JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 15, Number 3, Fall 2003

FREDHOLMNESS OF SINGULAR INTEGRAL OPERATORS WITH PIECEWISE CONTINUOUS COEFFICIENTS ON WEIGHTED BANACH FUNCTION SPACES

ALEXEI YU. KARLOVICH

ABSTRACT. We prove necessary conditions for the Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces. These conditions are formulated in terms of indices of submultiplicative functions associated with local properties of the space, of the curve, and of the weight. As an example, we consider weighted Nakano spaces $L_w^{p(\cdot)}$ (weighted Lebesgue spaces with variable exponent). Moreover, our necessary conditions become also sufficient for weighted Nakano spaces over nice curves whenever w is a Khvedelidze weight, and the variable exponent $p(t)$ satisfies the estimate

$$|p(\tau) - p(t)| \leq A/(-\log|\tau - t|).$$

1. Introduction. Let Γ be a Jordan curve, that is, a curve that homeomorphic to a circle. We suppose that Γ is rectifiable. We equip Γ with Lebesgue length measure $|d\tau|$ and the counter-clockwise orientation. The *Cauchy singular integral* of a measurable function $f : \Gamma \to \mathbf{C}$ is defined by

$$(Sf)(t) := \lim_{R \to 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t,R)} \frac{f(\tau)}{\tau - t} d\tau, \quad t \in \Gamma,$$

where the "portion" $\Gamma(t, R)$ is

$$\Gamma(t, R) := \{\tau \in \Gamma : |\tau - t| < R\}, \quad R > 0.$$

2000 AMS *Mathematics Subject Classification.* Primary 45E05, 46E30, Secondary 47B35, 47A53, 47A68.
Key words and phrases. Weighted Banach function space, Nakano space, singular integral operator, Fredholmness, Carleson curve, indices of submultiplicative function.
The author is partially supported by F.C.T. (Portugal) grants POCTI 34222/MAT/2000 and PRAXIS XXI/BPD/22006/99.
Received by the editors on February 18, 2003, and in revised form on April 29, 2003.

It is well known that $(Sf)(t)$ exists almost everywhere on Γ whenever f is integrable, see [**11**, Theorem 2.22]. A measurable function $w:\Gamma\to[0,\infty]$ is referred to as a *weight* if $0<w(t)<\infty$ almost everywhere on Γ. The Cauchy singular integral generates a bounded linear operator S on the weighted Lebesgue space L^p_w, $1<p<\infty$, with the norm

$$\|f\|_{L^p_w} := \left(\int_\Gamma |f(\tau)|^p w^p(\tau)|d\tau|\right)^{1/p}$$

if and only if w is a Muckenhoupt weight, $w\in A_p(\Gamma)$, that is,

$$\sup_{t\in\Gamma}\sup_{R>0}\left(\frac{1}{R}\int_{\Gamma(t,R)} w^p(\tau)|d\tau|\right)^{1/p}\left(\frac{1}{R}\int_{\Gamma(t,R)} w^{-p'}(\tau)|d\tau|\right)^{1/p'}<\infty,$$
$$\frac{1}{p}+\frac{1}{p'}=1,$$

see, e.g., [**3**, Theorem 4.15]. By Hölder's inequality, if $w\in A_p(\Gamma)$, then Γ is a *Carleson*, or *Ahlfors-David regular, curve*, that is,

$$C_\Gamma := \sup_{t\in\Gamma}\sup_{R>0}\frac{|\Gamma(t,R)|}{R}<\infty, \tag{1.1}$$

where $|\Omega|$ denotes the measure of a measurable set $\Omega\subset\Gamma$. The constant C_Γ is said to be the *Carleson constant*. We denote by PC the Banach algebra of all *piecewise continuous* functions on the curve Γ: by definition, a is in PC if and only if a is in L^∞ and the one-sided limits

$$a(t\pm 0) := \lim_{\tau\to t\pm 0} a(\tau)$$

exist for every $t\in\Gamma$.

A bounded linear operator A on a Banach space is said to be *semi-Fredholm* if its image is closed and at least one of the so-called defect numbers

$$n(A) := \dim\ker A, \quad d(A) := \dim\ker A^*$$

is finite. A semi-Fredholm operator A is called *Fredholm* if both $n(A)$ and $d(A)$ are finite. In this case the difference $n(A)-d(A)$ is referred to as the *index* of the operator A. Basic properties of (semi)-Fredholm operators are discussed in [**5, 16, 40**] and in many other monographs.

The study of Fredholmness of one-dimensional singular integral operators of the form

$$R_a := aP_+ + P_-, \quad a \in PC, \quad P_\pm := (I \pm S)/2$$

on Lebesgue spaces with power (Khvedelidze) weights

$$\varrho(t) := \prod_{k=1}^{n} |t - \tau_k|^{\lambda_k}, \quad \tau_k \in \Gamma, \quad k \in \{1, \dots, n\}, \quad n \in \mathbf{N}, \tag{1.2}$$

over Lyapunov curves started in the fifties with Khvedelidze [**27**] and was continued in the sixties by Widom, Simonenko, Gohberg and Krupnik, and others. The history and corresponding references can be found, e.g., in [**3, 16, 21, 28, 40**]. In the beginning of the nineties, Spitkovsky proved Fredholm criteria for singular integral operators with piecewise continuous coefficients on Lebesgue spaces with Muckenhoupt weights over smooth curves [**52**]. In the middle of nineties, Böttcher and Yu. Karlovich accomplished the Fredholm theory for the algebra of singular integral operators with piecewise continuous coefficients on Lebesgue spaces with Muckenhoupt weights over general Carleson curves. These results are documented in [**3**]; see also the brief but nice presentation in [**4**].

Lebesgue spaces $L^p, 1 \le p \le \infty$, are the simplest examples of so-called *Banach function spaces* introduced by Luxemburg in 1955. This scale of spaces includes Orlicz, Lorentz, and all other *rearrangement-invariant spaces*. By analogy with weighted Lebesgue spaces, for a Banach function space X and a weight w, it is possible to define the *weighted Banach function space*

$$X_w := \Big\{ f \text{ is measurable on } \Gamma \text{ and } fw \in X \Big\}.$$

Under some restrictions on the weight w, the space X_w is itself a Banach function space, although if X is a rearrangement-invariant Banach function space, then X_w is not necessarily rearrangement-invariant (even if X is a Lebesgue space). Another interesting class of Banach function spaces which are not rearrangement-invariant are constituted by Nakano spaces $L^{p(\cdot)}$ (generalized Lebesgue spaces with variable exponent). For details and references, see Section 2.

Unfortunately, little known about the boundedness of S on general weighted Banach function spaces X_w. As far as we know, even a criterion for the boundedness of S on Orlicz spaces L^φ_w with general weights w over general Carleson curves is unknown at the moment (February, 2003). We proved necessary conditions for the boundedness of S on weighted rearrangement-invariant Banach function spaces [**24**, Theorem 3.2] in terms of an analog of the Muckenhoupt class. On the other hand, if a weight w belongs to the Muckenhoupt classes $A_{1/\alpha_X}(\Gamma)$ and $A_{1/\beta_X}(\Gamma)$ where $\alpha_X, \beta_X \in (0,1)$ are the Boyd indices of a rearrangement-invariant Banach function space X, then S is bounded on the weighted rearrangement-invariant Banach function space X_w, see [**26**, Theorem 4.5].

On the basis of these boundedness results, following the approach of Böttcher, Yu. Karlovich, and Spitkovsky, the author proved separately necessary and sufficient conditions for Fredholmness of singular integral operators with piecewise continuous coefficients on weighted rearrangement-invariant Banach function spaces [**25, 26**]. Under some restrictions on spaces, curves, and weights, these conditions coincide, that is, become criteria. In those cases, the Banach algebra of singular integral operators with piecewise continuous coefficients is also studied [**26**].

Very recently Kokilashvili and Samko have proved criteria for the boundedness of S on Nakano spaces $L^{p(\cdot)}_\varrho$ with Khvedelidze weights ϱ over Lyapunov curves or Radon curves without cusps provided the variable exponent p satisfies the estimate

$$|p(\tau) - p(t)| \le A/(-\log|\tau - t|), \quad \tau, t \in \Gamma, \quad |\tau - t| \le 1/2, \tag{1.3}$$

see [**30**, Theorem 2] or Theorem 6.2. With the help of this key result, they have proved Fredholm criteria for the operator $aP_+ + bP_-$ with piecewise continuous functions a, b having finite numbers of jumps on (non-weighted) Nakano spaces $L^{p(\cdot)}$, see [**31**, Theorem A].

For an arbitrary weight w and an arbitrary Banach function space X, we define the weighted Banach function space X_w. Assume that

(B) the Cauchy singular integral operator S is bounded on X_w;

(R) X_w is reflexive.

We show that property (B) implies the condition $A_X(\Gamma)$ of Muckenhoupt type. In that case X_w is itself a Banach function space. Under the assumptions (B) and (R) we prove necessary conditions for Fredholmness of singular integral operators R_a with piecewise continuous coefficients a in the weighted Banach function spaces X_w. This result generalizes corresponding necessary conditions in [**25**, Theorem 4.2]. As an example, we consider these necessary conditions in Nakano spaces $L^{p(\cdot)}_w$ with general weights w. They have almost the same form as in the case of Lebesgue spaces L^p_w with Muckenhoupt weights over Carleson curves, see [**3**, Proposition 7.3]. We need only replace the constant p (for weighted Lebesgue spaces L^p_w) by the value $p(t)$ of the variable exponent $p(\cdot)$ at each point $t \in \Gamma$ (for weighted Nakano spaces $L^{p(\cdot)}_w$). Our approach is based on a local principle of Simonenko type, the Wiener-Hopf factorization of local representatives, and the theory of submultiplicative functions associated with local properties of the curve, of the weight, and of the space. Using the local principle allows us to consider coefficients a having a countable number of jumps (in contrast to [**31**], where only a finite number of jumps is allowed).

The paper is organized as follows. In Section 2 we collect necessary preliminaries on weighted Banach function spaces X_w and Nakano spaces $L^{p(\cdot)}$. In Section 3 we define an analog of the Muckenhoupt class $A_p(\Gamma)$, replacing the norm in L^p by the norm in a Banach function space X. We denote this class by $A_X(\Gamma)$. We show that if $w \in A_X(\Gamma)$ and $1 \in A_X(\Gamma)$, then $\log w$ has bounded mean oscillation. In Section 4 we recall the definitions and some properties of submultiplicative functions associated with the local behavior of the curve, of the weight, and of the space. In Section 5 we study inequalities between the indices of submultiplicative functions defined in Section 4. We investigate so-called *indicator functions* α^*_t, β^*_t and α_t, β_t of the triple (Γ, X, w) and of the pair (Γ, w), respectively. In particular, we show that if X is a Nakano space $L^{p(\cdot)}$ with a variable exponent $p(\cdot)$ satisfying (1.3), then we can separate the influence of the space from the influence of the weight and the curve, that is,

$$\alpha^*_t(x) = \frac{1}{p(t)} + \alpha_t(x), \quad \beta^*_t(x) = \frac{1}{p(t)} + \beta_t(x)$$

for $x \in \mathbf{R}$ such that $|(\tau - t)^{y+ix}|w(\tau) \in A_{L^{p(\cdot)}}(\Gamma, t)$, where $A_{L^{p(\cdot)}}(\Gamma, t)$ is the local analog of $A_{L^{p(\cdot)}}(\Gamma)$. So, weighted Nakano spaces satisfy the "disintegration condition" in the terminology of [**24, 26**].

In Section 6 we prove that the condition $w \in A_X(\Gamma)$ is necessary for the boundedness of the Cauchy singular integral operator S on the weighted Banach function spaces X_w. Further we extend basic results on the Fredholmness of singular integral operators with bounded measurable coefficients (the local principle, the theorem about a Wiener-Hopf factorization, etc.) to weighted Banach function spaces satisfying Axioms (B) and (R). These results are natural extensions of the classical theory for Lebesgue spaces with Khvedelidze weights over Lyapunov curves, see, e.g., [**16**, Chapters 7–8] or [**40**, Chapter 4]. A canonical local representative $g_{t,\gamma}$ for a piecewise continuous function is constructed in Section 7. We prove separately necessary and sufficient conditions for factorability of $g_{t,\gamma}$ in the weighted Banach function space X_w. On the basis of our necessary conditions for factorability, with the help of the results of Section 6, we prove necessary conditions for Fredholmness of the singular integral operator $R_a = aP_+ + P_-$ with $a \in PC$ in X_w. These conditions are formulated in terms of the indicator functions α_t^* and β_t^* defined in Section 5. In Section 8 we reformulate these necessary conditions for weighted Nakano spaces $L_w^{p(\cdot)}$ with general weights w and variable exponents satisfying (1.3) in terms of simpler indicator functions α_t and β_t. With the help of the boundedness criteria by Kokilashvili and Samko [**30**, Theorem 2], we prove that the latter necessary conditions become also sufficient if $w = \varrho$ is a Khvedelidze weight and Γ is either a Lyapunov Jordan curve or a Radon Jordan curve without cusps.

2. Weighted Banach function spaces.

2.1 Banach function spaces. Let Γ be a rectifiable Jordan (i.e., homeomorphic to a circle) curve equipped with Lebesgue length measure $|d\tau|$. The set of all measurable complex-valued functions on Γ is denoted by $\mathcal{M}$. Let $\mathcal{M}^+$ be the subset of functions in $\mathcal{M}$ whose values lie in $[0,\infty]$. The characteristic function of a measurable set $E \subset \Gamma$ is denoted by χ_E.

Definition 2.1 (Luxemburg, 1955, see [**1**, Chapter 1, Definition 1.1]). A mapping $\rho : \mathcal{M}^+ \to [0,\infty]$ is called a *Banach function norm* if, for all functions f, g, f_n ($n \in \mathbf{N}$) in $\mathcal{M}^+$, for all constants $a \ge 0$, and for all measurable subsets E of Γ, the following properties hold:

(A1) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$, $\rho(f+g) \le \rho(f) + \rho(g)$,
(A2) $0 \le g \le f$ a.e. $\Rightarrow \rho(g) \le \rho(f)$ (the lattice property),
(A3) $0 \le f_n \uparrow f$ a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$ (the Fatou property),
(A4) $\rho(\chi_E) < \infty$,
(A5) $\int_E f(\tau)|d\tau| \le C_E \rho(f)$
with $C_E \in (0, \infty)$ may depend on E and ρ but is independent of f.

When functions differing only on a set of measure zero are identified, the set X of all functions $f \in \mathcal{M}$ for which $\rho(|f|) < \infty$ is called a *Banach function space*. For each $f \in X$, the norm of f is defined by

$$\|f\|_X := \rho(|f|).$$

The set X under the natural linear space operations and under this norm becomes a Banach space [**1**, Chapter 1, Theorems 1.4 and 1.6].

If ρ is a Banach function norm, its associate norm ρ' is defined on $\mathcal{M}^+$ by

$$\rho'(g) := \sup\left\{ \int_\Gamma f(\tau)g(\tau)|d\tau| \ : \ f \in \mathcal{M}^+, \ \rho(f) \le 1 \right\}, \quad g \in \mathcal{M}^+.$$

It is a Banach function norm itself [**1**, Chapter 1, Theorem 2.2]. The Banach function space X' determined by the Banach function norm ρ' is called the *associate space* (*Köthe dual*) of X. The associate space X' is a subspace of the dual space X^*. The construction of the associate space implies the following Hölder inequality for Banach function spaces.

Lemma 2.2 (see [**1**, Chapter 1, Theorem 2.4]). *Let X be a Banach function space and X' be its associate space. If $f \in X$ and $g \in X'$, then fg is integrable and $\|fg\|_{L^1} \le \|f\|_X \|g\|_{X'}$.*

2.2 Rearrangement-invariant Banach function spaces. Let $\mathcal{M}_0$ and $\mathcal{M}_0^+$ be the classes of almost everywhere finite functions in $\mathcal{M}$ and $\mathcal{M}^+$, respectively. Two functions $f, g \in \mathcal{M}_0$ are said to be equimeasurable if

$$\Big|\{\tau \in \Gamma : |f(\tau)| > \lambda\}\Big| = \Big|\{\tau \in \Gamma : |g(\tau)| > \lambda\}\Big| \quad \text{for all } \lambda \ge 0.$$

A Banach function norm $\rho : \mathcal{M}^+ \to [0,\infty]$ is called rearrangement-invariant if for every pair of equimeasurable functions $f, g \in \mathcal{M}_0^+$ the equality $\rho(f) = \rho(g)$ holds. In that case, the Banach function space X generated by ρ is said to be a *rearrangement-invariant Banach function space* (or simply rearrangement-invariant space). Lebesgue, Orlicz, Lorentz, and Lorentz-Orlicz spaces are classical examples of rearrangement-invariant Banach function spaces, see, e.g., [**1**] and the references therein.

If X is an arbitrary rearrangement-invariant Banach function space and X' is its associate space, then for a measurable set $E \subset \Gamma$,

$$\|\chi_E\|_X \|\chi_E\|_{X'} = |E|, \tag{2.1}$$

see, e.g., [**1**, Chapter 2, Theorem 5.2].

2.3 Nakano spaces $L^{p(\cdot)}$. Function spaces $L^{p(\cdot)}$ of Lebesgue type with variable exponent p were studied for the first time probably by Orlicz [**45**] in 1931. Inspired by the successful theory of Orlicz spaces, Nakano defined in the late forties [**43, 44**] so-called *modular spaces*. He considered the space $L^{p(\cdot)}$ as an example of modular spaces. Musielak and Orlicz [**42**] extended Nakano's definition of modular spaces in 1959. Actually, that paper was the starting point for the theory of Musielak-Orlicz spaces (generalized Orlicz spaces generated by Young functions with a parameter), see [**41**].

Let $p : \Gamma \to [1,\infty)$ be a measurable function. Consider the convex modular (see [**41**, Chapter 1] for definitions and properties)

$$m(f,p) := \int_\Gamma |f(\tau)|^{p(\tau)} |d\tau|.$$

Denote by $L^{p(\cdot)}$ the set of all measurable complex-valued functions f on Γ such that $m(\lambda f, p) < \infty$ for some $\lambda = \lambda(f) > 0$. This set becomes a Banach space with respect to the *Luxemburg-Nakano norm*

$$\|f\|_{L^{p(\cdot)}} := \inf\Big\{\lambda > 0 : \ m(f/\lambda, p) \le 1\Big\},$$

see, e.g., [**41**, Chapter 2]. So, the spaces $L^{p(\cdot)}$ are a special case of Musielak-Orlicz spaces. Sometimes the spaces $L^{p(\cdot)}$ are referred to as

Nakano spaces, see, e.g., [**13**, p. 151], [**19**, p. 179]. We will follow this tradition. Clearly, if $p(\cdot) = p$ is constant, then the Nakano space $L^{p(\cdot)}$ is isometrically isomorphic to the Lebesgue space L^p. Therefore, sometimes $L^{p(\cdot)}$ are called generalized Lebesgue spaces with variable exponent.

Lemma 2.3 (see, e.g., [**12**, Proposition 1.3]). *Let $p : \Gamma \to [1,\infty)$ be a measurable function. The Nakano space $L^{p(\cdot)}$ is a Banach function space.*

It is not difficult to show that $L^{p(\cdot)}$ is not rearrangement-invariant, in general.

The following result on the reflexivity and duality of Nakano spaces was precisely stated in [**32**, Theorem 2.3 and Corollary 2.7], although it can be obtained from more general results for Musielak-Orlicz spaces [**41**, Chapters 1–2], see also [**45**].

Lemma 2.4. *Let $p : \Gamma \to [1,\infty)$ be a measurable function. If*

$$1 < \operatorname*{ess\,inf}_{t\in\Gamma} p(t) \le \operatorname*{ess\,sup}_{t\in\Gamma} p(t) < \infty,$$

then the Nakano space $L^{p(\cdot)}$ is reflexive. Its associate space coincides (up to the equivalence of the norms) with the Nakano space $L^{p'(\cdot)}$, where

$$p'(\tau) := \frac{p(\tau)}{p(\tau)-1}.$$

Finally, Nakano spaces are important in applications to fluid dynamics [**48**].

2.4 Weighted Banach function spaces. Let X be a Banach function space generated by a Banach function norm ρ and let $w : \Gamma \to [0,\infty]$ be a weight. Define the mapping $\rho_w : \mathcal{M}^+ \to [0,\infty]$ and the set X_w by

$$\rho_w(f) := \rho(fw) \quad (f \in \mathcal{M}^+), \qquad X_w := \Big\{ f \in \mathcal{M}^+ :\ fw \in X \Big\}.$$

Lemma 2.5. (a) ρ_w *satisfies Axioms* (A1)–(A3) *in Definition* 2.1 *and* X_w *is a linear normed space with respect to the norm*

$$\|f\|_{X_w} := \rho_w(|f|) = \rho(|fw|) = \|fw\|_X;$$

(b) *if* $w \in X$ *and* $1/w \in X'$, *then* ρ_w *is a Banach function norm and* X_w *is a Banach function space generated by* ρ_w. *Moreover,*

$$L^\infty \subset X_w \subset L^1;$$

(c) *if* $w \in X$ *and* $1/w \in X'$, *then* $X'_{1/w}$ *is the associate space for the Banach function space* X_w.

Proof. Part (a) follows from Axioms (A1)–(A3) for the Banach function norm ρ and the fact that $0 < w(\tau) < \infty$ almost everywhere on Γ.

(b) If $w \in X$, then by Axiom (A2) for ρ, we get $w\chi_E \in X$ for every measurable set E of Γ. Therefore, $\rho_w(\chi_E) = \rho(w\chi_E) < \infty$. Thus, ρ_w satisfies Axiom (A4). By Hölder's inequality, see Lemma 2.2, and Axiom (A2) for ρ, we have

$$\begin{aligned} \int_E f(\tau)|d\tau| &= \int_\Gamma \Big(f(\tau)w(\tau)\chi_E(\tau)\Big)\frac{\chi_E(\tau)}{w(\tau)}\,|d\tau| \\ &\le \rho(fw\chi_E)\rho'(\chi_E/w) \le \rho(fw)\rho'(\chi_E/w) \\ &=: C_E\rho_w(f), \end{aligned} \tag{2.2}$$

where $C_E := \rho'(\chi_E/w) \in (0,\infty)$. This constant, clearly, depends on ρ, w, and E, but it is independent of f. Therefore, ρ_w satisfies Axiom (A5). Thus, ρ_w is a Banach function norm and X_w is a Banach function space.

From (2.2) and Axiom (A2) for X' it follows that

$$\|f\|_{L^1} \le \|f\|_{X_w}\|1/w\|_{X'}, \quad f \in X_w.$$

Hence, $X_w \subset L^1$, in view of $1/w \in X'$. On the other hand, for $f \in L^\infty$,

$$0 \le |f(\tau)| \le \|f\|_\infty \quad \text{a.e. on} \quad \Gamma.$$

By Axioms (A2) and (A1) for ρ_w, we have

$$\|f\|_{X_w} = \rho_w(|f|) \le \rho_w(\|f\|_\infty) = \|f\|_\infty \rho_w(1) = \|f\|_\infty \|w\|_X.$$

Thus, $L^\infty \subset X_w$, in view of $w \in X$. Part (b) is proved.

(c) For $g \in \mathcal{M}^+$, we have

$$\begin{aligned}(\rho_w)'(g) &= \sup\left\{ \int_\Gamma f(\tau)g(\tau)|d\tau| : \ f \in \mathcal{M}^+, \ \rho_w(f) \le 1 \right\} \\ &= \sup\left\{ \int_\Gamma \Big(f(\tau)w(\tau)\Big)\left(\frac{g(\tau)}{w(\tau)}\right)|d\tau| : \ f \in \mathcal{M}^+, \ \rho(fw) \le 1 \right\} \\ &= \sup\left\{ \int_\Gamma h(\tau)\left(\frac{g(\tau)}{w(\tau)}\right)|d\tau| : \ h \in \mathcal{M}^+, \ \rho(h) \le 1 \right\} \\ &= \rho'(g/w).\end{aligned}$$

Hence, $(X_w)' = X'_{1/w}$. □

We will refer to the normed space X_w as a *weighted Banach function space* generated by the Banach function space X and the weight w. From Lemma 2.5(b) it follows that the weighted Banach function space X_w is a Banach function space itself whenever $w \in X$ and $1/w \in X'$.

For other definitions (different from ours) of weighted Banach function spaces, see, e.g., [**34, 37**].

2.5 Separability and reflexivity of weighted Banach function spaces. A function f in a Banach function space X is said to have *absolutely continuous norm* in X if $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^\infty$ of measurable sets on Γ satisfying $\chi_{E_n} \to 0$ almost everywhere on Γ as $n \to \infty$. If all functions $f \in X$ have this property, then the space X itself is said to have *absolutely continuous norm*, see [**1**, Chapter 1, Section 3].

In this subsection we assume that X is a Banach function space and w is a weight such that $w \in X$ and $1/w \in X'$. Then, by Lemma 2.5(b), the weighted Banach function space X_w is itself a Banach function space.

Proposition 2.6. *If X has absolutely continuous norm, then X_w has absolutely continuous norm too.*

Proof. If $f \in X_w$, then $fw \in X$ has absolutely continuous norm in X. Therefore, $\|f\chi_{E_n}\|_{X_w} = \|fw\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\}_{n=1}^{\infty}$ of measurable sets on Γ satisfying $\chi_{E_n} \to 0$ almost everywhere on Γ as $n \to \infty$. Thus, $f \in X_w$ has absolutely continuous norm in X_w. □

From Lemma 2.5 and [**1**, Chapter 1, Corollaries 4.3, 4.4] we obtain the following.

Lemma 2.7. (a) *The Banach space dual $(X_w)^*$ of the weighted Banach function space X_w is isometrically isomorphic to the associate space $X'_{1/w}$ if and only if X_w has absolutely continuous norm. If X_w has absolutely continuous norm, then the general form of a linear functional on X_w is given by*

$$G(f) := \int_\Gamma f(\tau)\overline{g(\tau)}|d\tau|, \quad g \in X'_{1/w}, \quad \text{and} \quad \|G\|_{(X_w)^*} = \|g\|_{X'_{1/w}}.$$

(b) *The weighted Banach function space X_w is reflexive if and only if both X_w and $X'_{1/w}$ have absolutely continuous norm.*

Corollary 2.8. *If X is reflexive, then X_w is reflexive.*

Proof. If X is reflexive, then, by [**1**, Chapter 1, Corollary 4.4], both X and X' have absolutely continuous norm. In that case, due to Proposition 2.6, both X_w and $X'_{1/w}$ have absolutely continuous norm. By Lemma 2.7(b), X_w is reflexive. □

Since Lebesgue length measure $|d\tau|$ is separable (for the definition of a separable measure, see, e.g., [**1**, p. 27] or [**20**, Section 6.10]), from Lemma 2.5 and [**1**, Chapter 1, Corollary 5.6] we immediately get the following criterion.

Lemma 2.9. *The weighted Banach function space X_w is separable if and only if it has absolutely continuous norm.*

We denote by C the set of all continuous functions on Γ and by $\mathcal{R}$ the set of all rational functions without poles on the curve Γ. With the help of Lemmas 2.7 and 2.9, literally repeating the proof of [**25**, Lemma 1.3], one can get the following.

Lemma 2.10. *The weighted Banach function space X_w is separable if and only if C is dense in X_w.*

Corollary 2.11. *If X_w (or X) is reflexive, then $\mathcal{R}$ is dense in X_w and in its associate space $X'_{1/w}$.*

Proof. If X_w is reflexive, then by Lemmas 2.7(b) and 2.9, both X_w and $X'_{1/w}$ are separable. This implies that C is dense in X_w and in $X'_{1/w}$, due to Lemma 2.10. In view of the Mergelyan theorem (see, e.g., [**14**, Chapter III, Section 2]), every function in C may uniformly be approximated by functions in $\mathcal{R}$. Thus, $\mathcal{R}$ is dense in X_w and in $X'_{1/w}$. If X is reflexive, we need first apply Corollary 2.8 and then repeat the above arguments. $\square$

3. Analogs of the Muckenhoupt class.

3.1 Definitions. Let X be a Banach function space. Fix $t \in \Gamma$. For a weight $w : \Gamma \to [0, \infty]$, put

$$B_{t,R}(w) := \frac{1}{R} \|w\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}/w\|_{X'}, \quad R > 0,$$

where $\chi_{\Gamma(t,R)}$ is the characteristic function of the portion $\Gamma(t, R)$. Consider the following classes of weights:

$$A_X(\Gamma, t) := \Big\{ w : \sup_{R>0} B_{t,R}(w) < \infty \Big\},$$
$$A_X(\Gamma) := \Big\{ w : \sup_{t\in\Gamma} \sup_{R>0} B_{t,R}(w) < \infty \Big\}.$$

Obviously, $A_X(\Gamma) \subset A_X(\Gamma, t)$ for $t \in \Gamma$. If X is a Lebesgue space $L^p, p \in (1, \infty)$, then $A_X(\Gamma)$ is the Muckenhoupt class $A_p(\Gamma)$. For a detailed discussion of Muckenhoupt weights on curves, see, e.g., [**3**]. The classes $A_X(\Gamma, t)$ and $A_X(\Gamma)$ were defined in [**24**] (see also [**22, 25**]) for rearrangement-invariant spaces X. Here we assume only that

X is a Banach function space. Our definition is similar to a definition in [**2**]. For other generalizations (different from ours) of the Muckenhoupt class $A_p(\Gamma)$ in the setting of Orlicz and Lorentz spaces, see, e.g., [**15, 29**] and in the setting of Banach function spaces, see [**34**].

With the help of Hölder's inequality (see Lemma 2.2), it is easy to show that $w \in A_X(\Gamma, t)$ implies

$$C_{\Gamma,t} := \sup_{R>0} \frac{|\Gamma(t,R)|}{R} < \infty. \tag{3.1}$$

We say that a rectifiable Jordan curve Γ is *locally a Carleson curve at the point* $t \in \Gamma$ if (3.1) is satisfied. In that case the constant $C_{\Gamma,t}$ is referred to as the *local Carleson constant at the point* $t \in \Gamma$. Analogously, if $w \in A_X(\Gamma)$, then

$$C_\Gamma = \sup_{t\in\Gamma} C_{\Gamma,t} < \infty,$$

that is, Γ is a Carleson curve.

3.2 Bounded and vanishing mean oscillation. Let Γ be a rectifiable Jordan curve. Let $f : \Gamma \to [-\infty, \infty]$ and $f \in L^1$. Suppose $t \in \Gamma, \delta \in (0, \infty]$, and $R \in (0, \infty)$. Put

$$\Omega_t(f,R) := \frac{1}{|\Gamma(t,R)|} \int_{\Gamma(t,R)} f(\tau)|d\tau|,$$
$$M_{\delta,t}(f) := \sup_{0<R<\delta} \frac{1}{|\Gamma(t,R)|} \int_{\Gamma(t,R)} |f(\tau) - \Omega_t(f,R)||d\tau|.$$

A function f is said to be of *bounded mean oscillation at the point* $t \in \Gamma$ if $\|f\|_{*,t} := M_{\infty,t}(f) < \infty$. In this case we will write $f \in BMO(\Gamma, t)$. A function $f \in BMO(\Gamma, t)$ has *vanishing mean oscillation at the point* $t \in \Gamma$ if

$$\lim_{\delta\to 0} M_{\delta,t}(f) = 0.$$

In that case we will write $f \in VMO(\Gamma, t)$.

One says that a function $f : \Gamma \to [-\infty, \infty]$ is of *bounded mean oscillation on* Γ if $f \in BMO(\Gamma, t)$ for all $t \in \Gamma$ and

$$\|f\|_* := \sup_{t\in\Gamma} \|f\|_{*,t} < \infty.$$

The class of functions of bounded mean oscillation on Γ is denoted by $BMO(\Gamma)$. A function $f \in BMO(\Gamma)$ is said to be of *vanishing mean oscillation on* Γ if

$$\lim_{\delta\to 0} \sup_{t\in\Gamma} M_{\delta,t}(f) = 0.$$

The class of functions of vanishing mean oscillation on Γ is denoted by $VMO(\Gamma)$. Clearly, $BMO(\Gamma) \subset BMO(\Gamma, t)$ and $VMO(\Gamma) \subset VMO(\Gamma, t)$ for every $t \in \Gamma$.

3.3 Bounded mean oscillation of logarithms of weights. Let

$$d_t := \max_{\tau\in\Gamma} |\tau - t|.$$

For a weight $w : \Gamma \to [0, \infty]$ such that $w \in X$ and $1/w \in X'$, we have $w, 1/w \in L^1$. Then, taking into account the obvious inequality $|\log x| \le x + 1/x$ for $x \in (0, \infty)$, we deduce that $\log w \in L^1$. For $t \in \Gamma$ and $R > 0$, put

$$C(w, t, R) := \exp(-\Omega_t(\log w, R)) \frac{\|w\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}\|_{X'}}{|\Gamma(t, R)|},$$
$$C'(w, t, R) := \exp(\Omega_t(\log w, R)) \frac{\|\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}/w\|_{X'}}{|\Gamma(t, R)|}.$$

Clearly, these quantities are well defined.

Lemma 3.1. (a) *If* $w \in A_X(\Gamma, t)$ *and* $1 \in A_X(\Gamma, t)$*, then*

$$1 \le \sup_{R>0} C(w, t, R) < \infty, \quad 1 \le \sup_{R>0} C'(w, t, R) < \infty. \tag{3.2}$$

(b) *If* $w \in A_X(\Gamma)$ *and* $1 \in A_X(\Gamma)$*, then*

$$1 \le \sup_{t\in\Gamma} \sup_{R>0} C(w, t, R) < \infty, \quad 1 \le \sup_{t\in\Gamma} \sup_{R>0} C'(w, t, R) < \infty. \tag{3.3}$$

Proof. The proof is developed by similarity to [**25**, Lemma 1.5]. Applying Jensen's inequality (see, e.g., [**33**, p. 78]) and Hölder's inequality (see Lemma 2.2), we obtain

$$\begin{aligned}\exp(\Omega_t(\log w, R)) &\le \frac{1}{|\Gamma(t,R)|}\int_{\Gamma(t,R)} w(\tau)|d\tau| \\ &\le \frac{\|w\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}\|_{X'}}{|\Gamma(t,R)|}.\end{aligned}$$

Hence,

$$1 \le C(w,t,R), \quad R > 0. \tag{3.4}$$

Analogously,

$$1 \le C'(w,t,R), \quad R > 0. \tag{3.5}$$

Inequalities (3.4) and (3.5) imply that (3.2) is equivalent to

$$\sup_{R>0}\Big(C(w,t,R)C'(w,t,R)\Big) < \infty \tag{3.6}$$

and (3.3) is equivalent to

$$\sup_{t\in\Gamma}\sup_{R>0}\Big(C(w,t,R)C'(w,t,R)\Big) < \infty. \tag{3.7}$$

Since $\Gamma(t,R) = \Gamma$ for $R > d_t$, we have for every $t \in \Gamma$,

$$\begin{aligned}\sup_{R>0} B_{t,R}(w) &= \sup_{R\in(0,2d_t]} B_{t,R}(w), \\ \sup_{R>0} B_{t,R}(1) &= \sup_{R\in(0,2d_t]} B_{t,R}(1),\end{aligned} \tag{3.8}$$

$$\sup_{R>0}\Big(C(w,t,R)C'(w,t,R)\Big) = \sup_{0<R\le 2d_t}\Big(C(w,t,R)C'(w,t,R)\Big). \tag{3.9}$$

Evidently, $R/2 \le |\Gamma(t,R)|$ for $R \in (0, 2d_t]$. Taking into account the latter inequality and the definitions of $C(w,t,R), C'(w,t,R)$, we get for $t \in \Gamma$ and $R \in (0, 2d_t]$,

$$\begin{aligned}C(w,t,R)C'(w,t,R) &\le \frac{\|w\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}/w\|_{X'}}{|\Gamma(t,R)|} \\ &\quad\times \frac{\|\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}\|_{X'}}{|\Gamma(t,R)|} \\ &\le 4B_{t,R}(w)B_{t,R}(1).\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{R\in(0,2d_t]}\Big(C(w,t,R)C'(w,t,R)\Big)\\ \le 4\Big(\sup_{R\in(0,2d_t]}B_{t,R}(w)\Big)\Big(\sup_{R\in(0,2d_t]}B_{t,R}(1)\Big).\end{aligned}\tag{3.10}$$

From (3.8)–(3.10) it follows that

$$\begin{aligned}\sup_{R>0}\Big(C(w,t,R)C'(w,t,R)\Big)\\ \le 4\Big(\sup_{R>0}B_{t,R}(w)\Big)\Big(\sup_{R>0}B_{t,R}(1)\Big),\end{aligned}\tag{3.11}$$

$$\begin{aligned}\sup_{t\in\Gamma}\sup_{R>0}\Big(C(w,t,R)C'(w,t,R)\Big)\\ \le 4\Big(\sup_{t\in\Gamma}\sup_{R>0}B_{t,R}(w)\Big)\Big(\sup_{t\in\Gamma}\sup_{R>0}B_{t,R}(1)\Big).\end{aligned}\tag{3.12}$$

(a) If $w\in A_X(\Gamma,t)$ and $1\in A_X(\Gamma,t)$, then (3.11) implies (3.6), but we have shown that (3.6) is equivalent to (3.2). Part (a) is proved. Part (b) is proved similarly by using (3.12) and the equivalence of (3.7) and (3.3). ◻

Lemma 3.2. (a) *If $w\in A_X(\Gamma,t)$ and $1\in A_X(\Gamma,t)$, then $\log w\in BMO(\Gamma,t)$.*

(b) *If $w\in A_X(\Gamma)$ and $1\in A_X(\Gamma)$, then $\log w\in BMO(\Gamma)$.*

Proof. This statement is proved by analogy with [**25**, Lemma 1.6], see also [**3**, Proposition 2.4]. Put $\Omega_t(R):=\Omega_t(\log w,R)$,

$$\Gamma_+(t,R):=\Big\{\tau\in\Gamma(t,R):\log w(\tau)\ge\Omega_t(R)\Big\},$$
$$\Gamma_-(t,R):=\Big\{\tau\in\Gamma(t,R):\log w(\tau)<\Omega_t(R)\Big\}.$$

Due to Jensen's inequality [**33**, p. 78],

$$\begin{aligned}(3.13)\quad &\exp\left(\frac{1}{|\Gamma(t,R)|}\int_{\Gamma(t,R)}|\log w(\tau)-\Omega_t(R)||d\tau|\right)\\ &\le \frac{1}{|\Gamma(t,R)|}\int_{\Gamma_+(t,R)}\exp\Big(\log w(\tau)-\Omega_t(R)\Big)|d\tau|\\ &\quad+\frac{1}{|\Gamma(t,R)|}\int_{\Gamma_-(t,R)}\exp\Big(-(\log w(\tau)-\Omega_t(R))\Big)|d\tau|\\ &\le \frac{1}{|\Gamma(t,R)|}\int_{\Gamma}\exp\Big(\log w(\tau)-\Omega_t(R)\Big)\chi_{\Gamma(t,R)}(\tau)|d\tau|\\ &\quad+\frac{1}{|\Gamma(t,R)|}\int_{\Gamma}\exp\Big(-(\log w(\tau)-\Omega_t(R))\Big)\chi_{\Gamma(t,R)}(\tau)|d\tau|.\end{aligned}$$

Applying Hölder's inequality (see Lemma 2.2) to the first term on the right of (3.13), we get

$$\begin{aligned}(3.14)\quad &\frac{1}{|\Gamma(t,R)|}\int_{\Gamma}\exp\Big(\log w(\tau)-\Omega_t(R)\Big)\chi_{\Gamma(t,R)}(\tau)|d\tau|\\ &\le \Big\|\exp\Big(\log w(\cdot)-\Omega_t(R)\Big)\chi_{\Gamma(t,R)}(\cdot)\Big\|_X\frac{\|\chi_{\Gamma(t,R)}\|_{X'}}{|\Gamma(t,R)|}\\ &= e^{-\Omega_t(R)}\frac{\|\chi_{\Gamma(t,R)}\|_X\|\chi_{\Gamma(t,R)}\|_{X'}}{|\Gamma(t,R)|}=C(w,t,R).\end{aligned}$$

Analogously,

(3.15)

$$\frac{1}{|\Gamma(t,R)|}\int_{\Gamma}\exp\Big(-(\log w(\tau)-\Omega_t(R))\Big)\chi_{\Gamma(t,R)}(\tau)|d\tau|\le C'(w,t,R).$$

Combining (3.13)–(3.15), we see that for every $t\in\Gamma$ and $R>0$,

$$\exp\left(\frac{1}{|\Gamma(t,R)|}\int_{\Gamma(t,R)}|\log w(\tau)-\Omega_t(R)||d\tau|\right)\le C(w,t,R)+C'(w,t,R).$$

Consequently,

$$(3.16)\quad \|\log w\|_{*,t}\le\log\left(\sup_{R>0}C(w,t,R)+\sup_{R>0}C'(w,t,R)\right),\quad t\in\Gamma,$$

$$(3.17)\quad \|\log w\|_{*}\le\log\left(\sup_{t\in\Gamma}\sup_{R>0}C(w,t,R)+\sup_{t\in\Gamma}\sup_{R>0}C'(w,t,R)\right).$$

Statement (a) follows from Lemma 3.1(a) and (3.16). Statement (b) follows from Lemma 3.1(b) and (3.17). □

For rearrangement-invariant Banach function spaces X, by using (2.1), we infer that $w \in A_X(\Gamma)$ implies $1 \in A_X(\Gamma)$. In that case, by Lemma 3.2(b), if $w \in A_X(\Gamma)$, then $\log w \in BMO(\Gamma)$. This result was obtained in [**25**, Lemma 1.6]. Note that for Lebesgue spaces L^p, $1 < p < \infty$, and Muckenhoupt classes $A_p(\Gamma)$ this fact is well known, see, e.g., [**3**, Proposition 2.4].

4. Indices of submultiplicative functions associated with weighted Banach function spaces.

4.1 Submultiplicative functions and their indices. Following [**3**, Section 1.4], we say a function $\Phi : (0,\infty) \to (0,\infty]$ is *regular* if it is bounded in an open neighborhood of 1. A function $\Phi : (0,\infty) \to (0,\infty]$ is said to be submultiplicative if

$$\Phi(xy) \le \Phi(x)\Phi(y) \quad \text{for all } x, y \in (0,\infty).$$

It is easy to show that if Φ is regular and submultiplicative, then Φ is bounded away from zero in some open neighborhood of 1. Moreover, in this case $\Phi(x)$ is finite for all $x \in (0,\infty)$. Given a regular and submultiplicative function $\Phi : (0,\infty) \to (0,\infty)$, one defines

$$\alpha(\Phi) := \sup_{x\in(0,1)} \frac{\log \Phi(x)}{\log x}, \quad \beta(\Phi) := \inf_{x\in(1,\infty)} \frac{\log \Phi(x)}{\log x}.$$

Clearly, $-\infty < \alpha(\Phi)$ and $\beta(\Phi) < \infty$.

Theorem 4.1 (see [**3**, Theorem 1.13]). *If $\Phi : (0,\infty) \to (0,\infty)$ is regular and submultiplicative, then*

$$\alpha(\Phi) = \lim_{x\to 0} \frac{\log \Phi(x)}{\log x}, \quad \beta(\Phi) = \lim_{x\to\infty} \frac{\log \Phi(x)}{\log x}$$

and $-\infty < \alpha(\Phi) \le \beta(\Phi) < +\infty$.

The quantities $\alpha(\Phi)$ and $\beta(\Phi)$ are called the *lower* and *upper indices of the regular and submultiplicative function* Φ, respectively.

4.2 Spirality indices. In this subsection we mainly follow [**3**, Chapter 1]. Fix $t \in \Gamma$. Suppose $\psi : \Gamma \setminus \{t\} \to (0, \infty)$ is a continuous function. Put

$$F_\psi(R_1, R_2) := \max_{\tau \in \Gamma, |\tau - t| = R_1} \psi(\tau) \Big/ \min_{\tau \in \Gamma, |\tau - t| = R_2} \psi(\tau), \quad R_1, R_2 \in (0, d_t].$$

By [**3**, Lemma 1.15], the function

$$(W_t\psi)(x) := \begin{cases} \sup\limits_{0 < R \le d_t} F_\psi(xR, R), & x \in (0, 1], \\ \sup\limits_{0 < R \le d_t} F_\psi(R, x^{-1}R), & x \in (1, \infty). \end{cases}$$

is submultiplicative. For $t \in \Gamma$, we have,

$$\tau - t = |\tau - t| e^{i \arg(\tau - t)}, \quad \tau \in \Gamma \setminus \{t\},$$

and the argument $\arg(\tau - t)$ may be chosen to be a continuous function of $\tau \in \Gamma \setminus \{t\}$. Consider

$$\eta_t(\tau) := e^{-\arg(\tau - t)}.$$

Using the local Carleson constant $C_{\Gamma,t}$ instead of the global Carleson constant C_Γ, we can obtain the following local versions of [**3**, Theorem 1.10 and Lemma 1.17].

Lemma 4.2. *If* Γ *is locally a Carleson curve at* $t \in \Gamma$*, then*

$$\arg(\tau - t) = O(-\log|\tau - t|) \quad \text{as} \quad \tau \to t.$$

Lemma 4.3. *If* Γ *is locally a Carleson curve at* $t \in \Gamma$*, then the submultiplicative function* $W_t\eta_t$ *is regular.*

Under the assumptions of Lemma 4.3, by Theorem 4.1, there exist the *spirality indices*

$$\delta_t^- := \alpha(W_t\eta_t), \quad \delta_t^+ := \beta(W_t\eta_t)$$

of the curve Γ *at the point* t, see [**3**, Chapter 1]. If, in addition,

$$\arg(\tau - t) = -\delta_t \log|\tau - t| + O(1) \quad \text{as } \tau \to t,$$

where $\delta_t \in \mathbf{R}$, then $\delta_t^- = \delta_t^+ = \delta_t$, see [**3**, Section 1.6]. Examples of Carleson curves with distinct spirality indices are also given there.

On a rectifiable Jordan curve we have $d\tau = e^{i\theta_\Gamma(\tau)}|d\tau|$ where $\theta_\Gamma(\tau)$ is the angle between the positively oriented real axis and the naturally oriented tangent of Γ at τ (which exists almost everywhere). A rectifiable Jordan curve Γ is said to be a *Lyapunov curve* if

$$|\theta_\Gamma(\tau) - \theta_\Gamma(t)| \le c|\tau - t|^\mu$$

for some constants $c > 0$, $\mu \in (0, 1)$ and all $\tau, t \in \Gamma$. If θ_Γ is a function of bounded variation on Γ, then the curve Γ is called a *Radon curve* (or a *curve of bounded rotation*). It is very well known that Lyapunov curves are smooth, but Radon curves may have at most countable set of corner points (or even cusps). All Lyapunov curves and Radon curves without cusps are Carleson curves, see, e.g., [**28**, Section 2.3]. The next statement is well known.

Proposition 4.4. *If* Γ *is either a Lyapunov Jordan curve or a Radon Jordan curve, then for every* $t \in \Gamma$,

$$\arg(\tau - t) = O(1) \quad \textit{as } \tau \to t,$$

and, therefore, $\delta_t^- = \delta_t^+ = 0$.

4.3 Indices of powerlikeness. To investigate whether the weight $|(\tau - t)^\gamma| w(\tau)$ with arbitrary $\gamma \in \mathbf{C}$ belongs to the Muckenhoupt class $A_p(\Gamma)$, Böttcher and Yu. Karlovich introduced submultiplicative functions $V_t w$ and $V_t^0 w$ associated with local properties of the weight w at the point $t \in \Gamma$, see [**3**, Chapter 3].

Let w be a weight on Γ such that $\log w \in L^1(\Gamma(t, R))$ for every $R \in (0, d_t]$. Put

$$H_w(R_1, R_2) := \exp(\Omega_t(\log w, R_1)) / \exp(\Omega_t(\log w, R_2)), \quad R_1, R_2 \in (0, d_t].$$

Consider the functions

$$(V_t w)(x) := \begin{cases} \sup\limits_{0<R\le d_t} H_w(xR, R), & x \in (0,1], \\ \sup\limits_{0<R\le d_t} H_w(R, x^{-1}R), & x \in (1,\infty), \end{cases}$$
$$(V_t^0 w)(x) := \limsup_{R\to 0} H_w(xR, R), \quad x \in (0,\infty).$$

Lemma 4.5. *The function $V_t w$ is submultiplicative. If $V_t w$ is regular, then $V_t^0 w$ is regular and submultiplicative. Moreover,*

$$\alpha(V_t^0 w) = \alpha(V_t w), \quad \beta(V_t^0 w) = \beta(V_t w).$$

Lemma 4.6. *If Γ is locally a Carleson curve at $t \in \Gamma$ and $\log w \in BMO(\Gamma, t)$, then $V_t w$ and $V_t^0 w$ are regular.*

Lemmas 4.5 and 4.6 are proved by analogy with [**3**, Lemma 3.5(a)] and [**3**, Lemma 3.2(a)]. These statements are stated in [**3**] under the assumption that Γ is a Carleson curve. But Lemma 4.5 is valid for arbitrary rectifiable curves Γ. Since Lemma 4.6 has a "local nature", we may use the "local" Carleson constant $C_{\Gamma,t}$ instead of the "global" Carleson constant C_Γ in its proof. Under the assumptions of Lemma 4.6, in view of Theorem 4.1, for the weight w, there exist the *indices of powerlikeness*

(4.1) $$\mu_t := \alpha(V_t^0 w) = \alpha(V_t w), \quad \nu_t := \beta(V_t^0 w) = \beta(V_t w)$$

at the point $t \in \Gamma$.

Obviously, for a power weight $w(\tau) = |\tau - t|^{\lambda_t}$, the indices of powerlikeness equal $\mu_t = \nu_t = \lambda_t$. Nontrivial examples of weights with distinct indices of powerlikeness are given in [**3**, Examples 3.24–3.28].

Lemma 4.7 (see [**25**, Lemma 2.4]). *If Γ is locally a Carleson curve at $t \in \Gamma$ and $\log w \in VMO(\Gamma, t)$, then $\mu_t = \nu_t = 0$.*

4.4 Submultiplicative functions associated with weighted Banach function spaces. Let Γ be a rectifiable Jordan curve and let

X be a Banach function space. Fix $t \in \Gamma$ and consider the portion of the curve Γ in the annulus

$$\Delta(t,R) := \Gamma(t,R) \setminus \Gamma(t,R/2), \quad R > 0.$$

Clearly,

$$R/2 \le |\Delta(t,R)|, \quad R \in (0,d_t]. \tag{4.2}$$

On the other hand, if Γ is locally a Carleson curve at $t \in \Gamma$, then

$$|\Delta(t,R)| \le |\Gamma(t,R)| \le C_{\Gamma,t} R, \quad R > 0. \tag{4.3}$$

Suppose $w : \Gamma \to [0,\infty]$ is a weight such that $w\chi_{\Delta(t,R)} \in X$ and $\chi_{\Delta(t,R)}/w \in X'$ for all $R \in (0,d_t]$. We denote

$$G_w(R_1,R_2) := \frac{\|w\chi_{\Delta(t,R_1)}\|_X \|\chi_{\Delta(t,R_2)}/w\|_{X'}}{|\Delta(t,R_2)|}, \quad R_1, R_2 \in (0,d_t].$$

Define the following functions, see [**24**, Section 5]:

$$(Q_t w)(x) := \begin{cases} \sup\limits_{0<R\le d_t} G_w(xR,R), & x \in (0,1], \\ \sup\limits_{0<R\le d_t} G_w(R,x^{-1}R), & x \in (1,\infty), \end{cases}$$

$$(Q_t^0 w)(x) := \limsup_{R\to 0} G_w(xR,R), \quad x \in (0,\infty).$$

Lemma 4.8. *The function $Q_t w$ is submultiplicative. If $Q_t w$ is regular, then $Q_t^0 w$ is regular and submultiplicative. Moreover,*

$$\alpha(Q_t^0 w) = \alpha(Q_t w), \quad \beta(Q_t^0 w) = \beta(Q_t w).$$

Lemma 4.9. *If $w \in A_X(\Gamma,t)$, then $Q_t w$ and $Q_t^0 w$ are regular. Moreover,*

$$0 \le \alpha(Q_t w) = \alpha(Q_t^0 w) \le \beta(Q_t^0 w) = \beta(Q_t w) \le 1.$$

These statements are proved in [**24**, Lemmas 5.1–5.2] and [**24**, Theorem 5.3], respectively, under the assumption that X is rearrangement-invariant. But, actually, we did not use this assumption in those proofs. So we can literally repeat the proofs for arbitrary Banach function spaces.

5. Relations between indices.

5.1 Case of general Banach function spaces. Let Γ be a rectifiable Jordan curve, let X be a Banach function space, and let $t \in \Gamma$.

Theorem 5.1. *Suppose $w : \Gamma \to [0,\infty]$ is a weight such that $\log w \in L^1(\Gamma(t,R))$ for every $R \in (0,d_t]$ and $\psi : \Gamma \setminus \{t\} \to (0,\infty)$ is a continuous function. If the functions $V_t w$ and $W_t\psi$ are regular, then the function $V_t(\psi w)$ is regular too. Moreover,*

$$\begin{aligned}\alpha(V_t w)+\alpha(W_t\psi) &\le \alpha(V_t(\psi w))\\ &\le \min\Big\{\alpha(V_t w)+\beta(W_t\psi),\beta(V_t w)+\alpha(W_t\psi)\Big\},\\ \beta(V_t w)+\beta(W_t\psi) &\ge \beta(V_t(\psi w))\\ &\ge \max\Big\{\alpha(V_t w)+\beta(W_t\psi),\beta(V_t w)+\alpha(W_t\psi)\Big\}.\end{aligned}$$

This statement is proved similarly to [**3**, Lemma 3.17].

Theorem 5.2. *Suppose $w : \Gamma \to [0,\infty]$ is a weight such that $w\chi_{\Delta(t,R)} \in X$ and $\chi_{\Delta(t,R)}/w \in X'$ for every $R \in (0,d_t]$ and $\psi : \Gamma \setminus \{t\} \to (0,\infty)$ is a continuous function. If the functions $Q_t w$ and $W_t\psi$ are regular, then the function $Q_t(\psi w)$ is regular too. Moreover,*

$$\begin{aligned}\alpha(Q_t w)+\alpha(W_t\psi) &\le \alpha(Q_t(\psi w))\\ &\le \min\Big\{\alpha(Q_t w)+\beta(W_t\psi),\beta(Q_t w)+\alpha(W_t\psi)\Big\},\\ \beta(Q_t w)+\beta(W_t\psi) &\ge \beta(Q_t(\psi w))\\ &\ge \max\Big\{\alpha(Q_t w)+\beta(W_t\psi),\beta(Q_t w)+\alpha(W_t\psi)\Big\}.\end{aligned}$$

This theorem is proved in [**24**, Theorem 5.8] for rearrangement-invariant Banach function spaces. The proof given there does not use the rearrangement-invariant property of the space, so it works for an arbitrary Banach function space.

Lemma 5.3. *If Γ is locally a Carleson curve at $t \in \Gamma$ and $\log w \in BMO(\Gamma, t)$, then for every $R \in (0, d_t]$,*

$$\exp(\Omega_t(\log w, R)) \le \frac{C_t}{|\Delta(t,R)|} \int_{\Delta(t,R)} w(\tau)|d\tau|$$

where $C_t := \exp(2C_{\Gamma,t}\|\log w\|_{*,t}) < \infty$.

The proof is actually given in [**3**, Lemma 3.2(b)].

Theorem 5.4. *Let Γ be locally a Carleson curve at $t \in \Gamma$ and let $w : \Gamma \to [0, \infty]$ be a weight such that $w\chi_{\Delta(t,R)} \in X, \chi_{\Delta(t,R)}/w \in X'$ for every $R \in (0, d_t]$ and $\log w \in BMO(\Gamma, t)$. If $Q_t w$ and $Q_t 1$ are regular, then*

$$\alpha(Q_t w) \le \mu_t + \beta(Q_t 1), \quad \nu_t + \alpha(Q_t 1) \le \beta(Q_t w). \tag{5.1}$$

Proof. The proof is developed by analogy with [**25**, Theorem 2.6]. From Lemma 5.3 and Hölder's inequality (see Lemma 2.2) we see that for every $R \in (0, d_t]$,

$$\exp(\Omega_t(\log w, R)) \le C_t \, \frac{\|w\chi_{\Delta(t,R)}\|_X \|\chi_{\Delta(t,R)}\|_{X'}}{|\Delta(t,R)|}, \tag{5.2}$$

$$\exp(-\Omega_t(\log w, R)) \le C_t \, \frac{\|\chi_{\Delta(t,R)}\|_X \|\chi_{\Delta(t,R)}/w\|_{X'}}{|\Delta(t,R)|}. \tag{5.3}$$

From (5.2) and (5.3) it follows that for $x \in (0, 1]$ and $R \in (0, d_t]$,

$$\begin{aligned} H_w(xR, R) &= \exp(\Omega_t(\log w, xR)) \exp(-\Omega_t(\log w, R)) \\ &\le C_t^2 \frac{\|w\chi_{\Delta(t,xR)}\|_X \|\chi_{\Delta(t,R)}/w\|_{X'}}{|\Delta(t,R)|} \\ &\quad \times \frac{\|\chi_{\Delta(t,R)}\|_X \|\chi_{\Delta(t,xR)}\|_{X'}}{|\Delta(t,xR)|} \\ &= C_t^2 G_w(xR, R) G_1(R, xR). \end{aligned} \tag{5.4}$$

Then, taking the supremum over all $R \in (0, d_t]$, we obtain for $x \in (0, 1]$,

$$(V_t w)(x) \le C_t (Q_t w)(x)(Q_t 1)(x^{-1}). \tag{5.5}$$

Analogously, for $x \in (1, \infty)$ and $R \in (0, d_t]$,

$$H_w(R, x^{-1}R) \le C_t^2 G_w(R, x^{-1}R) G_1(x^{-1}R, R). \tag{5.6}$$

Taking the supremum over all $R \in (0, d_t]$, we arrive at (5.5) for $x \in (1, \infty)$. By Lemmas 4.5–4.6, the function $V_t w$ is regular and submultiplicative. By Lemma 4.8, the functions $Q_t w$ and $Q_t 1$ are submultiplicative, they are regular, due to the assumption of the theorem. Therefore, in view of Theorem 4.1, the indices $\alpha(Q_t w), \beta(Q_t w)$; $\alpha(Q_t 1), \beta(Q_t 1)$; and $\alpha(V_t w), \beta(V_t w)$ exist and are well defined.

From (5.5) it follows that

$$\frac{\log(V_t w)(x)}{\log x} \ge \frac{\log C_t^2}{\log x} + \frac{\log(Q_t w)(x)}{\log x} - \frac{\log(Q_t 1)(x^{-1})}{\log x^{-1}}, \quad x \in (0, 1],$$
$$\frac{\log(V_t w)(x)}{\log x} \le \frac{\log C_t^2}{\log x} + \frac{\log(Q_t w)(x)}{\log x} - \frac{\log(Q_t 1)(x^{-1})}{\log x^{-1}}, \quad x \in (1, \infty).$$

Passing to the limit in the latter inequalities as $x \to 0$, respectively as $x \to \infty$, we obtain, respectively,

$$\mu_t = \alpha(V_t w) \ge \alpha(Q_t w) - \beta(Q_t 1), \quad \nu_t = \beta(V_t w) \le \beta(Q_t w) - \alpha(Q_t 1).$$

So, we arrive at (5.1). □

Theorem 5.5. *If $w \in A_X(\Gamma, t)$ and $1 \in A_X(\Gamma, t)$, then*

$$\alpha(Q_t 1) + \mu_t \le \alpha(Q_t w) \le \min\Big\{\alpha(Q_t 1) + \nu_t, \beta(Q_t 1) + \mu_t\Big\}, \tag{5.7}$$
$$\beta(Q_t 1) + \nu_t \ge \beta(Q_t w) \ge \max\Big\{\alpha(Q_t 1) + \nu_t, \beta(Q_t 1) + \mu_t\Big\}. \tag{5.8}$$

Proof. The idea of the proof is borrowed from [**25**, Theorems 2.6 and 2.7]. From Lemmas 4.8–4.9 it follows that the functions $Q_t w$ and $Q_t 1$ are regular and submultiplicative. On the other hand, by

Lemma 3.2(a), $\log w \in BMO(\Gamma, t)$. Therefore, by Lemmas 4.5–4.6, the function $V_t w$ is regular and submultiplicative. Thus, all the indices

$$\alpha(Q_t 1), \quad \beta(Q_t 1), \quad \alpha(Q_t w), \quad \beta(Q_t w), \quad \mu_t = \alpha(V_t w), \quad \nu_t = \beta(V_t w)$$

are well defined. By Theorem 5.4,

$$\alpha(Q_t w) \le \mu_t + \beta(Q_t 1), \quad \nu_t + \alpha(Q_t 1) \le \beta(Q_t w). \tag{5.9}$$

If $1 \in A_X(\Gamma, t)$, then from the lattice property it follows that for every $R > 0$,

$$\begin{aligned} \frac{1}{R}\|\chi_{\Delta(t,R)}\|_X \|\chi_{\Delta(t,R)}\|_{X'} &\le \frac{1}{R}\|\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}\|_{X'} \\ &\le \sup_{R>0} B_{t,R}(1) =: B_t(1). \end{aligned} \tag{5.10}$$

Combining (5.10) and (4.2), we arrive at

$$\|\chi_{\Delta(t,R)}\|_X \|\chi_{\Delta(t,R)}\|_{X'} \le 2B_t(1)|\Delta(t,R)|, \quad R \in (0, d_t].$$

Then we have for $x \in (0, 1]$,

$$\begin{aligned} \frac{1}{G_1(R, xR)} &= \frac{|\Delta(t, xR)|}{\|\chi_{\Delta(t,R)}\|_X \|\chi_{\Delta(t,xR)}\|_{X'}} \ge \frac{\|\chi_{\Delta(t,xR)}\|_X \|\chi_{\Delta(t,R)}\|_{X'}}{(2B_t(1))^2 |\Delta(t,R)|} \\ &= (2B_t(1))^{-2} G_1(xR, R). \end{aligned} \tag{5.11}$$

Analogously, we deduce that for $x \in (1, \infty)$,

$$\frac{1}{G_1(x^{-1}R, R)} \ge (2B_t(1))^{-2} G_1(R, x^{-1}R). \tag{5.12}$$

From (5.4) and (5.11) we obtain for $x \in (0, 1]$,

$$\begin{aligned} (2B_t(1))^{-2} G_1(xR, R) &\le \frac{1}{G_1(R, xR)} \le C_t^2 \frac{G_w(xR, R)}{H_w(xR, R)} \\ &= C_t^2 G_w(xR, R) H_w(R, xR). \end{aligned} \tag{5.13}$$

Similarly, from (5.6) and (5.11) we obtain for $x \in (1, \infty)$,

$$(2B_t(1))^{-2} G_1(R, x^{-1}R) \le C_t^2 G_w(xR, R) H_w(x^{-1}R, R).$$

Taking the supremum over $R \in (0, d_t]$ in (5.13) and (5.14), we get

$$(Q_t 1)(x) \le (2C_t B_t(1))^2 (Q_t w)(x)(V_t w)(x^{-1}), \quad x \in (0, \infty).$$

From this inequality it follows that for $x \in (0, 1]$,

$$\frac{\log(Q_t 1)(x)}{\log x} \ge \frac{\log(2C_t B_t(1))^2}{\log x} + \frac{\log(Q_t w)(x)}{\log x} - \frac{\log(V_t w)(x^{-1})}{\log x^{-1}} \tag{5.15}$$

and, analogously, for $x \in (1, \infty)$,

$$\frac{\log(Q_t 1)(x)}{\log x} \le \frac{\log(2C_t B_t(1))^2}{\log x} + \frac{\log(Q_t w)(x)}{\log x} - \frac{\log(V_t w)(x^{-1})}{\log x^{-1}}. \tag{5.16}$$

Passing to the limit in (5.15) as $x \to 0$ and in (5.16) as $x \to \infty$, we obtain, respectively,

$$\alpha(Q_t 1) \ge \alpha(Q_t w) - \beta(V_t w), \quad \beta(Q_t 1) \le \beta(Q_t w) - \alpha(V_t w). \tag{5.17}$$

By Lemma 3.1(a), there exist constants $C_1(t), C_2(t) > 0$ such that for every $R > 0$,

$$\exp(-\Omega_t(\log w, R)) \frac{\|w\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}\|_{X'}}{|\Gamma(t,R)|} \le C_1(t), \tag{5.18}$$

$$\exp(\Omega_t(\log w, R)) \frac{\|\chi_{\Gamma(t,R)}\|_X \|\chi_{\Gamma(t,R)}/w\|_{X'}}{|\Gamma(t,R)|} \le C_2(t). \tag{5.19}$$

On the other hand, from the lattice property, the Hölder inequality (see Lemma 2.2), (3.1) and (4.2) it follows that for $R \in (0, d_t]$,

$$\begin{aligned} \frac{|\Gamma(t,R)|}{\|\chi_{\Gamma(t,R)}\|_{X'}} &\le \frac{|\Gamma(t,R)|}{\|\chi_{\Delta(t,R)}\|_{X'}} = \frac{|\Gamma(t,R)| \cdot \|\chi_{\Delta(t,R)}\|_X}{\|\chi_{\Delta(t,R)}\|_X \|\chi_{\Delta(t,R)}\|_{X'}} \\ &\le \frac{|\Gamma(t,R)|}{|\Delta(t,R)|} \|\chi_{\Delta(t,R)}\|_X \le \frac{C_{\Gamma,t} R}{R/2} \|\chi_{\Delta(t,R)}\|_X \\ &= 2C_{\Gamma,t} \|\chi_{\Delta(t,R)}\|_X. \end{aligned} \tag{5.20}$$

Analogously, for $R \in (0, d_t]$,

$$\frac{|\Gamma(t,R)|}{\|\chi_{\Gamma(t,R)}\|_X} \le 2C_{\Gamma,t} \|\chi_{\Delta(t,R)}\|_{X'}. \tag{5.21}$$

From (5.18)–(5.21) and the lattice property it follows that for $R \in (0, d_t]$ and $x \in (0, 1]$,

$$\begin{aligned} G_w(xR, R) &= \frac{\|w\chi_{\Delta(t,xR)}\|_X \|\chi_{\Delta(t,R)}/w\|_{X'}}{|\Delta(t,R)|} \\ &\le \frac{\|w\chi_{\Gamma(t,xR)}\|_X \|\chi_{\Gamma(t,R)}/w\|_{X'}}{|\Delta(t,R)|} \\ &\le \frac{C_1(t)C_2(t)}{|\Delta(t,R)|} \exp(\Omega_t(\log w, xR)) \exp(-\Omega_t(\log w, R)) \\ &\qquad \times \frac{|\Gamma(t,xR)|}{\|\chi_{\Gamma(t,xR)}\|_{X'}} \cdot \frac{|\Gamma(t,R)|}{\|\chi_{\Gamma(t,R)}\|_X} \\ &\le (2C_{\Gamma,t})^2 C_1(t)C_2(t) H_w(xR, R) \frac{\|\chi_{\Delta(t,xR)}\|_X \|\chi_{\Delta(t,R)}\|_{X'}}{|\Delta(t,R)|} \\ &= (2C_{\Gamma,t})^2 C_1(t)C_2(t) H_w(xR, R) G_1(xR, R) \end{aligned} \tag{5.22}$$

and, similarly, for $R \in (0, d_t]$ and $x \in (1, \infty)$,

$$G_w(R, x^{-1}R) \le (2C_{\Gamma,t})^2 C_1(t)C_2(t) H_w(R, x^{-1}R) G_1(R, x^{-1}R). \tag{5.23}$$

Taking the supremum over all $R \in (0, d_t]$ in (5.22) and (5.23), we obtain

$$(Q_t w)(x) \le (2C_{\Gamma,t})^2 C_1(t)C_2(t)(V_t w)(x)(Q_t 1)(x), \quad x \in (0, \infty).$$

Therefore,

$$\alpha(Q_t w) \ge \alpha(V_t w) + \alpha(Q_t 1), \quad \beta(Q_t w) \le \beta(V_t w) + \beta(Q_t 1). \tag{5.24}$$

Combining (5.9), (5.17), and (5.24), we arrive at (5.7)–(5.8). □

If X is a rearrangement-invariant Banach function space, then from (2.1) it follows that the conditions $1 \in A_X(\Gamma, t)$ and $1 \in A_X(\Gamma)$ are equivalent to (3.1) and (1.1), respectively. Hence, $w \in A_X(\Gamma, t)$ implies $1 \in A_X(\Gamma, t)$ whenever X is rearrangement-invariant. This property allows us to simplify the formulation of Theorem 5.5 for rearrangement-invariant Banach function spaces, see [**25**, Theorems 2.6 and 2.7].

Note that $\alpha(Q_t 1)$ and $\beta(Q_t 1)$ can be considered as a generalization of the Zippin (fundamental) indices p_X and q_X of a rearrangement-invariant Banach function space X [**53**]. If X is rearrangement-invariant, then $\alpha(Q_t 1) = p_X$ and $\beta(Q_t 1) = q_X$, see [**24**, Theorem 5.4].

On the other hand, the Zippin indices for an Orlicz space L^φ coincide with the reciprocals of the Matuszewska-Orlicz indices, which control the growth of the Young function φ, see, e.g., [**39**] and the references given there. The notion of Matuszewska-Orlicz indices of Orlicz spaces was extended to the case of Musielak-Orlicz spaces in [**18, 19**]. Recall that Orlicz spaces are always rearrangement-invariant, but Musielak-Orlicz spaces are not rearrangement-invariant, in general.

5.2 Case of Nakano spaces. Suppose Γ is a rectifiable Jordan curve. Assume that $p : \Gamma \to (1,\infty)$ is a continuous function. Then

$$1 < p_* := \min_{t\in\Gamma} p(t) \le \max_{t\in\Gamma} p(t) := p^* < \infty, \tag{5.25}$$

due to the compactness of Γ. We will say that a continuous function $p : \Gamma \to (1,\infty)$ belongs to the class $\mathcal{P}_t$ if there is a constant $A_t > 0$ such that

$$|p(\tau) - p(t)| \le \frac{A_t}{-\log|\tau - t|} \quad \text{for all } \tau \in \Gamma(t, 1/2). \tag{5.26}$$

The class of all continuous functions $p : \Gamma \to (1,\infty)$ such that $p \in \mathcal{P}_t$ for every $t \in \Gamma$ and

$$\sup_{t\in\Gamma} A_t =: A < \infty$$

is denoted by $\mathcal{P}$. Clearly, $\mathcal{P} \subset \mathcal{P}_t$ for every $t \in \Gamma$.

The class $\mathcal{P}$ plays a very important role in questions on the boundedness of maximal functions and singular integrals on (weighted) Nakano spaces, see [**10, 30, 46**], the references therein, and also Theorem 6.2.

Proposition 5.6. *A function p belongs to $\mathcal{P}_t$ (respectively, to $\mathcal{P}$) if and only if the function $p'(\tau) := p(\tau)/(p(\tau) - 1)$ belongs to $\mathcal{P}_t$ (respectively, to $\mathcal{P}$).*

Proof. The statement immediately follows from the obvious inequality

$$|p'(\tau) - p'(t)| = \left|\frac{p(\tau) - p(t)}{(p(\tau) - 1)(p(t) - 1)}\right| \le \frac{|p(\tau) - p(t)|}{(p_* - 1)^2}, \quad \tau, t \in \Gamma,$$

and the reflexive relation $(p')' = p$. □

Lemma 5.7. *Let Γ be locally a Carleson curve at $t \in \Gamma$ and $p \in \mathcal{P}_t$. Then there exist constants $M_1(t), M_2(t), C_1(t), C_2(t) \in (0,\infty)$ such that*

$$\|\chi_{\Delta(t,R)}\|_{L^{p(\cdot)}} \ge M_1(t) R^{1/p(t)} \quad \text{for all } R \in (0, C_1(t)), \tag{5.27}$$

$$\|\chi_{\Gamma(t,R)}\|_{L^{p(\cdot)}} \le M_2(t) R^{1/p(t)} \quad \text{for all } R \in (0, C_2(t)). \tag{5.28}$$

Proof. From (5.26) it follows that for $\tau \in \Gamma(t, 1/2)$,

$$-p(t) - \frac{A_t}{-\log|\tau - t|} \le -p(\tau) \le -p(t) + \frac{A_t}{-\log|\tau - t|}. \tag{5.29}$$

Since $|\tau - t| \le R$ for $\tau \in \Gamma(t,R)$, we have

$$\frac{A_t}{-\log|\tau - t|} \le \frac{A_t}{-\log R}, \quad \tau \in \Gamma(t,R), \quad R \in (0, 1/2). \tag{5.30}$$

From (5.29) and (5.30) we get for $\tau \in \Gamma(t,R)$ and $R \in (0,1/2)$,

$$-p(t) + \frac{A_t}{\log R} \le -p(\tau) \le -p(t) - \frac{A_t}{\log R}. \tag{5.31}$$

For $R \in (0, e^{-A_t})$, taking into account that $p(t) \in (1,\infty)$, we obtain

$$p(t) + \frac{A_t}{\log R} = (p(t) - 1) + \left(1 + \frac{A_t}{\log R}\right) > p(t) - 1 > 0. \tag{5.32}$$

From (5.31) we get for $\lambda \in (0,1]$ and $R \in (0, \min\{1/2, e^{-A_t}\})$,

$$\begin{aligned} \exp\left(-\left[p(t) + \frac{A_t}{\log R}\right] \log\lambda\right) &\le \exp(-p(\tau)\log\lambda) \\ &\le \exp\left(-\left[p(t) - \frac{A_t}{\log R}\right] \log\lambda\right). \end{aligned} \tag{5.33}$$

Analogously, for $\lambda \in (1,\infty)$ and $R \in (0, \min\{1/2, e^{-A_t}\})$,

$$\begin{aligned} \exp\left(-\left[p(t) - \frac{A_t}{\log R}\right] \log\lambda\right) &\le \exp(-p(\tau)\log\lambda) \\ &\le \exp\left(-\left[p(t) + \frac{A_t}{\log R}\right] \log\lambda\right). \end{aligned} \tag{5.34}$$

Let us prove (5.27). From the first inequality in (5.33) and (4.2) it follows that for $\lambda \in (0,1]$ and $R \in (0, \min\{1/2, e^{-A_t}, d_t\})$,

$$\begin{aligned} m(\chi_{\Delta(t,R)}/\lambda, p) &= \int_{\Delta(t,R)} \exp(-p(\tau)\log\lambda)|d\tau| \\ &\ge \exp\left(-\left[p(t) + \frac{A_t}{\log R}\right]\log\lambda\right)|\Delta(t,R)| \\ &\ge \exp\left(\log\frac{R}{2} - \left[p(t) + \frac{A_t}{\log R}\right]\log\lambda\right). \end{aligned}$$

Put $C_1(t) := \min\{1/2, e^{-A_t}, d_t\}$. Therefore, taking into account (5.32), we obtain for $R \in (0, C_1(t))$,

$$\begin{aligned} &\Big\{\lambda \in (0,1] : \ m(\chi_{\Delta(t,R)}/\lambda, p) \le 1\Big\} \\ &\qquad \subset \left\{\lambda \in (0,1] : \ \log\frac{R}{2} - \left[p(t) + \frac{A_t}{\log R}\right]\log\lambda \le 0\right\} \\ &\qquad = \left\{\lambda : \ \exp\left(\frac{\log(R/2)}{p(t) + A_t/\log R}\right) \le \lambda \le 1\right\}. \end{aligned}$$

Thus, for $R \in (0, C_1(t))$,

(5.35)

$$N_1 := \inf\Big\{\lambda \in (0,1] : \ m(\chi_{\Delta(t,R)}/\lambda, p) \le 1\Big\} \ge \exp\left(\frac{\log(R/2)}{p(t)+A_t/\log R}\right).$$

Analogously, from the first inequality in (5.34) we obtain

$$\Big\{\lambda \in (1,\infty) : \ m(\chi_{\Delta(t,R)}/\lambda, p) \le 1\Big\} \subset (1,\infty)$$

because

$$\exp\left(\frac{\log(R/2)}{p(t) - A_t/\log R}\right) < 1 \quad \text{for } R \in (0, C_1(t)). \tag{5.36}$$

Thus, for $R \in (0, C_1(t))$,

$$N_2 := \inf\Big\{\lambda \in (1,\infty) : \ m(\chi_{\Delta(t,R)}/\lambda, p) \le 1\Big\} \ge 1. \tag{5.37}$$

From (5.35)–(5.37) we obtain for $R \in (0, C_1(t))$,
(5.38)

$$\begin{aligned}\|\chi_{\Delta(t,R)}\|_{L^{p(\cdot)}} &= \inf\left\{\lambda > 0 : \ m(\chi_{\Delta(t,R)}/\lambda, p) \le 1\right\} \\ &= \min\{N_1, N_2\} \ge \min\left\{1, \exp\left(\frac{\log(R/2)}{p(t) + A_t/\log R}\right)\right\} \\ &= \exp\left(\frac{\log(R/2)}{p(t) + A_t/\log R}\right).\end{aligned}$$

From (5.32) it follows that for $R \in (0, C_1(t))$,

$$\begin{aligned}\frac{\log(R/2)}{p(t) + A_t/\log R} - \frac{\log(R/2)}{p(t)} &= \frac{-A_t + A_t \log 2/\log R}{(p(t) + A_t/\log R)p(t)} \\ &\ge \frac{-A_t + A_t \log 2/\log R}{(p(t) - 1)p(t)} \\ &\ge \frac{A_t + \log 2}{(1 - p(t))p(t)}.\end{aligned}$$

From the latter inequality we deduce that

$$\begin{aligned}(5.39)\quad \exp\left(\frac{\log(R/2)}{p(t) + A_t/\log R}\right) \\ &= \exp\left(\frac{\log(R/2)}{p(t) + A_t/\log R} - \frac{\log(R/2)}{p(t)}\right)\left(\frac{R}{2}\right)^{1/p(t)} \\ &\ge \exp\left(\frac{A_t + \log 2}{(1 - p(t))p(t)} - \frac{\log 2}{p(t)}\right) R^{1/p(t)}.\end{aligned}$$

Combining (5.38) and (5.39), we arrive at (5.27) with

$$C_1(t) := \min\{1/2, e^{-A_t}, d_t\}, \quad M_1(t) := \exp\left(\frac{A_t + \log 2}{(1 - p(t))p(t)} - \frac{\log 2}{p(t)}\right).$$

Taking into account (3.1), one can prove that (5.28) is valid with

$$\begin{aligned}C_2(t) &:= \min\{1/2, 1/C_{\Gamma,t}, e^{-A_t}, d_t\}, \\ M_2(t) &:= \exp\left(\frac{A_t}{(p(t))^2} + \frac{\log C_{\Gamma,t}}{p(t)}\right).\end{aligned}$$

The proof of (5.28) is similar to the proof of (5.27) and it is omitted. □

Lemma 5.8. *Suppose* Γ *is locally a Carleson curve at* $t \in \Gamma$ *and* $p \in \mathcal{P}_t$*. Then* $1 \in A_{L^{p(\cdot)}}(\Gamma, t)$ *and*

$$\alpha(Q_t 1) = \beta(Q_t 1) = 1/p(t). \tag{5.40}$$

Proof. From Lemma 5.7 we deduce that there exist constants $C_i(t)$, $M_i(t)$ $(i = 1, 2)$ such that

$$\|\chi_{\Delta(t,R)}\|_{L^{p(\cdot)}} \ge M_1(t) R^{1/p(t)} \quad \text{for all } R \in (0, C_1(t)), \tag{5.41}$$

$$\|\chi_{\Gamma(t,R)}\|_{L^{p(\cdot)}} \le M_2(t) R^{1/p(t)} \quad \text{for all } R \in (0, C_2(t)). \tag{5.42}$$

By Proposition 5.6, $p' \in \mathcal{P}_t$. Analogously, applying Lemma 5.7 to $L^{p'(\cdot)}$ and taking into account that the latter space coincide with $(L^{p(\cdot)})'$ up to the equivalence of the norms (see Lemma 2.4), we infer that there exist constants $C_i'(t), M_i'(t)$, $i = 1, 2$, such that

$$\|\chi_{\Delta(t,R)}\|_{(L^{p(\cdot)})'} \ge M_1'(t) R^{1/p'(t)} \quad \text{for all } R \in (0, C_1'(t)), \tag{5.43}$$

$$\|\chi_{\Gamma(t,R)}\|_{(L^{p(\cdot)})'} \le M_2'(t) R^{1/p'(t)} \quad \text{for all } R \in (0, C_2'(t)). \tag{5.44}$$

From (5.42), (5.44) it follows that for $R \in (0, \min\{C_2(t), C_2'(t)\})$,

$$\begin{aligned} B_{t,R}(1) &= \frac{1}{R} \|\chi_{\Gamma(t,R)}\|_{L^{p(\cdot)}} \|\chi_{\Gamma(t,R)}\|_{(L^{p(\cdot)})'} \\ &\le \frac{1}{R} M_2(t) M_2'(t) R^{1/p(t)} R^{1/p'(t)} = M_2(t) M_2'(t). \end{aligned} \tag{5.45}$$

On the other hand, for $R \ge \min\{C_2(t), C_2'(t)\}$,

$$B_{t,R}(1) = \frac{1}{R} \|\chi_{\Gamma(t,R)}\|_{L^{p(\cdot)}} \|\chi_{\Gamma(t,R)}\|_{(L^{p(\cdot)})'} \le \frac{\|1\|_{L^{p(\cdot)}} \|1\|_{(L^{p(\cdot)})'}}{\min\{C_2(t), C_2'(t)\}}. \tag{5.46}$$

From (5.45) and (5.46) it follows that

$$\sup_{R>0} B_{t,R}(1) \le \max\left\{ M_2(t) M_2'(t), \frac{\|1\|_{L^{p(\cdot)}} \|1\|_{(L^{p(\cdot)})'}}{\min\{C_2(t), C_2'(t)\}} \right\} < \infty.$$

Thus, $1 \in A_{L^{p(\cdot)}}(\Gamma, t)$.

Put $C(t) := \min\{C_1(t), C_2(t), C_1'(t), C_2'(t)\}$. From (5.42), (5.44), (4.2), and the lattice property we obtain for $x \in (0, \infty)$ and $R \in (0, C(t)\min\{1, 1/x\})$,

$$
\begin{aligned}
G_1(xR, R) &:= \frac{\|\chi_{\Delta(t,xR)}\|_{L^{p(\cdot)}} \|\chi_{\Delta(t,R)}\|_{(L^{p(\cdot)})'}}{|\Delta(t, R)|} \\
&\le M_2(t) M_2'(t) \frac{(xR)^{1/p(t)} R^{1/p'(t)}}{|\Delta(t, R)|} \\
&\le M_2(t) M_2'(t) \frac{x^{1/p(t)} R}{R/2} = 2M_2(t) M_2'(t) x^{1/p(t)}.
\end{aligned} \tag{5.47}
$$

Combining (5.43), (5.45), and (4.3), we get for the same x and R,

$$
\begin{aligned}
G_1(xR, R) &\ge M_1(t) M_1'(t) \frac{(xR)^{1/p(t)} R^{1/p'(t)}}{|\Delta(t, R)|} \\
&\ge M_1(t) M_1'(t) \frac{x^{1/p(t)} R}{C_{\Gamma,t} R} = \frac{M_1(t) M_1'(t)}{C_{\Gamma,t}} x^{1/p(t)}.
\end{aligned} \tag{5.48}
$$

From (5.47) and (5.48) it follows that

$$
\frac{M_1(t) M_1'(t)}{C_{\Gamma,t}} x^{1/p(t)} \le (Q_t^0 1)(x) \le 2M_2(t) M_2'(t) x^{1/p(t)}, \quad x \in (0, \infty).
$$

Since $1 \in A_{L^{p(\cdot)}}(\Gamma)$, the function $Q_t^0 1$ is regular and submultiplicative, see Lemmas 4.8 and 4.9. From the latter inequality it follows that

$$
\alpha(Q_t^0 1) = \beta(Q_t^0 1) = 1/p(t).
$$

Combining the latter equalities with Lemma 4.8, we arrive at (5.40). □

Theorem 5.9. *Let Γ be locally a Carleson curve at $t \in \Gamma$, let $w : \Gamma \to [0, \infty]$ be a weight, and let $p \in \mathcal{P}_t$. If $w \in A_{L^{p(\cdot)}}(\Gamma, t)$, then $\log w \in BMO(\Gamma, t)$ and*

$$
\alpha(Q_t w) = 1/p(t) + \alpha(V_t w), \quad \beta(Q_t w) = 1/p(t) + \beta(V_t w). \tag{5.49}
$$

Proof. Since $p \in \mathcal{P}_t$ and Γ is locally a Carleson curve at t, in view of Lemma 5.8, $1 \in A_{L^{p(\cdot)}}(\Gamma, t)$. By Lemma 3.2(a), $\log w \in BMO(\Gamma, t)$. From Theorem 5.5 and (5.40) we get

$$\begin{aligned} 1/p(t) + \alpha(V_t w) \le \alpha(Q_t w) &\le \min\{1/p(t) + \alpha(V_t w), 1/p(t) + \beta(V_t w)\} \\ &= 1/p(t) + \alpha(V_t w), \\ 1/p(t) + \beta(V_t w) \ge \beta(Q_t w) &\ge \max\{1/p(t) + \alpha(V_t w), 1/p(t) + \beta(V_t w)\} \\ &= 1/p(t) + \beta(V_t w), \end{aligned}$$

that is, equalities (5.49) hold. □

Lemma 5.10. *Let Γ be a Carleson curve, let $w : \Gamma \to [0, \infty]$ be a weight, and let $p \in \mathcal{P}$. If $w \in A_{L^{p(\cdot)}}(\Gamma)$, then $\log w \in BMO(\Gamma)$.*

Proof. By analogy with Lemma 5.7 one can show that there exist constants $C > 0$ and $M, M' \in (0, \infty)$ such that

$$\|\chi_{\Gamma(t,R)}\|_{L^{p(\cdot)}} \le M R^{1/p(t)}, \quad \|\chi_{\Gamma(t,R)}\|_{L^{p'(\cdot)}} \le M' R^{1/p'(t)}$$

for all $R \in (0, C)$ and all $t \in \Gamma$. Taking into account Lemma 2.4, as in Lemma 5.8 from the latter inequalities we obtain $1 \in A_{L^{p(\cdot)}}(\Gamma)$. Therefore, $\log w \in BMO(\Gamma)$, due to Lemma 3.2(b). □

5.3 Indicator functions. In this subsection we generalize the notion of indicator functions (see [**3**, Chapter 3] and also [**24**, Section 7.2], [**25**, Section 2.5], [**26**, Section 3.3]) to the case of weighted Banach function spaces.

Suppose Γ is a rectifiable Jordan curve, $w : \Gamma \to [0, \infty]$ is a weight, X is a Banach function space.

Lemma 5.11. *Let Γ be locally a Carleson curve at $t \in \Gamma$. For every $x \in \mathbf{R}$, the function $W_t \eta_t^x$ is regular, submultiplicative, and*

$$\begin{aligned} \alpha_t^0(x) &:= \alpha(W_t \eta_t^x) = \min\{\delta_t^- x, \delta_t^+ x\}, \\ \beta_t^0(x) &:= \beta(W_t \eta_t^x) = \max\{\delta_t^- x, \delta_t^+ x\}. \end{aligned}$$

This statement follows from local analogs of [**3**, Lemmas 1.15, 1.16, and Proposition 3.1].

For a complex number $\gamma \in \mathbf{C}$, we define a continuous function $\varphi_{t,\gamma}$ on $\Gamma \setminus \{t\}$ by

$$\begin{aligned}\varphi_{t,\gamma}(\tau) := |(\tau - t)^\gamma| &= |\tau - t|^{\operatorname{Re}\gamma} e^{-\operatorname{Im}\gamma \arg(\tau - t)} \\ &= |\tau - t|^{\operatorname{Re}\gamma} (\eta_t(\tau))^{\operatorname{Im}\gamma}.\end{aligned} \tag{5.50}$$

Lemma 5.12. *If $w \in A_X(\Gamma, t)$, then for every $\gamma \in \mathbf{C}$, the function $Q_t(\varphi_{t,\gamma} w)$ is regular, submultiplicative, and*

$$\alpha(Q_t(\varphi_{t,\gamma} w)) = \operatorname{Re}\gamma + \alpha(Q_t(\eta_t^{\operatorname{Im}\gamma} w)), \tag{5.51}$$

$$\beta(Q_t(\varphi_{t,\gamma} w)) = \operatorname{Re}\gamma + \beta(Q_t(\eta_t^{\operatorname{Im}\gamma} w)). \tag{5.52}$$

Proof. This statement is proved similarly to [**24**, Lemma 7.2]. By a local analog of [**3**, Proposition 3.1], the function $W_t \varphi_{t,\operatorname{Re}\gamma}$ is regular and submultiplicative for every $\gamma \in \mathbf{C}$ and

$$\alpha(W_t \varphi_{t,\operatorname{Re}\gamma}) = \beta(W_t \varphi_{t,\operatorname{Re}\gamma}) = \operatorname{Re}\gamma. \tag{5.53}$$

On the other hand, by Lemmas 4.8–4.9, the function $Q_t w$ is regular and submultiplicative. Then, by Theorem 5.2, the function $Q_t(\varphi_{t,\gamma} w)$ is regular and submultiplicative for every $\gamma \in \mathbf{C}$. In particular, the function $Q_t(\eta_t^{\operatorname{Im}\gamma} w)$ is regular and submultiplicative for every $\gamma \in \mathbf{C}$. From Theorem 5.2 and (5.53) it follows that

$$\begin{aligned}&\alpha(Q_t(\eta_t^{\operatorname{Im}\gamma} w)) + \operatorname{Re}\gamma \le \alpha(Q_t(\varphi_{t,\gamma} w)) \\ &\le \min\{\alpha(Q_t(\eta_t^{\operatorname{Im}\gamma} w)) + \operatorname{Re}\gamma, \beta(Q_t(\eta_t^{\operatorname{Im}\gamma} w)) + \operatorname{Re}\gamma\}, \\ &\beta(Q_t(\eta_t^{\operatorname{Im}\gamma} w)) + \operatorname{Re}\gamma \ge \beta(Q_t(\varphi_{t,\gamma} w)) \\ &\ge \max\{\alpha(Q_t(\eta_t^{\operatorname{Im}\gamma} w)) + \operatorname{Re}\gamma, \beta(Q_t(\eta_t^{\operatorname{Im}\gamma} w)) + \operatorname{Re}\gamma\}.\end{aligned}$$

From the latter inequalities we immediately obtain (5.51)–(5.52). ◻

Lemma 5.13. *If $w \in A_X(\Gamma, t)$ and $1 \in A_X(\Gamma, t)$, then for every $\gamma \in \mathbf{C}$, the function $V_t(\varphi_{t,\gamma} w)$ is regular, submultiplicative, and*

$$\alpha(V_t(\varphi_{t,\gamma} w)) = \operatorname{Re}\gamma + \alpha(V_t(\eta_t^{\operatorname{Im}\gamma} w)), \tag{5.54}$$

$$\beta(V_t(\varphi_{t,\gamma} w)) = \operatorname{Re}\gamma + \beta(V_t(\eta_t^{\operatorname{Im}\gamma} w)). \tag{5.55}$$

Proof. By Lemma 3.2(a), $\log w \in BMO(\Gamma, t)$. Then by Lemma 4.6, the function $V_t w$ is regular. The rest is proved by analogy with Lemma 5.12 with the help of Theorem 5.1. □

If $w \in A_X(\Gamma, t)$, then for every $x \in \mathbf{R}$, the function $Q_t(\eta_t^x w)$ is regular and submultiplicative, in view of Lemma 5.12. From Theorem 4.1 and Lemma 4.9 we deduce that the following functions are well defined for $x \in \mathbf{R}$:

$$\alpha_t^*(x) := \alpha(Q_t(\eta_t^x w)) = \alpha(Q_t^0(\eta_t^x w)),$$
$$\beta_t^*(x) := \beta(Q_t(\eta_t^x w)) = \beta(Q_t^0(\eta_t^x w)).$$

If, in addition, $1 \in A_X(\Gamma, t)$, then the function $V_t(\eta_t^x w)$ is regular and submultiplicative for each $x \in \mathbf{R}$, due to Lemma 5.13. Then Theorem 4.1 and Lemma 4.5 imply that the functions

$$\alpha_t(x) := \alpha(V_t(\eta_t^x w)) = \alpha(V_t^0(\eta_t^x w)),$$
$$\beta_t(x) := \beta(V_t(\eta_t^x w)) = \beta(V_t^0(\eta_t^x w))$$

are well defined for all $x \in \mathbf{R}$.

The functions α_t^*, β_t^* are called the *indicator functions of the triple* (Γ, X, w) *at* $t \in \Gamma$. The functions α_t, β_t are referred to as the *indicator functions of the pair* (Γ, w) *at* $t \in \Gamma$. The functions α_t^*, β_t^* were introduced in [**25**] (see also [**24, 26**]) for rearrangement-invariant Banach function spaces. The functions α_t, β_t were defined in [**3**, Chapter 3] in the context of Lebesgue spaces and Muckenhoupt weights.

Lemma 5.14. *The functions α_t, α_t^* are concave, the functions β_t, β_t^* are convex. In particular, α_t, α_t^* and β_t, β_t^* are continuous on* $\mathbf{R}$.

Proof. By [**35**, Section 2.2, Property 6],

$$\left\| |f|^\theta |g|^{1-\theta} \right\|_X \le \|f\|_X^\theta \|g\|_X^{1-\theta}, \quad \theta \in [0,1],$$

for every $f, g \in X$. With the help of this property, one can prove concavity of α_t^* and convexity of β_t^* similarly to [**3**, Proposition 3.20]. Concavity of α_t and convexity of β_t are already proved there. □

The following statement generalizes [**26**, Lemma 3.5].

Lemma 5.15. (a) *If* $w \in A_X(\Gamma, t)$*, then for* $x, y \in \mathbf{R}$*,*

$$\begin{aligned}\alpha_t^*(x) + \alpha_t^0(y) \le \alpha_t^*(x+y) \le \min\{\alpha_t^*(x) + \beta_t^0(y), \beta_t^*(x) + \alpha_t^0(y)\},\\ \beta_t^*(x) + \beta_t^0(y) \ge \beta_t^*(x+y) \ge \max\{\alpha_t^*(x) + \beta_t^0(y), \beta_t^*(x) + \alpha_t^0(y)\}.\end{aligned}$$

(b) *If* $w \in A_X(\Gamma, t)$ *and* $1 \in A_X(\Gamma, t)$*, then for* $x, y \in \mathbf{R}$*,*

$$\begin{aligned}\alpha_t(x) + \alpha_t^0(y) \le \alpha_t(x+y) \le \min\{\alpha_t(x) + \beta_t^0(y), \beta_t(x) + \alpha_t^0(y)\},\\ \beta_t(x) + \beta_t^0(y) \ge \beta_t(x+y) \ge \max\{\alpha_t(x) + \beta_t^0(y), \beta_t(x) + \alpha_t^0(y)\}.\end{aligned}$$

Proof. (a) From Lemmas 5.11 and 5.13 it follows that the functions α_t^*, β_t^* and α_t^0, β_t^0 are well defined. Applying Theorem 5.2 to the weights $w := \eta_t^x w$ and $\psi := \eta_t^y$, we get Part (a). Part (b) is proved analogously with the help of Theorem 5.1 and Lemma 5.13. □

Corollary 5.16. *Let* Γ *be locally a Carleson curve at* $t \in \Gamma$ *such that* $\delta_t^- = \delta_t^+ =: \delta_t$*.*

(a) *If* $w \in A_X(\Gamma, t)$*, then*

$$\alpha_t^*(x) = \alpha(Q_t w) + \delta_t x, \quad \beta_t^*(x) = \beta(Q_t w) + \delta_t x \quad (x \in \mathbf{R}). \tag{5.56}$$

(b) *If* $w \in A_X(\Gamma, t)$ *and* $1 \in A_X(\Gamma, t)$*, then*

$$\alpha_t(x) = \mu_t + \delta_t x, \quad \beta_t(x) = \nu_t + \delta_t x \quad (x \in \mathbf{R}). \tag{5.57}$$

Proof. (a) Since $\delta_t^- = \delta_t^+ = \delta_t$, we have $\alpha_t^0(x) = \beta_t^0(x) = \delta_t x$. In that case from Lemma 5.15(a) we deduce that

$$\alpha_t^*(y) + \delta_t x = \alpha_t^*(x+y), \quad \beta_t^*(y) + \delta_t x = \beta_t^*(x+y) \tag{5.58}$$

for every $x, y \in \mathbf{R}$. Setting $y = 0$ in (5.58), we arrive at (5.56). Part (b) is proved similarly. □

5.4 Indicator functions for Nakano spaces. Let Γ be a rectifiable Jordan curve, let $L^{p(\cdot)}$ be a Nakano space. Fix $t \in \Gamma$. For a weight $w \in A_{L^{p(\cdot)}}(\Gamma, t)$, put

$$N_t := \Big\{\gamma \in \mathbf{C} : \quad \varphi_{t,\gamma} w \in A_{L^{p(\cdot)}}(\Gamma, t)\Big\}.$$

Lemma 5.17. *Let Γ be locally a Carleson curve at $t \in \Gamma$, let $p \in \mathcal{P}_t$, and let $w \in A_{L^{p(\cdot)}}(\Gamma, t)$. Then for every $\gamma \in N_t$,*

$$\alpha_t^*(\operatorname{Im}\gamma) = 1/p(t) + \alpha_t(\operatorname{Im}\gamma), \quad \beta_t^*(\operatorname{Im}\gamma) = 1/p(t) + \beta_t(\operatorname{Im}\gamma). \tag{5.59}$$

Proof. Let $\gamma \in N_t$. By Theorem 5.9,

$$\alpha(Q_t(\varphi_{t,\gamma} w)) = 1/p(t) + \alpha(V_t(\varphi_{t,\gamma} w)), \tag{5.60}$$

$$\beta(Q_t(\varphi_{t,\gamma} w)) = 1/p(t) + \beta(V_t(\varphi_{t,\gamma} w)). \tag{5.61}$$

Note that by Lemma 5.8, $1 \in A_{L^{p(\cdot)}}(\Gamma, t)$. Therefore, we can apply Lemma 5.13. From (5.60)–(5.61), (5.51)–(5.52) and (5.54)–(5.55) it follows that

$$\alpha(Q_t(\eta_t^{\operatorname{Im}\gamma} w)) = 1/p(t) + \alpha(V_t(\eta_t^{\operatorname{Im}\gamma} w)),$$
$$\beta(Q_t(\eta_t^{\operatorname{Im}\gamma} w)) = 1/p(t) + \beta(V_t(\eta_t^{\operatorname{Im}\gamma} w)),$$

that is, equalities (5.59) hold. □

Lemma 5.18. *Let Γ be locally a Carleson curve at $t \in \Gamma$ such that $\delta_t^- = \delta_t^+ = 0$, let $p \in \mathcal{P}_t$, and let $w \in A_{L^{p(\cdot)}}(\Gamma, t)$. Then for every $x \in \mathbf{R}$,*

$$\begin{aligned} \alpha_t(x) &= \mu_t, \quad \alpha_t^*(x) = 1/p(t) + \mu_t, \\ \beta_t(x) &= \nu_t, \quad \beta_t^*(x) = 1/p(t) + \nu_t, \end{aligned} \tag{5.62}$$

where μ_t, ν_t are the indices of powerlikeness of the weight w at t defined by (4.1).

Proof. By Lemma 5.8, $1 \in A_{L^{p(\cdot)}}(\Gamma, t)$. From Corollary 5.16 we get for every $x \in \mathbf{R}$,

$$\begin{aligned} \alpha_t^*(x) &= \alpha(Q_t w), \quad \alpha_t(x) = \mu_t, \\ \beta_t^*(x) &= \beta_t(Q_t w), \quad \beta_t(x) = \nu_t. \end{aligned} \tag{5.63}$$

On the other hand, by Theorem 5.9,

$$\alpha(Q_t w) = 1/p(t) + \mu_t, \quad \beta(Q_t w) = 1/p(t) + \nu_t. \tag{5.64}$$

Combining (5.63) and (5.64), we arrive at (5.62). □

6. Fredholm theory for singular integral operators with bounded measurable coefficients.

6.1 The Cauchy singular integral operator. Let Γ be a rectifiable Jordan curve. We provide Γ with the counter-clockwise orientation. The curve Γ divides the complex plane $\mathbf{C}$ into a bounded connected component D^+ and an unbounded connected component D^-. Without loss of generality we suppose that $0 \in D^+$. Let X be a Banach function space and $w : \Gamma \to [0, \infty]$ be a weight. Then the weighted Banach function space X_w is a linear normed space which becomes a Banach function space whenever $w \in X$ and $1/w \in X'$, see Lemma 2.5.

Theorem 6.1. *Let Γ be a rectifiable Jordan curve, let $w : \Gamma \to [0, \infty]$ be a weight, and let X be a Banach function space. If the Cauchy singular integral operator S is bounded on the weighted Banach function space X_w, then $w \in A_X(\Gamma)$.*

This theorem was proved for weighted rearrangement-invariant Banach function spaces in a slightly different form in [**24**, Theorem 3.2], see also [**22**, Theorem 4.3] and [**3**, Theorem 4.8]. First, as in [**24**, Lemma 3.3], by using the Landau lemma for the Banach function space X (see [**1**, Chapter 1, Lemma 2.7]), we show that $w \in X$ and $1/w \in X'$. Then, by Lemma 2.5(b), the weighted Banach function space X_w is itself a Banach function space. The proof of [**24**, Theorem 3.2], see also [**23**, Section 3], does not use the rearrangement-invariant property of the space X, so it works for arbitrary weighted Banach function spaces.

The question about the sufficiency of the condition $w \in A_X(\Gamma)$ for the boundedness of the Cauchy singular integral operator S on weighted Banach function spaces X_w is open. We know only that this condition is sufficient for the boundedness in the case of Lebesgue spaces $X = L^p, 1 < p < \infty$, that is, when $A_X(\Gamma) = A_p(\Gamma)$ is the Muckenhoupt class, see, e.g., [**3**, Theorem 4.15].

However, criteria for the boundedness of S on Nakano spaces with Khvedelidze weights $L^{p(\cdot)}_\varrho$ were recently proved by Kokilashvili and Samko [**30**] under the condition that the contour Γ is sufficiently nice.

Theorem 6.2 (see [**30**, Theorem 2]). *Let Γ be either a Lyapunov Jordan curve or a Radon Jordan curve without cusps, let ϱ be a Khvedelidze weight* (1.2)*, and let $p \in \mathcal{P}$. The Cauchy singular integral operator S is bounded on the weighted Nakano space $L^{p(\cdot)}_\varrho$ if and only if*

$$0 < \frac{1}{p(\tau_k)} + \lambda_k < 1 \quad \textit{for all } k \in \{1, \dots, n\}. \tag{6.1}$$

For weighted Lebesgue spaces L^p_ϱ this result is classic, for Lyapunov curves it was proved by Khvedelidze [**27**] and for Radon curves without cusps by Danilyuk and Shelepov [**9**, Theorem 2]. The proofs and history can be found in [**8, 16, 28, 40**].

6.2 Singular integral operators. In the following we will assume that Γ is a rectifiable Jordan curve, X is a Banach function space, $w : \Gamma \to [0, \infty]$ is a weight such that

(B) the Cauchy singular integral operator S is bounded on the weighted Banach function space X_w;

(R) the weighted Banach function space X_w is reflexive.

Axiom (B) guarantees that, by Theorem 6.1, $w \in A_X(\Gamma)$. Therefore, $w \in X$ and $1/w \in X'$. Hence, X_w is a Banach function space with the associate space $X'_{1/w}$ and

$$L^\infty \subset X_w \subset L^1.$$

On the other hand, if $w \in A_X(\Gamma)$, then Γ is a Carleson curve. Axiom (R) implies that the Banach dual $(X_w)^*$ of X_w coincides with its associate space $X'_{1/w}$ and the set $\mathcal{R}$ of all rational functions without poles on Γ is dense in both X_w and $X'_{1/w}$ (for details, see Subsection 2.4).

The above mentioned properties of weighted Banach functions spaces satisfying axioms (B) and (R) allow us to prove the following statements as in the case of weighted Lebsegue spaces, see, e.g., [**16**, Chapter 1] and [**3**, Chapter 6]. Detailed proofs can be found in [**23**, Chapter 2] (see also [**24, 25**]) for weighted rearrangement-invariant Banach function spaces X_w. Note that the assumption that X is rearrangement-invariant is not essential and can be omitted there.

We denote by $\mathcal{K}(X_w)$ the closed two-sided ideal of all compact operators on X_w in the Banach algebra $\mathcal{B}(X_w)$ of all bounded linear operators on X_w. As usual, I is the identity operator on X_w and aI denotes the operator of multiplication by a measurable function $a : \Gamma \to \mathbf{C}$.

Lemma 6.3. *If $a \in L^\infty$, then $aI \in \mathcal{B}(X_w)$ and $\|aI\|_{\mathcal{B}(X_w)} \le \|a\|_\infty$.*

Lemma 6.4. *The operators*

$$P_+ := (I + S)/2, \quad P_- := (I - S)/2$$

are bounded projections on both X_w and $X'_{1/w}$.

Lemma 6.5. *If $a \in C$, then $aS - SaI \in \mathcal{K}(X_w)$.*

On the weighted Banach function space X_w, or on its dual $(X_w)^* = X'_{1/w}$, define the operator H_Γ by $(H_\Gamma\varphi)(\tau) := e^{-i\theta_\Gamma(\tau)}\overline{\varphi(\tau)}$. Note that the operator H_Γ is additive but $H_\Gamma(\alpha\varphi) = \overline{\alpha}\cdot H_\Gamma\varphi$ for $\alpha \in \mathbf{C}$. Evidently, $H_\Gamma^2 = I$.

Lemma 6.6. *The adjoint of $S \in \mathcal{B}(X_w)$ is*

$$S^* = -H_\Gamma S H_\Gamma \in \mathcal{B}(X'_{1/w}).$$

For $a \in L^\infty$, put

$$T_a := P_+ a P_+ + P_-, \quad R_a := aP_+ + P_-.$$

Lemma 6.7. *Let $a \in L^\infty$. If one of the operators T_a, R_a is semi-Fredholm, Fredholm, left-invertible, right-invertible, invertible, then the second operator has the same property. If the operators T_a and R_a are semi-Fredholm, then*

$$n(T_a) = n(R_a), \quad d(T_a) = d(R_a).$$

Proof. By Lemmas 6.3–6.4, the operators aI and $P_\pm$ are bounded on X_w. The rest follows from [**21**, Lemma 1.21]. □

So, it is sufficient to study only one of the operators T_a, R_a. We will formulate our main results for the operator R_a. This operator is usually called a *singular integral operator with the coefficient a*. It is well known that Fredholm properties of this operator are closely connected with the solvability of the Riemann-Hilbert boundary value problem, see, e.g., [**6, 16, 38**].

6.3 Hardy type subspaces. In view of Lemma 6.4, one can define the following subspaces of X_w:

$$(X_w)_+ := P_+ X_w, \quad (X_w)^0_- := P_- X_w, \quad (X_w)_- := (X_w)^0_- \dot{+} \mathbf{C};$$

the corresponding subspaces $(X'_{1/w})_+, (X'_{1/w})^0_-, (X'_{1/w})_-$ of $X'_{1/w}$ are defined analogously. Also put

$$\begin{aligned} L^1_+ &:= \left\{ f \in L^1 : \int_\Gamma f(\tau)\tau^n d\tau = 0 \quad \text{for } n \geq 0 \right\}, \\ (L^1)^0_- &:= \left\{ f \in L^1 : \int_\Gamma f(\tau)\tau^n d\tau = 0 \quad \text{for } n < 0 \right\}, \\ L^1_- &:= (L^1)^0_- \dot{+} \mathbf{C}. \end{aligned}$$

Lemma 6.8. (see [**47**, pp. 202–206]). *We have $L^1_+ \cap (L^1)^0_- = \{0\}$ and $L^1_+ \cap L^1_- = \mathbf{C}$.*

Lemma 6.9. (a) *If* $f \in (X_w)_\pm$ *and* $g \in (X'_{1/w})_\pm$, *then* $fg \in L^1_\pm$. *If, in addition,* $f \in (X_w)^0_-$ *or* $g \in (X'_{1/w})^0_-$, *then* $fg \in (L^1)^0_-$.

(b) *We have*

$$(X_w)_+ = L^1_+ \cap X_w, \quad (X_w)^0_- = (L^1)^0_- \cap X_w, \quad (X_w)_- = L^1_- \cap X_w.$$

This lemma is proved by analogy with [**3**, Corollary 6.8] and [**3**, Lemma 6.11]. Here we essentially use Cauchy's theorem, Hölder's inequality for the weighted Banach function space X_w, and the density of $\mathcal{R}$ in X_w and in $X'_{1/w}$, see Corollary 2.11.

Lemma 6.10. *Suppose* $f_\pm$ *is analytic in* $D^\pm$ *and continuous on* $D^\pm \cup \Gamma$ *with the possible exception of finitely many points* $t_1, \dots, t_m \in \Gamma$. *Suppose that* $f_\pm|\Gamma \in X_w$ *and that* $f_\pm$ *admits the estimate*

$$|f_\pm(z)| \le M|z - t_k|^{-\mu}, \quad k = 1, \dots, m,$$

with some $M > 0, \mu > 0$ *for all* $z \in D^\pm$ *sufficiently close to* t_k. *Then* $f_\pm \in (X_w)_\pm$.

This result goes back to Grudsky [**17**, Proposition 1.5] for Lebesgue spaces. To prove this statement, we should repeat the proof of [**3**, Lemma 6.10], replacing $L^p(\Gamma, w)$ by X_w and using Lemma 6.9. For $\mu \in (0, 1]$ and Lebesgue spaces this result was known for a long time [**16**, Chapter 2, Theorem 4.8]. We remark that for our purposes (see Lemma 7.1) we really need this analog of Grudsky's lemma allowing also the case $\mu > 1$.

6.4 Two basic theorems. Let GL^∞ denote the set of all functions in L^∞ which are invertible in L^∞, that is, the set of functions $a \in L^\infty$ such that

$$\operatorname*{ess\,inf}_{\tau \in \Gamma} |a(\tau)| > 0.$$

Theorem 6.11. *Let* $a, b \in L^\infty$. *If the operator* $aP_+ + bP_-$ *is semi-Fredholm in* X_w, *then* $a, b \in GL^\infty$.

Theorem 6.12. *If* $a \in GL^\infty$*, then* $\min\{n(R_a), d(R_a)\} = 0$.

Theorem 6.12 was proved by Coburn [**7**] for Toeplitz operators on $L^2(\mathbf{T})$. In the form presented here Theorems 6.11 and 6.12 were proved by Simonenko in [**51**] for Lebesgue spaces with Khvedelidze weights over Lyapunov curves. For a detailed discussion of these theorems for weighted Lebesgue spaces, see [**3**, Section 6.6] and [**16**, Sections 7.4 and 7.5]. In our case the proofs are developed analogously on the basis of the results of subsections 6.2–6.3 and the Lusin-Privalov theorem, see, e.g., [**47**, p. 292].

6.5 The local principle of Simonenko type. Two functions $a, b \in L^\infty$ are said to be *locally equivalent at a point* $t \in \Gamma$ if

$$\inf\Big\{\|(a-b)c\|_\infty : \ c \in C, \quad c(t) = 1\Big\} = 0.$$

Theorem 6.13. *Let* $a \in L^\infty$*. Suppose for each* $t \in \Gamma$ *we are given a function* $a_t \in L^\infty$ *which is locally equivalent to* a *at* t*. If the operators* R_{a_t} *are Fredholm in* X_w *for all* $t \in \Gamma$*, then* R_a *is Fredholm in* X_w*.*

For weighted Lebesgue spaces, this theorem is known as Simonenko's local principle [**50**]. More information about localization techniques can be found, e.g., in [**3, 5, 16, 36**]. Theorem 6.13 can be proved similarly to [**3**, Theorem 6.30] with the help of Lemmas 6.5 and 6.7.

6.6 Wiener-Hopf factorization. We say that a function $a \in L^\infty$ admits a *Wiener-Hopf factorization in the weighted Banach function space* X_w if $1/a \in L^\infty$ and a can be written in the form

$$a(t) = a_-(t)t^\kappa a_+(t) \quad \text{a.e. on } \Gamma, \tag{6.2}$$

where $\kappa \in \mathbf{Z}$, and the factors $a_\pm$ enjoy the following properties:

(i) $a_- \in (X_w)_-$, $1/a_- \in (X'_{1/w})_-$, $a_+ \in (X'_{1/w})_+$, $1/a_+ \in (X_w)_+$,

(ii) the operator $(1/a_+)Sa_+I$ is bounded on X_w.

One can prove that the number κ is uniquely determined.

Theorem 6.14. *A function $a \in L^\infty$ admits a Wiener-Hopf factorization* (6.2) *in the reflexive weighted Banach function space X_w if and only if the operator R_a is Fredholm in X_w. If R_a is Fredholm, then its index is equal to $-\kappa$.*

This theorem goes back to Simonenko [**49, 51**]. For more about this topic we refer to [**3**, Section 6.12], [**5**, Section 5.5], [**16**, Section 8.3] and also to [**6, 38**] in the case of weighted Lebesgue spaces. Simonenko's result was generalized by the author to the case of reflexive Orlicz spaces [**22**, Theorem 5.6] and to the case of reflexive rearrangement-invariant spaces [**24**, Theorem 6.10]. In the case of reflexive weighted Banach function spaces the proof is developed by analogy. The proof is essentially based on the density of $\mathcal{R}$ in X_w and in $X'_{1/w}$, Lemmas 6.8–6.9 and Theorems 6.11–6.12. Detailed proofs for the results of this section can be found in [**23**, Chapter 2] for weighted Banach function spaces X_w provided X is rearrangement-invariant. We remind the reader that this assumption can be simply omitted.

7. Fredholmness of singular integral operators in weighted Banach function spaces.

7.1 Local representatives. Fix $t \in \Gamma$. For a function $a \in PC \cap GL^\infty$ we construct a "canonical" function $g_{t,\gamma}$ which is locally equivalent to a at the point $t \in \Gamma$. The interior and the exterior of the unit circle can be conformally mapped onto D^+ and D^- of Γ, respectively, so that the point 1 is mapped to t, and the points $0 \in D^+$ and $\infty \in D^-$ remain fixed. Let Λ_0 and Λ_∞ denote the images of $[0,1]$ and $[1,\infty) \cup \{\infty\}$ under this map. The curve $\Lambda_0 \cup \Lambda_\infty$ joins 0 to ∞ and meets Γ at exactly one point, namely t. Let $\arg z$ be a continuous branch of argument in $\mathbf{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. For $\gamma \in \mathbf{C}$, define the function $z^\gamma := |z|^\gamma e^{i\gamma \arg z}$, where $z \in \mathbf{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. Clearly, z^γ is an analytic function in $\mathbf{C} \setminus (\Lambda_0 \cup \Lambda_\infty)$. The restriction of z^γ to $\Gamma \setminus \{t\}$ will be denoted by $g_{t,\gamma}$. Obviously, $g_{t,\gamma}$ is continuous and nonzero on $\Gamma \setminus \{t\}$.

Since $a(t \pm 0) \neq 0$, we can define $\gamma_t = \gamma \in \mathbf{C}$ by the formulas

$$\text{(7.1)} \qquad \operatorname{Re}\gamma_t := \frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)}, \quad \operatorname{Im}\gamma_t := -\frac{1}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right|,$$

where we can take any value of $\arg(a(t-0)/a(t+0))$, which implies that any two choices of $\operatorname{Re}\gamma_t$ differ by an integer only. Clearly, there is a constant $c_t \in \mathbf{C}\setminus\{0\}$ such that $a(t\pm 0) = c_t g_{t,\gamma_t}(t\pm 0)$, which means that a is locally equivalent to $c_t g_{t,\gamma_t}$ at the point $t\in\Gamma$.

7.2 Sufficient conditions for factorability of the local representative.

Lemma 7.1. *If, for some $k \in \mathbf{Z}$ and $\gamma \in \mathbf{C}$, the operator $\varphi_{t,k-\gamma} S \varphi_{t,\gamma-k} I$ is bounded on the weighted Banach function space X_w, then*

$$g_{t,\gamma}(\tau) = (1-t/\tau)^{k-\gamma}\tau^k(\tau-t)^{\gamma-k}, \quad \tau\in\Gamma\setminus\{t\} \tag{7.2}$$

is a Wiener-Hopf factorization of the function $g_{t,\gamma}$ in X_w.

Proof. Since the operator $\varphi_{t,k-\gamma} S \varphi_{t,k-\gamma}^{-1} I$ is bounded on X_w, the operator S is bounded on the weighted Banach function space $X_{\varphi_{t,k-\gamma}w}$. By Theorem 6.1, $\varphi_{t,k-\gamma}w \in A_X(\Gamma)$. In that case Γ is a Carleson curve and $\varphi_{t,k-\gamma}w \in X$, whence $\varphi_{t,k-\gamma} \in X_w$.

Let us show that $(\tau-t)^{k-\gamma}\in(X_w)_+$. The function $f(z) := (z-t)^{k-\gamma}$ is analytic in D^+ and continuous on $D^+\cup(\Gamma\setminus\{t\})$. For $z\in D^+$,

$$|f(z)| = |(z-t)^{k-\gamma}| = |z-t|^{k-\operatorname{Re}\gamma-\Theta_t(z)\operatorname{Im}\gamma},$$

where $\Theta_\tau(z) := \arg(z-t)/(-\log|z-t|)$. As in [**24**, Theorem 7.7] and [**3**, Lemma 7.1] with the help of Lemma 4.9 one can show that there is a constant $M_t\in(0,\infty)$ such that

$$|f(z)| \le |z-t|^{k-\operatorname{Re}\gamma} e^{M_t|\operatorname{Im}\gamma|(-\log|z-t|)} = |z-t|^{k-\operatorname{Re}\gamma-M_t|\operatorname{Im}\gamma|}$$

for all z in a small neighborhood of t. By Lemma 6.10, $(\tau-t)^{k-\gamma}\in(X_w)_+$. Analogously one can prove that

$$(\tau-t)^{\gamma-k}\in(X'_{1/w})_+,\quad (1-t/\tau)^{k-\gamma}\in(X_w)_-,\quad (1-t/\tau)^{\gamma-k}\in(X'_{1/w})_-.$$

These facts together with the boundedness of $\varphi_{t,k-\gamma} S\varphi_{t,\gamma-k} I$ on the space X_w show that (7.2) is indeed a Wiener-Hopf factorization of the function $g_{t,\gamma}$. $\square$

7.3 Necessary conditions for factorability of the local representative.

Theorem 7.2. *If the function $g_{t,\gamma}$ admits a Wiener-Hopf factorization in the weighted Banach function space X_w, then*

$$-\operatorname{Re}\gamma + \theta\alpha_t^*(-\operatorname{Im}\gamma) + (1-\theta)\beta_t^*(-\operatorname{Im}\gamma) \notin \mathbf{Z} \tag{7.3}$$

for all $\theta \in [0,1]$. Moreover, there exists an $l \in \mathbf{Z}$ such that $\varphi_{t,l-\gamma}w$ belongs to $A_X(\Gamma)$.

Proof. The idea of the proof (in the case of weighted Lebesgue spaces) goes back to Spitkovsky [**52**] and it was further developed by Böttcher and Yu. Karlovich [**3**, Proposition 7.2]. This idea was applied to the proof in the case of reflexive rearrangement-invariant Banach function spaces (with weights) in [**24**, Theorem 7.6] and [**25**, Theorem 4.1]. Since, for our (more general) case, the arguments are the same, we point out only the main steps.

By Theorem 6.14, the operator $g_{t,\gamma}P_+ + P_-$ is Fredholm. Then there exists a $c > 0$ such that the operators $g_{t,\gamma-\varepsilon}P_+ + P_-$ are Fredholm for all $\varepsilon \in (-c,c)$. Applying Theorem 6.14 again, we infer that all functions $g_{t,\gamma-\varepsilon}$ admit a Wiener-Hopf factorization in X_w. By using its definition, one can show that there exists an $l \in \mathbf{Z}$ such that the operators $\varphi_{t,l-\gamma+\varepsilon}S\varphi_{t,l-\gamma+\varepsilon}^{-1}I$ are bounded on X_w for all $\varepsilon \in (-c,c)$. In that case, by Theorem 6.1, $\varphi_{t,l-\gamma+\varepsilon}w \in A_X(\Gamma) \subset A_X(\Gamma,t)$. By Lemma 4.9,

$$0 \le (Q_t(\varphi_{t,l-\gamma+\varepsilon}w)) \le \beta(Q_t(\varphi_{t,l-\gamma+\varepsilon}w)) \le 1. \tag{7.4}$$

From Lemma 5.12 and (7.4) it follows that

$$0 \le l + \varepsilon - \operatorname{Re}\gamma + \alpha_t^*(-\operatorname{Im}\gamma) \le l + \varepsilon - \operatorname{Re}\gamma + \beta_t^*(-\operatorname{Im}\gamma) \le 1$$

for all $\varepsilon \in (-c,c)$. Hence,

$$-l < -\operatorname{Re}\gamma + \theta\alpha_t^*(-\operatorname{Im}\gamma) + (1-\theta)\beta_t^*(-\operatorname{Im}\gamma) < l - 1$$

for every $\theta \in [0,1]$. Thus, (7.3) holds for every $\theta \in [0,1]$. □

7.4 Necessary conditions for Fredholmness. Now we are in a position to state the main result of this paper.

Theorem 7.3. *Let Γ be a rectifiable Jordan curve, let $w : \Gamma \to [0, \infty]$ be a weight, and let X be a Banach function space. Suppose the Cauchy singular integral operator S is bounded on the weighted Banach function space X_w and X_w is reflexive. If the operator $aP_+ + P_-$, where $a \in PC$, is Fredholm in X_w, then $a \in GL^\infty$ and*

$$-\frac{1}{2\pi}\arg\frac{a(t-0)}{a(t+0)} + \theta\alpha_t^*\left(\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right) + (1-\theta)\beta_t^*\left(\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right) \notin \mathbf{Z} \tag{7.5}$$

for all $t \in \Gamma$ and all $\theta \in [0, 1]$.

Proof. The proof is developed by analogy with the proof of necessity part of [**24**, Theorem 7.8], see also [**3**, Proposition 7.3].

If R_a is Fredholm, then, by Theorem 6.11, $a \in GL^\infty$. Fix an arbitrary $t \in \Gamma$. Choose $\gamma = \gamma_t \in \mathbf{C}$ as in (7.1). Then the function a is locally equivalent to $c_t g_{t,\gamma_t}$ at the point t, where $c_t \in \mathbf{C} \setminus \{0\}$ is some constant. If $\tau \in \Gamma \setminus \{t\}$, then g_{t,γ_t} is continuous and nonzero at τ. Hence, it is locally equivalent to the nonzero constant $b_\tau := g_{t,\gamma_t}(\tau)$ at τ. Clearly, the operator $R_{b_\tau} := b_\tau P_+ + P_-$ is invertible, $(b_\tau P_+ + P_-)^{-1} = b_\tau^{-1} P_+ + P_-$. Therefore, the operator R_{b_τ} is Fredholm for every $\tau \in \Gamma \setminus \{t\}$. Remind that the function g_{t,γ_t} is locally equivalent to the function $c_t^{-1} a$. Since

$$R_{c_t^{-1}} R_a = P_+ c_t^{-1} a P_+ + P_- = T_{c_t^{-1} a} \tag{7.6}$$

and the operator $R_{c_t^{-1}}$ is invertible, from Lemma 6.7 and (7.6) it follows that R_a is Fredholm if and only if $R_{c_t^{-1} a}$ is Fredholm. Therefore, applying Theorem 6.14, we infer that the operator $R_{g_{t,\gamma_t}}$ is Fredholm. By Theorem 6.14, the function g_{t,γ_t} admits a Wiener-Hopf factorization in X_w. From Theorem 7.2 it follows that

$$-\operatorname{Re}\gamma_t + \theta\alpha_t^*(-\operatorname{Im}\gamma_t) + (1-\theta)\beta_t^*(-\operatorname{Im}\gamma_t) \notin \mathbf{Z} \tag{7.7}$$

for all $\theta \in [0,1]$. Since $t \in \Gamma$ is arbitrary, from (7.1) and (7.7) we conclude that (7.5) holds for every $t \in \Gamma$ and every $\theta \in [0,1]$. □

7.5 Lower estimates for essential norms. For an operator $A \in \mathcal{B}(X_w)$, let

$$|A|_{X_w} := \inf_{K \in \mathcal{K}(X_w)} \|A+K\|_{\mathcal{B}(X_w)}$$

be its *essential norm* in X_w.

Theorem 7.4. *Let Γ be a rectifiable Jordan curve, let $w : \Gamma \to [0,\infty]$ be a weight, and let X be a Banach function space. If the Cauchy singular integral operator S is bounded on the weighted Banach function space X_w and X_w is reflexive, then*

$$|S|_{X_w} \ge \cot\Big(\pi\Lambda_{\Gamma,X,w}/2\Big), \quad |P_\pm|_{X_w} \ge 1/\sin(\pi\Lambda_{\Gamma,X,w}),$$

where

$$\Lambda_{\Gamma,X,w} := \inf_{t\in\Gamma} \min\Big\{\alpha(Q_t w), 1-\beta(Q_t w)\Big\}.$$

This statement is proved by a literal repetition of the proof of [**25**, Theorem 4.5] using the scheme of [**16**, Chapter 9, Theorem 9.1]. One can find more information about estimates of (essential) norms on weighted Lebesgue spaces in [**16**, Chapter 13] and [**36**, Chapter 2].

8. Fredholmness of singular integral operators in weighted Nakano spaces.

8.1 Necessary conditions for Fredholmness. The necessary conditions for the Fredholmness of R_a in weighted Nakano spaces have a simpler form than in the general case because we can replace the indicator functions α_t^* and β_t^* by the indicator functions $1/p(t)+\alpha_t$ and $1/p(t)+\beta_t$, respectively. More precisely, the next theorem is true.

Theorem 8.1. *Let Γ be a rectifiable Jordan curve, let $w : \Gamma \to [0,\infty]$ be a weight, and let $p \in \mathcal{P}$. Suppose the Cauchy singular integral* *operator is bounded on the weighted Nakano space* $L^{p(\cdot)}_w$. *If the operator* $aP_+ + P_-$, *where* $a \in PC$, *is Fredholm in* $L^{p(\cdot)}_w$, *then* $a \in GL^\infty$ *and*

$$(8.1)\quad -\frac{1}{2\pi}\arg\frac{a(t-0)}{a(t+0)} + \frac{1}{p(t)} + \theta\alpha_t\left(\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right) + (1-\theta)\beta_t\left(\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right) \notin \mathbf{Z}$$

for all $t \in \Gamma$ *and all* $\theta \in [0,1]$.

Proof. Since $p \in \mathcal{P}$, inequalities (5.25) are satisfied. In that case, by Lemma 2.4, the nonweighted Nakano space $L^{p(\cdot)}$ is reflexive. On the other hand, by Theorem 6.1, $w \in L^{p(\cdot)}$ and $1/w \in (L^{p(\cdot)})'$. Then the weighted Nakano space $L^{p(\cdot)}_w$ is also reflexive, due to Corollary 2.8. Thus, all assumptions of Theorem 7.3 are satisfied and we can repeat its proof. In view of Theorem 7.2, there exists an $l \in \mathbf{Z}$ such that $\varphi_{t,l-\gamma_t} w \in A_{L^{p(\cdot)}}(\Gamma)$, where γ_t is given by (7.1). In that case, by Lemma 5.17,

$$\begin{aligned} &-\operatorname{Re}\gamma_t + \theta\alpha^*_t(-\operatorname{Im}\gamma_t) + (1-\theta)\beta^*_t(-\operatorname{Im}\gamma_t) \\ &\qquad = -\operatorname{Re}\gamma_t + 1/p(t) + \theta\alpha_t(-\operatorname{Im}\gamma_t) + (1-\theta)\beta_t(-\operatorname{Im}\gamma_t). \end{aligned}$$

Therefore, we can replace condition (7.5) by condition (8.1) in the case of weighted Nakano spaces. □

For Lebesgue spaces L^p_w with Muckenhoupt weights w (that is, in the case when $p(\cdot)$ is constant), condition (8.1) becomes also sufficient for the Fredholmness of R_a, see [**3**, Proposition 7.3].

8.2 Lower estimates for essential norms.

Theorem 8.2. *Let* Γ *be a rectifiable Jordan curve, let* $w : \Gamma \to [0,\infty]$ *be a weight, and let* $p \in \mathcal{P}$. *If the Cauchy singular integral operator is bounded on the weighted Nakano space* $L^{p(\cdot)}_w$, *then*

$$|S|_{L^{p(\cdot)}_w} \ge \cot\Big(\pi\Lambda_{\Gamma,p,w}/2\Big), \quad |P_\pm|_{L^{p(\cdot)}_w} \ge 1/\sin(\pi\Lambda_{\Gamma,p,w}),$$

where

$$\Lambda_{\Gamma,p,w} := \inf_{t\in\Gamma} \min\left\{\frac{1}{p(t)} + \mu_t, 1 - \frac{1}{p(t)} - \nu_t\right\}.$$

By Theorem 6.1, $w \in A_{L^{p(\cdot)}}(\Gamma)$. Therefore, the latter theorem immediately follows from Theorem 7.4 and Theorem 5.9.

If $\log w \in VMO(\Gamma)$ (in particular, if $w = 1$), then from Lemma 4.7 and (5.25) it follows that

$$\begin{aligned}\Lambda_{\Gamma,p,w} &= \inf_{t\in\Gamma} \min\left\{\frac{1}{p(t)}, 1 - \frac{1}{p(t)}\right\} = \min\left\{\inf_{t\in\Gamma}\frac{1}{p(t)}, 1 - \sup_{t\in\Gamma}\frac{1}{p(t)}\right\}\\ &= \min\left\{1/p^*, 1 - 1/p_*\right\}.\end{aligned}$$

8.3 Fredholm criterion.

Theorem 8.3. *Let* Γ *be either a Lyapunov Jordan curve or a Radon Jordan curve without cusps, let* $p \in \mathcal{P}$*, and let* ϱ *be a Khvedelidze weight* (1.2) *satisfying* (6.1). *Then the operator* $aP_+ + P_-$*, where* $a \in PC$*, is Fredholm in the weighted Nakano space* $L^{p(\cdot)}_\varrho$ *if and only if*

$$a(t \pm 0) \neq 0, \quad -\frac{1}{2\pi}\arg\frac{a(t-0)}{a(t+0)} + \frac{1}{p(t)} + \lambda(t) \notin \mathbf{Z} \tag{8.2}$$

for all $t \in \Gamma$*, where*

$$\lambda(t) = \begin{cases} \lambda_k, & \text{if } t = \tau_k,\ k \in \{1,\dots,n\},\\ 0, & \textit{if } t \notin \Gamma \setminus \{\tau_1,\dots,\tau_n\}.\end{cases} \tag{8.3}$$

Proof. By Theorem 6.2, the operator S is bounded on the (reflexive) weighted Nakano space $L^{p(\cdot)}_\varrho$.

Necessity. By Proposition 4.4, for Lyapunov curves and Radon curves without cusps, we have $\delta_t^- = \delta_t^+ = 0$ whenever $t \in \Gamma$. By Lemma 5.18, the indicator functions of the pair (Γ, ϱ) are constants $\alpha_t(x) = \mu_t, \beta_t(x) = \nu_t$ for $x \in \mathbf{R}$, where the indices of powerlikeness

μ_t, ν_t of the Khvedelidze weight (1.2) coincide with $\lambda(t)$ given by (8.3). Thus,

$$\theta\alpha_t\left(\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right)+(1-\theta)\beta_t\left(\frac{1}{2\pi}\log\left|\frac{a(t-0)}{a(t+0)}\right|\right)=\lambda(t)$$

for every $\theta\in[0,1]$ and every $t\in\Gamma$. Therefore, the necessity of conditions (8.2) follows from Theorem 8.1. The necessity part is proved.

Sufficiency. From (8.2) it follows that for every $t\in\Gamma$, there exists an $m_t\in\mathbf{Z}$ such that

$$0<m_t-\operatorname{Re}\gamma_t+\frac{1}{p(t)}+\lambda(t)<1,$$

where γ_t is given by (7.1). By Theorem 6.2, the operator S is bounded on the weighted Nakano space $L^{p(\cdot)}_{\widetilde{\varrho}_t}$, where

$$\widetilde{\varrho}_t(\tau):=|\tau-t|^{m_t-\operatorname{Re}\gamma_t}\varrho(\tau),\quad \tau\in\Gamma.$$

In view of (5.50) and Proposition 4.4, there exist constants $C_1(t)$, $C_2(t)\in(0,\infty)$ such that

$$C_1(t)\widetilde{\varrho}_t(\tau)\le\varphi_{t,m_t-\gamma_t}(\tau)\le C_2(t)\widetilde{\varrho}_t(\tau),\quad \tau\in\Gamma\setminus\{t\}.$$

Therefore, $S\in\mathcal{B}(L^{p(\cdot)}_{\widetilde{\varrho}_t})$ if and only if $\varphi_{t,m_t-\gamma_t}S\varphi_{t,\gamma_t-m_t}I\in\mathcal{B}(L^{p(\cdot)}_{\varrho})$. By Lemma 7.1, the function g_{t,γ_t} admits a Wiener-Hopf factorization in the weighted Nakano space $L^{p(\cdot)}_{\varrho}$. Due to Theorem 6.14, for every $t\in\Gamma$, the operator $g_{t,\gamma_t}P_++P_-$ is Fredholm. Then the operator $cg_{t,\gamma_t}P_++P_-$ is Fredholm for $c\in\mathbf{C}\setminus\{0\}$ (see the proof of Theorem 7.3).

Since the function c_tg_{t,γ_t} with a specially chosen constant $c_t\in\mathbf{C}\setminus\{0\}$ is locally equivalent to the function $a\in PC$ at every point $t\in\Gamma$, in view of Theorem 6.13, the operator $R_a=aP_++P_-$ is Fredholm in the weighted Nakano space $L^{p(\cdot)}_{\varrho}$. □

In Theorem 8.3 the coefficient a can have a countable set of jumps. If a has only a finite number of jumps and $\varrho=1$, this result was obtained in [**31**, Theorem A] (as well as a formula for the index of the operator R_a). Note that the transition from finitely many to infinitely many jumps is more or less standard (see [**16**, Section 9.8] for Lebesgue spaces with Khvedelidze weights over Lyapunov curves), using the stability of Fredholm operators and localization techniques, see Section 6.5. We give the proof of Theorem 8.3 here for completeness. For Lebesgue spaces with Khvedelidze weights over Lyapunov curves the corresponding result was obtained in the late sixties by Gohberg and Krupnik [**16**, Chapter 9].

Acknowledgments. I would like to express my deep gratitude to Professor Lech Maligranda (Luleå University of Technology, Sweden) for historical remarks concerning Nakano spaces and to Professor Albrecht Böttcher (Chemnitz Technical University, Germany) for useful discussions.

REFERENCES

1. C. Bennett and R. Sharpley, *Interpolation of operators*, Pure Appl. Math., vol. 129, Academic Press, Boston, 1988.

2. E.I. Berezhnoi, *Two-weighted estimations for the Hardy-Littlewood maximal function in ideal Banach spaces*, Proc. Amer. Math. Soc. **127** (1999), 79–87.

3. A. Böttcher and Yu.I. Karlovich, *Carleson curves, Muckenhoupt weights, and Toeplitz operators*, Progr. Math., vol. 154, Birkhäuser Verlag, Basel, 1997.

4. ———, *Cauchy's singular integral operator and its beautiful spectrum*, in *Systems, approximation, singular integral operators, and related topics* (Bordeaux, 2000), Oper. Theory Adv. Appl., vol. 129, Birkhäuser Verlag, Basel, 2001, pp. 109–142.

5. A. Böttcher and B. Silbermann, *Analysis of Toeplitz operators*, Springer-Verlag, Berlin, 1990.

6. K.F. Clancey and I. Gohberg, *Factorization of matrix functions and singular integral operators*, Oper. Theory Adv. Appl., vol. 3, Birkhäuser Verlag, Basel, 1981.

7. L.A. Coburn, *Weyl's theorem for nonnormal operators*, Michigan Math. J. **13** (1966), 285–288.

8. I.I. Danilyuk, *Nonregular boundary value problems in the plane*, Nauka, Moscow, 1975 (in Russian).

9. I.I. Danilyuk and V.Yu. Shelepov, *Boundedness in L_p of a singular operator with Cauchy kernel along a curve of bounded rotation*, Dokl. Akad. Nauk SSSR **174** (1967), 514–517 (in Russian). English transl.: Soviet Math. Dokl. **8** (1967), 654–657.

10. L. Diening, *Maximal functions on generalized Lebesgue spaces $L^{p(x)}$*, Mathematische Fakultät, Albert-Ludvigs-Universität Freiburg (2002), Preprint Nr. 02/2002–16.01.2002. Math. Inequal. Appl., to appear.

11. E.M. Dynkin, *Methods of the theory of singular integrals.* (*Hilbert transform and Calderón-Zygmund theory*), Itogi nauki i tehniki VINITI, Ser. Sovrem. probl. mat., **15** (1987), 197–292 (in Russian). English transl.: *Commutative harmonic analysis* I. *General survey. Classical aspects*, Encyclopaedia Math. Sci. **15** (1991), 167–259.

12. D.E. Edmunds, J. Lang and A. Nekvinda, *On $L^{p(x)}$ norms*, Proc. Roy. Soc. London Ser. A **455** (1999), 219–225.

13. R.J. Fleming, J.E. Jamison and A. Kamińska, *Isometries of Musielak-Orlicz spaces*, in *Function spaces*, Edwardsville, IL, 1990, pp. 139–154; Lecture Notes in Pure and Appl. Math., vol. 136, Dekker, New York, 1992.

14. D. Gaier, *Lectures on complex approximation*, Birkhäuser Boston, Inc., Boston, MA, 1987.

15. I. Genebashvili, A. Gogatishvili, V. Kokilashvili and M. Krbec, *Weight theory for integral transforms on spaces of homogeneous type*, Pitman Monographs Surveys Pure Appl. Math., vol. 92, Addison Wesley Longman, Harlow, 1998.

16. I. Gohberg and N. Krupnik, *One-dimensional linear singular integral equations*, Vols. 1, 2, Oper. Theory Adv. Appl., vols. 53, 54, Birkhäuser Verlag, Basel, 1992. Russian original: Shtiintsa, Kishinev, 1973.

17. S.M. Grudsky, *Singular integral equations and the Riemann boundary value problem with an infinite index in the space $L_p(\Gamma, \omega)$*, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), 55–80 (in Russian). English transl.: Math. USSR-Izv. **26** (1986), 53–76.

18. A. Kamińska, *Indices, convexity and concavity in Musielak-Orlicz spaces*, Funct. Approx. Comment. Math. **26** (1998), 67–84.

19. A. Kamińska and B. Turett, *Type and cotype in Musielak-Orlicz spaces*, in *Geometry of Banach spaces*, Strobl, 1989, pp. 165–180; London Math. Soc. Lecture Note Ser., vol. 158, Cambridge Univ. Press, Cambridge, 1990.

20. L.V. Kantorovich and G.P. Akilov, *Functional analysis*, Nauka, Moscow, 3rd ed., 1984 (in Russian). English transl.: Pergamon Press, 2nd ed., Oxford, 1982.

21. N. Karapetiants and S. Samko, *Equations with involutive operators*, Birkhäuser Boston, Inc., Boston, MA, 2001.

22. A.Yu. Karlovich, *Algebras of singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces*, Math. Nachr. **179** (1996), 187–222.

23. ———, *Algebras of singular integral operators with piecewise continuous coefficients in rearrangement-invariant spaces with weight on Carleson curves*, Ph.D. Thesis, Odessa, Ukraine, 1998 (in Russian). Available at `http://www.math.ist.utl.pt/~akarlov/theses.html`.

24. ———, *Singular integral operators with piecewise continuous coefficients in reflexive rearrangement-invariant spaces*, Integral Equations Operator Theory **32** (1998), 436–481.

25. ———, *On the essential norm of the Cauchy singular integral operator in weighted rearrangement-invariant spaces*, Integral Equations Operator Theory **38** (2000), 28–50.

26. ———, *Algebras of singular integral operators with PC coefficients in rearrangement-invariant spaces with Muckenhoupt weights*, J. Operator Theory **47** (2002), 303–323.

27. B.V. Khvedelidze, *Linear discontinuous boundary problems in the theory of functions, singular integral equations and some of their applications*, Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **23** (1956), 3–158 (in Russian).

28. ———, *The method of the Cauchy type integrals for discontinuous boundary value problems of the theory of holomorphic functions of one complex variable*, Itogi nauki i tehniki VINITI, Ser. Sovrem. probl. mat. **7** (1975), 5–162 (in Russian). English transl.: J. Soviet Math. **7** (1977), 309–414.

29. V. Kokilashvili and M. Krbec, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific, New Jersey, 1991.

30. V. Kokilashvili and S. Samko, *Singular integrals in weighted Lebesgue spaces with variable exponent*, Georgian Math. J. **10** (2003), 145–156.

31. ———, *Singular integral equations in the Lebesgue spaces with variable exponent*, Proc. A. Razmadze Math. Inst. **131** (2003), 61–78.

32. O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), 592–618.

33. M.A. Krasnoselskii and Ya.B. Rutickii, *Convex functions and Orlicz spaces*, Fizmatgiz, Moscow, 1958 (in Russian). English transl.: Noordhoff Ltd., Groningen, 1961.

34. M. Krbec, B. Opic, L. Pick and J. Rákosník, *Some recent results on Hardy type operators in weighted function spaces and related topics*, in *Function spaces, differential operators and nonlinear analysis*, Friedrichroda, 1992, pp. 158–184; Teubner-Texte Math., vol. 133, Teubner, Stuttgart, 1993.

35. S.G. Krein, Ju.I. Petunin and E.M. Semenov, *Interpolation of linear operators*, Nauka, Moscow, 1978 (in Russian). English transl.: Amer. Math. Soc. Transl. Ser. 2, vol. 54, Providence, RI, 1982.

36. N.Ya. Krupnik, *Banach algebras with symbol and singular integral operators*, Oper. Theory Adv. Appl., vol. 26, Birkhäuser Verlag, Basel, 1987.

37. J. Lang, A. Nekvinda and L. Pick, *Boundedness and compactness of general kernel integral operators from a weighted Banach function space into L_∞*, Dept. of Math. Analysis (KMA), Faculty of Math. and Phys., Charles University, Praha, Preprint MATH-KMA-2003/94. Available at `http://adela.karlin.mff.cuni.cz/~rokyta/preprint/`.

38. G.S. Litvinchuk and I.M. Spitkovsky, *Factorization of measurable matrix functions*, Oper. Theory Adv. Appl., vol. 25, Birkhäuser Verlag, Basel, 1987.

39. L. Maligranda, *Indices and interpolation*, Dissert. Math. (Rozprawy Mat.) **234** (1985), 1–49.

40. S.G. Mikhlin and S. Prössdorf, *Singular integral operators*, Springer-Verlag, Berlin, 1986.

41. J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin, 1983.

42. J. Musielak and W. Orlicz, *On modular spaces*, Studia Math. **18** (1959), 49–65.

43. H. Nakano, *Modulared semi-ordered linear spaces*, Maruzen Co., Ltd., Tokyo, 1950.

44. ———, *Topology of linear topological spaces*, Maruzen Co., Ltd., Tokyo, 1951.

45. W. Orlicz, *Über konjugierte Exponentenfolgen*, Studia Math. **3** (1931), 200–211. Reprinted in *Wladyslaw Orlicz, Collected Papers*, PWN, Warsaw, 1988, pp. 200–213.

46. L. Pick and M. Ružička, *An example of a space $L^{p(x)}$ on which the Hardy-Littlewood maximal operator is not bounded*, Exposition. Math. **19** (2001), 369–371.

47. I.I. Privalov, *Boundary properties of analytic functions*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1950 (in Russian).

48. M. Ružička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Math., vol. 1748, Springer-Verlag, Berlin, 2000.

49. I.B. Simonenko, *The Riemann boundary value problem for n pairs functions with measurable coefficients and its application to the investigation of singular integral operators in the spaces L^p with weight*, Izv. AN SSSR, Ser. Matem. **28** (1964), 277–306 (in Russian).

50. ———, *A new general method of investigating linear operator equations of singular integral equation type*, I–II, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965), 567–586 (Part I), 757–782 (Part II) (in Russian).

51. ———, *Some general questions in the theory of the Riemann boundary value problem*, Izv. AN SSSR, Ser. Matem. **32** (1968), 1138–1146 (in Russian). English transl.: Math. USSR Izv. **2** (1968), 1091–1099.

52. I. Spitkovsky, *Singular integral operators with PC symbols on the spaces with general weights*, J. Funct. Anal. **105** (1992), 129–143.

53. M. Zippin, *Interpolation of operators of weak type between rearrangement invariant spaces*, J. Funct. Anal. **7** (1971), 267–284.

Departamento de Mathemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
E-mail address: `akarlov@math.ist.utl.pt`