JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 15, Number 2, Summer 2003

AN ASYMPTOTIC SERIES APPROACH TO QUALOCATION METHODS

VÍCTOR DOMÍNGUEZ AND FRANCISCO-JAVIER SAYAS

ABSTRACT. In this paper we develop an asymptotic analysis of qualocation methods when applied to a particular class of pseudodifferential equations. The main result is the existence of an error expansion between the numerical solution and an optimal projection over the splines. As by-products of this expansion we obtain some estimations of pointwise convergence and an asymptotic expansion between the exact and the numerical solution under the action of regularizing operators. In addition to this, using the error expansion we deduce sufficient conditions to obtain qualocation methods of higher order for some particular equations. Finally, we give a numerical experiment which corroborates the theoretical results.

1. Introduction. This paper is devoted to a full asymptotic analysis of qualocation methods for a certain class of periodic pseudodifferential equations. The class of equations (associated to what we will call expandable operators) we will be working on includes boundary integral equations [**3**] on smooth closed curves of the plane. In fact, it can be proved that the set of operators in our equations is that of periodic classical pseudodifferential of integer order [**11, 13, 19**].

Qualocation methods (qualocation is a compression of quadrature-modified collocation) form a recent family of numerical methods for the approximation of pseudodifferential equations. This family consists of three different groups of methods. The first group is that of semi-discretizations of spline-spline Petrov-Galerkin schemes via substitution of the test integration process by a suitable quadrature rule [**8, 28**]. The second one includes quadrature approximation of outer integrals in a spline-trigonometric Petrov-Galerkin method [**22, 26**]. Finally, the so-called "tolerant" qualocation only discretizes the test integral in the bilinear form but not in the righthand side (cf. [**30, 31**]). The paper [**25**] surveys recent advances on the analysis of this family of methods.

AMS *Mathematics Subject Classification.* 65R20.
Key words and phrases. Qualocation methods, boundary integral equations, pseudodifferential operators, Richardson extrapolation.
Research partially supported by MCYT/FEDER Project BFM2001-2521.

Although these methods can be seen as a further step towards full discretization of Petrov-Galerkin methods, both their analysis and properties (and also the look of the discrete equations) are closer to those of collocation methods. In fact, collocation methods are enclosed in this wider set. The main point for this proximity is that an apparent additional discretization of a Petrov-Galerkin method can give better approximation properties than the original scheme. The class of qualocation methods has been steadily developing in the nineties with the study of quadrature rules valid for more general equations. However, stability (especially when the operator is neither strongly nor oddly elliptic) is an open question in many cases, although some advances have been made in this area in recent years [**15, 20**].

The present paper contributes to the error analysis of qualocation methods by adopting a different approach. We basically expand the error as an asymptotic series in powers of the discretization parameter h (i.e., a possibly divergent series where remainders of partial sums can be bounded by higher order terms). In addition to the inherent interest these asymptotic series may have (including justification of Richardson extrapolation for convergence acceleration and a posteriori error estimation), this approach gives several by-products. On the one hand, we are able to find sets of superconvergence points in strong norms. On the other hand, by choosing adequate rules that cancel coefficients in the series we show how convergence can be improved. Moreover, the existence of a nonvanishing term proves optimality of the error bounds. This can be used to construct new qualocation methods (notice, nevertheless, that the existing catalogue of these is quite wide for the most used equations) and to prove that some of the already known can be applied to more general classes of equations without losing their good properties. We deal with this in Section 9.

The asymptotic series approach has however the drawback of assuming more regularity on the solutions than it is actually needed for some of the order inequalities to hold. In fact, in some preceding work (see [**10, 13**]) the solution was assumed to be infinitely often differentiable and no track on the precise regularity was kept. The advantages of this kind of analysis are not on this side, but on the possibility of developing the complete asymptotic character of the error and extracting from it information that is not apparent when dealing exclusively with inequalities.

The mathematical tools used for this analysis include a wide set of Fourier techniques. The numerical solution is not compared with the exact one but with an optimal order projection onto spline spaces D_h^d whose properties had already been studied [**2, 12, 13**]. This has the advantage of concentrating all the error terms due to the trial space in one simple operator and pushing the first term due to the whole numerical scheme to the weakest norm convergence order. The projection operator D_h^d is a matching of central Fourier coefficients. The consistency error $A(D_h^d u - u)$ is then studied and afterwards tests are applied to develop a full asymptotic series of the tested error. This is done by using error expansions of compound quadrature rules applied to the L^2 product of splines by smooth functions and supposes a sort of nonstandard Euler-Maclaurin formula. Once stability is formulated as an inf-sup condition (via identification of the couple of test spline and quadrature rule with the set of functionals it defines), expansions are easily converted into asymptotic series of the error in several norms. These results established, we apply the series to show how terms can be eliminated in particular situations. A final section is devoted to a single numerical example corroborating the theoretical results. In this example adequate numerical quadrature has been applied to preserve the good properties of the method.

Since the method includes collocation schemes, this paper generalizes results in [**6**] applied to logarithmic integral equations of the first kind with variable coefficients. In comparison with this work, novelties are plenty, not only in the much wider scope of equations and methods but also in the different mathematical techniques. A previous paper [**18**] had also treated asymptotic expansions of collocation methods for pseudodifferential equations. The comparison there with the exact solution (instead of going through the projection D_h^d) imposed the error analysis under the action of a regularizing functional, which limited the strength of the results and did not lend itself to a further study of numerical integration.

2. The functional setting and the equation. In this section we introduce the periodic Sobolev spaces as well as the set of periodic pseudodifferential operators and equations, which constitute the frame where we will be working in the sequel. In the space of 1-periodic infinitely often differentiable complex-valued functions of a real variable

$$\mathcal{D} := \left\{ \varphi : \mathbf{R} \to \mathbf{C} \;\middle|\; \varphi \in \mathcal{C}^\infty(\mathbf{R}),\ \varphi = \varphi(1+\cdot) \right\},$$

we consider the metric

$$\sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|\varphi_1^{(k)} - \varphi_2^{(k)}\|_\infty}{1 + \|\varphi_1^{(k)} - \varphi_2^{(k)}\|_\infty}$$

($\|\cdot\|_\infty$ denotes the maximum norm). There is a well-known identification of 1-periodic complex-valued distributions on the real line with the elements of $\mathcal{D}'$, the dual space of $\mathcal{D}$. For instance, $f \in L^1(0,1)$ defines an element of $\mathcal{D}'$ through

$$\mathcal{D} \ni \varphi \longmapsto \langle f, \varphi \rangle := \int_0^1 f(t)\varphi(t)\, dt.$$

The corresponding periodic distribution is simply the 1-periodization of f to the whole of $\mathbf{R}$. Therefore, we will refer to $T \in \mathcal{D}'$ as a 1-periodic distribution. For these we can calculate the distributional Fourier coefficients

$$\widehat{T}(m) := \langle T, \phi_{-m} \rangle, \quad m \in \mathbf{Z},$$

where

$$\phi_m(x) := \exp(2\pi \imath m\, x).$$

Obviously, distributional and classical Fourier coefficients of $f \in L^1(0,1)$ coincide.

We can define the periodic Sobolev spaces (see [**16**, Chapter 8], [**33**])

$$H^s := \left\{ u \in \mathcal{D}' \;\middle|\; \sum_{m \neq 0} |m|^{2s} |\widehat{u}(m)|^2 < \infty \right\}, \quad s \in \mathbf{R}$$

endowed with the norms

$$\|u\|_s := \left[|\widehat{u}(0)|^2 + \sum_{m \neq 0} |m|^{2s} |\widehat{u}(m)|^2 \right]^{1/2}.$$

Then for all s, H^s is a Hilbert space. If $s > t$, $H^s \subset H^t$ with dense and compact injection. Clearly $H^0 = L^2(0,1)$ and the norms coincide by Parseval's identity.

The inner product in H^0

$$(u,v)_0 := \widehat{u}(0)\overline{\widehat{v}(0)} + \sum_{m\neq 0} \widehat{u}(m)\overline{\widehat{v}(m)} = \int_0^1 u(t)\overline{v(t)}\,dt$$

can be extended to give a reciprocal representation of the duality between H^s and H^{-s} for all $s \in \mathbf{R}$, with the equality

$$\|u\|_{-s} = \sup_{0\neq v\in H^s} \frac{|(u,v)_0|}{\|v\|_s}.$$

A linear map $A : \mathcal{D}' \to \mathcal{D}'$ such that $A : H^s \longrightarrow H^{s-n}$ is bounded for all $s \in \mathbf{R}$ with a given $n \in \mathbf{Z}$ is called a pseudodifferential operator (ψdo in short) of order (at most) n. The differential and inverse-differential operators

$$D^j u := \sum_{m\neq 0} (2m\pi\imath)^j\, \widehat{u}(m)\phi_m, \quad j \in \mathbf{Z} \tag{1}$$

are ψdo of order j, satisfying $D^j D^k = D^{j+k}$ for all j,k, $D^0 u = u - \widehat{u}(0)$. In addition, D^1 is the differentiation operator in $\mathcal{D}'$. The periodic Hilbert transform

$$Hu := \sum_{m\neq 0} \operatorname{sign}(m)\widehat{u}(m)\phi_m = \sum_{m>0} \widehat{u}(m)\phi_m - \sum_{m<0} \widehat{u}(m)\phi_m$$

is a ψdo of order zero, which admits a singular integral representation when restricted to H^0 (see [**33**]) and satisfies $H^2 = D^0$. The operators D^j and HD^j will be taken now as monomials of the algebra of expandable operators.

We say that A is an expandable pseudodifferential operator of order n and we write $A \in \mathcal{E}(n)$, if there exist two sequences of functions $(a_j)_{j=-\infty}^n, (b_j)_{j=-\infty}^n \subset \mathcal{D}$ such that for all $M \geq -n$ integer

$$A = \sum_{j=-M}^{n} a_j D^j + \sum_{j=-M}^{n} b_j HD^j + K_M \tag{2}$$

with K_M a ψdo of order $-M-1$. For $A \in \mathcal{E}(n)$ we denote its expansion by

$$A \overset{\text{exp}}{=} \sum_{j=-\infty}^{n} a_j D^j + \sum_{j=-\infty}^{n} b_j H D^j.$$

On the other hand, the first term of the expansion $a_n D^n + b_n H D^n$ is called the principal part of A. Expandable ψdo's can be characterized as classical periodic pseudodifferential operators of integer order (cf. [**19**]).

Following classical denominations (see [**28**]) we say that A is strongly elliptic if there exists $\varphi \in \mathcal{D}$ such that

$$\operatorname{Re}\left[\varphi(a_n + b_n)\right] > 0, \qquad \operatorname{Re}\left[(-1)^n \varphi(a_n - b_n)\right] > 0.$$

If HA is strongly elliptic, we say that A is oddly elliptic. Let us remark that these concepts are not exclusive: an operator can be strongly and oddly elliptic at the same time.

If $a \in \mathcal{D}$ we equally denote by a the order zero ψdo of multiplication by a. Then

$$aD^n - D^n a \in \mathcal{E}(n-1). \tag{3}$$

In case $n \geq 1$, this is just the Leibnitz rule, whereas the cases $n \leq 0$ are equally simple. Also

$$aH - Ha \in \mathcal{E}(-\infty), \tag{4}$$

which means that this operator is a ψdo of arbitrarily low degree and as such admits a trivial expansion ($a_j = b_j = 0$ for all j).

The main purpose of this work is the study of qualocation methods for the approximate solution of the equation

$$Au = f, \tag{5}$$

where $A \in \mathcal{E}(n)$. These equations include a wide range of boundary integral operators such as integro-differential equations of Cauchy type and logarithmic integral equations of the first and second kind [**3**], as can be seen in [**13**, Appendix].

3. A class of qualocation methods. Qualocation methods include two different families of discrete methods to solve (5). On the one hand we have semi-discretizations of the exterior integral of a spline-spline Petrov-Galerkin method by means of compound quadrature rule as in [**8, 28**]. On the other hand, we have the same kind of semi-discretization of Petrov-Galerkin methods with splines as trial functions and trigonometric polynomials as tests [**22, 26**]. We will concentrate on the first and leave the second for a final section.

We begin by introducing the discrete spaces. Let N be a positive integer, $h := 1/N$ and $x_i := ih$ for all $i \in \mathbf{Z}$. The space of 1-periodic smoothest splines of degree $d \geq 1$ over the grid $\{x_i\}_{i\in\mathbf{Z}}$ is defined by

$$S_h^d := \left\{ u \in \mathcal{C}^{d-1}(\mathbf{R}) \,\middle|\, u = u(1+\,\cdot\,),\ u|_{[x_i,x_{i+1}]} \in \mathbf{P}_d,\, \forall\, i \right\}$$

where $\mathbf{P}_d$ is the set of polynomials of degree at most d. The space of periodic piecewise constant functions will be consequently denoted S_h^0. It is well known that $S_h^d \subset H^s$ for all $s < d + 1/2$.

Consider now a basic quadrature rule

$$Lu := \sum_{j=1}^{J} \omega_j u(\xi_j) \approx \int_0^1 u(t)\, dt,$$

where $0 \leq \xi_1 < \xi_2 < \ldots < \xi_J < 1$, $\omega_j > 0$ for all j and $\sum_j \omega_j = 1$. Then we define the corresponding composite rule

$$L_N u := h \sum_{i=0}^{N-1} \sum_{j=1}^{J} \omega_j u(x_i + h\xi_j) \approx \int_0^1 u(t)\, dt$$

and the discrete inner product

$$\langle u, v\rangle_N := L_N(u\overline{v}) \approx (u, v)_0.$$

The qualocation method to solve (5) is

$$(6) \qquad u_h \in S_h^d, \quad \text{s.t.} \quad \langle Au_h, r_h\rangle_N = \langle f, r_h\rangle_N, \quad \forall\, r_h \in S_h^{d'}.$$

The method depends on the degrees of the spline spaces involved (d and d') and on the quadrature rule. We restrict ourselves to the case

$$d \geq n+1.$$

With this restriction we can ensure that for all $u_h \in S_h^d$, the Fourier series of Au_h is absolutely (and hence uniformly) convergent and therefore Au_h is continuous. Part of the forthcoming analysis can be extended to cover some cases when $d = n$, namely those where Au_h can be evaluated in all the nodes $x_i + \xi_j h$ (see [**8**] for a discussion). For the sake of simplicity, we will omit any reference to this 'singular' case (see [**11**] for full details).

In case S_h^0 is the test space we admit $\xi_1 = 0$ as node of the basic quadrature rule understanding that

$$r_h(x_i) := \frac{1}{2}\left(r_h(x_i^+) + r_h(x_i^-)\right), \quad r_h \in S_h^0,$$

which is the limit value of the Cauchy sums of the Fourier series of r_h at x_i.

For some requirements of our analysis we must impose the following restriction:

Hypothesis 1. *The quadrature rule satisfies that if $J = 1$ then for d' even $\xi_1 \neq 0$, whereas for d' odd, $\xi_1 \neq 1/2$.*

This hypothesis will be discussed in the following section in terms of whether the set of tests $\langle \cdot, r_h \rangle_N$, with $r_h \in S_h^{d'}$ is $N-$dimensional or not (see Proposition 3).

If $J = 1$ and Hypothesis 1 holds, the qualocation method is equivalent to ξ_1-collocation

$$u_h \in S_h^d \quad \text{s.t.} \quad Au_h(x_i + \xi_1 h) = f(x_i + \xi_1 h), \quad i = 0, \dots, N-1.$$

Conversely, we can consider $\xi-$collocation methods as a particular class of qualocation methods with trial spaces $S_h^{d'}$ where we take $d' = d$ or $d' = d+1$ so that Hypothesis 1 is satisfied. The quadrature rule is defined by taking $J = 1$, $\xi_1 = \xi$ and $\omega = 1$.

We will extensively deal with two important sets of quadrature rules and qualocation methods which have been treated in the literature. Simpson-type rules are defined by taking $J = 2$, $\xi_1 = 0$ and $\xi_2 = 1/2$. Symmetric rules are those with basic quadrature points distributed symmetrically with respect to $1/2$: $\omega_r = \omega_{J-r}$, $\xi_r = 1 - \xi_{J-r}$ for all r.

Notice that if $\xi_1 = 0$ and the rest of points are symmetrically disposed with respect to $1/2$ with two symmetric points having the same weight, we can construct a symmetric rule with an additional node in $\xi_{J+1} = 1$, giving the same composite rule L_N. Hence Simpson rules can be understood as particular cases of symmetric rules. However, because of their special properties (and also, since they have been separately treated in the literature), we keep them apart in the forthcoming study.

4. Dual forms of the tests. For $r_h \in S_h^{d'}$ the map $\langle \cdot, r_h\rangle_N : H^1 \to \mathbf{C}$ is linear and bounded. Therefore, there exists a unique $\Pi_h r_h \in H^{-1}$ such that

$$\langle \cdot, r_h\rangle_N = (\cdot, \Pi_h r_h)_0.$$

Moreover, Π_h is a conjugate-linear map and its image $T_h := \Pi_h S_h^{d'}$ is a finite-dimensional subset of H^s for all $s < -1/2$, since

$$T_h \subset \operatorname{span} \left\langle\, \delta_{x_i+\xi_j h} \;\middle|\; i = 0, \ldots, N-1,\ j = 1, \ldots, J \right\rangle$$

being δ_z the periodic Dirac delta distribution on z.

We first introduce a well-known basis of $S_h^{d'}$. Let us consider the following set of representatives of $\mathbf{Z}$ modulo N

$$\Lambda_N := \left\{ \mu \in \mathbf{Z} \;\middle|\; -\frac{N}{2} < \mu \le \frac{N}{2} \right\}.$$

The set $\{\widehat{r}_h(\mu) \mid \mu \in \Lambda_N\}$ determines uniquely the spline r_h [**2, 4**]. Hence there exists a set $\{\varphi_\mu^{d'} \in S_h^{d'} \mid \mu \in \Lambda_N\}$ satisfying

$$\widehat{\varphi}_\mu^{d'}(\nu) = \delta_\mu^\nu, \quad \mu, \nu \in \Lambda_N,$$

where δ_μ^ν is the Kronecker delta symbol. Moreover for $s < d' + 1/2$, there exists $C = C(s, d')$ such that

$$C\|r_h\|_s^2 \le |\widehat{r}_h(0)|^2 + \sum_{0 \ne \mu \in \Lambda_N} |\mu|^{2s} |\widehat{r}_h(\mu)|^2 \le \|r_h\|_s^2, \quad \forall\, r_h \in S_h^{d'}. \tag{7}$$

The set $\{\varphi_\mu^{d'}\}$ is a basis for $S_h^{d'}$, yielding

$$r_h = \sum_{\mu\in\Lambda_N} \widehat{r}_h(\mu)\varphi_\mu^{d'}, \quad \forall\, r_h \in S_h^{d'}.$$

Finally, we can use the recurrence properties of Fourier coefficients of splines to show that

$$\varphi_\mu^{d'} = \left[1 + \Delta_{d'}\left(N\cdot, \frac{\mu}{N}\right)\right]\phi_\mu$$

where (see [**8**])

$$\Delta_{d'}(x,y) := y^{d'+1}\sum_{m\neq 0}\frac{1}{(m+y)^{d'+1}}\,\phi_m(x) : \mathbf{R}\times[-1/2,1/2]\to\mathbf{C}.$$

Proposition 1. *The sequence of maps $\Pi_h : S_h^{d'} \to T_h$ is uniformly bounded in H^s for all $s < -1/2$.*

Proof. Let $0 \neq \mu \in \Lambda_N$ and $m \in \mathbf{Z}$. By [**8**, Lemma 1] we can prove that

$$\begin{aligned}\widehat{\Pi_h r_h}(\mu+mN) &= (\Pi_h r_h, \phi_{\mu+mN})_0 = \langle r_h, \phi_{\mu+mN}\rangle_N \\ &= \sum_{\nu\in\Lambda_N}\widehat{r}_h(\nu)\langle\varphi_\nu^{d'}, \phi_{\mu+mN}\rangle_N = \widehat{r}_h(\mu)c_\mu^m\end{aligned} \tag{8}$$

where

$$c_\mu^m := \langle\varphi_\mu^{d'}, \phi_{\mu+mN}\rangle_N = \sum_{r=1}^J \omega_r\phi_{-m}(\xi_r)\left[1+\Delta_{d'}\left(\xi_r, \frac{\mu}{N}\right)\right]. \tag{9}$$

Since $\Delta_{d'}$ is bounded on $\mathbf{R}\times[-1/2,1/2]$ we obtain that

$$|\widehat{\Pi_h r_h}(\mu+mN)| \le C|\widehat{r}_h(\mu)|,$$

with C depending only on d'. On the other hand, $\widehat{\Pi_h r_h}(mN) = \widehat{r}_h(0)$. Then

$$\begin{aligned}\|\Pi_h r_h\|_s^2 &= |\widehat{r}_h(0)|^2 + \sum_{m\neq 0} |mN|^{2s} |\widehat{\Pi_h r_h}(mN)|^2 \\ &\quad + \sum_{0\neq\mu\in\Lambda_N} \sum_{m\neq 0} |\mu + mN|^{2s} |\widehat{\Pi_h r_h}(\mu + mN)|^2 \\ &\leq |\widehat{r}_h(0)|^2 \left[1 + \sum_{m\neq 0} |mN|^{2s}\right] + C \sum_{0\neq\mu\in\Lambda_N} |\mu|^{2s} |\widehat{r}_h(\mu)|^2 \\ &\quad \cdot \left[1 + \sum_{m\neq 0} \frac{|\mu/N|^{-2s}}{|m+\mu/N|^{-2s}}\right] \\ &\leq C'_s \left[|\widehat{r}_h(0)|^2 + \sum_{0\neq\mu\in\Lambda_N} |\mu|^{2s} |\widehat{r}_h(\mu)|^2\right] \leq C'_s \|r_h\|_s^2,\end{aligned}$$

since $|\mu/N| \leq 1/2$ and $-2s > 1$. □

Corollary 2. *Given $s > 1/2$, there exists $C > 0$ depending on s and d' such that for all $u \in H^s$, $r_h \in S_h^{d'}$*

$$|\langle u, r_h \rangle_N| \leq C \|u\|_s \|r_h\|_{-s}.$$

Proposition 3. *Hypothesis 1 is equivalent to Π_h having uniformly bounded inverses in H^s for all $s < -1/2$.*

Proof. We first remark that by [**8**], Re $(1 + \Delta_{d'}(x, y)) > 0$ except in the points $(1/2 + k, 1/2)$, $k \in \mathbf{Z}$ for d' odd and $(k, 1/2)$ for d' even, where it vanishes. Hence, Hypothesis 1 and the positivity of the weights ensure that

$$|c_\mu^0| \geq c > 0, \quad \forall\, \mu \in \Lambda_N,$$

with c_μ^0 given by (9). Then by (8) we obtain

$$C|\widehat{r}_h(\mu)| \leq |\widehat{\Pi_h r_h}(\mu)|, \quad \forall\, \mu \in \Lambda_N,$$

and consequently, by (7)

$$\|r_h\|_s \leq C_s \|\Pi_h r_h\|_s, \quad \forall\, r_h \in S_h^{d'} \tag{10}$$

which proves the uniform boundedness of Π_h^{-1}.

Assume now that the hypothesis does not hold. When $\mathrm{Re}\,(1+\Delta_{d'}(x,y))=0$, the imaginary part of this number is also zero, and we obtain for N even

$$c^m_{N/2}=0,\quad \forall\, m\in\mathbf{Z},$$

and therefore by (8),

$$\widehat{\Pi_h r_h}(N/2+mN)=0,\quad \forall\, m\in\mathbf{Z},\ \forall\, r_h\in S_h^{d'}.$$

Taking then $r_h=\varphi^{d'}_{N/2}$, we have $\Pi_h r_h=0$ which makes (10) false.

If N is odd, we have that

$$C^0_{(N-1)/2}\to 0,\quad \text{as } N\to\infty.$$

Thus for any $C>0$ there exists N odd such that $\|\varphi^{d'}_{(N-1)/2}\|_s\geq C\|\Pi_h\varphi^{d'}_{(N-1)/2}\|_s$. Hence Π_h is not uniformly bounded. $\square$

5. Asymptotics of the discrete inner product. In this section we work on the approximation error $\langle f,r_h\rangle_N=L_N(f\overline{r}_h)\approx(f,r_h)_0$. We begin by recalling some properties of Bernoulli functions. Let B_k be the Bernoulli polynomial of degree k ($k\geq 0$) (see [**1**]) and let $\underline{B}_k$ be its 1-periodization

$$\underline{B}_k(x+m)=\underline{B}_k(x),\quad \forall\, x\in[0,1),\ \forall\, m\in\mathbf{Z}.$$

It is well known that $\underline{B}_k$ is continuous for $k\geq 2$ since $B_k(0)=B_k(1)$. For $B_1(x)=x-1/2$ we will define $\underline{B}_1(0)=\underline{B}_1(1)=0$, i.e., the average of the side limits. Notice that for all $x\in\mathbf{R}$

$$\underline{B}_k(x)=(-1)^k\underline{B}_k(-x)=(-1)^k\underline{B}_k(1-x). \tag{11}$$

Moreover, $\underline{B}_k$ has the following Fourier expansion [**1**]

$$\underline{B}_k:=-\frac{k!}{(2\pi\imath)^k}\sum_{m\neq 0}\frac{1}{m^k}\phi_m.$$

We also introduce the constants

$$\gamma_k^{d'} := -\frac{1}{k!}\binom{-(d'+1)}{k-(d'+1)}.$$

Lemma 4. *Let $k \in \{0, \ldots, d'+1\}$. Then for all $f \in H^{k+1}$, $r_h \in S_h^{d'}$ and $\xi \in [0,1)$*

$$\left| h \sum_{i=0}^{N-1} f(x_i + \xi h)\overline{r_h(x_i + \xi h)} - (f, r_h)_0 \right| \leq C h^k \|f\|_{k+1} \|r_h\|_{-1}. \tag{12}$$

If $M \geq d'+1$, $f \in H^{M+2}$ and $r_h \in S_h^{d'}$

$$\begin{aligned} &h \sum_{i=0}^{N-1} f(x_i + \xi h)\overline{r_h(x_i + \xi h)} \\ &\qquad = (f, r_h)_0 + \sum_{k=d'+1}^{M} h^k (-1)^k \gamma_k^{d'} \underline{B}_k(\xi)(f^{(k)}, r_h)_0 \\ &\qquad\quad + \mathcal{O}(h^{M+1})\|f\|_{M+2}\|r_h\|_{-1}. \end{aligned} \tag{13}$$

Proof. Inequality (12) is a straightforward consequence of [**13**, Lemma 10]. Expansion (13) follows from [**13**, Lemma 11]. ◻

Proposition 5. *There exists a sequence $(\rho_k)_k \subset \mathbf{R}$, depending on d' and L, such that for all $M \geq d'+1$, $r_h \in S_h^{d'}$ and $f \in H^{M+2}$*

(14)

$$(f, r_h)_0 = \langle f, r_h \rangle_N + \sum_{k=d'+1}^{M} h^k \rho_k \langle f^{(k)}, r_h \rangle_N + \mathcal{O}(h^{M+1})\|f\|_{M+2}\|r_h\|_{-1}.$$

Moreover, if the rule L is symmetric or of Simpson type then $\rho_{2k+1} = 0$ for all k.

Proof. A simple reorganization of sums in L_N gives

$$\langle u, r_h \rangle_N = \sum_{j=1}^{J} \omega_j \left[h \sum_{i=1}^{N} u(x_i + \xi_j h)\overline{r_h(x_i + \xi_j h)} \right]. \tag{15}$$

We remark that the lefthand side sum of (13) is a particular case of discrete inner product with a single node $\xi = \xi_1$, and that by (15) we are down to a linear combination of formulae (13) for different values of ξ to obtain a full asymptotic study of $\langle f, r_h \rangle_N$:

$$\begin{aligned}\langle f, r_h\rangle_N = (f, r_h)_0 + \sum_{k=d'+1}^{M} h^k (-1)^k \gamma_k^{d'} L\underline{B}_k\, (f^{(k)}, r_h)_0 \\ + \mathcal{O}(h^{M+1}) \|f\|_{M+2} \|r_h\|_{-1}.\end{aligned} \tag{16}$$

Also, by (12) we have for $k \in \{0, \ldots, d' + 1\}$

$$|\langle f, r_h\rangle_N - (f, r_h)_0| \le C h^k \|f\|_{k+1} \|r_h\|_{-1}. \tag{17}$$

Hence, the proof of (14) is a simple reversal of expansion (16) by induction.

Notice that if the rule is symmetric or of Simpson type, $L\underline{B}_{2k+1} = 0$. Then in expansion (16) there are no odd powers of h. It is straightforward to check that the same result holds in expansion (14). □

6. Consistency error expansion. We introduce a projection onto the spline space which plays a central role in our analysis. Given $u \in \mathcal{D}'$, we define

$$D_h^d u := \sum_{\mu \in \Lambda_N} \widehat{u}(\mu) \varphi_\mu^d \in S_h^d.$$

This operator can be understood as the result of applying a spline-trigonometric Petrov-Galerkin projection to the identity, i.e., to the trivial equation $u = u$, see [**2**]. We also introduce the truncation operator for Fourier series

$$D_h^\infty u := \sum_{\mu \in \Lambda_N} \widehat{u}(\mu) \phi_\mu,$$

where the notation is suggested by the fact that the set of trigonometric polynomials spanned by $\{\phi_\mu \mid \mu \in \Lambda_N\}$ is the formal limit of S_h^d as d goes to infinity.

The operator D_h^d has optimal approximation properties in all available Sobolev norms [**2, 13**]: for $s \le t \le d + 1$ and $s < d + 1/2$

$$\|D_h^d u - u\|_s \le C h^{t-s} \|u\|_t, \quad \forall\, u \in H^t. \tag{18}$$

For $k \geq 1$ define

$$C_k := H\underline{B}_k = -\frac{k!}{(2\pi\imath)^k} \sum_{m\neq 0} \frac{\operatorname{sign}(m)}{m^k}\phi_m.$$

By (11) and since H maps odd functions into even functions and vice versa, it follows that for all $k \geq 1$

$$C_k(1-x) = (-1)^{k+1} C_k(x), \quad \forall\, x \in \mathbf{R}. \tag{19}$$

Following [**13**] we denote

$$\mathcal{L}_n := \left\{ \sum_{j=1}^{n} a_j D^j \;\middle|\; a_j \in \mathcal{D} \right\},$$

the set of differential operators of order n with smooth periodic coefficients and without zero term. For $A \in \mathcal{E}(n)$ with asymptotic series given by (2) we define two sequences of differential operators: for $k \geq d+1-n$

$$\begin{aligned} R_k &:= \sum_{j=d+1}^{n+k} \gamma_k^{d+k-j} a_{j-k} D^j \in \mathcal{L}_{n+k}, \\ T_k &:= \sum_{j=d+1}^{n+k} \gamma_k^{d+k-j} b_{j-k} D^j \in \mathcal{L}_{n+k}. \end{aligned} \tag{20}$$

Following step by step the proof of [**13**, Proposition 3] (the norms where the expansions hold are more demanding in this new situation) we can prove that for all $M \geq d+1-n$ and $u \in H^{M+n+1}$ the following expansion holds in H^1.

$$\begin{aligned} A(D_h^d u - u) = \sum_{k=d+1-n}^{M+1} & h^k \Big(\underline{B}_k(N\cdot) R_k D_h^\infty u + C_k(N\cdot) T_k D_h^\infty u \Big) \\ & + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2}. \end{aligned} \tag{21}$$

Remark 6. Expansion (21) can be written without the operator D_h^∞ if $d \geq n+2$. In the case $d = n+1$, D_h^∞ can be eliminated of all the addenda except $T_2 D_h^\infty u$, because of the unboundedness of C_2'. To avoid making this distinction, we keep (21) as it is.

The quantities

$$\begin{aligned} \alpha_k &:= L\underline{B}_k, \quad \alpha_{j,k} := L(\underline{B}_j\underline{B}_k),\\ \beta_k &:= LC_k, \quad \beta_{j,k} := L(\underline{B}_j C_k), \end{aligned} \tag{22}$$

will be relevant in the sequel.

Given $A \in \mathcal{E}(n)$, $d \geq n+1$ and $d' \geq 0$ we consider the following sequence of differential operators: for $k \geq d+1-n$

(23)

$$\begin{aligned} F_k &:= \alpha_k R_k + \beta_k T_k \\ &\quad + \sum_{j=d'+1}^{k-(d+1-n)} \gamma_j^{d'} (-1)^j \left[\alpha_{j,k-j} D^j R_{k-j} + \beta_{j,k-j} D^j T_{k-j}\right] \in \mathcal{L}_{n+k}. \end{aligned}$$

Theorem 7. *Let $A \in \mathcal{E}(n)$ and $(F_k)_k$ be the sequence of differential operators defined in (23). Then for all $r_h \in S_h^{d'}$ and $u \in H^{M+n+2}$*

(24)

$$\langle A(D_h^d u - u), r_h\rangle_N = \sum_{k=d+1-n}^{M} h^k (F_k u, r_h)_0 + \mathcal{O}(h^{M+1})\|u\|_{M+n+2}\|r_h\|_{-1}.$$

Proof. From (21) and Corollary 2 we obtain

$$\begin{aligned} &\langle A(D_h^d u - u), r_h\rangle_N \\ &\quad = \sum_{k=d+1-n}^{M+1} h^k \Big[\langle \underline{B}_k(N\cdot) R_k D_h^\infty u, r_h\rangle_N + \langle C_k(N\cdot) T_k D_h^\infty u, r_h\rangle_N\Big] \\ &\qquad + \mathcal{O}(h^{M+1})\|u\|_{M+n+2}\|r_h\|_{-1}. \end{aligned}$$

On the one hand, by (13),

$$\begin{aligned}
&\langle \underline{B}_k(N\cdot)f, r_h\rangle_N \\
&\quad = \sum_{r=1}^{J}\omega_r B_k(\xi_r)\Big[h\sum_{i=0}^{N-1} f((i+\xi_r)h)r_h((i+\xi_r)h)\Big] \\
&\quad = \sum_{r=1}^{J}\omega_r B_k(\xi_r)\Big[(f,r_h)_0 + \sum_{j=d'+1}^{\nu} h^j\gamma_j^{d'}(-1)^j\underline{B}_j(\xi_r)(f^{(j)},r_h)_0\Big] \\
&\qquad + \mathcal{O}(h^{\nu+1})\|f\|_{\nu+2}\|r_h\|_{-1}.
\end{aligned}$$

Let k be such that $d+1-n \le k \le M-d'-1$. Taking $\nu = M-k$ and $f = R_k D_h^\infty u$ above, we obtain the expansion

(25)

$$\begin{aligned}
&h^k\langle \underline{B}_k(N\cdot)R_k D_h^\infty u, r_h\rangle_N \\
&\quad = \alpha_k h^k (R_k D_h^\infty u, r_h)_0 + \sum_{j=d'+1}^{M-k} h^{j+k}\gamma_j^{d'}(-1)^j\alpha_{j,k}(D^j R_k D_h^\infty u, r_h)_0 \\
&\qquad + \mathcal{O}(h^{M+1})\|u\|_{M+n+2}\|r_h\|_{-1}.
\end{aligned}$$

For $M-d' \le k \le M+1$ we use (12) to obtain

$$\begin{aligned}
h^k\,|\langle \underline{B}_k(N\cdot)R_k D_h^\infty u, r_h\rangle_N - \alpha_k(R_k D_h^\infty u, r_h)_0| \\
\le C h^{M+1}\|u\|_{M+n+2}\|r_h\|_{-1}.
\end{aligned} \tag{26}$$

Similar arguments can be applied to prove that for $d+1-n \le k \le M-d'-1$

(27)

$$\begin{aligned}
&h^k\langle C_k(N\cdot)T_k D_h^\infty u, r_h\rangle_N \\
&\quad = h^k\beta_k (T_k D_h^\infty u, r_h)_0 + \sum_{j=d'+1}^{M-k} h^{j+k}\gamma_j^{d'}(-1)^j\beta_{j,k}(D^j T_k D_h^\infty u, r_h)_0 \\
&\qquad + \mathcal{O}(h^{M+1})\|u\|_{M+n+2}\|r_h\|_{-1}
\end{aligned}$$

whereas for $M-d' \le k \le M+1$ a bound similar to (26) holds. Collecting these expansions, we obtain

$$\begin{aligned}
&\langle A(D_h^d u - u), r_h\rangle_N \\
&\qquad = \sum_{k=d+1-n}^{M} h^k (F_k D_h^\infty u, r_h)_0 + \mathcal{O}(h^{M+1})\|u\|_{M+n+2}\|r_h\|_{-1}.
\end{aligned}$$

The bound

$$\begin{aligned}|(F_k(u-D_h^\infty u),r_h)_0| &\le C\|u-D_h^\infty u\|_{n+k+1}\|r_h\|_{-1}\\ &\le C' h^{M+1-k}\|u\|_{M+n+2}\|r_h\|_{-1},\end{aligned}$$

derived from the order of F_k and (18) finishes the proof. □

The next result is now a simple consequence of Theorem 7 and Proposition 5.

Corollary 8. *Let $A \in \mathcal{E}(n)$ and $d \ge n+1$. Then there exists a sequence $(G_k)_k$ with $G_k \in \mathcal{L}_{n+k}$ satisfying the following property: for all $r_h \in S_h^{d'}$ and $u \in H^{M+n+2}$*

$$\begin{aligned}(28)\quad &\langle A(D_h^d u-u),r_h\rangle_N\\ &\qquad = \sum_{k=d+1-n}^{M} h^k\langle G_k u,r_h\rangle_N + \mathcal{O}(h^{M+1})\|u\|_{M+n+2}\|r_h\|_{-1}.\end{aligned}$$

Remark 9. For the particular case of $\xi-$collocation methods, i.e., for single point rules, Corollary 8 is a direct consequence of (21), taking

$$G_k = \alpha_k R_k + \beta_k T_k = \underline{B}_k(\xi)R_k + C_k(\xi)T_k. \tag{29}$$

In the sequel, the cancellation of some of the first G_k will be relevant, since it provokes an improvement in the order of convergence of the method.

Definition 1. Let $b \ge 0$ be the greatest integer such that

$$G_k = 0, \quad k = d+1-n,\dots,d+b-n \tag{30}$$

($b=0$ means no cancellation of the first operator). We call this number, depending on d, d', L and A, the additional order of convergence of the method.

Notice that if $F_j = 0$ for $d+1-n \leq j \leq k$, then $G_j = 0$ for j in the same range. Since the expression of the operators F_j is simpler than that of G_j, we will study the additional order by means of the cancellation of F_k instead of using (30). A priori, the additional order is achieved for a single equation $Au = f$, although in most cases it will be proven for classes of operators having similar expansions.

7. Stability. We begin this section by writing the qualocation method in a standard projection form. To do that we introduce the artificial discrete projection P_h which assigns to $u \in H^s$, $s > 1/2$, the unique solution of

$$P_h u \in S_h^{d'+2}, \quad \text{s.t.} \quad \langle P_h u, r_h \rangle_N = \langle u, r_h \rangle_N, \quad \forall\, r_h \in S_h^{d'}.$$

Therefore, P_h is the qualocation operator associated to the identity (see also [**28**]). We use $d'+2$ to include the case $d'=0$; otherwise we would be allowed to take $P_h u \in S_h^{d'}$ without further complication. Then

$$\|P_h u\|_1 \leq C\|u\|_1, \qquad \|P_h u - u\|_1 \leq C h^{d'+1} \|u\|_{d'+2}.$$

Obviously u_h is the solution to

$$u_h \in S_h^d, \quad \text{s.t.} \quad \langle A u_h, r_h \rangle_N = \langle A u, r_h \rangle_N, \quad \forall\, r_h \in S_h^{d'},$$

if and only if $P_h A u_h = P_h A u$. Let then

$$A_h := P_h A\,|_{S_h^d} : S_h^d \to S_h^{d'+2}$$

which is a sequence of uniformly bounded operators from H^{n+1} to H^1. Moreover, if the qualocation equations have a unique solution then

$$Q_h := A_h^{-1} P_h A$$

is the operator mapping the exact to the numerical solution.

Proposition 10. *The following statements are equivalent*:

(a) *The qualocation method is H^{n+1}-stable, i.e.,*

$$\|Q_h u\|_{n+1} \leq C\|u\|_{n+1}, \quad \forall\, u \in H^{n+1}.$$

(b) *The uniform Babuška-Brezzi condition holds*

$$\inf_{0\neq g_h\in S_h^d}\ \sup_{0\neq r_h\in S_h^{d'}} \frac{|\langle A g_h, r_h\rangle_N|}{\|g_h\|_{n+1}\|r_h\|_{-1}} \geq \beta > 0. \tag{31}$$

(c) *The family A_h has uniformly bounded inverses, i.e.*

$$\|g_h\|_{n+1} \leq C\|A_h g_h\|_1, \quad \forall\, g_h \in S_h^d.$$

Proof. It is obvious that (c) implies (a). Let then $P_h^* : H^{-1} \to H^{-1}$ be the 0-adjoint operator to $P_h : H^1 \to H^1$,

$$(P_h^* v, u)_0 = (v, P_h u)_0, \quad \forall\, u \in H^1, v \in H^{-1}.$$

If $t_h \in T_h$ we have that

$$(u, P_h^* t_h)_0 = (P_h u, t_h)_0 = (u, t_h)_0$$

(recall that $T_h = \Pi_h S_h^{d'}$ and the definition of Π_h in Section 4). Therefore $P_h^* t_h = t_h$ for all $t_h \in T_h$. Since by transposition the image of P_h^* is at most $N-$dimensional, it follows that P_h^* is a projection onto T_h.

Assume now that (a) holds. Given $0 \neq u_h \in S_h^d$ we define $w_h = P_h A u_h$. Since

$$P_h A(A^{-1} w_h) = w_h = P_h A u_h, \quad u_h \in S_h^d,$$

then $Q_h(A^{-1}w_h) = u_h$. Therefore, (a) implies

$$\|u_h\|_{n+1} \leq C\|A^{-1} w_h\|_{n+1} \leq C'\|w_h\|_1. \tag{32}$$

Taking into account the identities

$$(w_h, v)_0 = (P_h w_h, v)_0 = (w_h, P_h^* v)_0,$$
$$(w_h, \Pi_h r_h)_0 = \langle P_h A u_h, r_h\rangle_N = \langle A u_h, r_h\rangle_N$$

for all $v \in H^{-1}$ and $r_h \in S_h^{d'}$ we have

$$\begin{aligned}
\|w_h\|_1 &= \sup_{0 \neq v \in H^{-1}} \frac{|(w_h, v)_0|}{\|v\|_{-1}} \leq C \sup_{0 \neq P_h^* v \in H^{-1}} \frac{|(w_h, P_h^* v)_0|}{\|P_h^* v\|_{-1}} \\
&= C \sup_{0 \neq r_h \in S_h^{d'}} \frac{|(w_h, \Pi_h r_h)_0|}{\|\Pi_h r_h\|_{-1}} \leq C' \sup_{0 \neq r_h \in S_h^{d'}} \frac{|\langle A u_h, r_h \rangle_N|}{\|r_h\|_{-1}}
\end{aligned} \tag{33}$$

where we have used the uniform boundedness of P_h^* and Π_h^{-1}. Obviously, (32) and (33) imply the Babuška-Brezzi condition.

If (b) holds, then

$$\begin{aligned}
\beta \|u_h\|_{n+1} &\leq \sup_{0 \neq r_h \in S_h^{d'}} \frac{|\Pi_h \langle A u_h, r_h \rangle|}{\|r_h\|_{-1}} \\
&= \sup_{0 \neq r_h \in S_h^{d'}} \frac{|\langle A_h u_h, r_h \rangle_N|}{\|r_h\|_{-1}} \leq C \|A_h u_h\|_1,
\end{aligned}$$

by Corollary 2, which proves (c). □

As a consequence of the preceding result we can easily prove the following lemma, showing preservation of stability under compact perturbations and multiplication operators.

Lemma 11. *Let $A \in \mathcal{E}(n)$ be invertible and such that the qualocation method is stable in H^{n+1}. Let $K : H^s \to H^{s-n}$ be compact for all s, such that $A+K$ is injective and $a \in \mathcal{D}$ satisfies $a(t) \neq 0$ for all t. Then the qualocation method for $A + K$ and aA is stable.*

Proof. The first result is a simple consequence of characterization (c) of stability with standard techniques (see [**16**, Chapter 13] and [**17**, Chapter 1]).

To see the second one, notice first that if $A \in \mathcal{E}(n)$ then by (3) and (4), $B := aA - Aa \in \mathcal{E}(n-1)$ is compact as an operator of order n. Let then

$$A_h := P_h A\,|_{S_h^d}\,, \qquad \widetilde{A}_h := P_h A a\,|_{S_h^d}\,.$$

By a commutator property of splines (see [**28**, Theorem 2.1]) we have for all $u_h \in S_h^d$

$$\|D_h^d(au_h) - au_h\|_{n+1} \le Ch\|u_h\|_{n+1}. \tag{34}$$

Then

$$\|A_h D_h^d(au_h) - \widetilde{A}_h u_h\|_1 \le C'h\|u_h\|_{n+1}. \tag{35}$$

Thus for h small enough

$$\begin{aligned}\|u_h\|_{n+1} &\le C_1\|au_h\|_{n+1} \le C_1\|D_h^d(au_h)\|_{n+1} + C_2h\|u_h\|_{n+1}\\ &\le C_3\|A_hD_h^d(au_h)\|_1 + C_2h\|u_h\|_{n+1}\\ &\le C_3\|\widetilde{A}_hu_h\|_1 + C_4h\|u_h\|_{n+1},\end{aligned}$$

where we have successively applied: the invertibility of the multiplication operator, the commutator property (34), the stability of the qualocation methods for A and (35). By Proposition 10, this proves that the method is stable for Aa. By the first result of this Lemma and the compactness of B, the method is stable for aA. □

8. Full expansion of the error and some consequences. We will prove in this part an error expansion in powers of h of the difference between the numerical solution and the Fourier spline projection of the exact solution. To prove it, we will use the infimum-supremum condition of Proposition 10 to transform the consistency expansion stated in Section 6 into this error expansion. As by-products we obtain an estimate of the order of the method, pointwise superconvergence in some particular points of the grid and a 'true' asymptotic expansion (i.e., one where all the coefficients accompanying powers of h are independent of h) of the error under the action of smooth operators (see Proposition 16).

Proposition 12. *Given $A \in \mathcal{E}(n)$ and a stable qualocation method with additional order b, there exists a sequence of pseudodifferential operators $(Q_k)_k$ with $Q_k \in \mathcal{E}(k)$ such that the following expansion holds*

(36)

$$\Big\| \big[D_h^d u - Q_h u\big] - \sum_{k=d-n+b+1}^{M} h^k D_h^d Q_k u\Big\|_{n+1} = \mathcal{O}(h^{M+1})\|u\|_{M+n+2}.$$

Proof. From Corollaries 8 and 2, we deduce

$$\Big\langle A(D_h^d u - Q_h u - \sum_{k=d-n+b+1}^{M} h^k Q_h A^{-1} G_k u), r_h \Big\rangle_N \\ = \mathcal{O}(h^{M+1}) \|u\|_{M+n+2} \|r_h\|_{-1}.$$

The infimum-supremum condition (31) yields

$$D_h^d u - Q_h u = \sum_{k=d-n+b+1}^{M} h^k Q_h A^{-1} G_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+n+2}$$

in H^{n+1}. Notice that we can apply the same argument to each term $Q_h A^{-1} G_k u$. Hence the result follows readily by induction. □

Theorem 13. *There exists $C > 0$ such that for all h and $u \in H^{d+b+2}$*

$$\|Q_h u - u\|_{n-b} \leq C h^{d-n+b+1} \|u\|_{d+b+2}.$$

Proof. From Proposition 12 and (18) we obtain

$$\begin{aligned} \|Q_h u - u\|_{n-b} &\leq \|Q_h u - D_h^d u\|_{n+1} + \|D_h^d u - u\|_{n-b} \\ &\leq C h^{d-n+b+1} \|u\|_{d+b+2}, \end{aligned} \tag{37}$$

and the assertion is proven. □

Therefore, we conclude that the order of the method is $d - n + b + 1$ in the norm $\|\cdot\|_{n-b}$, whenever the exact solution is smooth enough. Hence the order of the method equals the first power of h appearing in (36).

We point out that the analysis presented in [**27, 28**] proves the same order of convergence with somewhat weaker smoothness assumption on the solution, namely, it is enough that $u \in H^{d+b+1}$. The extra order of regularity on u is then a consequence of the asymptotic tools developed here. On the other hand, an appropriate use of inverse inequalities of splines, the convergence properties of D_h and the stability of qualocation methods in H^{n+1}, allow us to obtain convergence estimates in other norms and under weaker regularity conditions of the solution.

It is well known that for $n \geq 1$ even, the Bernoulli polynomial B_n has only two roots, ζ and $1-\zeta$, both in $(0,1)$. For n odd, the roots of B_n are $\{0, 1/2, 1\}$. Taking into account the exception of how we evaluate $\underline{B}_1$ at its breakpoints, we can say that $\underline{B}_n$ has two different roots in each period.

Proposition 14. *Assume that* $b \geq \max\{n, 1\}$ *and* ζ *any of the roots of* $\underline{B}_{d+1}$ *in* $[0,1)$. *Then*

$$\max_i |Q_h u(x_i + \zeta h) - u(x_i + \zeta h)| \leq C h^{d+2} \|u\|_{d+b+2}.$$

Proof. In [**12**, Corollary 3.2] it is proven that

$$\max_i |D_h^d u(x_i + \zeta h) - u(x_i + \zeta h)| \leq C h^{d+2} \|u\|_{d+3}.$$

If $n \leq -1$, the inverse inequalities for splines (see [**17**])

$$\|u_h\|_\infty \leq C h^{-s-1/2} \|u_h\|_s,$$

valid for all $s \leq 0$, and (36) with $M = d - n + b$ yield

$$\|D_h^d u - Q_h u\|_\infty \leq C h^{n+1/2} \|D_h^d u - Q_h u\|_{n+1} \leq C' h^{d+b+3/2} \|u\|_{d+b+2}.$$

When $n \geq 0$, we simply apply (36) and Sobolev's embedding theorem which states the continuous inclusion of H^1 into the space of continuous functions. $\square$

Remark 15. This pointwise superconvergence was shown in [**23**] for a particular class of qualocation methods, which includes collocation, and assuming that the operator has the form either $\alpha H D^n + K$ or $\alpha D^n + K$ with α constant and K an operator of order $-\infty$. Hence, Proposition 14 can be seen as a generalization of this result.

We finish this section by stating the existence of an asymptotic expansion of the error between the exact and numerical solution under the action of a smoothing operator. Such operators appear, for instance, in the boundary integral methods when we want to recover the solution of the differential problem by means of an integral representation formula or potential.

Proposition 16. *Let X be a normed space and $T : \mathcal{D}' \to X$ be bounded in the norm of H^s for all s, i.e., $\|Tu\|_X \le C_s \|u\|_s$ for all s. Then*

$$\Big\| T(u - Q_h u) - \sum_{k=d-n+b+1}^{M} h^k T Q_k u \Big\|_X = \mathcal{O}(h^{M+1}) \|u\|_{M+n+2}.$$

9. Particular equations and quadrature rules. In this section we will study the error expansion (36) of particular methods applied to some equations arising from practical situations. Furthermore, since we will give an exact expression of these terms, we will be able to deduce sufficient conditions to obtain quadrature rules which define methods of higher order by demanding the cancellation of the first terms of these expansions.

We remark that these conditions and the rules derived from them are already known and were stated in [**8, 27, 28**], although we arrive at the same results from a different way. Our approach enables us to prove that some methods keep their orders of convergence when they are applied to more general equations. For instance, we will extend the good properties of the Simpson-type rules to logarithmic, second kind and hypersingular equations with variable-coefficients. Moreover, we can prove that the first power of h appearing in the error expansion is multiplied by, in general, a non zero function. This implies that the order proved in (37) is optimal. On the other hand, our smoothness requirements over the exact solution are a little stronger than those in [**8, 27, 28**].

We will pay attention to four cases:

(a) collocation in midpoints or nodes,

(b) ξ-collocation with $\xi \neq 0$, $1/2$,

(c) qualocation methods with Simpson-type rules,

(d) qualocation methods with symmetric rules.

Since we are interested in studying when the operators F_k are null, we are concerned about coefficients $\alpha_j, \beta_j, \alpha_{j,k}$ and $\beta_{j,k}$ appearing in the definition of these operators. By Remark 9 in cases (a) and (b) only α_j and β_j have to be considered.

Lemma 17. *In cases* (a), (c) *and* (d)

$$\alpha_{2k+1} = \beta_{2k} = 0, \quad \forall\, k \in \mathbf{Z}. \tag{38}$$

For Simpson-type rules (*case* (c)) *it holds that*

$$\begin{cases} \alpha_{j,k} = 0, & \textit{if } j \textit{ or } k \textit{ is odd}, \\ \beta_{j,k} = 0, & \textit{if } j \textit{ is odd or } k \textit{ is even}, \end{cases} \tag{39}$$

whereas for symmetric rules (d) *we have*

$$\begin{cases} \alpha_{j,k} = 0, & \text{if } j+k \text{ is odd}, \\ \beta_{j,k} = 0, & \text{if } j+k \text{ is even}. \end{cases} \tag{40}$$

Proof. It is a straightforward consequence of (11) and (19). □

We will need to deal with the roots of C_{2k+1}. Similar to $\underline{B}_{2k}$, these functions have two roots in $[0,1)$ symmetrically disposed with respect to $1/2$ (see [**5**]). Besides, for $k \geq 0$,

$$B_{k+1}\Big(\frac{1}{2}\Big) = (2^{-k} - 1) B_{k+1}(0), \quad C_{k+1}\Big(\frac{1}{2}\Big) = (2^{-k} - 1) C_{k+1}(0). \tag{41}$$

These relations can be proved using the Fourier expansion of $\underline{B}_k$ and C_k, and applying a standard Zeta function trick [**24**].

Lemma 18. *Let*

$$A = aD^n + K \quad \textit{or} \quad A = aHD^n + K,$$

with $a \in \mathcal{D}$ such that $a(t) \neq 0$ for all $t \in \mathbf{R}$, and K of order n and compact. Let us consider the qualocation method with d, d' of the same *parity if A is strongly elliptic and d, d' of opposite parity if A is oddly elliptic. Then the method is stable in H^{n+1} if and only if Hypothesis 1 holds.*

Proof. For $a \equiv \alpha \in \mathbf{C} \setminus \{0\}$ and $K \in \mathcal{E}(-\infty)$ the result is proven in [**8**, Theorem 1]. The general case is now a straightforward consequence of Lemma 11. □

Finally, we introduce the following notation

$$p(m) := \begin{cases} m+2, & \text{if } m \text{ even}, \\ m+1, & \text{if } m \text{ odd}. \end{cases}$$

Logarithmic equations. We deal with equations of the form

$$\begin{aligned} Vu &:= \int_0^1 A(\cdot,t)\log(\sin^2\pi(\cdot - t))u(t)\,dt \\ &\quad + \int_0^1 K(\cdot,t)u(t)\,dt = f, \end{aligned} \tag{42}$$

where $A, K \in \mathcal{C}^\infty(\mathbf{R}^2)$ are 1-periodic in both variables with $A(t,t) \neq 0$ for all t. From [**13**, Appendix], we have

$$V \overset{\text{exp}}{=} \sum_{j=-\infty}^{-1} b_j H D^j, \tag{43}$$

with $b_{-1}(t) = -(2\pi\imath)A(t,t)$. Notice that by (20) and (43), $R_k = 0$ for all k.

Assuming that d and d' are of the same parity, Lemma 18 implies that methods (b)–(d) are stable. Moreover, collocation in midpoints for d odd and in the nodes for d even are the unique unstable collocation methods. We now briefly examine the error expansions and the order of the methods.

(a) From Remark 9 and (38), we conclude that $G_{2k} = 0$ for all k. Then expansion (28) has only odd powers of h. Hence stable collocation methods are of order $p(d)+1$. It follows readily that the error expansion of Proposition 12 takes the form

$$D_h^d u - Q_h u = \sum_{k=p(d)/2}^{p(d)} h^{2k+1} D_h^d Q_{2k} u + \sum_{k=2p(d)+2}^{M} h^k D_h^d Q_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+1}.$$

(b) By Remark 9 the first term in expansion (28) is related to the operator $G_{d+2} = C_{d+2}(\xi) T_{d+2}$. If d odd and ξ is one of the two roots of C_{d+2}, the $\xi-$collocation method is of order $d+3$, but with error expansion (28) containing even and odd powers of h. For other choices of ξ and d the method is of order $d+2$.

(c) Applying (38) and (39) we obtain that $F_{2k} = 0$ for all k and expansion (24) has only odd powers of h. By Proposition 5 this property is kept in (28), i.e, $G_{2k} = 0$ for all k. Thus, Simpson-type rules define methods of order at least $p(d)+1$. Moreover, since $G_{p(d)+1} = \beta_{p(d)+1} T_{p(d)+1}$, then by (41), $\beta_{p(d)+1} = 0$ if

$$\omega_1 = 1 - \omega_2 = \frac{2^{p(d)} - 1}{2^{p(d)+1} - 1}. \tag{44}$$

The Simpson-type rule with these weights defines a method of order $p(d)+3$. The error expansion in this case is

$$\begin{aligned} D_h^d u - Q_h u = &\sum_{k=p(d)/2+1}^{p(d)+2} h^{2k+1} D_h^d Q_{2k} u \\ &+ \sum_{k=2p(d)+6}^{M} h^k D_h^d Q_k u + \mathcal{O}(h^{M+1}) \|u\|_{M+1}. \end{aligned} \tag{45}$$

(d) As in (c), we obtain from (38), (40) and Proposition 5 that $F_{2k} = G_{2k} = 0$ for all k. Therefore expansions (24) and (28) have only odd powers of h. If b is such that $d+b+1$ is even and

$$F_{2k+1} = 0 \quad k = \frac{p(d)}{2}, \dots, \frac{d+b-1}{2},$$

then the method has additional order b. These cancellation of operators are for instance obtained if

$$\beta_{2k+1} = \beta_{j,2k+1-j} = 0, \quad k = \frac{p(d)}{2}, \dots, \frac{d+b-1}{2}, \\ d' + 1 \le j \le 2k - d - 1.$$

There is a simple choice [**8**] with a two-point symmetric rule if $d' \ge 1$, since in this case

$$G_{p(d)+1} = F_{p(d)+1} = \beta_{p(d)+1} T_{p(d)+1}.$$

Taking ξ_1, ξ_2 the roots of $C_{p(d)}$ and $\omega_1 = \omega_2 = 1/2$, we have that $\beta_{p(d)+1} = 0$, from where the order of the method becomes $p(d) + 3$. In the general case discussed above, this corresponds to $b = 2$ or 3 depending on whether d is odd or even, respectively. In fact, it can be proved that this is the unique way to make $\beta_{p(d)+1}$ vanish with two-point rules [**15**]. When $d' = 0$, and hence we take d even, the operator $G_{p(d)+1} = F_{p(d)+1}$ involves the coefficients β_{d+3} and $\beta_{1,d+2}$. Therefore cancellation of these is not so simple.

Equations of the second kind. Let $a \in \mathcal{D}$ be such that $a(t) \neq 0$ for all t, and $K \in \mathcal{C}^\infty(\mathbf{R}^2)$ be 1-periodic in both variables. We consider equations of the form

$$Au := au + \int_0^1 K(\cdot, t) u(t)\, dt = f. \tag{46}$$

The operator has the trivial expansion $A \overset{\exp}{=} aD^0$, and therefore $T_k = 0$, $R_k = \gamma_k^d a D^k$, for $k \ge d+1$. Cases (c) and (d) are stable if d and d' are of the same parity, whereas the unique unstable $\xi-$collocation methods are the collocation in midpoints for d odd and knot collocation for d is even.

We now study the order of convergence and the error expansion.

(a) Using Remark 9 and (38) we conclude that $G_{2k+1} = 0$ for all k. Moreover, by the proof of Proposition 12 we can show that $Q_{2k+1} = 0$ and expansion (36) only has even powers of h. Thus stable collocation methods are of order $p(d)$.

(b) By (29), the first term in expansion (28) is related to $G_{d+1} = B_{d+1}(\xi)R_{d+1}u$. For d odd and ξ one of the two roots of B_{d+1}, the corresponding $\xi-$collocation method is of order $d+2$. Other choices of ξ and d give methods of order $d+1$.

(c) Applying (38) and (39) we obtain that $F_{2k+1} = 0$ for all k and expansion (24) only has odd powers of h. By Proposition 5 it holds that in expansion (28) also $G_{2k+1} = 0$. Now it is straightforward to verify that expansion (36) has only even powers of h. On the other hand, the first power appearing in (28) is $p(d)$. From (41) we conclude that taking

$$\omega_1 = 1 - \omega_2 = \frac{2^{p(d)-1} - 1}{2^{p(d)} - 1},$$

$\alpha_{p(d)} = 0$ and therefore $G_{p(d)} = \alpha_{p(d)} R_{p(d)} = 0$. Hence this method has order $p(d) + 2$.

(d) As in (c), we obtain that in expansions (24), (28) and (36) there are no odd powers of h. Now the method is of even order $d + b + 1$ if

$$\alpha_{2k} = \alpha_{j,2k-j} = 0, \quad d' + 1 \leq j \leq 2k - d - 1,$$
$$k = \frac{p(d)}{2}, \dots, \frac{d+b-1}{2},$$

because of these cancellations imply that $F_{2k} = 0$ for $k = p(d)/2, \dots, (d + b - 1)/2$. If $d' \geq 1$, the symmetric rule of two points with ξ_1, ξ_2 the roots of $B_{p(d)}$ and the weights $\omega_1 = \omega_2 = 1/2$ gives a method of order $p(d) + 2$ with only even powers of h in expansion (36).

Equations of the second kind with a logarithmic part. Let

$$Au := au + \int_0^1 A(\cdot, t) \log(\sin^2 \pi(\cdot - t))u(t)\, dt + \int_0^1 K(\cdot, t)u(t)\, dt = f.$$

Then, A has the asymptotic expansion

$$A \overset{\exp}{=} aD^0 + \sum_{j \leq -1} b_j H D^j.$$

Thus, these equations are compact perturbations of equations of second kind and therefore, the requirements for ensuring stability are the same as in this case.

Collocation and $\xi-$collocation methods keep their order of convergence, but with even and odd powers of h appearing in their error expansions.

On the other hand, qualocation methods corresponding to Simpson-type rules and the symmetric rule with two points have now order $p(d)+1$. The conditions to obtain qualocation methods of higher order can be deduced following similar arguments to those applied above. These conditions involve now the coefficients $\alpha_{2k}, \beta_{2k+1}, \alpha_{j,2k-j}$ and $\beta_{j,2k+1-j}$ with k in a suitable range, which depends on the required order.

Singular integral equations. We consider equations of the form

$$Su = au + bHu + Ku = f,$$

K being a ψdo of order $-\infty$. The stability of collocation and $\xi-$collocation methods depends now on the balance between a and b and on the choice of the point of collocation (see [**21**, Theorem 3]). On the other hand, the stability of qualocation methods is not so clear, and cannot be deduced by the same ideas applied so far. We refer to [**28**] for a review of stability conditions for qualocation methods, and to [**20**] for very recent contributions to this problem. Provided that stability holds, an expansion of the error holds but now containing even and odd powers of h.

Some hypersingular equations. Let

$$\begin{aligned} Wu := \text{f.p.} \int_0^1 A_1(\cdot,t) \frac{1}{\sin^2(\pi(\cdot - t))}\, dt \\ + \text{p.v.} \int_0^1 A_2(\cdot,t)\cot(\pi(\cdot - t))u(t)\, dt \\ + \int_0^1 A_3(\cdot,t)\log(\sin^2\pi(\cdot - t))u(t)\, dt + \int_0^1 K(\cdot,t)u(t)\, dt = f, \end{aligned}$$

where f.p. denotes the Hadamard finite part and p.v. the Cauchy principal value. Here we must impose that $A_1(t,t) \neq 0$ for all t. By [**13**, Appendix],

$$W \overset{\text{exp}}{=} \sum_{j\leq 1} b_j HD^j \in \mathcal{E}(1),$$

with $b_1(t) = (\pi\imath)^{-1}A(t,t)$. Therefore, for d and d' of the same parity the unique unstable qualocation methods are those equivalent to collocation in midpoints for d odd and in the nodes for d even.

The stable collocation method gives the expansion

$$D_h^d u - Q_h u = \sum_{k=p(d)/2-1}^{p(d)-2} h^{2k+1} D_h^d Q_{2k+1} u + \sum_{k=2p(d)-2}^{M} h^k D_h^d Q_k u + \mathcal{O}(h^{M+1})\|u\|_{M+3}$$

Again, for d odd there exist two points ξ_1, ξ_2, the roots of C_d, such that the corresponding $\xi-$collocation is of order $d+1$.

The Simpson-type rule with

$$\omega_1 = 1 - \omega_2 = \frac{2^{p(d)-2}-1}{2^{p(d)-1}-1}$$

and the symmetric rule of two points ξ_1, ξ_2 with $d' \geq 1$ define methods with order $p(d)+1$. The error expansion is now

$$D_h^d u - Q_h u = \sum_{k=p(d)/2}^{p(d)} h^{2k+1} D_h^d Q_{2k+1} u + \sum_{k=2p(d)+2}^{M} h^k D_h^d Q_k u + \mathcal{O}(h^{M+1})\|u\|_{M+3}.$$

10. An example with numerical integration. In this section we illustrate with an example the convergence properties of the qualocation method of Simpson type with piecewise constant functions as trial and test space for logarithmic equations. Here we implement the method of higher order constructed by taking $\omega_1 = 3/7$ and $\omega_2 = 4/7$ according to (44). We show how this can be implemented with numerical integration applied in a way that preserves the order and the error expansion of the method. We remark that similar strategies can be developed to cover all qualocation methods for this kind of equations and can be extended to equations of second kind with logarithmic part.

Let us consider the equation

$$Vu := \int_0^1 V(\cdot,t)u(t)\,dt = \int_0^1 A(\cdot,t)\log(\sin^2\pi(\cdot - t))u(t)\,dt + \int_0^1 K(\cdot,t)u(t)\,dt = f,$$

where $A, K \in \mathcal{C}^\infty(\mathbf{R}^2)$ are 1-periodic in both variables. We consider the basis $\{\chi_i\}_{i=0}^{N-1}$ of S_h^0, χ_i being the 1-periodization of the characteristic function of the interval (x_i, x_{i+1}). Then the $S_h^0 \times S_h^0$ qualocation method is equivalent to solving

$$\sum_{j=0}^{N-1} u_j \langle V\chi_j, \chi_i\rangle = \langle f, \chi_i\rangle_N, \quad i = 0, \dots, N-1.$$

Notice that if $b_{i,j} = \langle V\chi_j, \chi_i\rangle_N$ and $a_{i,j}(\xi) = V\chi_j(x_i + \xi h)$, then

$$b_{i,j} = h\left[\frac{\omega_1}{2}\, a_{i,j}(0) + \omega_2\, a_{i,j}\left(\frac{1}{2}\right) + \frac{\omega_1}{2}\, a_{i+1,j}(0)\right]. \tag{47}$$

To approximate $a_{i,j}(\xi)$ we follow a variant of the strategies of [**7**], themselves based upon [**14**]. We first choose $C \in \mathcal{C}^\infty(\mathbf{R}^2)$ such that

$$C(s+1, t+1) = C(s,t), \quad \frac{\partial^k C}{\partial t^k}\,(s,s) = \frac{\partial^k A}{\partial t^k}\,(s,s), \quad k=0,1,2, \tag{48}$$

and remark that

$$F(s,t) := V(s,t) - C(s,t)\log(s-t)^2$$

admits a $\mathcal{C}^2$ extension to the diagonal $s = t$. For (i,j) such that $|i-j| \le N/2$ we approximate

$$\alpha_{i,j}^{(1)}(\xi) := \int_{z_j - h/2}^{z_j + h/2} F(x_i + \xi h, t)\, dt \approx h \mathrm{L} F(x_i + \xi h, z_j + h\,\cdot\,) =: \beta_{i,j}^{(1)}(\xi)$$

where $z_i = x_i + h/2$ and the quadrature rule with nodes outside the integration interval (see [**7**] and [**14**] for other situations where this strategy is used)

$$\begin{aligned}\mathrm{L}u &:= \frac{1}{5760}\,[-17u(-2) + 308u(-1) + 5178u(0) + 308u(1) - 17u(2)]\\ &\approx \int_{-1/2}^{1/2} u(t)\, dt,\end{aligned}$$

is exact for polynomials of degree not greater than 5. For the same pairs (i,j) we compute $\Pi_{i,j}^\xi \in \mathbf{P}_4$ such that

$$\Pi_{i,j}^\xi(z_j + kh) = C(x_i + \xi h, z_j + kh), \quad k = -2, -1, 0, 1, 2$$

and approximate

$$\alpha^{(2)}_{i,j}(\xi) := \int_{z_j-h/2}^{z_j+h/2} C(x_i+\xi h, t)\log(x_i+\xi h - t)^2\,dt$$
$$\approx \int_{z_j-h/2}^{z_j+h/2} \Pi^{\xi}_{i,j}(t)\log(x_i+\xi h - t)^2\,dt =: \beta^{(2)}_{i,j}(\xi).$$

The last quantities can be computed exactly (cf. [**7, 11**]). Finally we add $\beta_{i,j}(\xi) := \beta^{(1)}_{i,j}(\xi) + \beta^{(2)}_{i,j}(\xi)$, and reorganize

$$a^*_{i,j}(\xi) := \begin{cases} \beta_{i,j}(\xi) & \text{if } |i-j| < N, \\ \beta_{i,j+N}(\xi), & \text{if } i-j > N/2, \\ \beta^{(k)}_{i,j-N}(\xi), & \text{if } j-i > N/2, \\ \dfrac{1}{2}(\beta^{(k)}_{i,j-N}(\xi) + \beta^{(k)}_{i,j+N}(\xi)), & \text{if } |i-j| = N/2. \end{cases}$$

Hence (47) can be used to give an approximation $b^*_{i,j} \approx b_{i,j}$ to the matrix of our system. Notice that we have only used values of F and C at points $(x_i+\xi h, z_j)$, $|i-j| \le N/2+2$.

Let now u^*_h be the solution of the discrete qualocation method. The analysis given in [**7**] can be adapted to prove H^0-stability of the numerical method and that the following expansion

$$D^0_h u - u^*_h - \sum_{k=5}^{M} h^k D^0_h u^*_k = \mathcal{O}(h^{M+1})$$

holds in L^∞. Here, the functions u^*_k depend on the operator V and on the exact solution u. Notice that in particular the order of the method is preserved.

Example. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary Γ. We take $\mathbf{x} : [0,1] \to \Gamma$ a regular $\mathcal{C}^\infty$ parameterization of Γ. Then it is well known (cf. [**9**]) that the solution of

$$(49)\qquad \left|\begin{array}{ll} \Delta\omega - \lambda^2\omega = 0, & \text{in } \mathbf{R}^2\setminus\Gamma, \\ \omega|_\Gamma = f, & \\ \dfrac{\partial\omega}{\partial r}(\mathbf{y}) + \lambda\omega(\mathbf{y}) = o(|\mathbf{y}|^{-1/2}), & \text{as } |\mathbf{y}|\to\infty, \end{array}\right.$$

can be written in the form

$$\omega(\mathbf{y}) = \int_0^1 K_0(\lambda|\mathbf{y}-\mathbf{x}(t)|)u(t)\,dt, \quad \forall\,\mathbf{y}\in\mathbf{R}^2\setminus\Gamma,$$

if λ^2 is not a Dirichlet eigenvalue of the Laplace operator in Ω. Here K_0 is the modified Bessel function of the second kind and order 0 and u must be the solution of the periodic integral equation

$$\int_0^1 K_0(\lambda|\mathbf{x}(\cdot)-\mathbf{x}(t)|)u(t)\,dt = f(\mathbf{x}(\cdot)). \tag{50}$$

The kernel satisfies the decomposition (see [**1**])

$$K_0(\lambda|\mathbf{x}(s)-\mathbf{x}(t)|) = -\frac{1}{2}\,I_0(\lambda|\mathbf{x}(s)-\mathbf{x}(t)|) \cdot \log(\sin^2\pi(s-t)) + B(s,t),$$

$I_0\in\mathcal{C}^\infty(\mathbf{R})$ being the modified Bessel function of the first kind and order 0 and B smooth. Therefore, equation (50) is of logarithmic type. We take

$$C(s,t) := -\frac{1}{2} - \frac{\lambda^2}{8}\,|\mathbf{x}'(s)|^2(s-t)^2,$$

that satisfies (48). Notice that with this decomposition, the polynomials $\Pi^\xi_{i,j}$ appearing in the definition of $\beta^{(2)}_{i,j}(\xi)$ are of degree 2. If $u_h^* = \sum_{i=0}^{N-1} u_i^*\chi_i$ is the numerical solution, we define

$$\omega_h(\mathbf{y}) := h\sum_{i=0}^{N-1} u_i^*\mathrm{L}[K_0(\lambda|\mathbf{y}-\mathbf{x}(x_i+h\,\cdot)|)] \approx \omega(\mathbf{y}), \qquad \mathbf{y}\in\mathbf{R}^2\setminus\Gamma,$$

the approximate solution of problem (49). Using now Proposition 16 and [**7**, Theorem 8.1] we can prove that there exists a sequence of functions $(v_k)_k$ such that

$$\omega(\mathbf{y}) - \omega_h(\mathbf{y}) = \sum_{k=5}^{M} h^k v_k(\mathbf{y}) + \mathcal{O}(h^{M+1})$$

uniformly over compact sets of $\mathbf{R}^2 \setminus \Gamma$.

If the origin is contained in Ω, then for $f = K_0(\lambda|\mathbf{y}|)$ the unique solution of (49) in $\Omega' := \mathbf{R}^2 \setminus \Omega$ is $\omega(\mathbf{y}) := K_0(\lambda|\mathbf{y}|)$. In our experiment we have taken the ellipse Γ

$$\mathbf{x}(t) := (0.25 + 2\cos(2\pi t), 0.4 + \sin(2\pi t)),$$

$\lambda = 1/2$ and $N = 64$, 96, 144, 216, 324, 486.

We approximate the solution of the differential problem in $\mathbf{y}_0 = (2, 3)$, where $\omega(\mathbf{y}_0) \approx 0.145425$. Notice that the ratio of mesh sizes between two consecutive grids is 3/2. Then, using Richardson extrapolation (cf. [**29**]) we can define a new set of numerical solutions

$$\begin{cases} \omega_h^0 := \omega_h(\mathbf{y}_0) \\ \omega_h^j := \frac{(3/2)^{j+4}\omega_h^{j-1} - \omega_{3h/2}^{j-1}}{(3/2)^{j+4}-1}, \quad j > 0, \end{cases}$$

which converge to the exact solution with order $5 + j$. Richardson extrapolation can be used for many other step-sizes relations (basically, one can use any k values of N to cancel the first $k - 1$ terms of the asymptotic expansion). We take the ratio fixed (for instance, to 3/2) to be able to observe the asymptotic behavior of the result.

Finally, we denote $\varepsilon_h^j := |\omega_h^j - \omega(\mathbf{y}_0)|$. Table 1 shows the values of ε_h^j, and Table 2, the estimated convergence rates (e.c.r.) obtained by comparison between two consecutive grids, which confirms the theoretical rates written in the top row.

TABLE 1. Errors after extrapolation.

N	ε_h^0	ε_h^1	ε_h^2	ε_h^3	ε_h^4	ε_h^5
64	5.30E-09					
96	7.34E-10	4.19E-11				
144	1.00E-10	3.98E-12	3.28E-13			
216	1.35E-11	3.66E-13	1.81E-14	1.23E-15		
324	1.81E-12	3.31E-14	1.01E-15	4.59E-17	2.17E-18	
486	2.41E-13	2.95E-15	5.77E-17	1.73E-18	6.05E-20	4.20E-21

TABLE 2. E.c.r. for the errors in Table 1.

5	6	7	8	9
4.874				
4.913	5.807			
4.941	5.886	7.153		
4.961	5.930	7.103	8.110	
4.974	5.956	7.069	8.081	8.827

11. The other qualocation method. As mentioned in Section 3, there exists another class of qualocation methods. In a way, this can be considered as a limit case when $d' \to \infty$. To unify notations we write

$$S_h^\infty = \operatorname{span}\langle \phi_\mu \mid \mu \in \Lambda_N\rangle.$$

The qualocation method can then be written as

$$u_h \in S_h^d, \quad \text{s.t.} \quad \langle Au_h, r_h\rangle_N = \langle f, r_h\rangle_N, \quad \forall\, r_h \in S_h^\infty.$$

We remark that Hypothesis 1 makes no sense now since $d' = \infty$ is neither even nor odd. Moreover, Proposition 1, and hence Corollary 2, holds and Proposition 3 is always satisfied. To see that this is so we need only notice that the proof is valid by taking $\Delta_\infty = 0$.

Again it is possible to prove that (17) holds for all k, which makes the approximate inner product asymptotically optimal. The proof follows readily by doing the same trick (take $\Delta_\infty = 0$) in the proof of [**13**, Lemma 10]. Consequently, the proof of Theorem 7 is greatly simplified, and the expansion holds with $F_k = \alpha_k R_k + \beta_k T_k$ for all k. Moreover, by the good approximation properties of the discrete inner product, Corollary 8 holds with $G_k = F_k$. It is straightforward that Propositions 12 and 14 are also valid.

With respect to stability, the proof of Lemma 18 can be adapted to obtain that the qualocation method is stable for that kind of operators if and only if they are not equivalent to ill posed collocation: midpoint collocation for d odd and A strongly elliptic or d even and A oddly elliptic, and knot collocation for d even and A strongly elliptic or d odd and A oddly elliptic. For the particular equations examined in

Section 9, and the methods studied therein, the orders of convergence and the error expansions are also valid.

Acknowledgments. The authors want to thank referees for the valuable comments which helped to improve both the content and presentation of this paper.

REFERENCES

1. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, Dover, New York, 1970.

2. D.N. Arnold, *A spline-trigonometric Galerkin method and an exponentially convergent boundary integral method*, Math. Comp. **41** (1983), 383–397.

3. D.N. Arnold and W.L. Wendland, *On the asymptotic convergence of collocation methods*, Math. Comp. **41** (1983), 349–381.

4. ———, *The convergence of spline collocation for strongly elliptic equations on curves*, Numer. Math. **47** (1985), 349–381.

5. G. Brown, G.A. Chandler, I.H. Sloan and D.C. Wilson, *Properties of certain trigonometric series arising in numerical analysis*, J. Math. Anal. Appl. **162** (1991), 371–380.

6. R. Celorrio and F.-J. Sayas, *Extrapolation techniques and the collocation method for a class of boundary integral equations*, ANZIAM J. **42** (2001), 413–437.

7. ———, *Full collocation methods for some boundary integral equations*, Numer. Algorithms **22** (2000), 327–351.

8. G.A. Chandler and I.H. Sloan, *Spline qualocation methods for boundary integral equations*, Numer. Math. **58** (1990), 537–567.

9. G. Chen and J. Zhou, *Boundary element methods*, Academic Press, New York, 1992.

10. M. Crouzeix and F.-J. Sayas, *Asymptotic expansions of the error of spline Galerkin boundary element methods*, Numer. Math. **78** (1998), 523–547.

11. V. Domínguez, *Asymptotic analysis of projection methods for boundary integral equations*, Ph.D. Thesis, University of Zaragoza, 2001. (in Spanish)

12. V. Domínguez and F.-J. Sayas, *Local expansions of periodic spline interpolation with some applications*, Math. Nachr. **227** (2001), 43–62.

13. ———, *Full asymptotics of spline Petrov-Galerkin methods for some periodic pseudodifferential equations*, Adv. Comput. Math. **14** (2001), 75–101.

14. G.C. Hsiao, P. Kopp and W.L. Wendland, *A Galerkin-collocation method for some integral equations of the first kind*, Computing **25** (1980), 89–130.

15. M. Junges and C. Schneider, *Quadrature rules for qualocation*, in Proceedings of GAMM Annual Conference 2002.

16. R. Kress, *Linear integral equations*, 2nd ed., Springer-Verlag, Berlin, 1999.

17. S. Prössdorf and B. Silbermann, *Numerical analysis for integral and related operator equations*, Akademie-Verlag, Berlin, 1990.

18. J. Saranen, *Extrapolation methods for spline collocation solutions of pseudodifferential equations on curves*, Numer. Math. **56** (1989), 385–407.

19. J. Saranen and G. Vainikko, Periodic integral and pseudodifferential equations with numerical approximation, Springer-Verlag, Berlin, 2002.

20. C. Schneider, *Stability of qualocation methods for elliptic boundary integral equations*, preprint, University of Johannes Gutenberg, Mainz, 2002.

21. G. Schmidt, *On $\varepsilon-$collocation for pseudodifferential equations on a closed curve*, Math. Nachr. **126** (1986), 183–196.

22. I.H. Sloan, *A quadrature-based approach to improving the collocation method*, Numer. Math. **54** (1988), 41–56.

23. ———, *Superconvergence in the collocation and qualocation methods*, Internat. Ser. Numer. Math., vol. 86, 1988, pp. 429–441.

24. ———, *Unconventional methods for boundary integral equations in the plane*, in Numerical Analysis 1991 (D.F. Griffiths and G.A. Watson, eds.), Longman Sci. Tech., Harlow, 1992, pp. 194–218.

25. ———, *Qualocation*, J. Comput. Appl. Math. **125** (2000), 461–478.

26. I.H. Sloan and W.L. Wendland, *A quadrature-based approach to improving the collocation method for splines of even degree*, Z. Anal. Anwendungen **8** (1989), 361–376.

27. ———, *Qualocation methods for elliptic boundary integral equations*, Numer. Math. **79** (1998), 451–483.

28. ———, *Spline qualocation methods for variable-coefficients elliptic equations on curves*, Numer. Math. **83** (1999), 497–533.

29. J. Stoer and R. Bulirsch, *Introduction to numerical analysis*, Springer Verlag, New York, 1972.

30. T. Tran and I.H. Sloan, *Tolerant qualocation – a qualocation method for boundary integral equations with reduced regularity requirement*, J. Integral Equations Appl. **10** (1998), 85–115.

31. ———, *A tolerant qualocation method for variable-coefficient elliptic equations on curves*, J. Integral Equations Appl. **13** (2001), 73–98.

32. W. Wendland, *Boundary element methods for elliptic problems*, in *Mathematical theory of finite and boundary element methods* (A.H. Schatz, V. Thomée and W.L. Wendland, eds.), Birkhäuser, Berlin, 1990, pp. 219–276.

33. Y. Yan and I.H. Sloan, *On integral equations of the first kind with logarithmic kernels*, J. Integral Equations Appl. **1** (1988), 549–579.

Dep. Matemática e Informática, Universidad Pública de Navarra, Campus de Arrosadía, 31006 Pamplona, Spain
E-mail address: victor.dominguez@unavarra.es

Dep. Matemática Aplicada, Universidad de Zaragoza, Centro Politécnico Superior, María de Luna, 3 – 50018 Zaragoza, Spain
E-mail address: jsayas@posta.unizar.es