufficiently small, one can still apply, e.g., degree theory [**3**] (and in the linear case, the Fredholm alternative holds [**1**]).

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This is the question which we would like to tackle in this paper. For the case of linear operators such estimates have been obtained in [**2**]. However, we obtain different estimates in this paper which are usually better even in the linear case, although our emphasis is on the nonlinear case.

The philosophy for the compactness estimates is to approximate the given Urysohn operator $A(f)$ by a sequence of simpler Urysohn operators $A(f_n)$ which are "regular enough" to prove their compactness. If we would have that $A(f_n)$ converges uniformly to $A(f)$, this would imply the compactness of $A(f)$. However, if we do not require the growth assumptions on $A(f)$, we need not have uniform convergence. Nevertheless, we will establish a "convergence" theorem which yields that $A(f_n)$ is "uniformly close" to $A(f)$ for large n. This "convergence" theorem is the main novelty of this paper. Since this theorem is of independent interest, not only in connection with the compactness proof (we will also use it in the forthcoming paper [**7**]), we formulate it in larger generality in the next section.

2. A uniform convergence theorem for Urysohn operators. Let T and S be σ-finite measure spaces, with nonnegative measures. Let $(V, |\cdot|)$ be a Banach space, M a metric space and $M_n \uparrow M$ a sequence of Borel sets, once and for all fixed. In most applications $M = U$ will be a normed space and $M_n := \{u \in U : |u| < n\}$ but also other constellations are thinkable.

By a measurable function we will always understand a strongly measurable function, i.e., a function which can be approximated almost everywhere (in the sense of the Lebesgue extension of the measure space) by a sequence of functions which assume only finitely many values and which have measurable fibers.

Let B be a set of measurable functions $x: S \to M$. We call a function $f: T \times S \times M \to V$ a *B-function* if the superposition operator $F(f)x(t,s) := f(t,s,x(s))$ defines a measurable function for each $x \in B$. For example, if $f(\cdot,\cdot,u)$ is measurable for each $u \in M$ and $f(t,s,\cdot)$ is continuous for almost all $(t,s) \in T \times S$, i.e., if f is a *Carathéodory function*, then f is a B-function, see e.g., [**10**, Proposition 8.2]. For each B-function f, we define the corresponding Urysohn operator $\mathrm{A}(f)$

by

$$A(f)x(t) := \int_S f(t,s,x(s))\,ds = \int_S F(f)x(t,s)\,ds, \quad t \in T$$

and Krasnosel'skiĭ's cutting operator

$$C_n f(t,s,u) := \begin{cases} f(t,s,u) & \text{if } u \in M_n, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $C_n f$ is automatically a B-function, because M_n is assumed to be a Borel set.

Proposition 1. *Let f be a B-function and $x \in B$. Then $A(|f|)x$ is always measurable, and if $A(f)x(t)$ is defined, i.e., $A(|f|)x(t) < \infty$, for almost all t, then $A(f)x$ is measurable. Moreover, in this case for each B-function g with $|g| \le |f|$ also $A(g)x$ is almost everywhere defined and measurable. In particular, also $A(C_n f)x$ are defined and measurable for each n.*

Proof. The claim follows from the Fubini-Tonelli theorem in the form [**11**, Theorem 1.33]. □

Let $(Y, \|\cdot\|)$ be a pre-ideal space of functions $y: T \to V$, i.e., a normed space of (classes of) measurable functions with the property that, for each $y \in Y$ and each measurable $z: T \to V$ the relation $|z(s)| \le |y(s)|$ almost everywhere implies $z \in Y$ and $\|z\| \le \|y\|$. If Y is complete, then Y is called an *ideal space*. For surveys on ideal spaces, we refer to [**8, 12, 13**]. Actually, we require only that Y is quasi-normed, i.e., instead of the triangle inequality of the norm, we require only that

$$\|x+y\| \le q \cdot (\|x\| + \|y\|), \quad x, y \in X \tag{1}$$

for some finite constant q. We associate to Y its *real form* $Y_{\mathbf{R}}$ which is a space of scalar measurable functions $z: T \to \mathbf{R}$ defined in the obvious way by the relations

$$|y| \in Y_{\mathbf{R}} \Longleftrightarrow y \in Y, \qquad \||y|\|_{Y_{\mathbf{R}}} = \|y\|.$$

Henceforth, we will notationally not distinguish between Y and its real form.

For a measurable function y and a measurable set E, we denote by $P_E y$ the function $P_E y(s) := \chi_E(s) y(s)$. In a slight misuse of notation, we will also use this projection operator if y is a function of more variables than s. If E_n are measurable sets with $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_n E_n = \varnothing$, we write $E_n \downarrow \varnothing$. The notation $E_n \uparrow E$ is defined analogously.

Let now f_n be a sequence of B-functions which converges to a function f in the sense that there is a sequence of sets $R_k \uparrow T \times S$ (up to possibly some null set) such that $f_n \to f$ uniformly on each set of the form $R_k \times M_k$. Since we have in particular for each $u \in M$ that $f_n(t,s,u) \to f(t,s,u)$ almost everywhere and so $f_n(t,s,x(s)) \to f(t,s,x(s))$ almost everywhere, f is automatically a B-function.

Further, let g be a B-function which "dominates" the convergence in the sense that for each $x \in B$ the estimate

$$|f_n(t,s,x(s)) - f(t,s,x(s))| \leq g(t,s,x(s)) \tag{2}$$

holds for almost all (t,s). For each natural numbers $j \leq n$, let $F_{j,n}$ be an operator which maps $x \in B$ to a measurable function $F_{j,n}x\colon T \times S \to V$ such that $F_{j,n}$ lies "between" $F(C_j f_n)$ and $F(f)$ in the sense that

$$\begin{cases} F_{j,n}x(t,s) = f_n(t,s,x(s)) & \text{if } x(s) \in M_j, \\ |F_{j,n}x(t,s) - f(t,s,x(s))| \leq g(t,s,x(s)) & \text{if } x(s) \notin M_j, \end{cases} \tag{3}$$

for almost all $(t,s) \in T \times S$. In particular, the choices $F_{j,n} := F(f_n)$ and $F_{j,n} := F(C_j f_n)$ are possible. We are interested in a uniform convergence theorem of the double sequence

$$A_{j,n}x(t) := \int_S F_{j,n}x(t,s)\,ds, \quad t \in T$$

to the Urysohn operator $A(f)$. In particular, the sequences of Urysohn operators $A_{j,n} = A(f_n)$ or $A_{j,n} = A(C_j f_n)$ have this form. However, more general operator sequences $A_{j,n}$ are also admissible, which will turn out to be important in [**7**]. $A_{j,n}$ need not even be an Urysohn operator because we do not require that $F_{j,n}$ is a superposition operator.

Nevertheless, since $F_{j,n}$ lies "between" $F(C_j f_n)$ and $F(f)$, one could expect that, under reasonable mild assumptions, we have $A_{j,n}x \to A(f)x$ as $j, n \to \infty$. The crucial point, however, is that we want to obtain estimates which are uniform with respect to $x \in B$. The growth of the dominating function g will play the key role. More precisely, we will assume that the following quantities are defined and finite:

$$\gamma_S(g, B) := \sup_{S \supseteq D_n \downarrow \varnothing} \limsup_{n\to\infty} \sup_{x\in B} \left\| \int_{D_n} |g(\cdot, s, x(s))|\, ds \right\|, \tag{4}$$

$$\gamma_T(g, B) := \sup_{T \supseteq E_n \downarrow \varnothing} \limsup_{n\to\infty} \sup_{x\in B} \|P_{E_n} A(|g|)x\|. \tag{5}$$

Recall that, since S is σ-finite, there exists a normalized measure ν on S, i.e., a finite measure with the same measurable sets and null sets as the original measure space S. We call the set B *measure bounded*, for the sequence $M_n \uparrow M$, if

$$\lim_{n\to\infty} \sup_{x\in B} \nu(\{s \in S : x(s) \notin M_n\}) = 0.$$

Recall that this property is actually independent of the particular choice of ν, see e.g., [**10**, Proposition 9.4]. Moreover, for the case that $M = U$ is normed and $M_n := \{u \in U : |u| \le n\}$, the set B is measure bounded if and only if it is bounded in the topological (metric) vector space of all measurable functions $S \to U$, endowed with its usual topology with respect to the measure ν.

Theorem 1. *Let B in the above situation be measure bounded, and suppose that $A_{j,n}x$ and $A(f)x$ are almost everywhere defined for each $x \in B$. Suppose that there are sets $T_k \uparrow T$ and $S_k \uparrow S$, up to null sets, such that for each k, each $j \in \mathbf{N}$, and each $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \sup_{x\in B} \operatorname{mes}\left\{t \in T_k : \int_{S_k} \chi_{Q_n}(t,s)|C_j g(t, s, x(s))|\, ds > \varepsilon\right\} = 0 \tag{6}$$

where $Q_n := (T \times S) \setminus R_n \downarrow \varnothing$. Then there are sequences $j_1 < j_2 < \cdots$ and $p_1 < p_2 < \cdots$ of natural numbers with $j_n \le p_n$ such that $A_{j_n,p_n}x - A(f)x \in Y$ for all sufficiently large n and

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{x\in B} \|A_{j_n,p_n}x - A(f)x\| \\ &\quad \le q \min\{\gamma_S(g,B) + q\gamma_T(g,B), q\gamma_S(g,B) + \gamma_T(g,B)\} \end{aligned} \tag{7}$$

where q is the quantity from (1)*. The choice of j_n depends only on B and the sets M_n.*

Remark 1. For the case that $\gamma_S(g,B) = \gamma_T(g,B) = 0$ and $A_{j,n} = A(f_n)$, Theorem 1 brings to mind the convergence result [**10**, Theorem 9.21] where [**10**, Lemma 9.20] is already incorporated (the latter was necessary to obtain a sharper estimate in the case $\gamma_S(g,B) > 0$). However, in contrast to what one might expect by this remark, our proof requires some more careful estimates than the proof of [**10**, Theorem 9.21].

Before we prove the result, let us point out that the technical condition (6) is actually rather mild. It appears that all examples for which this condition fails are rather pathologic. For example, if $g(t,s,\cdot)$ is a scalar linear function, this condition is automatically satisfied, as we will see. Each of the earlier cited compactness results for Urysohn operators requires a similar condition as (6) (besides $\gamma_S(g,B) = \gamma_T(g,B) = 0$), and as discussed in [**9**], condition (6) is the mildest of those.

Proof. Since $\operatorname{supp} Y$ exists by [**10**, Proposition 3.6] and all functions in consideration vanish almost everywhere outside $\operatorname{supp} Y$, it is no loss of generality to assume $\operatorname{supp} Y = T$. By [**10**, Theorem 3.8], there is a sequence of sets $\tilde{T}_k \uparrow T = \operatorname{supp} Y$ with $\chi_{\tilde{T}_k} \in Y$. Replacing the sets T_k in the hypothesis of the theorem by $T_k \cap \tilde{T}_k$, if necessary, we may thus assume without loss of generality that $\chi_{T_n} \in Y$. Similarly, since S is σ-finite, we may assume that the sets S_k in the hypothesis of the theorem have finite measure.

Since B is measure bounded, there is a sequence $j_n \uparrow \infty$ of natural numbers such that for each sequence $x_n \in B$ the set

$$\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{s \in S : x_n(s) \notin M_{j_n}\} \tag{8}$$

is a null set, see [**10**, Proposition 9.4]. Applying (6), we find for each k some m_k with

$$\sup_{x \in B} \operatorname{mes} \left\{ t \in T_k : \int_{S_k} \chi_{Q_{m_k}}(t,s) |C_{j_k} g(t,s,x(s))| \, ds > \frac{1}{k} \right\} < \frac{1}{k^2}. \tag{9}$$

Without loss of generality, we may assume that $m_1 < m_2 < \cdots$. Let

$$P_{j,x}y(t,s) := \begin{cases} y(t,s) & \text{if } x(s) \in M_j, \\ 0 & \text{if } x(s) \notin M_j. \end{cases}$$

In view of (3), we have $P_{j,x}F_{j,n}x = P_{j,x}F(f_n)x$. Hence, since $f_n \to f$ uniformly on $R_{m_k} \times M_{j_k}$ and S_k has finite measure, we find some $p_k \geq j_k$ such that

(10)
$$\int_{S_k} \chi_{R_{m_k}}(t,s)|P_{j_k,x}(F_{j_k,p_k}x - F(f)x)(t,s)|\,ds \leq \frac{1}{k}, \quad x \in B,\ t \in T.$$

We claim that (7) holds with the above constructed sequences j_n and p_n. Assume by contradiction that this is not the case. Then there is a sequence $x_n \in B$, some $\delta > 0$, and an infinite set $N \subseteq \mathbf{N}$ with

(11)
$$\begin{aligned} \|A_{j_n,p_n}x_n - A(f)x_n\| \geq q(\min\{\gamma_S(g,B) + q\gamma_T(g,B), q\gamma_S(g,B) \\ + \gamma_T(g,B)\} + (1+q)\delta), \\ n \in N. \end{aligned}$$

Put now

$$D_k := (S \setminus S_k) \cup \left(\bigcup_{n=k}^{\infty} \{s \in S : x_n(s) \notin M_{j_n}\} \right).$$

Then $D_k \downarrow \varnothing$ (up to a null set, since (8) is a null set), and so we find by the definition of $\gamma_S(g,B)$ some index K with

(12)
$$\|A(|P_{D_k}g|)x_n\| < \gamma_S(g,B) + \delta, \quad n \in N,\ k \geq K.$$

The estimate (9) implies that the measure of the set

$$H_n := \left\{ t \in T_n : \int_{S_n} \chi_{Q_{m_n}}(t,s)|C_{j_n}g(t,s,x_n(s))|\,ds > \frac{1}{n} \right\}$$

is at most $1/n^2$, and so $E := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} H_n$ is a null set, because $\operatorname{mes} E \leq \sum_{n=k}^{\infty} \operatorname{mes} H_n \leq \sum_{n=k}^{\infty} 1/n^2$ for all k. Put now

$$E_k := (T \setminus T_k) \cup \left(\bigcup_{n=k}^{\infty} H_n \right).$$

Then $E_k \downarrow \varnothing$ (up to a null set, since E is a null set) and so we find by the definition of $\gamma_T(g,B)$ some index J such that

$$\|P_{E_J} A(|g|)x_n\| < \gamma_T(g,B) + \delta, \quad n \in N. \tag{13}$$

Observe that (3) implies in view of (2) that

$$|F_{j,n}x(t,s) - f(t,s,x(s))| \le g(t,s,x(s)) \tag{14}$$

almost everywhere for each $j \le n$ and each $x \in B$. We use this to estimate the nonnegative function

$$c_n(t) := \chi_{T \setminus E_J}(t) \int_{S \setminus D_K} |F_{j_n,p_n} x_n(t,s) - f(t,s,x_n(s))|\, ds$$

for $n \in N$ and $n \ge \max\{J,K\}$. Note that for $s \in S_K \setminus D_K$ the definition of D_K implies $x_n(s) \in M_{j_n}$ and so, using $T \setminus E_J \subseteq T_J \subseteq T_n$, $S \setminus D_K \subseteq S_K \subseteq S_n$, $H_n \subseteq E_J$, and (14), we obtain

$$\begin{aligned} c_n(t) &\le \chi_{T_J \setminus E_J}(t) \int_{S_K} |P_{j_n,x_n}(F_{j_n,p_n}x_n - F(f)x_n)(t,s)|\, ds \\ &\le \chi_{T_J}(t) \int_{S_K} \chi_{R_{m_n}}(t,s) |P_{j_n,x_n}(F_{j_n,p_n}x_n - F(f)x_n)(t,s)|\, ds \\ &\quad + \chi_{T_J}(t)\chi_{T_n \setminus H_n}(t) \int_{S_n} \chi_{Q_{m_n}}(t,s)|C_{j_n} g(t,s,x_n(s))|\, ds. \end{aligned}$$

From (10) and the definition of H_n we conclude that $c_n(t) \le \chi_{T_J}(t)(1/n + 1/n)$, and so

$$\|c_n\| \le \frac{2}{n}\|\chi_{T_J}\|, \quad \max\{J,K\} \le n \in N. \tag{15}$$

By (14), we have for almost all t that

$$\begin{aligned} |A_{j_n,p_n}x_n(t) - A(f)x_n(t) &\le \int_S |F_{j_n,p_n}x_n(t,s) - f(t,s,x_n(s))|\, ds \\ &\le |A(|P_{D_K} g|)x_n(t)| + |P_{E_J} A(|g|)x_n(t)| + c_n(t). \end{aligned}$$

Now we take the (quasi-)norm on the functions of both side of this inequality, using the triangle inequality twice on the right-hand side. Doing this in two different ways, we obtain from the three estimates (12), (13) and (15) for all sufficiently large $n \in N$ the estimate

$$\|A_{j_n,p_n} x_n - A(f)x_n\| < q(\min\{\gamma_S(g,B) + q\gamma_T(g,B), q\gamma_S(g,B) + \gamma_T(g,B)\} + (1+q)\delta)$$

which contradicts (11). □

3. An estimate for the measure of noncompactness of the Urysohn operator. We consider the situation described in the beginning of the previous section. For the fixed sequence of Borel sets $M_n \uparrow M$, we equip the space $\mathcal{B}(M,V)$ of all maps $h: M \to V$ with the uniform structure of uniform convergence on each M_n, i.e., we equip $\mathcal{B}(M,V)$ e.g., with the metric

$$d_{\mathcal{B}(M,V)}(h_1,h_2) = \sum_{n=1}^{\infty} \sup_{u \in M_n} \min\{d(h_1(u),h_2(u)), 2^{-n}\}$$

or some other equivalent metric.

As in the previous section, let B be a set of measurable functions $x: S \to M$, and f be a B-function. Then we call f a *strict B-function* if the function $g(t,s) := f(t,s,\cdot)$ is measurable as a function from $T \times S$ into $\mathcal{B}(M,V)$.

If M_n is separable, then a Carathéodory function f is a strict B-function if and only if for almost all (t,s) the value $f(t,s,\cdot)$ belongs to a separable subset of $\mathcal{B}(M,V)$, see [**10**, Theorem 8.5]. This is in particular the case for each Carathéodory function if M is locally compact and separable [**10**, Theorem 8.15]. Hence, as a rule, if M is a (not too pathological) subset of a finite-dimensional space, the assumption that f is a strict B-function is usually satisfied. The situation is different, in general, if M is a subset of an infinite-dimensional space. An exception of this rule is the case when each of the maps $f(t,s,\cdot)$ is a compact Urysohn operator in ideal spaces which is discussed in [**10**, Section 10].

As in the previous section, let V be a Banach space and Y a (quasi-normed) pre-ideal space of functions $y: T \to V$. The *regular part* Y_0 of

Y is the subspace of all functions $y \in Y$ with the property that for each sequence $E_n \downarrow \varnothing$ we have $\inf_n \|P_{E_n} y\| = 0$. The regular part is itself a pre-ideal space and even an ideal space if Y is an ideal space.

We want to estimate the measure of noncompactness of the image of B under the Urysohn operator $A(f)$. More precisely, we are interested in the *Hausdorff measure of noncompactness*. For a subset A of a metric (or quasinormed) space Z, we define this measure as

$$\chi_Z(A) := \inf\{\varepsilon > 0 : A \text{ has a finite } \varepsilon\text{-net in } Z\}.$$

Of course, $\chi_Z(A) = 0$ if and only if the completion of A is compact, i.e., if and only if each sequence in A has a Cauchy subsequence. Our main compactness result is the following:

Theorem 2. *Let B be measure bounded, and f be a strict B-function with $A(|f|): B \to Y$ and such that $g := f$ satisfies the assumptions of Theorem 1, i.e.,* (6) *holds. Suppose that there is a measurable function $r_n: T \times S \to [0, \infty]$ with*

$$\chi_V(f(t, s, M_n)) \le r_n(t, s) \tag{16}$$

for almost all (t, s). Assume that the regular part Y_0 of Y contains the image $A(f)(B)$ and each of the functions

$$I_n(t) := \int_S r_n(t, s)\, ds.$$

Then

$$\begin{aligned} &\chi_{Y_0}(A(f)(B)) \\ &\le q^2\big(q \sup_n \|I_n\| + \min\{\gamma_S(f, B) + q\gamma_T(f, B), q\gamma_S(f, B) + \gamma_T(f, B)\}\big), \end{aligned}$$

where γ_S and γ_T are defined by (4) *and* (5)*, respectively.*

Proof. As in the proof of Theorem 1, we may assume that $\operatorname{supp} Y_0 = T$ and thus that we find a sequence $T_n \uparrow T = \operatorname{supp} Y_0$ with $\chi_{T_n} \in Y_0$. Since $I_n \in Y_0$, the functions r_n are almost everywhere finite.

Condition (16) thus implies in particular that $f(t, s, M_k)$ is for almost all (t, s) a bounded subset of V, i.e., the function

$$h_k(t, s) := \sup_{u \in M_k} |f(t, s, u)|$$

is almost everywhere finite. Let us first prove that h_k is measurable.

In fact, since f is a *strict* B-function, it can be approximated by a sequence g_n of simple functions, i.e., each g_n is a finite sum of functions of the form $(t, s, u) \mapsto \chi_R(t, s)b(u)$, and for almost all (t, s), we have $g_n(t, s, \cdot) \to f(t, s, \cdot)$ in the space $\mathcal{B}(M, V)$, i.e., uniformly on each M_k. The latter implies that $\sup_{u \in M_k} |g_n(t, s, u)| \to h_k(t, s)$, and from the form of g_n it is clear that the function on the left-hand side is measurable. Hence, h_k is measurable, as claimed.

Choose sets $S_n \uparrow S$ of finite measure, and put

$$R_{k,j} := \{(t, s) \in T_j \times S_j : h_k(t, s) \leq j\}.$$

For each fixed k, we have $R_{k,j} \uparrow T \times S$. By [**11**, Lemma 1.4], we find a sequence $R_n \uparrow T \times S$ which is finer than any of these sequences, i.e. for each n and each k there is some j with $R_n \subseteq R_{k,j}$. Put $f_n(t, s, u) := \chi_{R_n}(t, s)f(t, s, u)$. Since R_n is increasing, we have $f_n \to f$ uniformly on each $R_k \times M$. Moreover, for each n and each k, we find some j such that the function $C_k f_n$ has its support in $T_j \times S_j$ and is bounded by j. In particular, for all measurable sets $D_m \subseteq S$ and $E_m \subseteq T$, we have

$$\int_{D_m} |C_k f_n(t, s, x(s))| \, ds \leq j \ \text{mes}\,(D_m \cap S_j)\chi_{T_j}(t),$$

and

$$|P_{E_m} A(|C_k f_n|)x(t)| \leq j \ \text{mes}\,(S_j) P_{E_m} \chi_{T_j}(t)$$

for each $x \in B$ for almost all $t \in T$. Since $\text{mes}\, S_j < \infty$ and since χ_{T_j} belongs to the regular part of Y, we conclude that $\gamma_S(C_k f_n, B) = \gamma_T(C_k f_n, B) = 0$. Moreover, for almost all (t, s), and fixed k and n, the family

$$\{C_k f_n(t, s, x(\cdot)) : x \in B\}$$

is uniformly dominated on each S_m by the integrable function $j\chi_{S_m}$, and so this family is bounded in $L_1(S_m, Y)$ with equicontinuous norm.

Hence, and in view of [**10**, Proposition 9.11] all conditions of the compactness result [**10**, Theorem 9.10] are satisfied for the Urysohn operator $A(C_k f_n)$. This result implies that

$$\chi_{Y_0}(A(C_k f_n)(B)) \le q^2 \sup_n \|I_n\|,$$

i.e., for each $\varepsilon > \sup_n \|I_n\|$, each k and each n, we find a finite ε-net $N_{k,n,\varepsilon} \subseteq Y_0$ for $A(C_k f_n)(B)$. By Theorem 1, we find some k and n with

$$\sup_{x\in B} \|A(C_k f_n)x - A(f)x\| \le q \min\{\gamma_S(f,B) + q\gamma_T(f,B), q\gamma_S(f,B) + \gamma_T(f,B)\},$$

and so $N_{k,n,\varepsilon} \subseteq Y_0$ is a finite $q(q \min\{\gamma_S(f,B) + q\gamma_T(f,B), q\gamma_S(f,B) + \gamma_T(f,B)\} + q^2 \sup_n \|I_n\|)$-net for $A(f)(B)$. □

Even if the reader should only be interested in the case of linear integral operators, it appears unavoidable to make use of the cutting operator. Thus, even if we start with a linear kernel, we end up with a nonlinear, although very simple, Urysohn operator in the proof. Let us formulate the special case of linear integral operators explicitly.

Thus, let U be a normed linear space, V a Banach space, and $\mathcal{B}(U,V)$ the space of all bounded linear operators from U into V, endowed with the uniform topology. Let S and T be σ-finite measure spaces, $k: T \times S \to \mathcal{B}(U,V)$, $M \subseteq U$ and $M_n := \{u \in M : |u| \le r_n\}$ with $r_n \uparrow \infty$. Let X be a (quasi-normed) pre-ideal space of functions $x: S \to U$ and Y a (quasi-normed) pre-ideal space of functions $y: T \to V$. Assume that $k(\cdot,\cdot)u$ is measurable for each $u \in U$. Let $B \subseteq X$ be bounded and such that each $x \in B$ attains almost all of its values in M. We consider the linear integral operators

$$Kx(t) = \int_S k(t,s)x(s)\,ds, \quad t \in T$$

$$|K|_0 x(t) = \int_S |k(t,s)x(s)|\,ds, \quad t \in T$$

and

$$|K|x(t) = \int_S k(t,s)x(s)\,ds, \quad t \in T.$$

Note that $|K|_0$ is a nonlinear operator and that $|K|_0x(t)$ is finite if and only if $Kx(t)$ is defined. If we consider scalar functions, we have of course $|K|_0x = |K||x|$, but in general $|K|_0x$ is smaller.

As previously remarked, condition (6) is automatically satisfied if we consider scalar functions. For non-scalar functions, we have to assume that also $|K||x|$ is finite:

Corollary 1. *In the above situation the following holds*:

1. $B \subseteq X$ *is measure bounded* (*with respect to* M_n).

2. *If* $|K|\chi_E$ *is almost everywhere finite for each* $\chi_E \in X$ *with* $\operatorname{mes} E < \infty$, *then condition* (6) *holds with* $g(t,s,u) := |k(t,s)||u|$ *for each* $B \subseteq X$.

This is in particular the case if $\dim U < \infty$ *and* Kx *is almost everywhere defined for each measurable* $x\colon S \to U$ *with* $|x(s)| = \chi_D(s) \in X$ *and* $\operatorname{mes} D < \infty$.

3. *Suppose in addition to the above assumption that* $|K|_0$ *sends* B *into the regular part* Y_0 *of* Y. *Assume also that* $k\colon T \times S \to \mathcal{B}(U,V)$ *is measurable* (*this is automatically the case if* $\dim U < \infty$).

Then $K\colon B \to Y_0$, *and if* $k(t,s)$ *is compact for almost all* $(t,s) \in T \times S$, *we have*

$$(17)\quad \chi_{Y_0}(K(B)) \leq q \min\{\gamma_S(k,B) + q\gamma_T(k,B), q\gamma_S(k,B) + \gamma_T(k,B)\}$$

where q *is the constant* (1) *for the space* Y, *and*

$$(18)\qquad \gamma_S(k,B) := \sup_{S \supseteq D_n \downarrow \varnothing} \limsup_{n\to\infty} \sup_{x \in B} \||K|_0 P_{D_n} x\|$$

$$(19)\qquad \gamma_T(k,B) := \sup_{T \supseteq E_n \downarrow \varnothing} \limsup_{n\to\infty} \sup_{x \in B} \|P_{E_n} |K|_0 x\|.$$

Proof. 1. Since convergence in X implies convergence in the normalized measure by [**10**, Theorem 3.4], we obtain from [**10**, Corollary 9.6] that each bounded $B \subseteq X$ is measure bounded.

2. We may assume that $\operatorname{supp} X = S$. By [**8**, Corollary 2.2.7], we find a sequence $S_n \uparrow S$ with $\chi_{S_n} \in X$ and $\operatorname{mes} S_n < \infty$, i.e., for almost all t the value $|K|\chi_{S_n}(t)$ is finite, i.e., for almost all $t \in T$ the

function $|k(t,\cdot)|$ is integrable over S_n. By [**10**, Proposition 9.13], and the subsequent remark (9.11), it follows that (6) holds.

If $\dim U < \infty$ and Kx is almost everywhere defined for each measurable x with $|x| = \chi_E$, then it follows from [**2**] that $|K|\chi_E$ is almost everywhere finite.

3. Put for a moment $M := U$ and $M_n := \{u \in U : |u| < n\}$. Then the Carathéodory function $(t,s,u) \mapsto k(t,s,u)$ is a strict B-function if and only if $k: T \times S \to \mathcal{B}(U,V) \subseteq \mathcal{B}(M,V)$ is measurable. If $\dim U < \infty$, then this is automatically the case by [**10**, Theorem 8.15], since U is locally compact and separable. The claim now follows from Theorem 2 with $g(t,s,u) = k(t,s)u$ and $r_n(t,s) \equiv 0$: The quantities (18) and (19) correspond to (4) and (5), respectively. □

We point out that the estimate (17) is stronger than the related result [**2**]. We emphasize in this connection also that it is not necessary for Corollary 1 that B is the full unit ball of X. As sketched in [**2**], this observation is particularly useful for Hammerstein operators

$$Hx(t) = \int_S k(t,s) f(s,x(s))\,ds$$

which can be written in the form $H = KF$ with the superposition operator $Fx(s) := f(s,x(s))$: Under mild assumptions on f, one obtains usually much better estimates for (18) and (19) on a bounded set of the form $F(B)$ than on the full (multiple of the) unit ball of X.

We close with a very simple application of Theorem 2.

Example 1. Let $S := T := [0,1]$, and let $f: S \times T \times \mathbf{R}^N \to \mathbf{R}^M$ be a Carathéodory function satisfying a linear growth condition of the form

$$|f(t,s,u)| \le \left(\frac{C_1}{t} + C_2\right)|u|, \quad 0 \le s \le t \le 1$$

with fixed numbers $C_1, C_2 \ge 0$. Then, for the unit ball B in the space $X := Y := L_p([0,1])$, $1 < p < \infty$, the Volterra-Urysohn operator

$$Ax(t) := \int_0^t f(t,s,x(s))\,ds, \quad t \in [0,1], \tag{20}$$

satisfies

$$\chi_Y(A(B)) \le \frac{2C_1 p}{p-1}. \tag{21}$$

In fact (6) holds, because for almost all t Hölder's inequality and the dominated convergence theorem imply

$$\begin{aligned} \sup_{x\in B} \int_0^t \chi_{Q_n}(t,s)|f(t,s,x(s))|\, ds \\ \le \left(\frac{C_1}{t} + C_2\right) \left\| \chi_{Q_n}(t,\cdot) \right\|_{L_{p/(p-1)}} \longrightarrow 0, \quad Q_n \downarrow \varnothing. \end{aligned}$$

Moreover, $\gamma_S(f,B)$ and $\gamma_T(f,B)$ are both at most $C_1 p/(p-1)$, because the linear integral operator

$$Kx(t) := \frac{C_1}{t} \int_0^t x(s)\, ds \quad t \in [0,1] \tag{22}$$

satisfies $\|Kx\|_Y \le C_1 p/(p-1)$ for $x \in B$ by Hardy's inequality, and so

$$\begin{aligned} \sup_{x\in B} \left\| \int_{D_n} |f(\cdot,s,x(s))|\, ds \right\|_Y &\le \sup_{x\in B} (\|Kx\|_Y + C_2 \text{mes}\,(D_n)) \\ &\longrightarrow \frac{C_1 p}{p-1}, \quad D_n \downarrow \varnothing \end{aligned}$$

(without loss of generality, we assume $f(t,s,u) = 0$ for $s > t$) and similarly

$$\begin{aligned} \sup_{x\in B} \left\| P_{E_n} \int_0^1 |f(\cdot,s,x(s))\, ds \right\|_Y &\le \sup_{x\in B} (\|P_{E_n} Kx\|_Y + C_2 \text{mes}\,(E_n)^{1/p}) \\ &\longrightarrow \frac{C_1 p}{p-1}, \quad E_n \downarrow \varnothing. \end{aligned}$$

Note that, since the well-known Hardy operator (22) is not compact, also (20) is not compact in general. Hence, the classical compactness results cannot be applied in Example 1 (not even indirectly, because in general A is *not* just a compact perturbation of some multiple of the Hardy operator). For the Hardy operator and, more general, when

$C_2 = 0$, one can obtain a better, by the factor 2, estimate than (21) in the trivial way by estimating $\|Ax\|$, i.e., by considering the finite net $\{0\} \subseteq Y$. However, for large C_2, the estimate (21) obtained by Theorem 2 is better, of course.

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