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DISCONTINUOUS BOUNDARY-VALUE PROBLEMS: EXPANSION AND SAMPLING THEOREMS

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ABSTRACT. This paper is devoted to the derivation of expansion and sampling theorems associated with nth order discontinuous eigenvalue problems defined on $[-1,1]$, illustrated with detailed examples. The problem consists of nth order differential expressions and n boundary and n compatibility conditions at $x=0$. The differential expressions are defined, in general, in two different ways throughout $[-1,1]$. We derive an eigenfunction expansion theorem for the Green's function of the problem as well as a theorem of uniform convergence of the Birkhoff series of a certain class of functions. Then we derive a sampling theorem for integral transforms whose kernels are the product of the Green's function and the characteristic determinant of the problem.

1. Introduction. In [**24**] a sampling theorem associated with the discontinuous Sturm-Liouville problem

$$l^{(2)}y := -y'' + q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.1}$$

$$hy(0) - y'(0) = hy(\pi) + y'(\pi) = 0, \tag{1.2}$$

with two symmetric discontinuities at $d_1 = d$, $0 < d < \pi/2$ and $d_2 := \pi - d$ is studied, where the following jump conditions are satisfied

$$y(d_1^+) = ay(d_1^-), \quad y'(d_1^+) = a^{-1}y'(d_1^-) + by(d_1^-), \tag{1.3}$$

$$y(d_2^-) = ay(d_2^-), \quad y'(d_2^-) = a^{-1}y'(d_2^+) - by(d_2^+). \tag{1.4}$$

Here h, a, b are real numbers with $a > 0$ and $q(\cdot)$ is an $L^1(0,\pi)$-real valued function. The eigenfunction expansion theorem associated with

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the above problem is derived by Kobayashi in [**13**]. In brief, Zayed-García's result is derived as follows. Let $\phi_0(\cdot,\lambda)$ be a solution of (1.1) that satisfies the jump conditions (1.3)–(1.4) and the initial condition

$$\phi_0(0,\lambda)-\phi_0'(0,\lambda)=0. \tag{1.5}$$

Then the eigenvalues of problem (1.1)–(1.4) are the zeros of the entire function

$$\omega(\lambda)=h\phi_0(\pi,\lambda)+\phi_0'(\pi,\lambda). \tag{1.6}$$

The eigenvalues constitute a sequence $\{\lambda_k\}_{k=0}^{\infty}$ with ∞ as the only accumulation point. According to the asymptotic behavior of these eigenvalues, the product

$$p_0(\lambda)=\begin{cases}\prod_{k=0}^{\infty}\left(1-(\lambda/\lambda_k)\right), & \text{if zero is not an eigenvalue,}\\ \lambda\prod_{k=1}^{\infty}\left(1-(\lambda/\lambda_k)\right), & \text{if } \lambda_0=0 \text{ is an eigenvalue,}\end{cases} \tag{1.7}$$

is convergent and defines an entire function of order 1/2 with simple zeros at λ_k, $k=0,1,\dots$. The main result of [**24**] states that the transform

$$F_0(\lambda)=\int_0^{\pi} f(x)\phi_0(x,\lambda)\,dx, \quad f\in L^2(0,\pi), \tag{1.8}$$

can be recovered using the sampling expansion

$$F_0(\lambda)=\sum_{k=0}^{\infty}F_0(\lambda_k)\,\frac{p_0(\lambda)}{(\lambda-\lambda_k)p_0'(\lambda_k)}, \quad \lambda\in\mathbf{C} \tag{1.9}$$

with uniform convergence on compact sets of $\mathbf{C}$.

During the sampling meeting SAMPTA '97 held in Aveiro, Portugal, 1997, June 16–19, after Zayed introduced the above mentioned result, Walter asked about the possibility of deriving a sampling theorem associated with the Green's function of problem (1.1)–(1.4). The aim of this paper is to derive a sampling theorem for transforms whose kernels are Green's functions of problems that are more general than problem (1.1)–(1.4) but still with only one point of discontinuity. In this class of problems the equation $(l^{(2)}-\lambda)y=0$ is replaced by an equation of the form $(l-\lambda)y=0$, where l is defined by two nth order differential expressions, $l = l_1$ on $[-1, 0)$ and $l = l_2$ on $(0, 1]$. Another generalization concerning the differential expression is that l is not necessarily assumed to be self-adjoint. In another direction we generalize the problem by assuming that the boundary conditions are not necessarily of separate type as conditions (1.2) above, neither are they self adjoint. To guarantee that the eigenfunctions are a Riesz basis, [**20**], we assume that the boundary conditions are strongly regular. The definition of strongly regular boundary conditions is given below.

In Section 2 we give a brief account of the role of Green's functions in sampling theory. Section 3 is devoted to defining the eigenvalue problem and some of its properties. The properties of the eigenvalues and eigenfunctions stated in this section are taken from [**15**]. In Section 4 we introduce the Green's function and then derive an expansion theorem associated with the eigenvalue problem and the sampling theorem. In the last section we give two examples exhibiting the sampling theorem.

2. Sampling and Green's function. The use of Green's functions in sampling theory was first considered by Haddad, Yao and Thomas, [**11**]. They used Green's function of first order problems to derive the celebrated sampling theorem of Whittaker and Shannon [**7, 17, 19**]. The second appearance of Green's functions in sampling theorems, as far as we know, was in [**22, 24**] followed by [**4**]. In [**4**] the authors derived sampling theorems associated with the nth order eigenvalue problem consisting of the differential equation

$$\sum_{k=0}^{n} p_k(x)\, y^{(n-k)}(x) = \lambda y(x), \tag{2.1}$$

and the strongly regular conditions

$$N_\mu(y) = \sum_{i=1}^{n} \alpha_{\mu i} y^{(i-1)}(a) + \beta_{\mu i} y^{(i-1)}(b) = 0, \quad \mu = 1, \dots, n, \tag{2.2}$$

where x lies in a closed finite interval $[a, b]$. If we let $\varphi_1(\cdot, \lambda), \dots, \varphi_n(\cdot, \lambda)$ be the fundamental set (FS) of solutions of (2.1) such that

$$\varphi_i^{(j-1)}(a, \lambda) = \delta_{ij} \quad \text{for all} \quad \lambda \in \mathbf{C}, \quad 1 \le i, \quad j \le n, \tag{2.3}$$

then the Green's function of (2.1)–(2.2) will have the form, see [**9, 16**]

$$K(x,\xi,\lambda)=\frac{h(x,\xi,\lambda)}{D_0(\lambda)},\quad \text{for } D_0(\lambda)\neq 0, \tag{2.4}$$

where

$$D_0(\lambda)=\begin{vmatrix} N_1(\varphi_1) & \dots & N_1(\varphi_n)\\ \vdots & & \vdots\\ N_n(\varphi_1) & \dots & N_n(\varphi_n)\end{vmatrix}; \tag{2.5}$$

$$h(x,\xi,\lambda)=\begin{vmatrix} \varphi_1(x,\lambda) & \dots & \varphi_n(x,\lambda) & g(x,\xi,\lambda)\\ N_1(\varphi_1) & \dots & N_1(\varphi_n) & N_1(g)\\ \vdots & \vdots & \vdots & \vdots\\ N_n(\varphi_1) & \dots & N_n(\varphi_n) & N_n(g)\end{vmatrix}, \tag{2.6}$$

$$g(x,\xi,\lambda)=\frac{\pm 1}{2W_0(\xi)}\begin{vmatrix} \varphi_1(x,\lambda) & \dots & \varphi_n(x,\lambda)\\ \varphi_1^{(n-2)}(\xi,\lambda) & \dots & \varphi_n^{(n-2)}(\xi,\lambda)\\ \vdots & \vdots & \vdots\\ \varphi_1(\xi,\lambda) & \dots & \varphi_n(\xi,\lambda)\end{vmatrix}. \tag{2.7}$$

Here the sign is positive if $x>\xi$ and is negative if $x<\xi$ and $W_0(\xi)$ is the Wronskian of $\varphi_1,\dots,\varphi_n$. In this setting, strong regularity implies that the eigenfunctions of the problem and its adjoint are Riesz bases [**5, 14**]. Moreover, since almost all eigenvalues are simple, it is assumed, without any loss of generality, that all eigenvalues are simple (geometrically and algebraically). Thus $K(x,\xi,\lambda)$ has simple poles only at the eigenvalues $\{\mu_k\}_{k=0}^{\infty}$. Let $p(\lambda)$, $\lambda\in\mathbf{C}$ be

$$p(\lambda):=\begin{cases}\prod_{k=0}^{\infty}\left(1-(\lambda/\mu_k)\right)e^{\lambda/\mu_k}, & \text{if } n=1,\\ \prod_{k=0}^{\infty}\left(1-(\lambda/\mu_k)\right), & \text{if } n>1,\end{cases} \tag{2.8}$$

if zero is not an eigenvalue and

$$p(\lambda):=\begin{cases}\lambda\prod_{k=1}^{\infty}\left(1-(\lambda/\mu_k)\right)e^{\lambda/\mu_k}, & \text{if } n=1,\\ \lambda\prod_{k=1}^{\infty}\left(1-(\lambda/\mu_k)\right), & \text{if } n>1,\end{cases}. \tag{2.9}$$

if $\mu_0 = 0$ is an eigenvalue. Due to the behavior of the eigenvalues, [**16**], $p(\lambda)$ is an entire function of order $1/n$ with simple zeros at the eigenvalues. Fix ξ_0 in $[a, b]$, then

$$\phi(x, \lambda) = p(\lambda) K(x, \xi_0, \lambda) \tag{2.10}$$

is an entire function of λ. The main result of [**4**] states that for $f \in L^2(a, b)$,

$$F(\lambda) = \int_a^b \overline{f}(x) \phi(x, \lambda)\, dx, \quad \lambda \in \mathbf{C} \tag{2.11}$$

is an entire function of order $1/n$ and type η, $0 \leq \eta \leq b - a$ which may be recovered via

$$F(\lambda) = \sum_{k=0}^{\infty} F(\mu_k) \frac{p(\lambda)}{(\lambda - \mu_k) p'(\mu_k)}, \tag{2.12}$$

where the convergence is uniform on compact subsets of $\mathbf{C}$.

The choice of ξ_0 in (2.10) is arbitrary. Thus we do not have a sampling theorem for just a single transform, but for a family of transforms. From properties of Green's function it is known that $\phi(x, \lambda) := \phi_{\xi_0}(x, \lambda)$ is a solution of (2.1)–(2.2) and that (2.12) is a sampling representation of the solution to the problem

$$\begin{gathered} p_0(x) y^{(n)}(x) + \ldots + p_n(x) y(x) - \lambda y(x) = \overline{f}(x), \\ N_\nu(y) = 0, \quad \nu = 1, \ldots, n. \end{gathered} \tag{2.13}$$

at the point ξ_0 when λ is not an eigenvalue.

The results of [**4**] are extended to Kamke problems in [**2**]. In [**1, 8**] sampling theorems associated with nth order eigenvalue problems were derived under the basic assumption that all the eigenvalues are simple.

3. The boundary-value problem. Let $D(L)$ denote the following subspace of $L^2(-1, 1)$

(3.1)
$$D(L) = \{ y \in L_2[-1, 1] \cap D_{n,0} \,\big|\, U_\nu(y) = 0,\ V_\nu(y) = 0,\ 1 \leq \nu \leq n \},$$

where $D_{n,0}$ is the set of all functions that have n continuous derivatives in $[-1,1]$ except possibly at zero, but the limits $y^{(k)}(0^-)$ and $y^{(k)}(0^+)$, $0 \le k \le n-1$, exist and are finite. The linear forms generating the boundary and compatibility conditions $U_\nu(y)=0$ and $V_\nu(y)=0$, $1 \le \nu \le n$, are defined by

$$
\begin{aligned}
U_\nu(y) &= U_{\nu,-1}(y_1) + U_{\nu,1}(y_2),\\
U_{\nu,-1}(y_1) &= \alpha_\nu y_1^{(k_\nu)}(-1) + \alpha_{\nu,k_\nu-1} y_1^{(k_\nu-1)}(-1) + \dots,\\
U_{\nu,1}(y_2) &= \beta_\nu y_2^{(k_\nu)}(1) + \beta_{\nu,k_\nu-1} y_2^{(k_\nu-1)}(1) + \dots;
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
V_\nu(y) &= \gamma_{\nu,0^-}(y_1) + \gamma_{\nu,0^+}(y_2),\\
V_{\nu,0^-}(y_1) &= \gamma_\nu y_1^{(l_\nu)}(0^-) + \gamma_{\nu,l_\nu-1} y_2^{(l_\nu-1)}(0^-) + \dots,\\
V_{\nu,0^+}(y_2) &= \delta_\nu y^{(l_\nu)}(0^+) + \delta_{\nu,l_\nu-1} y_2^{(l_\nu-1)}(0^+) + \dots.
\end{aligned}
\tag{3.3}
$$

We arrange U_ν and V_ν for which k_ν and l_ν, the orders of U_ν and V_ν, satisfy

$$n-1 \ge k_1 \ge k_2 \ge \dots \ge k_n, \quad k_\nu > k_{\nu+1}, \tag{3.4}$$

$$n-1 \ge l_1 \ge l_2 \ge \dots \ge l_n, \quad l_\nu > l_{\nu+1}. \tag{3.5}$$

Also α_ν, β_ν, γ_ν and δ_ν satisfy

$$|\alpha_\nu| + |\beta_\nu| > 0, \quad |\gamma_\nu| + |\delta_\nu| > 0, \quad 1 \le \nu \le n. \tag{3.6}$$

Since any set of n linearly independent forms can be set in normalized forms, see e.g. [**16**], we assume that the forms U_ν and V_ν, $1 \le \nu \le n$ are normalized. For example, this implies for U_ν that if $\widehat{U}_\nu$, $1 \le \nu \le n$ is any equivalent system with orders $\hat{k}_1 \ge \hat{k}_2 \ge \dots \ge \hat{k}_n$, then $\hat{k}_\nu \ge k_\nu$, $1 \le \nu \le n$. The differential expressions $l(\cdot)$ are defined by

$$
\begin{aligned}
l(y) &:= \begin{cases} l_1(y) & \text{for } -1 \le x < 0,\\ l_2(y) & \text{for } \quad 0 < x \le 1,\end{cases}\\
l_i(y) &= y^{(n)} + \sum_{j=1}^{n} p_{ij}(x) y^{(n-j)}(x), \qquad i = 1, 2.
\end{aligned}
\tag{3.7}
$$

The coefficients $p_{ij}(\cdot)$, $1 \leq j \leq n$ satisfy the following smoothness conditions,

$$p_{1j}(\cdot) \in C^{n-j}([-1,0]); \qquad p_{2j}(\cdot) \in C^{n-j}([0,1]). \tag{3.8}$$

By the differential operator L we mean the operator

$$L : D(L) \to L^2[-1,1], \quad y \longmapsto l(y). \tag{3.9}$$

Now we define the operator adjoint to L, which we denote by $L^\dagger$. Similar to [**9, 16**], we first derive the Lagrange identity associated with the operator L. We complement the boundary and compatibility linear forms U_ν and V_ν with linearly independent forms to obtain linearly independent systems U_ν and V_ν, $\nu = 1, \dots, 2n$. Using integration by parts, it is not hard to see that the generalized Lagrange's identity

$$\begin{aligned} \int_{-1}^{1} l(y)(x)\bar{z}(x)\,dx = \sum_{i=1}^{2n} V_i(y)V^\dagger_{2n-i+1}(\bar{z}) + \sum_{i=1}^{2n} U_i(y)U^\dagger_{2n-i+1}(\bar{z}) \\ + \int_{-1}^{1} y(x)\overline{l^\dagger(z)}(x)\,dx. \end{aligned} \tag{3.10}$$

holds for any functions y, z, in $D_{n,0}$. The differential expression $l^\dagger$ is the adjoint of l and the conditions $U^\dagger_\nu(y) = 0$ and $V^\dagger_\nu(y) = 0$, $\nu = 1, \dots, n$, are the adjoints of (3.2) and (3.3) respectively. The operator $L^\dagger$ is determined by the differential expression

$$l^\dagger(y) = \begin{cases} l^\dagger_1(y_1), & \text{for } -1 \leq x < 0, \\ l^\dagger_2(y_2), & \text{for } \quad 0 < x \leq 1 \end{cases} \tag{3.11}$$

and the boundary and compatibility conditions

$$U^\dagger_\nu(y) = 0, \quad V^\dagger_\nu(y) = 0, \quad \nu = 1, 2, \dots, n. \tag{3.12}$$

When $L = L^\dagger$, the problem is self adjoint. The nontrivial function $y(\cdot) \in D(L)$ given by

$$y(x) = \begin{cases} y_1(x) & -1 \leq x < 0, \\ y_2(x) & \quad 0 < x \leq 1, \end{cases} \tag{3.13}$$

is said to be an eigenfunction of the operator L, if there is $\lambda \in \mathbf{C}$ such that $l(y) = \lambda y$ and $y(\cdot)$ satisfies (3.2)–(3.3). In this case, λ is an eigenvalue of L with eigenfunction $y(\cdot)$. The eigenvalues are the zeros of the $2n \times 2n$ characteristic determinant.

(3.14)

$$\triangle(\lambda) = \begin{vmatrix} U_{1,-1}(y_{11}) & \dots & U_{1,-1}(y_{1n}) & U_{1,1}(y_{21}) & \dots & U_{1,1}(y_{2n}) \\ U_{2,-1}(y_{11}) & \dots & U_{2,-1}(y_{1n}) & U_{2,1}(y_{21}) & \dots & U_{2,1}(y_{2n}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) \end{vmatrix} = 0.$$

Here $\{y_{jm}\}_{m=1}^{n}$ is an FS set of solutions of $l_j(y) - \lambda y = 0$, $j = 1, 2$, which will be specified below. Next we introduce briefly the definition of strong regularity. Set $\lambda = -\rho^n$ and let S_k, $0 \le k \le 2n-1$ be the sectors of the complex ρ-plane determined by

$$\frac{k\pi}{n} \le \arg\rho \le \frac{(k+1)\pi}{n}, \tag{3.15}$$

where $\omega_1, \dots, \omega_n$ are the nth roots of -1 which are chosen so that

$$\operatorname{Re}(\rho\omega_1) \le \operatorname{Re}(\rho\omega_2) \le \dots \le \operatorname{Re}(\rho\omega_n), \quad \text{for } \rho \in S_0.$$

Let $y_{1j}(\cdot, \lambda)$ and $y_{2j}(\cdot, \lambda)$ be respectively the FS of $l_1 y = \lambda y$, $l_2 y = \lambda y$ defined by

$$y_{1j}^{(k-1)}(-1, \lambda) = \delta_{kj}; \quad y_{2j}^{(k-1)}(0, \lambda) = \delta_{kj}, \quad 1 \le k, j \le n. \tag{3.16}$$

From the asymptotics of y_{1j}, y_{2j} derived by Muravei, [**15**], there are numbers $\theta_{-1}, \theta_0, \theta_1$ such that for sufficiently large $\rho \in S_0$, if $n = 2\mu - 1$, then there exists a function $\triangle_0(\lambda)$ which is nonzero for $\lambda \neq 0$ such that for $\lambda \neq 0$

$$\frac{\triangle(\lambda)}{\triangle_0(\lambda)} = \theta_{-1} e^{-\rho\omega_\mu} + \theta_0 + \theta_1 e^{\rho\omega_\mu} + O\left(\frac{e^{-\rho\omega_\mu}}{\rho}\right); \tag{3.17}$$

and if $n = 2\mu$,

$$\frac{\triangle(\lambda)}{\triangle_0(\lambda)} = \theta_{-1} e^{-2\rho\omega_\mu} + \theta_0 + \theta_1 e^{2\rho\omega_\mu} + O\left(\frac{1 + e^{-2\rho\omega_\mu}}{\rho}\right). \tag{3.18}$$

Definition 3.1. The problem $Ly = \lambda y$ is called *regular* if $\theta_{-1} \neq 0 \neq \theta_1$, and is called *strongly regular* if $\theta_0^2 \neq 4\theta_1\theta_{-1} \neq 0$.

Lemma 3.2 [**15**]. *Let L be strongly regular. Then L has two sequences of eigenvalues $\{\lambda'_k\}_{k\geq 1}$ and $\{\lambda''_k\}_{k\geq 1}$ where $\lambda'_k = -(\rho'_k)^n$ and $\lambda''_k = -(\rho''_k)^n$ with*

(i) *for $n = 2\mu - 1$,*

$$\begin{aligned} \rho'_k &= \frac{1}{\omega_\mu}\left(2k\pi i + \ln\ \xi' + \frac{\eta'}{k} + 0\left(\frac{1}{k^2}\right)\right), \\ \rho''_k &= \frac{1}{\omega_\mu}\left(2k\pi i + \ln\ \xi'' + \frac{\eta''}{k} + 0\left(\frac{1}{k^2}\right)\right), \end{aligned} \tag{3.19}$$

(ii) *for $n = 2\mu$*

$$\begin{aligned} \rho'_k &= \frac{1}{\omega_\mu}\left(k\pi i + \frac{1}{2}\ln\ \xi' + \frac{\eta'}{k} + 0\left(\frac{1}{k^2}\right)\right), \\ \rho''_k &= \frac{1}{\omega_\mu}\left(k\pi i + \frac{1}{2}\ln\ \xi'' + \frac{\eta''}{k} + 0\left(\frac{1}{k^2}\right)\right), \end{aligned} \tag{3.20}$$

valid for $|k| \to \infty$, where η', η'' are constants independent of k and ξ', ξ'' are the zeros of the quadratic equation $0 = \theta_1,\ \xi^2 + \theta_0\xi + \theta_{-1} = \theta_1(\xi - \xi')(\xi - \xi'')$.

Because of strong regularity the eigenvalues are eventually simple geometrically and algebraically. For convenience, we assume that all eigenvalues are simple. We have the following theorem taken from [**15**].

Theorem 3.3 [**15**]. *The systems $\{\varphi_i\}_{i=1}^\infty$ and $\{\psi_i\}_{i=1}^\infty$ of eigenfunctions of a strongly regular operator L of the form* (3.9) *above and of its adjoint $L^\dagger$, respectively, are both Riesz-bases of $L^2[-1,1]$. Moreover,*

$$\int_{-1}^{1} \varphi_i(x)\overline{\psi}_j(x)\,dx = \delta_{ij}, \quad 1 \leq i,j \leq n. \tag{3.21}$$

From now on we assume that the eigenvalues are given by the sequence $\{\lambda_k\}_{k=0}^\infty$. Consequently the eigenvalues of the adjoint problem will be $\{\bar{\lambda}_k\}_{k=0}^\infty$.

4. Green's function and the main results. Let $W_j(\cdot,\lambda)$ denote the Wronskians

$$W_j(x) := W_j(x,\lambda) = \begin{vmatrix} y_{j1}(x,\lambda) & \dots & y_{jn}(x,\lambda) \\ \vdots & \vdots & \vdots \\ y_{j1}^{(n-1)}(x,\lambda) & \dots & y_{jn}^{(n-1)}(x,\lambda) \end{vmatrix}, \quad j=1,2, \tag{4.1}$$

where $-1 \le x < 0$ when $j = 1$ and $0 < x \le 1$ when $j = 2$. Let also $g_j(x,\xi,\lambda)$, $j = 1,2$, denote the functions

$$g_j(x,\xi,\lambda) = \pm \frac{1}{2W_j(\xi)} \begin{vmatrix} y_{j1}(x,\lambda) & \dots & y_{jn}(x,\lambda) \\ y_{j1}^{(n-2)}(\xi,\lambda) & \dots & y_{jn}^{(n-2)}(\xi,\lambda) \\ \vdots & \vdots & \vdots \\ y_{j1}(\xi,\lambda) & \dots & y_{jn}(\xi,\lambda) \end{vmatrix}. \tag{4.2}$$

$g_1(x,\xi,\lambda)$ is defined for $-1 \le x,\ \xi < 0$, $g_2(x,\xi,\lambda)$ is defined for $0 < x, \xi \le 1$. Here the positive sign is taken when $x \ge \xi$ and the minus sign is taken when $x \le \xi$. Notice that $W_j(\xi) \ne 0$ for all $\xi \in [-1,0)$ when $j = 1$ and for all $\xi \in (0,1]$ when $j = 2$. The function $H(x,\xi,\lambda)$, which plays the same role of $h(x,\xi,\lambda)$ above will be given as follows. For $-1 \le x, \xi < 0$,

$$(4.3) \quad H(x,\xi,\lambda)$$

$$= \begin{vmatrix} y_{11}(x,\lambda) & \dots & y_{1n}(x,\lambda) & 0 & \dots & 0 & g_1(x,\xi,\lambda) \\ U_{1,-1}(y_{11}) & \dots & U_{1,-1}(y_{1n}) & U_{1,1}(y_{21}) & \dots & U_{1,1}(y_{2n}) & U_{1,-1}(g_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) & V_{n,0^-}(g_1) \end{vmatrix},$$

when $-1 \le x < 0$ and $0 < \xi \le 1$, we have

$$(4.4) \quad H(x,\xi,\lambda)$$

$$= \begin{vmatrix} y_{11}(x,\lambda) & \dots & y_{1n}(x,\lambda) & 0 & \dots & 0 & 0 \\ U_{1,-1}(y_{11}) & \dots & U_{1,-1}(y_{1n}) & U_{1,1}(y_{21}) & \dots & U_{1,1}(y_{2n}) & U_{1,1}(g_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) & V_{n,0^+}(g_2) \end{vmatrix},$$

and if $0 < x \le 1$ and $-1 \le \xi < 0$, $H(x,\xi,\lambda)$ becomes

(4.5) $H(x,\xi,\lambda)$

$$= \begin{vmatrix} 0 & \dots & 0 & y_{21}(x,\lambda) & \dots & y_{2n}(x,\lambda) & 0 \\ U_{1,-1}(y_{11}) & \dots & U_{1,-1}(y_{1n}) & U_{1,1}(y_{21}) & \dots & U_{1,1}(y_{2n}) & U_{1,-1}(g_1) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) & V_{n,0^-}(g_1) \end{vmatrix},$$

and if $0 < x,\ \xi \le 1$, we obtain

(4.6) $H(x,\xi,\lambda)$

$$= \begin{vmatrix} 0 & \dots & 0 & y_{21}(x,\lambda) & \dots & y_{2n}(x,\lambda) & g_2(x,\xi,\lambda) \\ U_{1,-1}(y_{11}) & \dots & U_{1,-1}(y_{1n}) & U_{1,1}(y_{21}) & \dots & U_{1,1}(y_{2n}) & U_{1,1}(g_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n,0^-}(y_{11}) & \dots & V_{n,0^-}(y_{1n}) & V_{n,0^+}(y_{21}) & \dots & V_{n,0^+}(y_{2n}) & V_{n,0^+}(g_2) \end{vmatrix}.$$

Green's function of the operator $L - \lambda E$, λ is not an eigenvalue, is given by

$$G(x,\xi,\lambda) = \frac{H(x,\xi,\lambda)}{\triangle(\lambda)}, \quad -1 \le x, \quad \xi \le 1, \quad x \ne 0 \ne \xi. \tag{4.7}$$

Therefore, $G(x,\xi,\lambda)$ is a meromorphic function with simple poles at the zeros of $\triangle(\lambda)$. We know that if λ is not an eigenvalue, then for any continuous function $f(\cdot)$, the function

$$y(x,\lambda) = \int_{-1}^{1} G(x,\xi,\lambda) f(\xi)\, d\xi \tag{4.8}$$

uniquely solves the boundary value problem $(L-\lambda E)y = f$. Analogous to Naimark [**16**, Section 3.8] it follows that the residues of $G(x,\xi,\lambda)$ have the special form

$$\operatorname{Res}_{\lambda=\lambda_k} G(x,\xi,\lambda) = -\varphi_k(x)\overline{\psi_k(\xi)}, \quad k \in \mathbf{N}, \tag{4.9}$$

where $\varphi_k(\cdot)$ is an eigenfunction of L corresponding to λ_k and $\psi_k(\cdot)$ is an eigenfunction of $L^\dagger$ corresponding to $\overline{\lambda}_k$. In Section 2 of [**15**] it has been shown that in the complex λ-plane there exists a sequence of circles $\Gamma_k = \{\lambda \in \mathbf{C} : \ |\lambda| = R_k\}$, $k \in \mathbf{N}$, with increasing radii $R_k \to \infty$ for $k \to \infty$ and a constant $\delta > 0$ such that all eigenvalues λ_j of L lie at a distance greater than or equal to δ form each of these circles. Moreover, for $k \geq 0$ the interior of Γ_k contains exactly the eigenvalues $\lambda_1, \dots, \lambda_k$ and there is a positive constant M such that for x, $\xi \in [-1,1] - \{0\}$,

$$|G(x,\xi,\lambda)| \leq M|\lambda|^{-(n-1)/n} = M|\rho|^{-1/n} \tag{4.10}$$

for all $\lambda \in \Gamma_k$, $k \in \mathbf{N}$. Now we are ready to prove the following theorem.

Theorem 4.1. *Assume that L is strongly regular and that all the eigenvalues are simple. Then for all $\lambda \in \mathbf{C}$ with $\triangle(\lambda) \neq 0$*

$$G(x,\xi,\lambda) = \sum_{j=0}^{\infty} \frac{\varphi_j(x)\overline{\psi}_j(\xi)}{\lambda_j - \lambda}, \tag{4.11}$$

uniformly for $x, \xi \in [-1,1]$.

Proof. Let $\lambda^* \in \mathbf{C}$ be not an eigenvalue of L. Then we infer, using (4.9), for $k \geq 0$,

$$I_k := \frac{1}{2\pi i} \int_{\Gamma_k} \frac{G(x,\xi,\lambda)}{\lambda - \lambda^*}\, d\lambda = G(x,\xi,\lambda^*) - \sum_{j=0}^{k} \frac{\varphi_j(x)\overline{\psi}_k(\xi)}{\lambda_j - \lambda^*}.$$

Inequality (4.10) yields, $\lim_{k\to\infty} I_k = 0$, uniformly for x, $\xi \in [-1,1]$, which proves the assertion of Theorem 4.1. $\square$

Theorem 4.2. *Let $f(\cdot) \in D(L)$. That is, $f(\cdot)$ is a $D_{n,0}$-function that satisfies all boundary and compatibility conditions, $U_\nu(f) = V_\nu(f) = 0$, $1 \leq \nu \leq n$. Then*

$$f(x) = \sum_{\nu=0}^{\infty} \left(\int_{-1}^{1} f(t)\overline{\psi}_\nu(t)\, dt \right) \varphi_\nu(x) \tag{4.12}$$

uniformly on $[-1,1]$; this series is called the Birkhoff series of f.

Proof. It follows from (4.11) that

$$\sum_{\nu=0}^{\infty}\left(\int_{-1}^{1} f(\xi)\overline{\psi}_{\nu}(\xi)\,d\xi\right)\varphi_{\nu}(x) = \lim_{k\to\infty}\frac{-1}{2\pi i}\int_{\Gamma_k}\int_{-1}^{1} G(x,\xi,\lambda)f(x)\,dx\,d\lambda.$$

Since $f \in D(L)$ satisfies all boundary and compatibility conditions, we get, using (4.10) and the properties of the Green's function, see [**15**, p. 29] after n partial integrations

$$\begin{aligned}
J_k(x) &= \frac{-1}{2\pi i}\int_{\Gamma_k}\int_{-1}^{1} G(x,\xi,\lambda)f(\xi)\,d\xi\,d\lambda \\
&= \frac{-1}{2\pi i}\int_{\Gamma_k}\int_{-1}^{1} \frac{1}{\overline{\lambda}}\ell^*(G(x,\xi,\lambda))f(\xi)\,d\xi\,d\lambda \\
&= f(x) - \frac{1}{2\pi i}\int_{\Gamma_k}\int_{-1}^{1} \frac{1}{\overline{\lambda}}G(x,\xi,\lambda))l^*(f)(\xi)\,d\xi\,d\lambda \\
&= f(x) + O(|R_k|^{-(n-1)/n}).
\end{aligned}$$

Hence $\lim_{k\to\infty} J_k = f(x)$, uniformly on $[-1,1]$. □

Using a more precise estimate than (4.10), which could be derived after some lengthy calculations, and applying the same method as in the proof of [**10**, Theorem 4.2] one can derive the following expansion theorem.

Theorem 4.3. *Let $p_{11}(\cdot) = p_{21}(\cdot) = 1$, and let $f(\cdot)$ be continuous and of bounded variation in $[-1,1]$. If $f(0) = 0$ and if f satisfies all boundary conditions $U_\nu(f) = 0$ with $k_\nu = 0$, i.e., the boundary conditions of order zero (if any exist), then*

$$f(x) = \sum_{\nu=0}^{\infty}\left(\int_{-1}^{1} f(t)\overline{\psi}_{\nu}(t)\,dt\right)\varphi_{\nu}(x) \tag{4.13}$$

uniformly on $[-1,1]$.

It is worthwhile to mention that, moreover, it is possible to prove in the case $p_{11} = p_{21} = 1$ that on each compact set $K \subset (-1,0)\cup(0,1)$ the

Birkhoff series of f is uniformly equiconvergent and also equisummable by Riesz typical means to the trigonometric Fourier series of f, for the proof of these results one has to first derive a detailed estimate of $G(x,\xi,\lambda)$ and then apply the method presented in the Stone's paper [**18**]; we omit the technical details.

Let us define for some $\xi_0 \in [-1,1]$ the entire function

$$\Phi(x,\lambda) := \triangle(\lambda) G(x,\xi_0,\lambda), \quad \lambda \in \mathbf{C}. \tag{4.14}$$

This function is an entire function of λ since the simple poles of $G(x,\xi_0,\lambda)$, i.e., the eigenvalues of (3.9), will be canceled by the simple zeros of $\triangle(\lambda)$.

The main sampling result of this article is the following theorem.

Theorem 4.4. *Let $f \in L^2[-1,1]$ and let $F(\lambda)$ be the transformation*

$$F(\lambda) = \int_{-1}^{1} \overline{f}(x)\Phi(x,\lambda)\, dx. \tag{4.15}$$

Then $F(\lambda)$ is an entire function of order $1/n$ and type not exceeding 2 which admits the sampling representation

$$F(\lambda) = \sum_{k=1}^{\infty} F(\lambda_k) \frac{\triangle(\lambda)}{(\lambda-\lambda_k)\triangle'(\lambda_k)}, \quad \lambda \in \mathbf{C}. \tag{4.16}$$

The sampling series (4.16) converges uniformly on compact sets of the complex plane and absolutely on $\mathbf{C}$. In (4.16) $F(\lambda_k) = 0$ if $\psi_k(\xi_0) = 0$.

Proof. The assertions on the order and the type of $F(\lambda)$ are immediate consequences of the asymptotic estimates for $G(x,\xi,\lambda)$ proved in [**15**]. From the fact that the normalized eigenfunctions of problem (3.9) and of its adjoint, $\{\varphi_i\}_{i=1}^{\infty}$ and $\{\psi_i\}_{i=1}^{\infty}$, respectively, form Riesz bases of $L_2[-1,1]$ we have for $\triangle(\lambda) \neq 0$

$$\Phi(x,\lambda) = \sum_{k=0}^{\infty} \widehat{\Phi}(k,\lambda)\varphi_k(x), \tag{4.17}$$

$$f(x) = \sum_{k=0}^{\infty} \widehat{F}(k)\psi_k(x), \tag{4.18}$$

where

$$\widehat{\Phi}_k(k,\lambda) := \int_{-1}^{1} \Phi(x,\lambda)\overline{\psi}_k(x)\,dx, \tag{4.19}$$

$$\widehat{F}(k) := \int_{-1}^{1} f(x)\overline{\varphi}_k(x)\,dx. \tag{4.20}$$

These, together with the expansions (4.11) and (4.14), imply on account of the biorthogonality properties of the eigenfunctions (3.21) that

$$\widehat{\Phi}_k(k,\lambda) = \overline{\psi}_k(\xi_0)\,\frac{\triangle(\lambda)}{\lambda_k-\lambda}, \quad \lambda \neq \lambda_k. \tag{4.21}$$

We assume for a moment that $\bar{\psi}_k(\xi_0) \neq 0$ for all k. Using (4.11), (4.14)–(4.15) and (4.17)–(4.21) we get,

$$F(\lambda) = \sum_{k=0}^{\infty} \overline{\widehat{F}}(k)\bar{\psi}_k(\xi_0)\,\frac{\triangle(\lambda)}{\lambda_k-\lambda}, \quad \lambda \neq \lambda_k. \tag{4.22}$$

Let $k \in \{0,1,\dots\}$ be fixed. Since $G(x,\xi_0,\lambda_k)$ has a simple pole at λ_k, then there is a $\delta > 0$ such that

$$G(x,\xi_0,\lambda) = \frac{R(x,\xi_0)}{\lambda-\lambda_k} + G_k(x,\xi_0,\lambda), \quad \text{for } |\lambda-\lambda_k| < \delta, \tag{4.23}$$

where $G_k(x,\xi_0,\lambda)$ is analytic in $|\lambda-\lambda_k| < \delta$ and, according to (4.9),

$$R(x,\xi_0) = -\varphi_k(x)\overline{\psi}_k(\xi_0). \tag{4.24}$$

Therefore

$$\begin{aligned} F(\lambda_k) &= \lim_{\lambda\to\lambda_k} \int_{-1}^{1} \overline{f}(x)\Phi(x,\lambda)\,dx \\ &= \lim_{\lambda\to\lambda_k} \int_{-1}^{1} \overline{f}(x)\,\frac{\triangle(\lambda)\varphi_k(x)\overline{\psi}_k(\xi_0)}{\lambda-\lambda_k}\,dx \\ &= -\int_{-1}^{1} \triangle'(\lambda_k)\overline{f}(x)\varphi_k(x)\overline{\psi}_k(\xi_0)\,dx \\ &= -\triangle'(\lambda_k)\overline{\psi}_k(\xi_0)\overline{\widehat{F}}(k). \end{aligned} \tag{4.25}$$

Combining (4.22) and (4.25), we get the sampling series (4.16) with pointwise convergence on **C**. As for the proof of uniform convergence, one may use the same technique as in [**2**]. When $\psi_k(\xi_0) = 0$ for some k, then obviously the same results hold with $F(\lambda_k) = 0$.

5. Examples. In this section we derive two examples illustrating the sampling results. The first example is a first order self-adjoint problem that leads to the classical sampling theorem. The second example, which involves a second order self-adjoint problem, leads to a classical representation of a class of entire functions.

To simplify the computation, we may choose a different FS of solutions than that described in Section 3. As is known the choice of the FS depends on the base point, but does not affect the results of Sections 3 and 4. Notice the examples considered below can be transformed by the replacements $\lambda \mapsto i\lambda$ and $\lambda \mapsto -\lambda$, respectively, to the form (3.9).

Example 5.1. Let us consider the self-adjoint eigenvalue problem

$$l_1(y) = l_2(y) = -i\frac{dy}{dx}, \tag{5.1}$$

$$U_1(y) = y(-1) - y(1) = 0, \quad V_1(y) = y(0^-) - y(0^+) = 0. \tag{5.2}$$

A fundamental set of solutions of (5.1)–(5.2) is (in the notation of Section 3)

$$y_{11}(x,\lambda) = e^{i\lambda x}, \quad -1 \le x < 0, \quad y_{21}(x,\lambda) = e^{i\lambda x}, \quad 0 < x \le 1. \tag{5.3}$$

Hence $\triangle(\lambda) = 2i\sin\lambda$. Thus $\theta_1 = 1$, $\theta_{-1} = -1$ and $\theta_0 = 0$, implying strong regularity and the eigenvalues are $\lambda_k = k\pi$, $k \in \mathbf{Z}$. The normalized eigenfunctions are $\varphi_k(x) = e^{ik\pi x}/2$, $k \in \mathbf{Z}, \dots$. The functions $W_j(\xi)$, $g_j(x,\xi,\lambda)$, $j = 1, 2$ and $H(x,\xi,\lambda)$ are given as follows. $W_j(\xi) = e^{i\lambda\xi}$;

$$g_j(x,\xi,\lambda) = \frac{1}{2}\begin{cases} e^{i\lambda(x-\xi)} & x > \xi, \\ -e^{i\lambda(x-\xi)} & x < \xi; \end{cases} \tag{5.4}$$

$$H(x,\xi,\lambda) = \begin{cases} -\cos\lambda e^{i\lambda(x-\xi)} + 2i\sin\lambda\ g_1(x,\xi,\lambda) & \\ \qquad\qquad -1 \le x,\ \xi \le 0, & \\ -e^{i\lambda} e^{i\lambda(x-\xi)} \qquad -1 \le x \le 0,\ \le \xi \le 1, & \\ -e^{-i\lambda} e^{i\lambda(x-\xi)} \qquad -1 \le \xi \le 0,\ \le x \le 1, & \\ -\cos\lambda e^{i\lambda(x-\xi)} + 2i\sin\lambda\ g_2(x,\xi,\lambda) & \\ \qquad\qquad 0 \le x,\xi \le 1. & \end{cases} \tag{5.5}$$

The kernel $\Phi(x,\lambda)$ depends on the choice of $\xi_0 \in [-1,1]$. For instance if $\xi_0 \in (-1,0)$, then

$$\Phi(x,\lambda) = \begin{cases} -2\cos\lambda e^{i\lambda(x-\xi_0)} + 2i\sin\lambda g_1(x,\xi_0,\lambda) & -1 \le x \le 0, \\ -2e^{-i\lambda} e^{i\lambda(x-\xi_0)} & 0 \le x \le 1; \end{cases} \tag{5.6}$$

$$\Phi(x,\lambda) = \begin{cases} -2e^{-i\lambda} e^{i\lambda(x-\xi_0)} & -1 \le x \le 0, \\ -2\cos\lambda e^{i\lambda(x-\xi_0)} + 2i\sin\lambda g_2(x,\xi_0,\lambda), & 0 \le x \le 1, \end{cases} \tag{5.7}$$

for $\xi_0 \in (0,1)$.

Theorem 4.4 states that the transformation

$$F(\lambda) = \int_{-1}^{1} \overline{f}(x)\Phi(x,\lambda)\,dx, \tag{5.8}$$

can be recovered via the classical sampling representation

$$F(\lambda) = \sum_{k=-\infty}^{\infty} F(k\pi)\,\frac{\sin(\lambda - k\pi)}{\lambda - k\pi}. \tag{5.9}$$

Example 5.2. Let us consider the self adjoint problem, [**4**],

$$l_1(y) = l_2(y) = -\frac{d^2y}{dx^2}, \tag{5.10}$$

$$U_1(y) = y(-1) = 0, \quad U_2(y) = y(1) = 0. \tag{5.11}$$

$$V_1(y) = y(0^-) + y(0^+) = 0, \quad V_2(y) = y'(0^-) + y'(0^+) = 0. \tag{5.12}$$

A fundamental set of solutions of (5.10) is

$$y_{11}(x,\lambda)=e^{-i\rho x},\quad y_{12}(x,\lambda)=e^{i\rho x},\quad -1\le x<0, \tag{5.13}$$

$$y_{21}(x,\lambda)=e^{-i\rho x},\quad y_{22}(x,\lambda)=e^{i\rho x},\quad 0<x\le 1, \tag{5.14}$$

where $\rho=\sqrt{\lambda}$. In this case $\triangle(\lambda)=4\rho\sin 2\rho$. Hence $\theta_{-1}=2\neq 0$, $\theta_1=-2\neq 0$, $\theta_0=0$, i.e., the problem is strongly regular and the eigenvalues are $\lambda_k=k^2\pi^2/4$, $k\in\mathbf{N}$. We can see that zero is not an eigenvalue. In the above notations, $W_j(\xi)=2i\rho$,

$$g_j(x,\xi,\lambda)=\frac{1}{2\rho}\begin{cases}\sin\rho(\xi-x) & x\ge\xi,\\ -\sin\rho(\xi-x) & x\le\xi.\end{cases} \tag{5.15}$$

Here we use `Maple` to compute the function $H(x,\xi,\lambda)$. First for $-1\le x<0$, $-1\le\xi<0$, and $x\ge\xi$; $\xi\ge x$ we respectively have

$$\begin{aligned}2\rho\, H(x,\xi,\lambda) &= -4\,\sin(\rho(\xi+1))\rho\sin(\rho(x-1))\\ &\quad -4\sin(\rho(x-\xi))\rho\sin(2\rho)\\ &\quad -e^{-i\rho\,(x+2)}\cos(\rho\xi)-ie^{-i\rho(x+2)}\sin(\rho\xi)\rho\\ &\quad +e^{-i\rho x}\cos(\rho\xi)-ie^{-i\rho x}\sin(\rho\xi)\rho\\ &\quad +ie^{i\rho(x+2)}\sin(\rho\xi)\rho+e^{i\rho\, x}\cos(\rho\xi)\\ &\quad +ie^{i\rho x}\sin(\rho\xi)\rho-e^{i\rho(x+2)}\cos(\rho\xi);\end{aligned} \tag{5.16}$$

$$\begin{aligned}2\rho\, H(x,\xi,\lambda) &= -4\sin(\rho(\xi+1))\rho\sin(\rho(x-1))\\ &\quad +4\sin(\rho(x-\xi))\rho\sin(2\rho)\\ &\quad -e^{-i\rho(x+2)}\cos(\rho\xi)-ie^{-i\rho(x+2)}\sin(\rho\xi)\rho\\ &\quad +e^{-i\rho x}\cos(\rho\xi)\\ &\quad -ie^{-i\rho x}\sin(\rho\xi)\rho+ie^{i\rho(x+2)}\sin(\rho\xi)\rho\\ &\quad +e^{i\rho x}\cos(\rho\xi)\\ &\quad +ie^{i\rho x}\sin(\rho\xi)\rho-e^{i\rho(x+2)}\cos(\rho\xi),\end{aligned} \tag{5.17}$$

for $-1 \le x < 0$, $0 < \xi \le 1$, we have

$$
\begin{aligned}
H(x,\xi,\lambda) = 2\sin(\rho(\xi-1))\sin(\rho(x+1)) \\
&+ \frac{1}{2}\, e^{-i\rho(x+2)}\cos(\rho\xi) \\
&+ \frac{1}{2}\, ie^{-i\rho(x+2)}\sin(\rho\xi) - \frac{1}{2}\, ie^{i\rho x}\sin(\rho\xi) \\
&- \frac{1}{2}\, e^{-i\rho x}\cos(\rho\xi) - \frac{1}{2}\, ie^{-i\rho x}\sin(\rho\xi) \\
&+ \frac{1}{2}\, e^{i\rho x}\cos(\rho\xi) + \frac{1}{2}\, e^{i\rho(x+2)}\cos(\rho\xi) \\
&- \frac{1}{2}\, ie^{i\rho(x+2)}\sin(\rho\xi),
\end{aligned} \tag{5.18}
$$

now for $0 < x \le 1$, $-1 \le \xi < 0$, we have

$$
\begin{aligned}
H(x,\xi,\lambda) = 2\sin(\rho(\xi+1))\sin(\rho(x-1)) \\
&+ \frac{1}{2}\, e^{-i\rho(x-2)}\cos(\rho\xi) - \frac{1}{2}\, e^{i\rho x}\cos(\rho\xi) \\
&+ \frac{1}{2}\, i\sin(\rho\xi)e^{-i\rho(x-2)} + -\frac{1}{2}\, ie^{i\rho x}\sin(\rho\xi) \\
&- \frac{1}{2}\, e^{-i\rho x}\cos(\rho\xi) + \frac{1}{2}\, e^{i\rho(x-2)}\cos(\rho\xi) \\
&+ \frac{1}{2}\, ie^{-i\rho x}\sin(\rho\xi) - \frac{1}{2}\, ie^{i\rho(x-2)}\sin(\rho\xi),
\end{aligned} \tag{5.19}
$$

finally when $0 < x \le 1$, $0 < \xi \le 1$ and $x > \xi$; $\xi > x$ we respectively have

$$
\begin{aligned}
H(x,\xi,\lambda) = -2\sin(\rho(\xi-1))\sin(\rho(x+1)) \\
&- 2\sin(\rho(x-\xi))\sin(2\rho) - \frac{1}{2}\, e^{-i\rho(x-2)}\cos(\rho\xi) \\
&+ \frac{1}{2}\, e^{i\rho x}\cos(\rho\xi) + \frac{1}{2}\, e^{-i\rho x}\cos(\rho\xi) \\
&- \frac{1}{2}\, e^{i\rho(x-2)}\cos(\rho\xi) - \frac{1}{2}\, ie^{-i\rho(x-2)}\sin(\rho\xi) \\
&+ \frac{1}{2}\, ie^{i\rho x}\sin(\rho\xi) + \frac{1}{2}\, ie^{i\rho(x-2)}\sin(\rho\xi) \\
&- \frac{1}{2}\, ie^{-i\rho x}\sin(\rho\xi);
\end{aligned} \tag{5.20}
$$

and

$$\begin{aligned}H(x,\xi,\lambda) = &-2\sin(\rho(\xi-1))\sin(\rho(x+1))\\ &+2\sin(\rho(x-\xi))\sin(2\rho) - \frac{1}{2}\,e^{-i\rho(x-2)}\cos(\rho\xi)\\ &+\frac{1}{2}\,e^{i\rho x}\cos(\rho\xi) + \frac{1}{2}\,e^{-i\rho x}\cos(\rho\xi)\\ &-\frac{1}{2}\,e^{i\rho(x-2)}\cos(\rho\xi) - \frac{1}{2}\,ie^{-i\rho(x-2)}\sin(\rho\xi)\\ &+\frac{1}{2}\,ie^{i\rho x}\sin(\rho\xi) + \frac{1}{2}\,ie^{i\rho(x-2)}\sin(\rho\xi)\\ &-\frac{1}{2}\,ie^{-i\rho x}\sin(\rho\xi).\end{aligned} \tag{5.21}$$

Green's function associated with the eigenvalue problem (5.10)–(5.12) will be

$$G(x,\xi,\lambda) = \frac{H(x,\xi,\lambda)}{4\rho\sin 2\rho}. \tag{5.22}$$

For $\xi_0 \in [-1,1]$, $\Phi(x,\lambda) = H(x,\xi_0,\lambda)$ and according to Theorem 4.4, the integral transform

$$F(\lambda) = \int_{-1}^{1} \bar{g}(x)\,\Phi(x,\lambda)\,dx, \quad g(\cdot) \in L^2(-1,1) \tag{5.23}$$

can be reconstructed from its values at the eigenvalues via

$$F(\lambda) = \sum_{k=1}^{\infty} F\left(\frac{k^2\pi^2}{4}\right)\frac{\rho\,\sin(2\rho - k\pi)}{\rho^2 - (k^2\pi^2/4)}. \tag{5.24}$$

Example 5.3. Consider the problem that consists of the differential expressions

$$l_1(y) = -i\frac{dy}{dx}, \quad l_2(y) = \frac{dy}{dx} \tag{5.25}$$

and the boundary conditions (5.2). The eigenvalues will be the zeros of the entire function $\triangle(\lambda) = e^{\lambda} - e^{-i\lambda}$. That is, $\lambda_k = k\pi(1+i)$,

$k \in \mathbf{Z}$, i.e., an equidistant set of points on the line $y = x$. Hence, $W_1(\xi) = e^{i\lambda\xi}, \quad W_2(\xi) = e^{\lambda\xi}$,

$$\begin{aligned} g_1(x,\xi,\lambda) &= \frac{1}{2}\begin{cases} e^{i\lambda(x-\xi)} & x > \xi, \\ -e^{i\lambda(x-\xi)} & x < \xi, \end{cases} \\ g_2(x,\xi,\lambda) &= \frac{1}{2}\begin{cases} e^{\lambda(x-\xi)}, & x > \xi, \\ -e^{\lambda(x-\xi)}, & x < \xi. \end{cases} \end{aligned} \tag{5.26}$$

The function $H(x,\xi,\lambda)$ will have the form

(5.27)

$$H(x,\xi,\lambda) = \begin{cases} -1/2\left(e^{\lambda} + e^{i\lambda}\right) e^{i\lambda(x-\xi)} + \triangle(\lambda)\, g_1(x,\xi,\lambda) \\ \qquad\qquad\qquad\qquad -1 \le x,\xi \le 0, \\ -e^{\lambda} e^{i\lambda(x+i\xi)}, \qquad\qquad -1 \le x \le 0 \le \xi \le 1, \\ -e^{-i\lambda} e^{\lambda(x-i\xi)}, \qquad\qquad -1 \le \xi \le 0 \le x \le 1, \\ -1/2\left(e^{\lambda} + e^{i\lambda}\right) e^{\lambda(x-\xi)} + \triangle(\lambda)\, g_2(x,\xi,\lambda), \\ \qquad\qquad\qquad\qquad 0 \le x,\xi \le 1. \end{cases}$$

Choosing $\xi_0 \in [-1,1]$, the transform

$$F(\lambda) = \int_{-1}^{1} \overline{f}(x)\, H(x,\xi_0,\lambda)\, dx \tag{5.28}$$

has the sampling representation

$$F(\lambda) = \sum_{k=-\infty}^{\infty} F(\lambda_k)\, \frac{e^{\lambda} - e^{-i\lambda}}{(\lambda-\lambda_k)(e^{\lambda_k} + i\lambda_k e^{-i\lambda_k})}. \tag{5.29}$$

It should be mentioned that although the eigenvalue problem of the last example does not belong to the class investigated in [**15**], it can be shown that the expansion and sampling results are still valid.

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