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NONSTATIONARY WAVEFORM RELAXATION METHODS FOR ABEL INTEGRAL EQUATIONS

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ABSTRACT. In this paper the nonstationary waveform relaxation methods for Abel equations are introduced and their convergence analysis is performed. Then the fully parallel waveform relaxation methods are especially considered. Nonstationary Richardson methods are constructed in such a way to optimize the convergence rate, and a significant error estimate is proved.

1. Introduction. Large systems of Volterra integral equations with weakly singular kernels (of Abel type)

$$\begin{gathered} y(t) = f(t) + \int_0^t \frac{k(t,s,y(s))}{(t-s)^\alpha}\, ds \\ t \in [0,T], \quad 0 < \alpha < 1, \\ y, f, k \in R^d, \quad d \gg 1, \end{gathered} \tag{1.1}$$

arise in many branches of applications such as, for example, reaction-diffusion problems in small cells [**14**] as well as by the semi-discretization in space of Abel partial integral or integro-differential equations.

In order to get accurate solutions of these systems in a reasonable time frame, high performance numerical methods are required.

Methods of this kind are the waveform relaxation methods that have been recently developed by some of the authors for systems of Volterra equations both with regular kernels [**6, 8, 10–12, 13**] and with weakly singular kernel [**4**].

The waveform relaxation methods (WR methods) for the system (1.1) are introduced using a suitable function $\mathcal{G} = \mathcal{G}(t,s,u,v)$ such that

$$\mathcal{G}(t,s,u,u) = k(t,s,u). \tag{1.2}$$

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The *waveforms* $\{y^{(\nu)}(t)\}_{\nu\in N}$ are then obtained from

$$\begin{aligned} y^{(\nu+1)}(t) &= f(t) + \int_0^t \frac{\mathcal{G}(t,s,y^{(\nu)}(s),y^{(\nu+1)}(s))\,ds}{(t-s)^\alpha}\,ds \\ & t\in[0,T], \quad \nu = 0,1,\dots \\ & y^{(0)}(t) = f(t). \end{aligned} \tag{1.3}$$

Obviously, if the sequence $\{y^{(\nu)}(t)\}_{\nu\in N}$ is convergent, its limit is the solution of (1.1).

The convergence properties as well as the computational cost of each waveform depend heavily on the choice of the function $\mathcal{G}$. In particular, if the function $\mathcal{G}$ is such that the system (1.3) is decoupled into independent subsystems that can be solved in parallel, the corresponding WR method is a parallel method.

A fully parallel WR method is the Richardson method, which corresponds to

$$\mathcal{G}(t,s,u,v) = \mu\mathbf{I}u - \mu\mathbf{I}v + k(t,s,v) \tag{1.4}$$

where μ is a suitable chosen parameter and $\mathbf{I}$ is the identity matrix of order d.

If $\mu = 0$, then the Picard method is obtained. The Richardson method is then

$$\begin{aligned} y^{(\nu+1)}(t) = f(t) &+ \int_0^t \frac{\mu\mathbf{I}y^{(\nu+1)}(s)\,ds}{(t-s)^\alpha}\,ds \\ &+ \int_0^t \frac{[k(t,s,y^{(\nu)}(s)) - \mu\mathbf{I}y^{(\nu)}(s)]\,ds}{(t-s)^\alpha}\,ds \end{aligned} \tag{1.5}$$

and so the system of d equations is decoupled into d independent equations.

However, fully parallel methods are usually slowly convergent. Moreover, for the Abel equation, the presence of the singularity slows down the convergence rate [**4**]. Therefore, a crucial question is to develop fast convergent fully parallel methods.

With this aim, in this paper we introduce the nonstationary WR methods (NSWR methods), where the function $\mathcal{G}$ depends on the

iterate. Then we consider a sequence of functions $\{\mathcal{G}_\nu(t,s,u,v)\}_{\nu\in N}$ such that

$$\mathcal{G}_\nu(t,s,u,u) = k(t,s,u) \quad \text{for all } \nu. \tag{1.6}$$

The corresponding NSWR method is

$$y^{(\nu+1)}(t) = f(t) + \int_0^t \frac{\mathcal{G}_\nu(t,s,y^{(\nu)}(s),y^{(\nu+1)}(s))\,ds}{(t-s)^\alpha}\,ds. \tag{1.7}$$

We perform the convergence analysis of the NSWR methods. Firstly, we prove a local convergence result for a general nonlinear kernel under the hypothesis that $\mathcal{G}_\nu$ satisfies a Lipschitz type condition. Then we prove that, if the kernel is linear with respect to y and $\mathcal{G}_\nu$ is linear with respect to u and v, the convergence of the NSWR methods takes place in every finite integration interval, and it is superlinear.

The proved results generalize those obtained in [**4**] for the stationary methods. Then the fully parallel nonstationary WR Richardson methods are especially considered. The convergence analysis of these methods enable us to construct "fast" Richardson methods, in the sense that they optimize the convergence rate.

Finally, a significant error bound for these fast Richardson methods is proved. This allows us to predict the number of iterations required to obtain the desired precision and to determine the class of problems for which the method is more suitable.

2. Convergence analysis of nonstationary WR methods. In this section we perform the convergence analysis of the NSWR methods. In particular, we prove first a local convergence result, true under very general hypotheses. Then we consider the linear case and prove a superlinear convergence in every finite integration range.

To be more precise, let us consider the Banach space of the continuous vector-valued functions, defined in $[0,T]$, equipped with the maximum norm

$$\|y\|_T := \max_{0\le t\le T} \|y(t)\|$$

where $\|\cdot\|$ denotes any of the usual vector norms in R^d. Let us consider the NSWR method (1.7) with the function $\mathcal{G}_\nu(t,s,u,v)$ satisfying the

Lipschitz condition:

(2.1)
$$\|\mathcal{G}_\nu(t,s,u_1,v_1)-\mathcal{G}_\nu(t,s,u_2,v_2)\| \le l_1^{(\nu)}\|u_1-u_2\| + l_2^{(\nu)}\|v_1-v_2\|$$
$$\forall\, u_1,u_2,v_1,v_2 \in R^d;\ (t,s)\in S := \{(t,s) : 0\le s\le t\le T\}$$

Then the following local convergence theorem holds:

Theorem 2.1. *If the functions G_ν satisfy* (2.1) *with $l_1^{(\nu)}$ and $l_2^{(\nu)}$ uniformly bounded with respect to ν, there exists $T_1 > 0$ such that the* NSWR *method converges in* $[0,T_1]$.

Proof. Subtracting (1.7) from the Abel system (1.1), remembering that
$$\mathcal{G}_\nu(t,s,y,y) = k(t,s,y)$$
and, using (2.1), it follows:
$$\|y(t)-y^{(\nu+1)}(t)\|_{T_1} \le \frac{l_1^{(\nu)}T_1^{1-\alpha}}{1-\alpha-l_2^{(\nu)}T_1^{1-\alpha}}\,\|y(t)-y^{(\nu)}(t)\|_{T_1}.$$
Then, putting $l_1 = \sup_\nu l_1^{(\nu)}$, $l_2 = \sup_\nu l_2^{(\nu)}$, the method converges in $[0,T_1]$ with
$$0 < T_1 < \left(\frac{1-\alpha}{l_1+l_2}\right)^{1/(1-\alpha)}. \qquad \square$$

Now let us consider a linear Volterra system

(2.2)
$$y(t) = f(t) + \int_0^t \frac{k(t,s)}{(t-s)^\alpha}\cdot y(s)\,ds, \quad t\in[0,T],$$

and let us choose the functions $\mathcal{G}_\nu$ preserving the linearity, i.e.,

(2.3)
$$\mathcal{G}_\nu(t,s,u,v) = M_\nu(t,s)v + N_\nu(t,s)u.$$

Obviously, the condition (1.6) becomes
$$M_\nu(t,s) + N_\nu(t,s) = k(t,s)$$

and $M_\nu(t,s)$, $N_\nu(t,s)$ are said to be a splitting of the kernel $k(t,s)$.

The corresponding NSWR method is

$$\begin{aligned} y^{(\nu+1)}(t) &= f(t) + \int_0^t \frac{M_\nu(t,s)}{(t-s)^\alpha} \cdot y^{(\nu+1)}(s)\, ds \\ &\quad + \int_0^t \frac{N_\nu(t,s)}{(t-s)^\alpha} \cdot y^{(\nu)}(s)\, ds, \quad t \in [0,T]. \end{aligned} \tag{2.4}$$

Let $(t-s)^{-\alpha}\mathcal{Q}_\nu(t,s)$ be the resolvent of the kernel $(t-s)^{-\alpha}M_\nu(t,s)$ and let us put

$$\overline{\mathcal{Q}}_\nu := \max_{(t,s)\in S} |\mathcal{Q}_\nu(t,s)|$$
$$\overline{N}_\nu := \max_{(t,s)\in S} |N_\nu(t,s)|.$$

Then the following convergence result holds.

Theorem 2.2. *If $\mathcal{Q}_\nu$ and N_ν are uniformly bounded with respect to ν, then the method* (2.4) *converges superlinearly in* $[0,T]$.

Proof. Let us define the νth iterate error

$$e^{(\nu)}(t) := y(t) - y^{(\nu)}(t).$$

Obviously $e^{(\nu+1)}(t)$ and $e^{(\nu)}(t)$ are related by the following expression

$$\begin{aligned} e^{(\nu+1)}(t) &= \int_0^t \frac{N_\nu(t,s)}{(t-s)^\alpha} \cdot e^{(\nu)}(s)\, ds \\ &\quad + \int_0^t \frac{\mathcal{Q}_\nu(t,s)}{(t-s)^\alpha} \int_0^s \frac{N_\nu(s,\tau)}{(s-\tau)^\alpha} \cdot e^{(\nu)}(\tau)\, d\tau\, ds. \end{aligned}$$

By applying the Dirichlet formula and putting

$$\begin{aligned} H_\nu(t,s) &= N_\nu(t,s) + (t-s)^{1-\alpha} \\ &\quad \cdot \int_0^1 \frac{\mathcal{Q}_\nu(t,(t-s)u+s) N_\nu((t-s)u+s,s)}{u^\alpha (1-u)^\alpha}\, du \end{aligned}$$

it follows

$$e^{(\nu+1)}(t) = \int_0^t \frac{H_\nu(t,s)}{(t-s)^\alpha} \cdot e^{(\nu)}(s)\, ds. \tag{2.5}$$

On the other hand,

$$\max_{(t,s)\in S} |H_\nu(t,s)| \leq \overline{N}_\nu + T^{1-\alpha}\,\frac{\Gamma(1-\alpha)^2\overline{\mathcal{Q}}_\nu\overline{N}_\nu}{\Gamma|2(1-\alpha)|}.$$

From the hypothesis of uniform boundedness of $\overline{\mathcal{Q}}_\nu$ and $\overline{N}_\nu$, it follows that there exists B such that

$$\max_{(t,s)\in S} |H_\nu(t,s)| \leq B$$

uniformly with respect to ν. So

$$\|e^{(\nu+1)}\|_T \leq \frac{B^{\nu+1}T^{(\nu+1)(1-\alpha)}\Gamma(1-\alpha)^{\nu+1}}{\Gamma[(\nu+1)(1-\alpha)+1]}\,\|\,e^{(0)}\|_T.$$

By using the asymptotic formulas of the gamma function [**1**],

$$\Gamma(\nu(1-\alpha)+1) \sim \sqrt{2\pi}\,e^{-\nu(1-\alpha)}(\nu(1-\alpha))^{\nu(1-\alpha)+(1/2)}$$

it results

$$\lim_{\nu\to\infty} \frac{\|e^{(\nu+1)}\|_T}{\|e^{(\nu)}\|_T} = 0$$

and so the thesis follows. □

Remark 2.1. In the stationary case the theorems (2.1) and (2.2) reduce to the analogous theorems proved in [**4**].

3. Fully parallel NSWR methods. Fully parallel NSWR methods are obtained choosing $M_\nu(t,s)$ as a diagonal matrix. In this case the system (1.7) of order d is decoupled into d independent equations that can be solved in parallel. In particular, let us consider the NSWR Richardson methods which corresponds to

$$M_\nu(t,s) = \mu_\nu \mathbf{I}. \tag{3.1}$$

In order to construct the fast NSWR Richardson methods, let us apply the method to the Abel equation with constant kernel

$$y(t) = f(t) + \int_0^t \frac{A\cdot y(s)}{(t-s)^\alpha}\,ds, \quad t\in[0,T], \quad A\in R^{d\times d} \tag{3.2}$$

where A has eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_d < 0$, so that the stability of the equation is ensured [**5**]. Then the NSWR Richardson method is

$$
\begin{aligned}
y^{(\nu+1)}(t) = f(t) &+ \int_0^t \frac{\mu_\nu I}{(t-s)^\alpha} \cdot y^{(\nu+1)}(s)\, ds \\
&+ \int_0^t \frac{[A - \mu_\nu I]}{(t-s)^\alpha} \cdot y^{(\nu)}(s)\, ds.
\end{aligned} \tag{3.3}
$$

In order to ensure the stability of the equation (3.3), we assume $\mu_\nu < 0$, $\nu = 0, 1, \ldots$. Let us define the following polynomial matrix

$$
P_\nu(A) = \prod_{k=1}^{\nu} (A - \mu_k \mathbf{I}) \tag{3.4}
$$

and prove the following

Theorem 3.1. *The* NSWR *Richardson method* (3.3) *converges to the solution of the problem* (3.2) *and the following error bound holds*

$$
\|\varepsilon^{(\nu)}\|_T \le \|P_\nu(A)\| \cdot \frac{T^{\nu(1-\alpha)}\Gamma(1-\alpha)^\nu}{\Gamma[\nu(1-\alpha)+1]}\, \|\varepsilon^{(0)}\|_T. \tag{3.5}
$$

Proof. In this case the function $H_\nu(t,s)$ in (2.5) becomes

$$
\begin{aligned}
H_\nu(t,s) = (A - \mu_\nu I) \Bigg[& 1 + (t-s)^{1-\alpha} \\
& \cdot \int_0^1 \frac{\mu_\nu \Gamma(1-\alpha) E_{1-\alpha,1-\alpha}[(t-s)^{1-\alpha}(1-u)^{1-\alpha}\mu_\nu\Gamma(1-\alpha)]}{u^\alpha \cdot (1-u)^\alpha}\, du \Bigg]
\end{aligned}
$$

where $E_{1-\alpha,1-\alpha}$ is the generalized Mittag-Leffler function [**16**].

By substituting $u \to 1-u$, $\beta \to 1-\alpha$ and putting $\mathcal{M} := (t-s)^{1-\alpha}\mu_\nu\Gamma(1-\alpha)$, we can rewrite

$$
H_\nu(t,s) = (A - \mu_\nu I) \left[1 + \mathcal{M} \int_0^1 u^\beta \cdot (1-u)^\beta E_{\beta,\beta}[\mathcal{M}u^\beta]\, du \right].
$$

Then, by using a known equality for this integral [**20**], we obtain

$$H_\nu(t,s) = 1 + \mathcal{M} \cdot \Gamma(\beta) \cdot E_{\beta,2\beta}[\mathcal{M}],$$

and, by $E_{\alpha,\beta}(z) = 1/(\Gamma(\beta)) + zE_{\alpha,\alpha+\beta}(z)$ [**15**], we have

$$H_\nu(t,s) = 1 + \mathcal{M} \cdot \Gamma(\beta) \cdot \left[\frac{E_{\beta,\beta}[\mathcal{M}] - (1/\Gamma(\beta))}{\mathcal{M}}\right]$$

and from here,

$$H_\nu(t,s) = \Gamma(\beta) \cdot E_{\beta\beta}[\mathcal{M}].$$

So we obtain

$$\varepsilon^{(\nu+1)}(t) = [A - \mu_\nu I] \cdot \int_0^t \frac{\Gamma(1-\alpha)}{(t-s)^\alpha} E_{1-\alpha,1-\alpha}\left[\mu_\nu \Gamma(1-\alpha)(t-s)^{1-\alpha}\right] \varepsilon^{(\nu)}(s)\, ds.$$

Now, because $\mu_\nu < 0$, $E_{1-\alpha,1-\alpha}(0) = 1/(\Gamma[1-\alpha])$ and $E_{1-\alpha,1-\alpha}(-t)$ is completely monotone for $t > 0$ [**17**], we iterate on ν and for any vector norm, we have

$$\|\varepsilon^{(\nu)}(t)\| \le \|P_\nu(A)\| \cdot \frac{t^{\nu\cdot(1-\alpha)}\Gamma(1-\alpha)^\nu}{\Gamma[\nu(1-\alpha)+1]} \|\varepsilon^{(0)}(t)\|. \tag{3.6}$$

Then, if we take the infinity norm on $[0,T]$, it follows the thesis.

4. Fast convergent nonstationary Richardson methods. On the basis of the results proved in the previous section, we can construct the fast NSWR Richardson methods in the sense that they optimize the convergence rate. These methods are derived choosing the parameters μ_ν which minimize $\|P_\nu(A)\|$ in (3.6).

Firstly, for the sake of completeness, we prove the following result on the stationary Richardson method (1.5).

Theorem 4.1. *The best stationary Richardson method is obtained choosing μ as the mean value between the minimum and maximum eigenvalue of A.*

Proof. In this case $P_\nu(A) = (A - \mu\mathbf{I})^\nu$ and the thesis follows trivially observing that the eigenvalues of $P_\nu(A)$ are $(\lambda_i - \mu)^\nu$. □

The spectral radius of $P_\nu(A)$ holds the bound

$$\rho(P_\nu(A)) \leq \left[\frac{\lambda_d - \lambda_1}{2}\right]^\nu.$$

Remembering that, for every $\varepsilon > 0$, there exists a norm such that

$$\|P_\nu(A)\| \leq \rho[P_\nu(A)] + \varepsilon \tag{4.1}$$

and, moreover, that the norms of matrices are equivalent, we can conclude that there exists a constant $\gamma > 0$ such that

$$\|P_\nu(A)\| \leq \gamma\left[\left(\frac{\lambda_d - \lambda_1}{2}\right)^\nu + \varepsilon\right]. \tag{4.2}$$

Therefore, the error bound proved in Theorem 3.1 becomes

$$\|e^{(\nu)}\| \leq \gamma\left[\left(\frac{\lambda_d - \lambda_1}{2}\right)^\nu + \varepsilon\right] \frac{\Gamma^{\nu(1-\alpha)}\Gamma(1-\alpha)^\nu}{\Gamma[\nu(1-\alpha)+1]} \|e^{(0)}\|. \tag{4.3}$$

Now, let us dedicate to the NSWR methods. Firstly, we prove the following

Theorem 4.2. *If μ_i are the eigenvalues of A, the corresponding NSWR Richardson method gives the exact solution in a number of iterates equal to the dimension d of the system* (3.2).

Proof. From the Caley-Hamilton theorem, A is the solution of its characteristic polynomial. Therefore, if $\lambda_i = \mu_i$, then $P_d(A) = 0$ and $\|e^{(d)}\| = 0$. □

We observe that, if the dimension d of the system of integral equations is large, to perform d iterates can be too expensive, whereas an acceptable precision could be achieved by performing a number n of iterates substantially lower than d.

Then, let us fix the number of iterates n and look for a polynomial $P_n(x)$ which minimizes $\|P_n(A)\|$.

In order to perform the effective construction of this polynomial, let us proceed as follows. Firstly, on the basis of (4.1), we can minimize

$$\rho[P_\nu(A)] = \max_{1\le i\le d} |P_n(\lambda_i)|.$$

Then, let us define the virtual spectral radius [**18**] as

$$\bar\rho[P_\nu(A)] := \max_{\lambda_1\le\lambda\le\lambda_d} |P_n(\lambda)| \tag{4.4}$$

and look for the polynomial which minimizes it.

The following theorem holds

Theorem 4.3. *For every fixed n, the fast* NSWR *Richardson method is obtained choosing μ_i as the zeros of the Chebyshev polynomial shifted in the range $[\lambda_1, \lambda_d]$.*

Proof. The result follows by observing that, for the known minimax properties, the Chebyshev polynomial minimizes the virtual spectral radius (4.4). □

Denoting by $T_n(x)$ the Chebyshev polynomial defined in $[-1, 1]$, we obtain the results

$$P_n(x) = (-1)^n T_n\left(\frac{2x-\lambda_d-\lambda_1}{\lambda_d-\lambda_1}\right)\frac{[\lambda_d-\lambda_1]^n}{2^{2n-1}}. \tag{4.5}$$

Therefore,

$$\rho[P_\nu(A)] \le \bar\rho[P_\nu(A)] \le \frac{[\lambda_d-\lambda_1]^n}{2^{2n-1}} \tag{4.6}$$

and the error bound proved in Theorem 3.1 becomes

$$\|e^{(\nu)}\|_T \le \gamma'\left[(\lambda_d-\lambda_1)^\nu\cdot\frac{1}{2^{2\nu-1}}+\varepsilon\right]\frac{T^{\nu(1-\alpha)}\Gamma(1-\alpha)^\nu}{\Gamma[\nu(1-\alpha)+1]}\,\|e^{(0)}\|_T. \tag{4.7}$$

On these methods the following remarks can be made:

Remark 4.1. The construction of the fast NSWR Richardson methods doesn't require the knowledge of the whole spectrum of A, but only of the maximum and minimum eigenvalues.

Remark 4.2. The error estimate (4.7) is especially useful since it can be used both to bound the error for a fixed n, and to determine the number n of iterates necessary to achieve a desired precision.

Remark 4.3. The construction of these methods can be performed also when A has complex eigenvalues. In this case if the eigenvalues of A are known to lie in the ellipse

$$\left[\frac{x-r}{a}\right]^2 + \left[\frac{y}{b}\right]^2 = 1,$$

the μ_i are the zeros of the polynomial

$$T_n\left(\frac{z-r}{a^2-b^2}\right).$$

Remark 4.4. The fast NSWR Richardson methods have been derived for the system (3.2) with constant kernel. For the general linear system (2.2) "quite fast" NSWR methods of Richardson type can be obtained by approximating the kernel $k(t,s)$ with the first term of its Taylor development, i.e., $k(0,0)$ and then by choosing the parameters μ_i as the zeros of the Chebyshev polynomial shifted in $[\bar{\lambda}_1, \bar{\lambda}_d]$, where $\bar{\lambda}_1, \bar{\lambda}_d$ are the minimum and maximum eigenvalue of $k(0,0)$. The methods derived in this way have good performances if the integration range is small enough. Otherwise, it is possible to subdivide the integration range into subintervals, the said windows, and construct the methods in each of the windows.

5. Concluding remarks. We have introduced the nonstationary WR methods and proved that they can reach convergence rates bigger than the classical methods. We have derived the fully parallel Richardson methods with fast convergence properties and proved a useful error

estimate, that can be used also to determine a priori the number of iterates necessary to obtain the desired accuracy.

In a forthcoming paper, the nonstationary discrete time WR methods will be analyzed and strategies in order to develop an efficient parallel algorithm will be discussed [**9**].

A parallel code based on these methods on a distributed memory architecture is under construction.

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