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SOLUTIONS OF HAMMERSTEIN INTEGRAL EQUATIONS VIA A VARIATIONAL PRINCIPLE

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ABSTRACT. We study solutions of the nonlinear Hammerstein integral equation with changing-sign kernels by using a variational principle of Ricceri and critical points theory techniques. Combining the effects of a sublinear and superlinear nonlinear terms we establish new existence and multiplicity results for the equation. As an application we consider a semilinear Dirichlet problem for polyharmonic elliptic operators.

1. Introduction. We study the solvability of the nonlinear Hammerstein integral equation

$$u(x) = \int_\Omega k(x,y) f(y, u(y))\, dy, \tag{1}$$

where $\Omega \subset \mathbf{R}^N$ is a bounded domain, i.e., open connected set, $k(x,y) : \Omega \times \Omega \to \mathbf{R}$ is a measurable and symmetric kernel and $f(x,u) : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function, that is, $f(x,u)$ is measurable for each $u \in \mathbf{R}$ and continuous for almost all $x \in \Omega$.

The Hammerstein equation (1) appeared in the earlier 30s as a general model for study of semi-linear boundary-value problems. The kernel $k(x,y)$ typically arises as the Green function of a differential operator. Green functions of specific boundary value problems admit lots of specific properties like positivity, maximum principles, pointwise estimates, etc., depending on the structure of the differential expression in the data and boundary conditions. If the kernel $k(x,y)$ is positive, then methods of positive operators are applicable to study solutions of (1). The advantage of positivity methods is that in many cases they allow to obtain not only existence but also rather precise information about a location of solutions, for example, between explicitly constructed sub- and super-solutions of the equation, see, e.g., [**1, 8**].

Another classical method to study equation (1) is variational. If the kernel $k(x,y)$ is symmetric, one can associate to (1) a functional J on

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a suitable energy space $\mathbf{E}$ such that critical points of J correspond to solutions of (1). Then direct variational methods or methods of critical points theory could be used to study critical points of J. If in addition, the kernel $k(x, y)$ is positive, then variational and positivity methods could be combined together. This provides a powerful tool for the study of multiple solutions of (1), see, e.g. [**3**].

In the present paper we study (1) by variational methods, without assuming positivity of the kernel $k(x, y)$. Examples of changing-sign kernels arise from higher order elliptic boundary value problems, see e.g., [**4**]. Instead of using positivity of the kernel we are going to apply to (1) a new variational principle of Ricceri [**12**] which provides a powerful tool for localization of minima of variational problems. In some sense such localization information can compensate for the lack of positivity properties and could be used efficiently, in order to prove the existence of multiple solutions of (1). In a convenient for our purposes form the variational principle reads as follows, see [**12**, Theorem 2.5] for a more general statement.

Theorem 1. *Let $\mathbf{E}$ be a Hilbert space and $\Phi, \Psi : \mathbf{E} \to \mathbf{R}$ two sequentially weakly lower semi-continuous functionals. Assume that Ψ is strongly continuous and coercive on $\mathbf{E}$, that is, $\lim_{\|u\|\to+\infty} \Psi(u) = +\infty$. For each $\rho > \inf_{\mathbf{E}} \Psi$ set*

$$\varphi(\rho) := \inf_{\Psi^\rho} \frac{\Phi(u) - \inf_{\mathrm{cl}_w \Psi^\rho} \Phi}{\rho - \Psi(u)}, \tag{2}$$

where $\Psi^\rho := \{u \in \mathbf{E} : \Psi(u) < \rho\}$ and $\mathrm{cl}_w \Psi^\rho$ is the closure of Ψ^ρ in the weak topology of $\mathbf{E}$. Then, for each $\rho > \inf_{\mathbf{E}} \Psi$ and each $\mu > \varphi(\rho)$, the restriction of the functional $\Phi + \mu\Psi$ to Ψ^ρ has a global minimum point in Ψ^ρ.

Theorem 1 implies in particular that if there exists $\rho > 0$, such that $\varphi(\rho) < 1/2$, then the functional $J = \Psi/2 + \Phi$ has a local minimum point $v \in \Psi^\rho$, see [**12**, Theorem 2.5]. Using such information one can proceed further, by applying the Mountain Pass theorems to J. We will use this strategy to study (1) under various assumptions on $f(x, u)$. In the next section we introduce some notations and describe a variational setting of the problem. Section 3 is devoted to the proof of our main existence-localization theorem. In Sections 4 and 5 we apply this result to equation (1) with superlinear and combination of sub- and superlinear nonlinearities. Finally in the last section we describe applications to a polyharmonic boundary value problem.

2. Variational setting. In this section we define the energy space $\mathbf{E}$ generated by the integral kernel $k(x,y)$ and the energy functional J for the Hammerstein equation (1). These results are standard, see e.g., [**10**]. We present some proofs for completeness. In what follows $\mathbf{L}_p = \mathbf{L}_p(\Omega, \mathbf{R})$ stands for the Lebesgue space with the usual norm $\|\cdot\|_p$, $p' = p/(p-1)$ for the index conjugate to p and

$$(u,v) = \int_\Omega u(x)v(x)\,dx$$

for the standard inner product in $\mathbf{L}_2$. By $c, c_1, \dots$ we denote various positive constants whose values are irrelevant.

We write equation (1) in the operator form

$$u = \mathbf{Kf}u,$$

where

$$\mathbf{f}u = f(\cdot, u)$$

is a nonlinear superposition operator generated by the function f and $\mathbf{K}$ is a linear integral operator

$$\mathbf{K}u(x) = \int_\Omega k(x,y)u(y)\,dy$$

generated by the kernel k. Hereafter we assume that

- $\mathbf{K}$ *is a bounded compact operator from* $\mathbf{L}_{p'}$ *into* $\mathbf{L}_p$ *for some* $p \in (2, +\infty)$,
- $\mathbf{K}$ *is positive-definite on* $\mathbf{L}_2$, that is $(\mathbf{K}u, u) > 0$ for all $u \in \mathbf{L}_2$, $u \neq 0$.

By $\|\mathbf{K}\|$ we denote the norm of $\mathbf{K}$ in $\mathbf{L}_2$.

Since the kernel $k(x,y)$ is symmetric $\mathbf{K}$ is self-adjoint in $\mathbf{L}_2$, see cf., [**15**] for regular operators and [**14**] for the general case. Let $\mathbf{L}$ be the left inverse to $\mathbf{K}$. It is defined as an unbounded positive definite self-adjoint operator in $\mathbf{L}_2$ with a domain $D(L) \subset \mathbf{L}_2$ such that

$$\mathbf{LK}x = x \quad \text{for all } x \in \mathbf{L}_2.$$

Define a bilinear form $\langle u, v\rangle$ on $D(L)$ by means of the formula

$$\langle u, v\rangle = (u, \mathbf{L}v).$$

It is clear that $\langle \cdot, \cdot\rangle$ is symmetric and

$$\langle u, u\rangle \geq \|\mathbf{K}\|^{-1}\|u\|_2^2 \quad \text{for all } \ u \in D(L).$$

Therefore $\langle \cdot, \cdot\rangle$ is a closable form in $\mathbf{L}_2$. By $\mathbf{E}$ we denote the domain of the closure of $\langle \cdot, \cdot\rangle$ in $\mathbf{L}_2$. Thus $\mathbf{E}$ is a Hilbert space with a scalar product $\langle \cdot, \cdot\rangle$ and the corresponding norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot\rangle}$. The space $\mathbf{E}$ is said to be the *energy space* for operator $\mathbf{K}$.

By $\mathbf{E}'$ we denote the dual space to $\mathbf{E}$ obtained by the completion of $\mathbf{L}_2$ with respect to the norm $\|x\|_\star = (x, \mathbf{K}x)^{1/2}$. It is easy to see that the conjugate to $\mathbf{E}$ space $\mathbf{E}^\star$ can be identified with the space $\mathbf{E}'$. Namely for any $l \in \mathbf{E}^\star$ there exists $y \in \mathbf{E}'$ such that $l(x) = (y, x)$ for all $x \in \mathbf{E}$, moreover, $\|l\| = \|y\|_\star$. The energy space $\mathbf{E}$ can be represented as $\mathbf{E} \cong \mathbf{K}^{1/2}\mathbf{L}_2 \cong \tilde{\mathbf{K}}\mathbf{E}'$, where $\mathbf{K}^{1/2}$ is a square root of $\mathbf{K}$, $\tilde{\mathbf{K}}$ is a closure of $\mathbf{K}$ with respect to the $\|\cdot\|_\star$ and "$\cong$" means an isomorphism. Therefore the embeddings

$$\mathbf{E} \subset \mathbf{L}_p \subset \mathbf{L}_2 \subset \mathbf{L}_{p'} \subset \mathbf{E}'$$

are continuous and dense. Since $\mathbf{K}$ is compact from $\mathbf{L}_{p'}$ into $\mathbf{L}_p$ the embedding $\mathbf{E} \subset \mathbf{L}_p$ is compact.

Define the energy functional for Hammerstein equation (1) on the space $\mathbf{E}$ by the formula

$$J(u) = \frac{1}{2}\|u\|^2 + \Phi(u),$$

where

$$\Phi(u) = -\int_\Omega F(x, u(x))\, dx \quad \text{and} \quad F(x, u) = \int_0^u f(x, \xi)\, d\xi.$$

In what follows we make the following assumptions on the growth of the nonlinearity:

(f) there exist $r \in (1,2)$, $q \in (2,p)$ and $\alpha \in \mathbf{L}_{p/(p-r)}$, $\beta \in \mathbf{L}_{p/(p-q)}$, $\gamma \in \mathbf{L}_{p'}$ such that

$$|f(x,u)| \le \alpha(x)|u|^{r-1} + \beta(x)|u|^{q-1} + \gamma(x).$$

Lemma 2. *Assume (f) holds. Then Φ is well-defined and sequentially weakly continuous on $\mathbf{E}$.*

Proof. By (f) and since $\mathbf{E} \subset \mathbf{L}_p$ it is standard to see that the superposition operator $F(\cdot,u)$ maps $\mathbf{L}_p$ into $\mathbf{L}_1$. Therefore the functional Φ is well defined on $\mathbf{E}$. We are going to check that Φ is sequentially weakly continuous on $\mathbf{E}$. Let $(u_n) \subset \mathbf{E}$ weakly converge to $u \in \mathbf{E}$. Then (u_n) is bounded in $\mathbf{E}$. Since the embedding $\mathbf{E} \subset \mathbf{L}_p$ is compact (u_n) contains a subsequence, still denoted by (u_n), which strongly converges to u in $\mathbf{L}_p$. Therefore there exist $h \in \mathbf{L}_p$ and a subsequence (u_{n_k}) of (u_n) such that $|u_{n_k}(x)| \le h(x)$ and $u_{n_k}(x) \to u(x)$ a.e. in Ω. Hence

$$F(x,u_{n_k}(x)) \to F(x,u(x)) \quad \text{a.e. in } \Omega.$$

Moreover, by using (f) we obtain the uniform bound

$$|F(x,u_{n_k}(x))| \le H(x)$$

for some $H \in \mathbf{L}_1$. Therefore the Lebesgue dominated convergence theorem implies that $\Phi(u_{n_k}) \to \Phi(u)$, as required. $\square$

Lemma 3. *Assume (f) holds. Then the energy functional J is continuously differentiable on $\mathbf{E}$ with the derivative given by*

$$J'(u)(h) = (Lu,h) - \int_\Omega f(x,u(x))h(x)dx \quad \textit{for all } h \in \mathbf{E}.$$

Any critical point of J is a solution to the Hammerstein equation (1).

Proof. To prove the lemma it is enough to show that Φ is continuously differentiable on $\mathbf{E}$. To do this we prove first that Φ is Gâteaux differentiable on $\mathbf{L}_p$. Fix $u, h \in \mathbf{L}_p$. Let $0 < \tau < 1$. By the mean value theorem there exists a function θ, which can be chosen measurable, see [**15**] such that $0 \leq \theta(x) \leq 1$ and

$$F(x, u(x) + \tau h(x)) - F(x, u(x)) = f(x, u(x) + \tau\theta(x)h(x))\tau h(x).$$

Then

$$\frac{\Phi(u + \tau h) - \Phi(u)}{\tau} = -\int_\Omega f(x, u(x) + \tau\theta(x)h(x))h(x)\,dx,$$

where $u + \tau\theta h \in \mathbf{L}_p$. By the assumption (f) the superposition operator $f(\cdot, u)$ continuously maps $\mathbf{L}_p$ into $\mathbf{L}_{p'}$. Therefore the last integral is well defined. Moreover

$$f(x, u(x) + \tau\theta(x)h(x)) \longrightarrow f(x, u(x)) \quad \text{a.e. in } \Omega \text{ as } \tau \to 0.$$

By using (f) again we conclude that

$$|f(x, u(x) + \tau\theta(x)h(x))| \leq H(x)$$

for some $H \in \mathbf{L}_{p'}$. Thus the Lebesgue dominated convergence theorem implies that

$$\lim_{\tau \to 0} \frac{\Phi(u + \tau h) - \Phi(u)}{\tau} = -\int_\Omega f(x, u(x))h(x)\,dx.$$

Since the energy space $\mathbf{E}$ is continuously embedded into $\mathbf{L}_p$ we see that Φ is Gâteaux differentiable on $\mathbf{E}$ with the derivative given by

$$\Phi'(u)(h) = -\int_\Omega f(x, u(x))h(x)dx \quad \text{for all } h \in \mathbf{E}.$$

Finally, from the continuity of the superposition operator $f(\cdot, u)$ from $\mathbf{E}$ into $\mathbf{L}_{p'}$ we conclude that Φ is continuously (Frechet) differentiable on $\mathbf{E}$. $\square$

3. Existence of a minimum. In this section we apply a variational principle of Ricceri [**12**, Theorem 2.5] to the energy functional J on $\mathbf{E}$. For $\rho \geq 0$ define

$$\Lambda(\rho) := \kappa^r \|\alpha\|_{p/(p-r)}\rho^{r-1} + \kappa^q \|\beta\|_{p/(p-q)}\rho^{q-1} + \kappa\|\gamma\|_{p'},$$

where $\kappa > 0$ denotes the embedding constant of $\mathbf{E}$ into $\mathbf{L}_p$, that is,

$$\kappa := \sup\{\|u\|_p : u \in \mathbf{E}, \|u\| \le 1\}.$$

Our main existence-localization result reads as follows.

Theorem 4. *Assume (f) holds and there exists $\rho_* > 0$ such that*

$$\Lambda(\rho_*) < \rho_*. \tag{3}$$

Then the functional J has a local minimum u in $\mathbf{E}$ such that $\|u\| < \rho_*$.

Proof. We are going to apply Theorem 1 to the functionals $\Psi(u) := \|u\|^2$ and $\Phi(u)$ on $\mathbf{E}$. Rewriting (2) we deduce from Theorem 1 that, if there exists $\rho > 0$ such that

$$\varphi(\rho^2) = \inf_{\Psi^{\rho^2}} \frac{\Phi(u) - \inf_{cl_w \Psi^{\rho^2}} \Phi}{\rho^2 - \Psi(u)} = \inf_{\|u\|<\rho} \frac{\Phi(u) - \inf_{\|u\|\le\rho} \Phi}{\rho^2 - \|u\|^2} < \frac{1}{2}, \tag{4}$$

then the energy functional $J(u) = \|u\|^2/2 + \Phi(u)$ has a local minimum $u \in \mathbf{E}$ such that $\|u\| < \rho$. For $\rho > 0$ define

$$\phi(\rho) := \sup_{\|v\|\le\rho} \int_\Omega F(x, v(x))\, dx.$$

Thus (4) is equivalent to

$$\inf_{\rho>0} \inf_{\sigma<\rho} \frac{\phi(\rho) - \phi(\sigma)}{\rho^2 - \sigma^2} < \frac{1}{2},$$

which is fulfilled if there exists $\rho > 0$ such that

$$\limsup_{\tau\to 0} \frac{\phi(\rho+\tau) - \phi(\rho)}{\tau} < \rho. \tag{5}$$

We are going to estimate the lefthand side of (5). As in [**2**], if $\rho > 0$, $|\tau| < \rho$ and $\tau \ne 0$ then by using (f) we obtain

$$\begin{aligned}\frac{\phi(\rho+\tau) - \phi(\rho)}{\tau} &\le \frac{1}{|\tau|} \sup_{\|v\|\le 1} \int_\Omega \left| \int_{\rho v(x)}^{(\rho+\tau)v(x)} |f(x,u)|\, du \right| dx \\ &\le \frac{\kappa^r}{r} \|\alpha\|_{p/(p-r)} \left| \frac{(\rho+\tau)^r - \rho^r}{\tau} \right| \\ &\quad + \frac{\kappa^q}{q} \|\beta\|_{p/(p-q)} \left| \frac{(\rho+\tau)^q - \rho^q}{\tau} \right| + \kappa \|\gamma\|_{p'}.\end{aligned}$$

Therefore

$$\limsup_{\tau\to 0} \frac{\phi(\rho+\tau)-\phi(\rho)}{\tau} \le \kappa^r\|\alpha\|_{p/(p-r)}\rho^{r-1}+\kappa^q\|\beta\|_{p/(p-q)}\rho^{q-1}+\kappa\|\gamma\|_{p'}.$$

Therefore J has a local minimum $u \in \mathbf{E}$ such that $\|u\| < \rho_*$, provided that $\Lambda(\rho_*) < \rho_*$ for some $\rho_* > 0$. □

In the next sections we consider concrete examples of nonlinearities $f(x,u)$ such that explicit estimates on ρ_* are available and can be used for proving the existence of multiple solutions of Hammerstein equation (1).

4. Applications: a nonhomogeneous superlinear problem. In this section we apply Theorem 4 combined with the Mountain Pass theorem for the study of solutions of equation (1) with nonhomogeneous superlinear nonlinearity. First we consider the case

$$(f_1) \qquad f_\lambda(x,u) = \beta(x)g(u) + \lambda\gamma(x),$$

where $\lambda > 0$ is a real parameter, $0 \ne \gamma \in \mathbf{L}_{p'}$, $0 < \beta \in \mathbf{L}_{p/(p-q)}$ for some $q \in (2,p)$ and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function such that $g(0) = 0$ and g satisfies the assumptions:

(g_1) $|g(u)| \le b|u|^{q-1}$ for $u \in \mathbf{R}$;

(g_2) there exist $\sigma > 2$, $R_\sigma > 0$ such that

$$0 < \sigma G(u) < g(u)u \quad \text{for } |u| \ge R_\sigma, \quad \text{where } G(u) := \int_0^u g(\xi)\,d\xi.$$

We denote by J_λ the energy functional which corresponds to $f_\lambda(x,u)$.

Theorem 5. *Let (f_1), (g_1) and (g_2) hold. Then there exists $\lambda_* > 0$ such that for each $\lambda \in (0,\lambda_*)$ equation (1) has at least two solutions $u_\lambda, v_\lambda \in \mathbf{E}$. The solution u_λ is a local minimum of J_λ and $\|u_\lambda\| \to 0$ as $\lambda \to 0$.*

Proof. We shall apply Theorem 4 to J_λ in order to prove the existence of a local minimum of J_λ for small $\lambda > 0$. Then we obtain the second solution by using the Mountain Pass theorem.

Step 1. *J_λ has a local minimum.* It is clear that (f_1) and (g_1) implies (f). Hence, we can use Theorem 4. The function $\Lambda(\rho)$ becomes in this case

$$\Lambda(\rho) = \kappa^q b\|\beta\|_{p/(p-q)}\rho^{q-1} + \kappa\lambda\|\gamma\|_{p'}.$$

By the direct computations one can see that if

$$\lambda_* = \|\gamma\|_{p'}^{-1}\left(2^{q-1}b\|\beta\|_{p/(p-q)}\kappa^{2q-2}\right)^{-1/(q-2)}, \quad \rho_*(\lambda) = 2\lambda\kappa\|\gamma\|_{p'}$$

then for each $\lambda \in (0, \lambda_*)$ one has $\Lambda(\rho_*(\lambda)) < \rho_*(\lambda)$. Then by Theorem 4 we conclude that J_λ has a local minimum $u_\lambda \in \mathbf{E}$ such that $\|u_\lambda\| < \rho_*(\lambda)$.

Step 2. *J_λ is unbounded below.* The assumption (g_2) implies that

$$G(u) \geq c_1|u|^\sigma - c_2$$

for some $c_1, c_2 > 0$. Thus for $u \in \mathbf{E}$ and $\tau > 0$ we obtain

$$\begin{aligned}\int_\Omega F_\lambda(x, \tau u)\,dx &= \int_\Omega \beta(x)G(\tau u(x))\,dx + \lambda\tau\int_\Omega \gamma(x)u(x)\,dx \\ &\geq c_1\tau^\sigma\int_\Omega \beta(x)|u(x)|^\sigma\,dx - c_2\int_\Omega \beta(x)\,dx \\ &\quad + \lambda\tau\int_\Omega \gamma(x)u(x)\,dx.\end{aligned}$$

Since $\sigma > 2$ for a fixed $u \neq 0$ we see that

$$\begin{aligned}J_\lambda(\tau u) &= \frac{\tau^2}{2}\|u\|^2 - \int_\Omega F_\lambda(x, \tau u)\,dx \\ &\leq c_3\tau^2 - c_4\tau^\sigma + c_5 - \lambda c_6\tau \to -\infty \quad \text{as } \tau \to \infty.\end{aligned}$$

Step 3. *J_λ satisfies the Palais-Smale condition.* Let $(u_n) \subseteq \mathbf{E}$ be a Palais-Smale sequence for J_λ, that is,

$$J_\lambda(u_n) \leq M, \quad J'_\lambda(u_n) \to 0.$$

We need to show that (u_n) contains a convergent subsequence. Fix $\nu \in \mathbf{N}$ such that for all $n > \nu$ and $v \in \mathbf{E}$

$$|J'_\lambda(u_n)(v)| < \|v\|.$$

Therefore

$$\begin{aligned}\sigma M + \|u_n\| &\geq \left(\frac{\sigma}{2} - 1\right)\|u_n\|^2 - \sigma \int_\Omega F_\lambda(x, u_n)\,dx + \int_\Omega f_\lambda(x, u_n)u_n\,dx \\ &= \left(\frac{\sigma}{2} - 1\right)\|u_n\|^2 + \int_\Omega \beta(x)\left[g(u_n)u_n - \sigma G(u_n)\right]\,dx \\ &\quad - \lambda(\sigma - 1)\int_\Omega \gamma(x)u_n\,dx.\end{aligned}$$

By assumption (g_2) it follows that for $n > \nu$

$$\int_\Omega \beta(x)\left[\sigma g(u_n)u_n - \sigma G(u_n)\right]\,dx \geq \min_{|\xi| \leq R_\sigma}\left[g(\xi)\xi - \sigma G(\xi)\right]\|\beta\|_1.$$

Set $m = \min_{|\xi| \leq R_0}\left[\sigma G(\xi) - g(\xi)\xi\right]$. Then for $n > \nu$ we obtain

$$\left(\frac{\sigma}{2} - 1\right)\|u_n\|^2 \leq \left(1 + (\sigma - 1)\kappa\lambda\|\gamma\|_{p'}\right)\|u_n\| + \sigma M + m\|\beta\|_1,$$

which implies that (u_n) is bounded in $\mathbf{E}$.

We shall prove that (u_n) admits a strongly convergent subsequence in $\mathbf{E}$. Since (u_n) is bounded in $\mathbf{E}$ it contains a subsequence, that we still denote by (u_n), which is weakly convergent to some $u \in \mathbf{E}$. Choose a positive number M_0 such that $\|u_n - u\| < M_0$ for each $n \in \mathbf{N}$. Then, for any $\varepsilon > 0$, there exists $\nu_0 \in \mathbf{N}$ such that for all $v \in \mathbf{E}$ and $n > \nu_0$ we have

$$|J'(u_n)(v)| = \left|\frac{1}{2}\Psi'(u_n)(v) + \Phi'(u_n)(v)\right| < \frac{\varepsilon}{M_0}\|v\|.$$

Therefore for all $n > \nu_0$ we obtain

$$\left|\frac{1}{2}\Psi'(u_n)(u_n - u) + \Phi'(u_n)(u_n - u)\right| < \varepsilon,$$

that is,

$$\lim \left| \frac{1}{2}\Psi'(u_n)(u_n - u) + \Phi'(u_n)(u_n - u) \right| = 0.$$

Since $\mathbf{E}$ is compactly embedded in $\mathbf{L}_p$ the sequence (u_n) converges strongly to u in $\mathbf{L}_p$. Applying the Lebesgue dominated convergence theorem we conclude that

$$\lim |\Phi'(u_n)(u_n - u)| = \lim \left| \int_\Omega f_\lambda(x, u_n)\,(u_n - u)\; dx \right| = 0.$$

Therefore,

$$\lim |\Psi'(u_n)(u_n - u)| = 0. \tag{6}$$

Since $\Psi'(u)$ is a linear continuous functional on $\mathbf{E}$ we conclude that

$$\lim \Psi'(u)(u_n - u) = 0. \tag{7}$$

Then (6) and (7) imply

$$\Psi'(u_n)(u_n - u) - \Psi'(u)(u_n - u) = 2\|u_n - u\|^2 \to 0,$$

as claimed.

Now all conditions of the Mountain Pass theorem are satisfied, see, e.g., [**5, 11**], so we conclude that J_λ has a critical point v_λ, which is different from the local minimum u_λ, obtained at Step 1. □

Similar arguments can be used in the case

$$f_\lambda(x, u) = g(u + \lambda\gamma(x)), \tag{f_2}$$

where $\lambda > 0$ is a real parameter, $0 \neq \gamma \in \mathbf{L}_{(q-1)p'}$ and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function such that $g(0) = 0$ and g satisfies assumptions (g_1), (g_2). Nonlinearities of this kind arise after the reduction of a superlinear elliptic boundary value problem with nonzero Dirichlet boundary data to Hammerstein equation (1).

Theorem 6. *Assume (f_2), (g_1) and (g_2) hold. Then there exists $\lambda_* > 0$ such that for each $\lambda \in (0, \lambda_*)$ equation (1) has at least two* *solutions* $u_\lambda, v_\lambda \in \mathbf{E}$. *The solution* u_λ *is a local minimum of* J_λ *and* $\|u_\lambda\| \to 0$ *as* $\lambda \to 0$.

Proof. From (f_2) and (g_1) we deduce that

$$|f_\lambda(x,u)| \leq 2^{q-1} b |u|^{q-1} + 2^{q-1} b \lambda^{q-1} |\gamma(x)|^{q-1}.$$

Thus the condition (f) is satisfied. The function $\Lambda(\rho)$ becomes in this case

$$\Lambda(\rho) = 2^{q-1}\kappa^q b \rho^{q-1} + 2^{q-1}\kappa b \lambda^{q-1} \|\gamma\|^{q-1}_{(q-1)p'}.$$

Hence if we set

$$\lambda_* = \|\gamma\|^{-1}_{(q-1)p'} \left(2^q b \kappa^2\right)^{-1/(q-2)},$$

and

$$\rho_*(\lambda) = 2^q \lambda^{q-1} \kappa b \|\gamma\|^{q-1}_{(q-1)p'},$$

then for each $\lambda \in (0, \lambda_*)$, one has $\Lambda(\rho_*(\lambda)) < \rho_*(\lambda)$. Therefore by Theorem 4 we conclude that J_λ has a local minimum $u_\lambda \in \mathbf{E}$ such that $\|u_\lambda\| < \rho_*(\lambda)$.

Existence of a second solution v_λ to (1) for $\lambda \in (0, \lambda_*)$ can be derived via the Mountains Pass theorem in the same way as in the proof of Theorem 5. We omit the details here. □

5. Applications: a combination of sub and superlinear terms. In this section we prove the existence of multiple nontrivial solutions to equation (1) with the nonlinearity which is a combination of sublinear and superlinear terms. We consider

$$(f_3) \qquad f_\lambda(x,u) = \beta(x) g(u) + \lambda \alpha(x) u |u|^{r-2},$$

where $\lambda > 0$ is a real parameter, $0 < \alpha \in \mathbf{L}_{p/(p-r)}$ for some $r \in (1,2)$, $\beta \in \mathbf{L}_{p/(p-q)}$ for some $q \in (2,p)$ and $g : \mathbf{R} \to \mathbf{R}$ is a continuous function such that $g(0) = 0$ and g satisfies assumptions (g_1), (g_2). In this case $u = 0$ is a trivial solution to (1). We prove the following.

Theorem 7. *Assume* (f_3), (g_1) *and* (g_2) *hold. Then there exists* $\lambda_* > 0$ *such that for each* $\lambda \in (0, \lambda_*)$ *equation (1) has at least two* *nontrivial solutions $u_\lambda, v_\lambda \in \mathbf{E}$. The solution u_λ is a local minimum of J_λ and $\|u_\lambda\| \to 0$ as $\lambda \to 0$.*

Proof. We shall apply Theorem 4 to J_λ in order to prove the existence of a local minimum u_λ of J_λ for small $\lambda > 0$. Then we obtain the second solution v_λ by using the Mountain Pass theorem. Note that $f_\lambda(x,0) = 0$ and $u \equiv 0$ is a trivial solution to (1). So we must ensure that u_λ and v_λ are different from zero.

Step 1. *J_λ has a local minimum.* From (f_3) and (g_1) we deduce that

$$|f_\lambda(x,u)| \le b\beta(x)|u|^{q-1} + \lambda\alpha(x)|u|^{r-1}.$$

Thus the condition (f) is satisfied. The function $\Lambda(\rho)$ becomes in this case

$$\Lambda(\rho) = \kappa^q b\|\beta\|_{p/(p-q)}\rho^{q-1} + \lambda\kappa^r\|\alpha\|_{p/(p-r)}\rho^{r-1}.$$

By the direct computations one can see that if we set

$$\begin{aligned}\lambda_* = {}&(\kappa^2(q-r))^{-(q-r)/(q-2)}\|\alpha\|^{-1}_{p/(p-r)}(b\|\beta\|_{p/(p-q)})^{-(2-r)/(q-2)}\\ &\times (q-2)(2-r)^{(2-r)/(q-2)}\end{aligned}$$

and

$$\rho_*(\lambda) = \frac{1}{\kappa}\left(\frac{\lambda\|\alpha\|_{p/(p-r)}}{b\|\beta\|_{p/(p-q)}}\frac{(2-r)}{(q-2)}\right)^{1/(q-r)}$$

then for each $\lambda \in (0,\lambda_*)$ one has $\Lambda(\rho_*(\lambda)) < \rho_*(\lambda)$. By Theorem 4 we conclude that J_λ has a local minimum $u_\lambda \in \mathbf{E}$ such that $\|u_\lambda\| < \rho_*(\lambda)$.

Step 2. *u_λ is different from zero.* In order to prove that $u_\lambda \ne 0$, we observe that $u = 0$ is not a local minimum of J_λ for any $\lambda \in (0,\lambda_*)$. Indeed, fix $w \in \mathbf{E}$, $w \ne 0$. Let $\tau \in \mathbf{R}$. Then (g_1) implies that

$$J_\lambda(\tau w) \le \frac{1}{2}\tau^2\|w\|^2 + \frac{b}{q}|\tau|^q\int_\Omega \beta(x)|w(x)|^q\,dx - \frac{\lambda}{r}|\tau|^r\int_\Omega \alpha(x)|w(x)|^r\,dx.$$

Since $\alpha(x) > 0$ in Ω we conclude that $J_\lambda(\tau w) < 0$ for τ close to zero. Therefore $u_\lambda \ne 0$.

Step 3. *J_λ has a second critical point $v_\lambda \neq 0$.* Let $w \in \mathbf{E}$ be a critical point of J_λ. Let $c = J_\lambda(w)$ and $J_\lambda^c = \{u \in \mathbf{E} : J_\lambda(u) \leq c\}$. By $B_\rho(w)$ we denote the open ball of radius $\rho > 0$ centered at w. Recall that w is called a *mountain pass type critical point* of J_λ if there exists arbitrary small $\rho > 0$ such that the set

$$\{J_\lambda^c \cap B_\rho(w)\} \setminus \{w\}$$

is nonempty and not path connected, for topological notions mentioned in the paper we refer the readers to [**13**].

Let $\lambda \in (0, \lambda_*)$. By the previous step J_λ has a local minimum $u_\lambda \neq 0$. By the arguments similar to those used in the proof of Theorem 5 one can show that J_λ is unbounded from below and satisfies the Palais-Smale condition (PS). Thus by a version of the Mountain Pass theorem, see, e.g., [**5**], J_λ has a mountain pass type critical point v_λ or the set of critical points of J_λ is infinite. Therefore to prove that J_λ has at least two nontrivial critical points we need only to show that $u = 0$ is not a mountain pass type critical point of J_λ. This will be proved in the next two lemmas, where we follow the arguments from [**9**] with mild modifications.

Lemma 8. *There exists $\rho > 0$ such that*

$$\frac{d}{d\tau} J_\lambda(\tau u)|_{\tau=1} > 0 \tag{8}$$

for any $u \in M_\rho := \{u \in B_\rho(0) : J_\lambda(u) \geq 0\}$.

Proof. By direct computation we have

$$\begin{aligned} \frac{1}{r}\frac{d}{d\tau} J_\lambda(\tau u) = \tau \|u\|^2 &- \int_\Omega \beta(x) g(\tau u(x)) u(x)\, dx \\ &- \lambda \tau^{r-1} \int_\Omega \alpha(x) |u(x)|^r \, dx \end{aligned}$$

for $\tau > 0$. Hence,

$$\begin{aligned} \frac{1}{r}\frac{d}{d\tau} J_\lambda(\tau u)|_{\tau=1} = &\left(\frac{1}{r} - \frac{1}{2}\right)\|u\|^2 + J_\lambda(u) \\ &+ \int_\Omega \beta(x) \left[G(u(x)) - \frac{1}{r} g(u(x))u(x)\right] dx. \end{aligned}$$

Thus, by the hypotheses on g, we obtain

$$\left| \int_\Omega \beta(x) \left[G(u(x)) - \frac{1}{r} g(u(x)) u(x) \right] dx \right| \\ \leq \left(\frac{1}{q} + \frac{1}{r} \right) b\kappa^q \|\beta\|_{p/(p-q)} \|u\|^q.$$

Hence,

$$\frac{1}{r} \frac{d}{d\tau} J_\lambda(\tau u)|_{\tau=1} \geq \left(\frac{1}{r} - \frac{1}{2} - b\kappa^q \|\beta\|_{p/(p-q)} \|u\|^{q-2} \right) \|u\|^2 + J_\lambda(u).$$

Now, if $J_\lambda(u) \geq 0$, it is easily seen that as $\|u\|$ tends to zero, the righthand side of the previous inequality is positive, as we claimed. $\square$

Lemma 9. *Let $\rho > 0$ be taken from Lemma 8. Then the set $\{J_\lambda^0 \cap B_\rho(0)\} \setminus \{0\}$ is pathwise connected.*

Proof. First, we prove that the set $J_\lambda^0 \cap B_\rho(0)$ is starshaped with respect to the origin. Assume, by contradiction, that there exists $u_0 \in J_\lambda^0 \cap B_\rho(0)$ and $\tau_0 \in (0,1)$ such that $J_\lambda(\tau_0 u_0) > 0$. Then from (8) it follows that

$$\frac{d}{d\tau} J_\lambda(\tau_0 u_0) > 0.$$

By the monotonicity arguments this implies that $J_\lambda(\tau u_0) > 0$ for all $\tau \in [\tau_0, 1]$. This contradicts to definition of u_0.

Now we prove that $\{J_\lambda^0 \cap B_\rho(0)\} \setminus \{0\}$ is a retract of $B_\rho(0) \setminus \{0\}$. From Lemma 8 it follows that for each $u \in M_\rho$ there exists a positive solution $\tau(u) \in (0,1]$ of the equation

$$J_\lambda(\tau u) = 0.$$

Since the set $J_\lambda^0 \cap B_\rho$ is star-shaped with respect to the origin it follows that such a solution $\tau(u)$ is unique. Fix $u \in M_\rho$. By (8) we have

$$\frac{d}{d\tau} J_\lambda(\tau(u)u) > 0.$$

Hence the implicit function theorem implies the continuity of the function $\tau(u)$ in a neighborhood of u in M_ρ. Therefore $\tau : M_\rho \to (0,1]$ is continuous. Let $r : B_\rho(0) \to \{J^0_\lambda \cap B_\rho(0)\} \setminus \{0\}$ be defined by the formula

$$r(u) = \begin{cases} \tau(u)u, & u \in M_\rho, \\ u, & u \in \{J^0_\lambda \cap B_\rho(0)\} \setminus \{0\}. \end{cases}$$

The continuity of r follows from the continuity of τ. Moreover

$$r(u) = u \quad \text{for} \quad u \in \{J^0_\lambda \cap B_\rho(0)\} \setminus \{0\}.$$

Thus r is a retraction of $\{B_\rho\} \setminus \{0\}$ into $\{J^0_\lambda \cap B_\rho\} \setminus \{0\}$. But $\{B_\rho\} \setminus \{0\}$ is contractible in itself. By [**13**] the retract of a contractible in itself set is also contractible in itself. Therefore $\{J^0_\lambda \cap B_\rho\} \setminus \{0\}$ is contractible in itself. In particular, $\{J^0_\lambda \cap B_\rho\} \setminus \{0\}$ is pathwise connected. This concludes the proof of Lemma 9 and Theorem 7. $\square$

6. Example: a polyharmonic elliptic problem. As an example of applications of Theorems 5, 6 and 7 let us consider a semi-linear polyharmonic problem

$$\begin{cases} (-\Delta)^m u = f(x,u) & \text{in } \Omega, \\ \mathcal{D}_m u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

where $m \in \mathbf{N}$ is an integer, $(-\Delta)^m$ is the m-harmonic Laplace operator, $\mathcal{D}_m u := (D^k u)_{k \in \mathbf{N}^n}$, $0 \le |k| \le m-1$, is the boundary operator, $\Omega \subset \mathbf{R}^N$ is a bounded domain with the boundary $\partial\Omega$ of the class C^{2m+1} and $f(x,u) : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function.

The problem (9) is equivalent to Hammerstein equation (1) with the kernel generated by the Green function for the polyharmonic operator $(-\Delta)^m$ in Ω with the boundary conditions $\mathcal{D}_m u = 0$ on $\partial\Omega$. It is known, see, e.g., Sections 3.5 and 4.4.4 of [**6**], that such Green function $G_{m,N}(x,y)$ exists, symmetric and satisfies the estimate

$$|G_{m,N}(x,y)| \le \begin{cases} c|x-y|^{2m-N} & \text{if } m < (N/2), \\ |\log|x-y|| + c & \text{if } m = (N/2), \\ c & \text{if } m > (N/2). \end{cases}$$

This implies that integral operator $\mathbf{K}$ is compact as operator from $\mathbf{L}_{p'}$ into $\mathbf{L}_p$, where

$$p < \begin{cases} (2N)/(N-2m) & \text{if } m < (N/2), \\ \infty & \text{if } m \ge (N/2). \end{cases}$$

Moreover, the operator $\mathbf{K}$ is positive definite in $\mathbf{L}_2$, that is $(\mathbf{K}u, u) > 0$ for all $0 \neq u \in \mathbf{L}_2$ [**6**, Section 4.3]. Notice that the Green function $G_{m,N}(x,y)$ changes sign on many model domains, see, e.g., [**4**]. Thus the classical positivity methods can not be used to study (9). On the other hand, the results of the present paper are directly applicable to (9).

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