JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 14, Number 4, Winter 2002

THE CONSERVED PENROSE-FIFE PHASE FIELD MODEL WITH SPECIAL HEAT FLUX LAWS AND MEMORY EFFECTS

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ABSTRACT. In this paper a phase-field model of Penrose-Fife type is considered for diffusive phase transitions with conserved order parameter. Different motivations lead to investigate the case when the heat flux is the superposition of two different contributions; one part is the gradient of a function of the absolute temperature ϑ, behaving like $1/\vartheta$ as ϑ approaches to 0 and like $-\vartheta$ as $\vartheta \nearrow +\infty$, while the other is given by the Gurtin-Pipkin law introduced in the theory of materials with thermal memory. An existence result for a related initial-boundary value problem is proven. Strengthening some assumptions on the data, the uniqueness of the solution is also achieved.

1. Introduction. This note is concerned with the study of the following initial-boundary value problem in the cylindrical domain $Q := \Omega \times (0,T)$, where $\Omega \subset \mathbf{R}^N$ $(N \le 3)$ is a bounded connected domain with a smooth boundary Γ and $T > 0$. Find a pair $(\vartheta, \chi) : Q \to \mathbf{R}^2$ satisfying

$$\partial_t\big(\vartheta + \lambda\chi\big) - \Delta\big(\psi(\vartheta) + k * \alpha(\vartheta)\big) = g \quad \text{in } Q, \tag{1.1}$$

$$-\,\partial_\nu\big(\psi(\vartheta) + k * \alpha(\vartheta)\big) = \gamma\big(\psi(\vartheta) + k * \alpha(\vartheta) - h\big) \quad \text{on } \Sigma := \Gamma \times (0,T), \tag{1.2}$$

$$\vartheta(\cdot,0) = \vartheta^0 \quad \text{in } \Omega, \tag{1.3}$$

$$\partial_t\chi - \Delta\left(-\,\Delta\chi + \xi + \sigma'(\chi) + \frac{\lambda}{\vartheta}\right) = 0 \quad \text{in } Q, \tag{1.4}$$

$$\xi \in \beta(\chi), \quad \text{in } Q, \tag{1.5}$$

$$\partial_\nu\chi = 0, \quad \partial_\nu\left(-\,\Delta\chi + \xi + \sigma'(\chi) + \frac{\lambda}{\vartheta}\right) = 0 \quad \text{on } \Sigma, \tag{1.6}$$

$$\chi(\cdot,0) = \chi^0 \quad \text{in } \Omega, \tag{1.7}$$

Received by the editors on December 10, 2001, and in revised form on July 1, 2002.

with $\partial_t = \partial/\partial t$, the symbol Δ denoting the Laplacian with respect to the space variables, and ∂_ν denoting the outward normal derivative on Γ.

In (1.1) there is a memory term given by the convolution product with respect to time, that is,

$$(a * b)(t) := \int_0^t a(s)b(t-s)\,ds, \quad t \in [0,T], \tag{1.8}$$

where a and b may also depend on the space variables.

The system given by the partial differential equations (1.1) and (1.4) provides a quite general version of the phase-field model proposed by Penrose and Fife in [**28, 29**] for the kinetics of phase transitions.

This model describes the evolution of a material, with constant latent heat of fusion-solidification process, being λ, exhibiting two different phases (e.g., solid-liquid in melting phenomena), in terms of the absolute temperature $\vartheta : Q \to (0,+\infty)$ and the order parameter $\chi : Q \to \mathbf{R}$ (representing, for instance, the fraction of one of the phases). The data g and h stand for the heat supply and the outer temperature; the function σ comes from the smooth part of the free energy, while the multi-valued map β is derived from its nonsmooth but convex part (usually β is the inverse of the Heaviside graph). To be more precise, the sum $\beta + \sigma'$ stands for the derivative of the double-well part of a Ginzburg-Landau free energy potential (see e.g. [**4, 29**]).

We may observe that we have coupled the second equation, which rules the evolution of the order parameter χ, with Neumann homogeneous boundary condition on χ and on the chemical potential $w := -\Delta\chi + \xi + \sigma'(\chi) + (\lambda/\vartheta)$, but we can study an analogous system in the case of the Dirichlet homogeneous or Neumann nonhomogeneous boundary condition for w. This seems indeed to be of some physical interest. In any case, we have to take the "natural" homogeneous Neumann boundary condition for the concentration χ (see [**27**] for a justification).

Finally, $k : [0,T] \to \mathbf{R}$ is an integration kernel, $\alpha : (0,+\infty) \to \mathbf{R}$ is a concave function, which will be specified in the sequel, ψ is linked to α as detailed below, and γ is a positive constant coefficient.

The term $-\Delta\big(\psi(\vartheta) + k * \alpha(\vartheta)\big)$ in (1.1) represents the divergence of the heat flux, which is given by

$$\mathbf{q} = -\nabla\big(\psi(\vartheta) + k * \alpha(\vartheta)\big). \tag{1.9}$$

Several papers have been devoted to the investigation of variants of (1.1)–(1.7), under the common position that

$$\mathbf{q} = -\nabla\Big(-\frac{C_1}{\vartheta}\Big), \tag{1.10}$$

where C_1 is a positive constant (see e.g. [**18, 20, 21, 23**]). It is worth mentioning that the non-conserved phase-field model with (1.10) as heat flux law (which basically differs from (1.1), and (1.4) because of a second order dynamics for χ) has been deeply investigated (see e.g. [**10, 16, 22, 25, 32**]). However, the law (1.10) that turns out to be satisfactory for low and intermediate temperatures and offers some advantages from the mathematical point of view, does not look acceptable for high temperatures because it does not provide any coerciveness as ϑ becomes larger and larger. These considerations suggest to replace (1.10) by

$$\mathbf{q} = -\nabla\Big(-\frac{C_1}{\vartheta} + C_2\vartheta\Big), \tag{1.11}$$

for some $C_2 > 0$. Concerning this case, for the non-conserved model, an existence result is given in [**8**], where, more generally, the heat flux is given by

$$\mathbf{q} = -\nabla\alpha(\vartheta), \tag{1.12}$$

where α is a nonlinear function chosen in such a way that the system given by (1.1) and (1.4) is still consistent with the second principle of thermodynamics.

In [**9**] a more particular case is considered and a uniqueness result is proven, still permitting α to belong to a wide class of nonlinearities that includes (1.11) and other important cases.

In [**30**] existence and uniqueness of solution are proven, again in the non-conservative case, under the same assumptions on α of [**9**], but with the constitutive law (1.9), with $\alpha = \psi$.

The first work that couples the Penrose-Fife model with memory effects is [**11**], where the above-named inconveniences given by (1.10) are overcome by considering the following law

$$\mathbf{q} = -\nabla\Big(-\frac{C_1}{\vartheta} + k * \vartheta\Big). \tag{1.13}$$

In our paper we use the constitutive law (1.9) in the case of a conserved order parameter, with a special α, namely

$$\alpha(\vartheta) = -\frac{C_1}{\vartheta} + C_2\vartheta,$$

for every ϑ in $(0, +\infty)$ and for some positive constants C_1 and C_2. Moreover $\psi : (0, +\infty) \to \mathbf{R}$ is a maximal monotone function such that the compositions $\psi \circ \alpha^{-1}$ and $\alpha \circ \psi^{-1}$ are two Lipschitz continuous functions.

For a justification of (1.8) and for other related works where memory effects are concerned, we refer to [**3, 5, 7, 12–15, 19**].

In our paper we are going to prove the existence of a weak solution to (1.1)–(1.7), making use of an implicit time discretization procedure. More regularity on the data is required in order to prove the uniqueness of the solution.

To simplify the treatment of this problem, in the sequel we will suppose $\psi = \alpha$, but all the estimates may be repeated in the more general case of different ψ and α, with the necessary constraint recalled above.

Let us remark that the existence and uniqueness of the solution for the system (1.1)–(1.7) with possibly nonconstant latent heat of fusion-solidification process or with more general structure hypotheses on α are still open problems.

2. Main results.

Consider the initial-boundary value problem (1.1)–(1.7). We make the following general assumptions on the data of the system

(A1) β is the subdifferential of a nonnegative, proper, convex, and l.s.c. function $\hat{\beta} : \mathbf{R} \to [0, +\infty]$ satisfying $\hat{\beta}(0) = 0$, and $D(\beta)$ denotes its domain, (A2) $\sigma \in C^2(\mathbf{R})$, $\sigma'' \in L^\infty(\mathbf{R})$,

(A3) $\lambda \in \mathbf{R}$,

(A4) $\alpha : (0,+\infty) \to \mathbf{R}$, and $\alpha(r) = -(C_1/r) + C_2 r$, for some positive constants C_1, C_2 and $\forall\, r \in (0,+\infty)$,

(A5) $k \in W^{1,1}(0,T)$,

(A6) $g \in L^2(0,T;L^2(\Omega))$, $h \in L^2(0,T;H^{1/2}(\Gamma))$,

(A7) $\vartheta^0 \in L^2(\Omega)$, $\vartheta^0 > 0$ almost everywhere in Ω,

$u^0 := \alpha(\vartheta^0) \in L^2(\Omega)$,

(A8) $\chi^0 \in H^1(\Omega)$, $\hat\beta(\chi^0) \in L^1(\Omega)$.

Let us now remark on some properties of α such as the one in (A4) that will be useful in the sequel

$$\alpha' \geq C_2 > 0; \tag{2.1}$$

$$\lim_{r\searrow 0} r^2\alpha'(r) = C_1; \tag{2.2}$$

$$\lim_{r\searrow 0} \alpha(r) = -\infty \quad \text{and} \quad \lim_{r\to+\infty} \alpha(r) = +\infty. \tag{2.3}$$

Moreover, since α is invertible, we can set

$$\rho := \alpha^{-1} : \mathbf{R} \to (0,+\infty), \tag{2.4}$$

that is increasing and Lipschitz continuous, because (2.1) gives $\rho' \leq 1/C_2$.

Finally, we may observe that the following implications hold

$$\alpha(\vartheta^0) \in L^2(\Omega) \Longrightarrow \frac{1}{\vartheta^0} \in L^2(\Omega) \Longrightarrow \frac{1}{\vartheta^0} \in L^1(\Omega) \Longrightarrow \ln(\vartheta^0) \in L^1(\Omega);$$
$$u^0 \in L^2(\Omega) \Longrightarrow \hat\upsilon(u^0) \in L^1(\Omega),$$

where $\hat\upsilon : \mathbf{R} \to \mathbf{R}$ is such that $\hat\upsilon(s) = -\int_0^s d\tau/\rho(\tau)$, $s \in \mathbf{R}$.

Now let us give a variational formulation of (1.1)–(1.7). To this end, we denote by $(\cdot,\cdot)$ both the scalar product in $H := L^2(\Omega)$ and in $(L^2(\Omega))^N$, also denoted by H, and by $|\cdot|$ the corresponding norm. For the sake of convenience, $V := H^1(\Omega)$ will be endowed with the inner product $(\!(\cdot,\cdot)\!)$, defined by

$$(\!(v_1, v_2)\!) := \int_\Omega \nabla v_1 \nabla v_2 + \gamma \int_\Gamma v_1 v_2, \quad \forall\, v_1, v_2 \in V, \tag{2.5}$$

where γ is the positive constant appearing in the boundary condition (1.2). Define $W := H^2(\Omega)$, and let us also indicate by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V. We identify H with a subspace of V', as usual, so that $\langle u, v\rangle = (u, v)$ for all $u \in H$ and for all $v \in V$.

Next, we define the Riesz isomorphism $J : V \to V'$, and the scalar product in V', respectively, by

$$\langle J v_1, v_2\rangle := (\!(v_1, v_2)\!), \quad \forall\, v_1, v_2 \in V, \tag{2.6}$$

$$(\!(w_1, w_2)\!)_* := \langle w_1, J^{-1} w_2\rangle, \quad \forall\, w_1, w_2 \in V'. \tag{2.7}$$

Let us observe that the norm in V related to the inner product defined above (which will be indicated as $\|\cdot\|$) is equivalent to the usual norm in V. Similar considerations holds also for V', and we term $\|\cdot\|_*$ the norm in V' related to the inner product (2.7).

Remark 2.1. Let us observe that the special form of α, given in (A4) and (2.4), leads to the following inequalities that will be useful in the sequel

- $\displaystyle \int_\Omega \rho(u) u = \int_\Omega \rho(u)\Big(-\frac{C_1}{\rho(u)} + C_2 \rho(u)\Big) = -C_1|\Omega| + C_2|\rho(u)|^2,$
- $\displaystyle \Big(\rho(u) - \rho(v)\Big)(u - v) = (\vartheta' - \vartheta'')\Big(\alpha(\vartheta') - \alpha(\vartheta'')\Big)$
$$= C_2(\vartheta' - \vartheta'')^2 + C_1 \frac{(\vartheta' - \vartheta'')^2}{\vartheta'\vartheta''},$$
- $\displaystyle \int_\Omega \Big(\rho(u) - \rho(v)\Big) u \geq C_2 \int_\Omega \Big(\vartheta' - \vartheta''\Big)\vartheta' - C_1 \int_\Omega \ln(\vartheta') + C_1 \int_\Omega \ln(\vartheta''),$
- $\displaystyle \int_\Omega \Big(\rho(u) - \rho(v)\Big)\Big(u - C_2\rho(u)\Big) = \int_\Omega \Big(\vartheta' - \vartheta''\Big)\Big(-\frac{C_1}{\vartheta'}\Big)$
$$\geq -C_1 \int_\Omega \ln(\vartheta') + C_1 \int_\Omega \ln(\vartheta''),$$

$$\bullet\ ((u, u - C_2\rho(u))) \geq \int_\Omega \nabla\Big(-\frac{C_1}{\vartheta'} + C_2\vartheta'\Big)\nabla\Big(-\frac{C_1}{\vartheta'}\Big) + \gamma\int_\Gamma\Big(-\frac{C_1}{\vartheta'} + C_2\vartheta'\Big)\Big(-\frac{C_1}{\vartheta'}\Big) \geq -\gamma C_1 C_2|\Gamma|, \quad \forall\, u,\, v,\, \vartheta',\, \vartheta'' \in V,$$

where $\vartheta' := \rho(u)$ and $\vartheta'' := \rho(v)$.

Let us now introduce the following spaces

$$\mathcal{V} = \Big\{v \in V, \text{ such that } \int_\Omega v = 0\Big\},$$

$$\mathcal{H} = \Big\{v \in H, \text{ such that } \int_\Omega v = 0\Big\},$$

$$\mathcal{W} = \Big\{v \in W, \text{ such that } \partial_\nu v = 0 \text{ on } \Gamma \text{ and } \int_\Omega v = 0\Big\}.$$

We may define now the operator $\mathcal{N} : \mathcal{H} \to \mathcal{W}$ that maps $v \in \mathcal{H}$ into the unique function $\mathcal{N}v \in \mathcal{W}$ such that

$$-\Delta(\mathcal{N}v) = v \quad \text{a.e. in } \Omega, \quad \text{and} \quad \partial_\nu(\mathcal{N}v) = 0 \quad \text{a.e. on } \Gamma, \quad \int_\Omega \mathcal{N}v = 0.$$

Note that any solution ϕ to

$$-\Delta\phi = v \quad \text{a.e. in } \Omega \quad \text{and} \quad \partial_\nu\phi = 0 \quad \text{a.e. on } \Gamma, \tag{2.8}$$

corresponding to a $v \in \mathcal{H}$, can be written as $\phi = \mathcal{N}v + \mu$, where μ is the mean-value of ϕ.

The operator $\mathcal{N}$ is an isomorphism and it may be extended to a new operator (always called $\mathcal{N}$) from $\mathcal{V}' := \{v \in V' : \langle v, 1\rangle = 0\}$ to $\mathcal{V}$ (note the space $\mathcal{V}'$ may not be identified with the dual space of $\mathcal{V}$), such that

$$\mathcal{N}v \in \mathcal{V}, \quad \int_\Omega \nabla\big(\mathcal{N}v\big)\nabla z = \langle v, z\rangle \quad \forall\, z \in \mathcal{V}. \tag{2.9}$$

We note that $\mathcal{N}$ is also an isomorphism from $\mathcal{V}'$ to $\mathcal{V}$, so that, for $v \in \mathcal{V}'$, the norm

$$\Big(\int_\Omega |\nabla\big(\mathcal{N}v\big)|^2\Big)^{1/2} = \langle v, \mathcal{N}v\rangle^{1/2} \tag{2.10}$$

is equivalent to the norm $\|v\|_*$ and we will use this norm, when it is convenient.

Finally, let $f \in L^2(0,T;V')$ be defined by

(2.11)
$$\langle f(t), v\rangle := \int_\Omega g(t)v + \gamma \int_\Gamma h(t)v, \quad \forall\, v \in V, \text{ and for a.e. } t \in (0,T).$$

Remark 2.2. Suppose now, as we noted in the introduction, $\alpha = \psi$ in (1.1)–(1.2). Thanks to (A4), if we set $u := \alpha(\vartheta)$, it is possible to write the term λ/ϑ in (1.4) in the form $(\lambda/C_1)\big(u - C_2\rho(u)\big)$. Indeed, the role played by (A4) is fundamental in view of the resolution of (1.1)–(1.7) and, in the following variational formulation, it is convenient to write the equations in terms of u rather than of ϑ.

Then our problem can be stated as follows.

Problem (P). *Find a pair* (ϑ, χ) *and* (w, ξ) *such that*

(2.12)
$$\vartheta \in L^2(0,T;V) \cap \mathbf{C}^0([0,T];H), \quad \vartheta > 0 \textit{ a.e. in } Q;$$

(2.13)
$$u := \alpha(\vartheta) \in L^2(0,T;V), \quad k * u \in L^2(0,T;V);$$

(2.14)
$$\chi \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;W), \qquad \chi \in D(\beta) \textit{ a.e. in } Q;$$

(2.15)
$$\xi \in L^2(0,T;H);$$

(2.16)
$$w \in L^2(0,T;V);$$

(2.17)
$$\xi \in \beta(\chi) \textit{ a.e. in } Q;$$

(2.18)
$$\partial_t\big(\rho(u) + \lambda\chi\big) + Ju + J(k * u) = f \textit{ in } V', \qquad \textit{a.e. in } (0,T);$$

(2.19)
$$\langle \partial_t \chi, v\rangle + \int_\Omega \nabla w \nabla v = 0 \quad \forall\, v \in V,\ \textit{a.e. in } (0,T);$$

(2.20)
$$\langle w, v\rangle = \int_\Omega \nabla\chi\nabla v + \Big\langle \xi + \sigma'(\chi) - \frac{\lambda}{C_1}\big(u - C_2\rho(u)\big), v\Big\rangle, \\ \forall\, v \in V,\ \textit{a.e. in } (0,T);$$

(2.21)
$$\vartheta(\cdot,0) = \vartheta^0, \qquad \chi(\cdot,0) = \chi^0, \quad \textit{a.e. in } \Omega.$$

Let us now state our main results, which will be proven in the following sections.

Theorem 2.1. *Suppose that* (A1)–(A8) *are satisfied and assume that the mean value of* χ^0 *is an interior point of* $D(\beta)$*, i.e.,*

$$m_0 := \frac{1}{|\Omega|}\,\langle \chi^0, 1\rangle \in \operatorname{int}(D(\beta)). \tag{2.22}$$

Then, Problem (P) *admits at least one solution.*

Concerning the uniqueness of solution, we have the following result.

Theorem 2.2. *Suppose that* (A1)–(A8) *and* (2.22) *are satisfied. Assume in addition that*

(A9) $$f \in W^{1,1}(0,T;V'),$$

(A10) $$u^0 \in V,$$

(A11) $$\chi^0 \in H^3(\Omega),\ \partial_\nu\chi^0 = 0$$
$$\textit{on } \Gamma,\ \exists\, \xi^0 \in V \text{ s.t. } \xi^0 \in \beta(\chi^0)\ \textit{a.e. in } \Omega,$$

then there exists a solution (ϑ,χ)*,* (w,ξ) *to Problem* (P) *satisfying the* *further regularity*

(2.23)
$$\vartheta \in H^1(0,T;H),$$
(2.24)
$$u := \alpha(\vartheta) \in L^\infty(0,T;V),$$
(2.25)
$$\chi \in W^{1,\infty}(0,T;V') \cap H^1(0,T;V) \cap L^\infty(0,T;W),$$
(2.26)
$$\xi \in L^\infty(0,T;H),$$
(2.27)
$$w \in L^\infty(0,T;V),$$

and the components ϑ and χ of such a solution are unique.

Remark 2.3. Let us observe that (A6) and (A9) are satisfied if $g \in W^{1,1}(0,T;H)$ and $h \in L^2(0,T;H^{1/2}(\Omega)) \cap W^{1,1}(0,T;H^{-1/2}(\Gamma))$. Moreover (A10) yields $\vartheta^0 = \rho(u^0) \in V$, because ρ is Lipschitz continuous.

To conclude this section, let us recall these two formulas concerning the convolution product which hold whenever they make sense, namely the identity

$$(a*b)' = a(0)b + a' * b, \tag{2.28}$$
$$a*b = 1*(a*b)', \tag{2.29}$$

and the Young theorem

$$\|a*b\|_{L^r(0,T;X)} \le \|a\|_{L^p(0,T)} \|b\|_{L^q(0,T;X)}, \tag{2.30}$$

with $1 \le p,q,r \le \infty$, $1/r = (1/p) + (1/q) - 1$, where X is a normed space.

Moreover, we account for the compact embedding of V into $L^4(\Omega)$ and H, which implies (see e.g. [**26**, p. 102])

$$|v|^2 + \|v\|^2_{L^4(\Omega)} \le \zeta \|v\|^2 + C_\zeta \|v\|^2_* \quad \forall\, v \in V, \tag{2.31}$$
$$|v|^2 \le \|v\|\,\|v\|_*, \quad \forall\, v \in V, \tag{2.32}$$

for any $\zeta > 0$ and some constant $C_\zeta > 0$.

Let us recall that, as Ω is a smooth bounded domain in $\mathbf{R}^N$, $N \le 3$, there holds

$$\|v\|_{L^p(\Omega)} \le C\|v\|, \quad \forall\, v \in V, \quad 1 \le p \le 6, \tag{2.33}$$

for some constant C depending only on Ω and p.

We widely use also the elementary inequality

$$ab \le \eta a^2 + \frac{1}{4\eta}\, b^2, \quad \forall\, a, b \in \mathbf{R} \quad \forall\, \eta > 0. \tag{2.34}$$

Let us note that we denote several constants by the same symbol C in the sequel. Their values might be different from each other even in the same formula, but they are allowed to depend only on the quantities specified in the statements.

3. Time discretization. In this section we present an implicit time discretization scheme for (2.12)–(2.21). As a first step, we prepare some results in the direction of a discrete convolution procedure.

We start by fixing a partition of the time interval $[0, T]$. To this end, we choose a constant time step $\tau := T/n$, $n \in \mathbf{N}$. Let us assume $\tau \le 1$. Our next aim is to introduce a discrete version of the convolution product in $(0, t)$, for $t \in (0, T)$. Hence, we recall (cf., e.g., [**33**]) the following:

Definition 3.1. Let $\underline{a} = \{a\}_{i=1}^n \in E^n$ and $\underline{b} = \{b_i\}_{i=1}^n \in E^n$, where E stands for a real linear space. Then we define the vector $\{(\underline{a} *_\tau \underline{b})_i\}_{i=0}^n \in E^{n+1}$ as

$$(\underline{a} *_\tau \underline{b}) := \begin{cases} 0 & \text{if } i = 0 \\ \tau \displaystyle\sum_{j=1}^{i} a_{i-j+1} b_j & \text{if } i = 1, \dots, n. \end{cases} \tag{3.1}$$

We note that an equivalent definition is the one that calls $(\underline{a} *_\tau \underline{b})_i := \tau \sum_{j=1}^{i} a_{i-j+1} b_j$ for any $i = 0, \dots, n$, with the convention, widely used in the sequel, that it is equal to zero when the sum is done on an empty set of indices. We stress also that, in the definition of $(\underline{a} *_\tau \underline{b})_i$ only the values $\{a_j\}_{j=1}^i$ and $\{b_j\}_{j=1}^i$ are involved.

Other properties of our discrete convolution product with respect to the time step τ are

$$(\underline{a} *_\tau \underline{b}) = (\underline{b} *_\tau \underline{a}), \tag{3.2}$$

$$((\underline{a} *_\tau \underline{b}) *_\tau \underline{c}) = (\underline{a} *_\tau (\underline{b} *_\tau \underline{c})). \tag{3.3}$$

Let us now introduce some convenient notations.

For the $(n+1)$-tuple $\{z_i\}_{i=0}^n \in E^{n+1}$, let the functions $\overline{z}_\tau$, z_τ : $(0,T) \to E$ be specified by

$$\overline{z}_\tau(t) := z_i, \quad z_\tau(t) := \alpha_i(t) z_i + (1 - \alpha_i(t)) z_{i-1}, \tag{3.4}$$

where $\alpha_i(t) := (t - (i-1)\tau)/\tau$, for $t \in ((i-1)\tau, i\tau]$, $i = 1, \ldots, n$. Let us also set

$$\delta z_i := \frac{z_i - z_{i-1}}{\tau}, \quad \text{for } i = 1, \ldots, n. \tag{3.5}$$

Owing to the previous notation, it is not difficult to check the following equality

$$\overline{(\underline{a} *_\tau \underline{b})}_\tau(t) = (\overline{a}_\tau * \overline{b}_\tau)(i\tau), \quad \text{for } t \in ((i-1)\tau, i\tau], \tag{3.6}$$

and $i = 1, \ldots, n$.

For the sake of reproducing a discrete version of relation (2.28), it suffices to observe that, given $\{a_i\}_{i=0}^n \in \mathbf{R}^{n+1}$ and $\{b_i\}_{i=1}^n \in E^n$, we have

$$\begin{aligned} \delta(\underline{a} *_\tau \underline{b})_i &= \sum_{j=1}^{i} a_{i-j+1} b_j - \sum_{j=1}^{i-1} a_{i-j} b_j = a_1 b_i + \sum_{j=1}^{i-1} \delta a_{i-j+1} b_j \\ &= a_1 b_i + (\underline{\delta a} *_\tau \underline{b})_i - \tau \delta a_1 b_i = a_0 b_i + (\underline{\delta a} *_\tau \underline{b})_i, \\ &\qquad \text{for } i = 1, \ldots, n. \end{aligned} \tag{3.7}$$

Finally, we state a discrete Young lemma.

Lemma 3.1. *Let $\{a_i\}_{i=1}^n \in \mathbf{R}^n$ and $\{b_i\}_{i=1}^n \in E^n$, where E denotes a linear space endowed with the norm $\|\cdot\|_E$. Then the following inequalities hold*

$$\sum_{i=1}^n \tau \|(\underline{a} *_\tau \underline{b})_i\|_E \le \left(\sum_{i=1}^n \tau |a_i|\right)\left(\sum_{i=1}^n \tau \|b_i\|_E\right), \tag{3.8}$$

$$\left(\sum_{i=1}^n \tau \|(\underline{a} *_\tau \underline{b})_i\|_E^2\right)^{1/2} \le \left(\sum_{i=1}^n \tau |a_i|\right)\left(\sum_{i=1}^n \tau \|b_i\|_E^2\right)^{1/2}, \tag{3.9}$$

$$\|(\underline{a} *_\tau \underline{b})_i\|_E^2 \le \sum_{j=1}^n \tau |a_i|^2 \sum_{j=1}^i \tau \|b_j\|_E^2, \quad \text{for } i = 1, \dots, n. \tag{3.10}$$

For a proof of the two first inequalities, see, e.g., [**33**]. Instead, (3.10) follows from definition (3.1) and elementary properties of the sums.

Let us note that, given a real vector $\{k_i\}_{i=0}^n$ and a vector $\{\sigma_i\}_{i=1}^n \in E^n$, where E stands for a normed space, and according to the definitions (3.1), (3.4)–(3.5), we have that

$$\overline{k}_\tau * \overline{\sigma}_\tau \quad \text{is a piecewise linear continuous function.} \tag{3.11}$$

Indeed, in view of (3.4) it is a standard matter to check that

$$(\overline{k}_\tau * \overline{\sigma}_\tau)(t) = \alpha_i(t)(\underline{k} *_\tau \underline{\sigma})_i + (1 - \alpha_i(t))(\underline{k} *_\tau \underline{\sigma})_{i-1}, \\ \text{for } t \in ((i-1)\tau, i\tau], \text{ and } i = 1, \dots, n.$$

Now it is worth introducing our approximation of equations (2.18)–(2.21). Let us set

$$k_i := k(i\tau), \quad \text{for } i = 0, \dots, n, \tag{3.12}$$

whence we may say that, thanks to (A5), we have

$$\|\overline{\delta k}_\tau\|_{L^1(0,T)} = \sum_{i=1}^n \tau |\delta k_i| \le \|k'\|_{L^1(0,T)}, \tag{3.13}$$

where k' stands for the time derivative of k and

$$\|k-\overline{k}_\tau\|_{L^1(0,T)} \le \tau \mathrm{Var}_{[0,T];\mathbf{R}}[k],$$

where $\mathrm{Var}_{[0,T];\mathbf{R}}[k]$ denotes the total variation on the interval $[0,T]$ in $\mathbf{R}$ of the function k (see e.g. [**2**]).

Moreover, let us recall [**33**, Proposition 4.4] and state it for the reader's convenience.

Lemma 3.2. *Let* (A5) *hold and* $\{\sigma_i\}_{i=1}^n \in E^n$ *where* E *denotes a linear space endowed with the norm* $\|\cdot\|_E$. *Moreover, let* $\{k_i\}_{i=0}^n$, $\overline{\sigma}_\tau$, *and* $\{(\underline{k} *_\tau \underline{\sigma})_i\}_{i=1}^n$ *be defined as in* (3.12), (3.4) *and* (3.1), *respectively. Then, it holds*

$$\begin{aligned}\|\overline{(\underline{k} *_\tau \underline{\sigma})}_\tau - k * \overline{\sigma}_\tau\|_{L^1(0,T;E)}& \\ \le \tau\big(2\mathrm{Var}_{[0,T];\mathbf{R}}[k] &+ \|k\|_{L^\infty(0,T)}\big)\|\overline{\sigma}_\tau\|_{L^1(0,T;E)}.\end{aligned}$$

Regarding f, we set

$$f_i := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} f(t)\,dt \in V', \quad \text{for } i = 1\ldots,n. \tag{3.15}$$

Note that

$$\|\overline{f}_\tau\|_{L^2(0,T;V')} \le \|f\|_{L^2(0,T;V')}. \tag{3.16}$$

Then, the approximation scheme may be formulated by making use of an auxiliary unknown $\xi_i = \beta_\tau(\chi_i)$, where

(3.17)

$$\beta_\tau,\ \text{for } \tau>0,\ \text{ is the Yosida approximation of } \beta,\ \text{with constant } \tau^{1/4},$$

so that β_τ is Lipschitz continuous with constant $\tau^{-1/4}$.

Then, the approximated problem takes the form

(3.18)

$$\tau^{1/4}\,\frac{u_i-u_{i-1}}{\tau} + \frac{\rho(u_i)-\rho(u_{i-1})}{\tau} + \lambda\frac{\chi_i-\chi_{i-1}}{\tau} + Ju_i + J(\underline{k} *_\tau \underline{u})_i = f_i, \quad \text{in } V', \quad \text{for } i = 1,\ldots,n;$$

$$\Big\langle \frac{\chi_i - \chi_{i-1}}{\tau}, v\Big\rangle + \int_\Omega \nabla w_i \nabla v = 0, \qquad (3.19)$$
$$\forall\, v \in V, \quad \text{for } i = 1, \dots, n;$$

(3.20)
$$\langle w_i, v\rangle = \int_\Omega \nabla \chi_i \nabla v + \int_\Omega \xi_i v + \int_\Omega \sigma'(\chi_i) v - \frac{\lambda}{C_1}\int_\Omega \big(u_i - C_2 \rho(u_i)\big) v,$$
$$\forall\, v \in V, \quad \text{for } i = 1, \dots, n;$$

$$\xi_i = \beta_\tau(\chi_i), \quad \text{for } i = 1, \dots, n; \qquad (3.21)$$

$$u_0 = u^0, \qquad \chi_0 = \chi^0. \qquad (3.22)$$

Next we state and prove an existence and uniqueness result for the solution to scheme (3.18)–(3.22).

Theorem 3.1. *Let assumptions* (A1)–(A8) *and* (3.15) *hold, and let the time step τ be small enough. Then a unique quadruplet of vectors $\{\vartheta_i, \chi_i, w_i, \xi_i\}_{i=0}^n \in H^{4(n+1)}$ exist, which fulfill relations* (3.18)–(3.22).

In order to give a proof of this theorem, let us introduce some notations. Let

(3.23)
$$g_i := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} g(t)\, dt \in H, \quad \text{for } i = 1, \dots, n;$$

(3.24)
$$h_i := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} h(t)\, dt \in H^{1/2}(\Gamma), \quad \text{for } i = 1, \dots, n,$$

where g and h are the same as in (A6).

Let us now rewrite equation (3.18) in the following form

$$A_i(u_i) = -\lambda \chi_i, \quad \text{a.e. in } \Omega, \quad \forall\, i = 1, \dots, n,$$

where the operator A_i, for $i = 1, \dots, n$, is defined as follows

$$\begin{aligned} A_i(u) = {} & \tau^{1/4} u + \rho(u) - \tau \Delta u - \tau^2 k_1 \Delta u - \tau^2 \Delta s_i \\ & - \tau g_i - \tau^{1/4} u_{i-1} - \rho(u_{i-1}) - \lambda \chi_{i-1}, \end{aligned}$$

for all $u \in D(A_i)$, where $s_i = \sum_{j=1}^{i-1} k_{i-j+1} u_j$, and

$$D(A_i) = \{u \in W : -\partial_\nu(u+\tau k_1 u) = \gamma(u+\tau k_1 u - h_i^*), \text{ a.e. on } \Gamma\}, \quad \text{with} \quad h_i^* = h_i - \frac{\tau}{\gamma}\,\partial_\nu s_i - \tau s_i. \tag{3.25}$$

Note that we will prove in the sequel that A_i^{-1} is a well-defined and Lipschitz continuous operator (see Remarks 3.1–3.2). So we can say that (3.18) is satisfied if and only if

$$u_i = A_i^{-1}(-\lambda\chi_i), \quad \forall\, i = 1, \dots, n.$$

Now, multiplying (3.19) by τ, using (3.20) and applying to the resulting equation the operator $\mathcal{N}$ defined in (2.9), we obtain that (3.18)–(3.22) are satisfied if and only if we find $\chi_i \in H$ such that

$$\mathcal{N}(\chi_i - \chi_{i-1}) - \tau\Delta\chi_i + \tau\beta_\tau(\chi_i) + \tau G_i(\chi_i) + \tau E_i(\chi_i) = \tau\mu_i(\chi_i), \quad \text{a.e. in } \Omega \quad \text{and} \quad \forall\, i = 1, \dots, n; \tag{3.26}$$

$$\partial_\nu\chi_i = 0, \qquad \int_\Omega \chi_i = \int_\Omega \chi_{i-1}, \quad \forall\, i = 1, \dots, n; \tag{3.27}$$

$$\mu_i(\chi) := \frac{1}{|\Omega|}\int_\Omega \big(\beta_\tau(\chi) + E_i(\chi) + G_i(\chi)\big)\,dx, \quad \forall\, i = 1, \dots, n, \tag{3.28}$$

where the constants C_1, C_2 and λ are the same as in (A3)–(A4),

$$E_i(\chi) = -\frac{\lambda}{C_1}\,A_i^{-1}(-\lambda\chi), \quad \forall\, \chi \in H, \tag{3.29}$$

and $G_i : H \to H$ is the Lipschitz mapping

$$G_i(\chi) = \frac{C_2}{C_1}\,\lambda\rho(A_i^{-1}(-\lambda\chi)) + \sigma'(\chi), \quad \forall\, \chi \in H.$$

Suppose now that

$$\tau k_1 > -1, \tag{3.30}$$

note that it is always true for τ sufficiently small.

Now we can see that A_i is a maximal monotone operator. Indeed, A_i is the subdifferential of a convex function $\phi : H \to (-\infty, +\infty]$, defined by conditions

$$\phi(u) = \begin{cases} \widehat{\rho}(u) + \dfrac{\tau^{1/4}}{2} \displaystyle\int_\Omega |u|^2 \, dx \\ \quad + \dfrac{\tau + \tau^2 k_1}{2} \Big(\displaystyle\int_\Omega |\nabla u|^2 \, dx + \gamma \int_\Gamma u^2 \, d\sigma \Big) \\ \quad - \displaystyle\int_\Omega g_i^* u \, dx - \gamma \int_\Gamma h_i^* u \, d\sigma & \text{if } u \in V \\ +\infty & \text{otherwise,} \end{cases}$$

where $\widehat{\rho} : \mathbf{R} \to \mathbf{R}$ is defined by $\widehat{\rho}(s) = \int_0^s \rho(r) \, dr$, for all $s \in \mathbf{R}$ and $g_i^* = \tau^2 s_i + \tau g_i + \tau^{1/4} u_{i-1} + \rho(u_{i-1}) + \lambda \chi_{i-1}$.

Note that h_i^* has been defined in (3.25). Note also that this map ϕ is convex thanks to the hypothesis (3.30).

Let us now give one preliminary lemma and some remarks that are needed to prove Theorem 3.1. Henceforth, let C denote any constant dependent on the data, but not on the time step τ. Of course, C may vary from line to line. Symbols like C_ε denote constants which may also depend on ε.

Lemma 3.3. *The operator A_i is coercive in H, i.e.,*

$$\lim_{|u| \to +\infty} \frac{\int_\Omega (A_i u) u \, dx}{|u|} = +\infty; \tag{3.31}$$

moreover, it is also injective.

Proof. Let $u \in D(A_i)$ be arbitrary. Applying (3.1), the definition of A_i and the hypothesis (3.30), using also (2.34), the first equality in

Remark 2.1 and (A6), we can see that

$$\begin{aligned}\int_\Omega (A_i u)u\,dx &= \tau^{1/4}|u|^2 - C_1|\Omega| + C_2|\rho(u)|^2 \\ &\quad + \tau(|\nabla u|^2 + \tau k_1|\nabla u|^2) \\ &\quad + \tau\int_\Gamma \Big[\gamma(u^2+\tau k_1 u^2) - \gamma(1+\tau k_1)h_i u\Big]\,d\sigma \\ &\quad - \int_\Omega (\lambda\chi_{i-1} + \rho(u_{i-1}) \\ &\quad + \tau^2\Delta s_i + \tau g_i + \tau^{1/4}u_{i-1})u\,dx \\ &\geq (\tau+\tau^2 k_1)|\nabla u|^2 + \tau\gamma(1+\tau k_1)\|u\|^2_{L^2(\Gamma)} \\ &\quad - \frac{\tau\gamma}{2}(1+\tau k_1)\|h_i\|^2_{L^2(\Gamma)} \\ &\quad - \frac{\tau\gamma}{2}(1+\tau k_1)\|u\|^2_{L^2(\Gamma)} - C_1|\Omega| \\ &\quad - |\rho(u_{i-1}) + \tau g_i + \lambda\chi_{i-1} + \tau^2\Delta s_i + \tau^{1/4}u_{i-1}||u|.\end{aligned}$$

It follows that

$$\int_\Omega (A_i u)u\,dx \geq \frac{\tau}{2}\big(1+\tau k_1\big)\|u\|^2 - C(1+|u|), \quad \forall\, u\in D(A_i),$$

which implies (3.31).

To prove now the injectivity of A_i, let us assume $u, v \in D(A_i)$, with $A_i u = A_i v$. Then, applying the second equality in Remark 2.1, (3.21) and (3.30), we get

$$\begin{aligned}0 &= \int_\Omega (A_i u - A_i v)(u-v)\,dx \\ &\geq \tau^{1/4}\int_\Omega |u-v|^2\,dx \\ &\quad + C_1\int_\Omega \frac{\big(\rho(u)-\rho(v)\big)^2}{\rho(u)\rho(v)}\,dx + C_2\int_\Omega \big(\rho(u)-\rho(v)\big)^2\,dx \\ &\quad + \tau\big(1+\tau k_1\big)\Big(|\nabla(u-v)|^2 + \gamma\int_\Gamma (u-v)^2\,d\sigma\Big).\end{aligned} \tag{3.32}$$

Now, thanks to the definition of $D(A_i)$, we immediately get $u = v$, and the proof of Lemma 3.3 is completed.

Now let us give some remarks that will be useful in the sequel.

Remark 3.1. With the previous considerations, we have obtained that A_i is a maximal-monotone, one-to-one, coercive operator. Hence, also E_i is a maximal monotone operator from H to H.

Remark 3.2. The operator E_i, defined in (3.29) is also Lipschitz continuous (of constant $(\lambda^2/C_1\tau^{1/4})$). Indeed, for all $u, v \in H$ such that $u = E_i(x)$ and $v = E_i(y)$, with $x, y \in H$, applying (3.32), we have

$$\begin{aligned} C_1|u-v||x-y| &\geq C_1\langle u-v, x-y\rangle \\ &= -\frac{C_1}{\lambda}\left\langle u-v, A_i\left(-\frac{C_1}{\lambda}u\right) - A_i\left(-\frac{C_1}{\lambda}\,v\right)\right\rangle \\ &\geq \tau^{1/4}\,\frac{C_1^2}{\lambda^2}\,|u-v|^2. \end{aligned}$$

Note that in the previous inequality the choice (3.25) for the domain $D(A_i)$ is crucial.

Moreover, also the above defined function G_i is Lipschitz, with constant $(C/\tau^{1/4}) + C_{\sigma'}$, where $C_{\sigma'}$ is the Lipschitz constant of σ'.

Remark 3.3. The operator $\mathcal{N} : \mathcal{H} \to \mathcal{H}$ defined in (2.9) is obviously a monotone and continuous operator. Consequently, also the operator

$$\mathcal{N}_i : \{u \in H : u - \chi_{i-1} \in \mathcal{H}\} \longrightarrow \mathcal{W}$$

$$u \longmapsto \mathcal{N}(u - \chi_{i-1})$$

is continuous and monotone.

Now we are ready to give the

Proof of Theorem 3.1. Let L_i be the operator defined as

$$\begin{aligned} &L_i(u) = \mathcal{N}_i(u) - \tau\Delta u + \tau E_i(u), \quad \text{with domain} \\ &D(L_i) = D(\mathcal{N}_i) \cap \{v \in W \text{ such that } \partial_\nu v = 0 \text{ on } \Gamma\}. \end{aligned}$$

Since the operator $\mathcal{N}_i - \tau\Delta$ is maximal monotone in $D(L_i)$, thanks to Lemma 3.3, Remark 3.1 and [**1**, p. 46], we can say that L_i is a maximal monotone operator.

Moreover, applying [**1**, p. 48], the operator L_i is surjective in H because it is coercive on H. Indeed, applying the Poincaré inequality, (2.10), (2.31) and using the monotonicity of E_i, we get

$$\begin{aligned}\langle L_i(u) - L_i(v), u - v\rangle &= \langle \mathcal{N}(u-v), u-v\rangle + \tau\langle E_i(u) - E_i(v), u - v\rangle \\ &\quad - \tau\langle -\Delta(u-v), u-v\rangle \\ &\ge \|u-v\|_*^2 + C\tau\|u-v\|^2 \\ &\ge C\tau^{1/2}|u-v|^2, \quad \forall\, u, v \in D(L_i),\end{aligned}$$

whence the coerciveness and the injectivity of L_i follows immediately. So, the operator $L_i^{-1} : H \to H$ is well defined. Now, we can see that it is Lipschitz continuous of constant $C\tau^{-1/2}$. Indeed, for all $x, y \in D(L_i)$, if $u = L_i(x)$, $v = L_i(y)$, using the Poincaré inequality (note that $x-y$ is a zero mean-valued function), (2.10), (2.32), (2.34), and the monotonicity of E_i, we have

$$\begin{aligned}|u-v||x-y| \ge (u-v, x-y) &= (\mathcal{N}_i(x) - \mathcal{N}_i(y), x-y) \\ &\quad - \tau(-\Delta x + \Delta y, x-y) + \tau(E_i(x) - E_i(y), x-y) \\ &\ge \|x-y\|_*^2 + C\tau\|x-y\|^2 \\ &\ge 2C\tau^{1/2}|x-y|^2,\end{aligned}$$

whence it comes immediately that L_i^{-1} is Lipschitz continuous, with constant $C/\tau^{1/2}$.

Now it is possible to define the operator $S : H \to H$, which maps χ into the unique solution $S(\chi) \in H$ to the equation

$$L_i\big(S(\chi)\big) \ni \tau\mu_i(\chi) - \tau\big(\beta_\tau(\chi) + G_i(\chi)\big),$$

where μ_i is the function defined in (3.28). In order to apply the contraction mapping principle to S, we let $\chi_1, \chi_2 \in H$. Then, by virtue of the previous considerations and the Lipschitz continuity of G_i, μ_i, and of β_τ (defined in (3.17)) we have

$$\begin{aligned}|S(\chi_1) - S(\chi_2)| &\le C\tau^{1/2}\Big(|\mu_i(\chi_1) - \mu_i(\chi_2)| + |\beta_\tau(\chi_1) - \beta_\tau(\chi_2)| \\ &\qquad\qquad + |G_i(\chi_1) - G_i(\chi_2)|\Big) \\ &\le C\tau^{1/4}|\chi_1 - \chi_2| + C_{\sigma'}\tau^{1/2}|\chi_1 - \chi_2| \le C_S|\chi_1 - \chi_2|,\end{aligned}$$

with $0 < C_S < 1$, for τ sufficiently small. Thus, S turns out to be a contraction mapping on H, whence Theorem 3.1 follows from the contraction mapping principle.

Now, for the sake of clarity, and due to the last Theorem 3.1, we may rewrite (3.18)–(3.22) as follows

$$\tau^{1/4}\partial_t u_\tau + \partial_t \rho(u_\tau) + \lambda \partial_t \chi_\tau + J\overline{u}_\tau + J\overline{(\underline{k} *_\tau \underline{u})}_\tau = \overline{f}_\tau, \quad \text{in } V', \text{ a.e. in } (0,T); \tag{3.33}$$

$$\langle \partial_t \chi_\tau, v\rangle + \int_\Omega \nabla \overline{w}_\tau \nabla v = 0, \quad \forall\, v \in V, \text{ a.e. in } (0,T); \tag{3.34}$$

$$\begin{aligned}\langle \overline{w}_\tau, v\rangle = \int_\Omega \nabla \overline{\chi}_\tau \nabla v + \int_\Omega \overline{\xi}_\tau v \\ + \int_\Omega \sigma'(\overline{\chi}_\tau) v - \frac{\lambda}{C_1}\int_\Omega \big(\overline{u}_\tau - C_2\rho(\overline{u}_\tau)\big) v, \\ \forall\, v \in V, \quad \text{a.e. in } (0,T);\end{aligned} \tag{3.35}$$

$$\overline{\xi}_\tau = \beta_\tau(\overline{\chi}_\tau), \quad \overline{u}_\tau(0) = u^0, \quad \overline{\chi}_\tau(0) = \chi^0 \quad \text{a.e. in } \Omega, \tag{3.36}$$

where the notations (3.4) are taken into account.

4. Existence. This section concludes the proof of Theorem 2.1. Let us first give a lemma that will be useful in the sequel.

Lemma 4.1. *Given $a, b > 0$, a positive constant C exists such that*

$$\frac{a}{2}\, s^2 + b|\ln s| \le as^2 - b\ln s + C, \quad \forall\, s > 0, \tag{4.1}$$

$$\frac{a}{2}\, s^2 + b|\widehat{\upsilon}(s)| \le as^2 + b\widehat{\upsilon}(s) + C, \quad \forall\, s \in \mathbf{R}, \tag{4.2}$$

where $\widehat{\upsilon} : \mathbf{R} \to \mathbf{R}$ is such that $\widehat{\upsilon}(s) = -\int_0^s d\tau/\rho(\tau)$, $s \in \mathbf{R}$.

Proof. The first inequality is obvious; the second is due to the properties of the function ρ, cf. (2.4). Indeed, taking (A4) into account, we have that $\rho(s) = (s + \sqrt{s^2 + 4C_1C_2}/2C_2)$, and so $\upsilon := -1/\rho$ is of the form $-C'/\big(s + \sqrt{s^2 + C''}\big)$ for some positive constants C', C''. So it follows immediately that the function $\widehat{\upsilon}$ defined above satisfies (4.2).

Now, in view of giving some boundedness estimates, uniform with respect to τ, let us state a discrete integration by parts formula (whose proof is obvious): let $\{a_i\}_{i=0}^n \in \mathbf{R}^{n+1}$ and $\{b_i\}_{i=1}^n \in \mathbf{R}^n$; then, the following equality holds

$$\sum_{i=1}^{n}(a_i - a_{i-1})b_i = a_n b_n - a_0 b_1 - \sum_{i=1}^{n-1} a_i(b_{i+1} - b_i). \tag{4.3}$$

First estimate. Multiplying equation (3.18) by a positive constant ε and then testing it by τu_i, one can get

$$\varepsilon\tau^{1/4}\int_\Omega (u_i - u_{i-1})u_i + \varepsilon\int_\Omega \big(\rho(u_i) - \rho(u_{i-1})\big)u_i \\ + \varepsilon\lambda\langle \chi_i - \chi_{i-1}, u_i\rangle + \varepsilon\tau\|u_i\|^2 = \varepsilon\,\tau\langle f_i, u_i\rangle - \varepsilon\tau(\!((\underline{k} *_\tau \underline{u})_i, u_i)\!).$$

We can observe that, by the third inequality in Remark 2.1, we have that

$$\varepsilon\int_\Omega \big(\rho(u_i) - \rho(u_{i-1})\big)u_i \geq \varepsilon C_2\int_\Omega \big(\rho(u_i) - \rho(u_{i-1})\big)\rho(u_i) \\ - \varepsilon C_1\int_\Omega \ln(\rho(u_i)) + \varepsilon C_1\int_\Omega \ln(\rho(u_{i-1})). \tag{4.5}$$

Now, using (2.34), the Schwarz inequality, and (3.5), we get

$$-\lambda\varepsilon\tau\langle \delta\chi_i, u_i\rangle \leq \frac{\varepsilon\lambda^2}{4\zeta}\,\tau\|\delta\chi_i\|_*^2 + \varepsilon\zeta\tau\|u_i\|^2, \quad \forall\,\zeta > 0. \tag{4.6}$$

Moreover, thanks to (3.10), (2.34) and the Schwarz inequality, we can also say that

$$-\varepsilon\tau(\!((\underline{k} *_\tau \underline{u})_i, u_i)\!) \leq \varepsilon\frac{\tau}{4}\|u_i\|^2 + \varepsilon\tau\sum_{j=1}^{n}\tau|k_j|^2\sum_{j=1}^{i}\tau\|u_j\|^2. \tag{4.7}$$

Now, taking (4.7)–(4.7) into account, summing up (4.4) for $i = 1, \dots, m$, and using Lemma 4.1, we have

$$\begin{aligned}
&\frac{\varepsilon\tau^{1/4}}{2}\sum_{i=1}^{m}|u_i-u_{i-1}|^2+\frac{\varepsilon\tau^{1/4}}{2}\,|u_m|^2+\frac{C_2\varepsilon}{2}\sum_{i=1}^{m}|\rho(u_i)-\rho(u_{i-1})|^2\\
&\quad+\frac{C_2\varepsilon}{4}\,|\rho(u_m)|^2+\varepsilon C_1\|\ln(\rho(u_m))\|_{L^1(\Omega)}+\frac{\varepsilon}{4}\sum_{i=1}^{m}\tau\|u_i\|^2\\
&\le C\varepsilon+\frac{\varepsilon\tau^{1/4}}{2}\,|u^0|^2+\frac{C_2\varepsilon}{4}\,|\rho(u^0)|^2+\varepsilon C_1\|\ln(\rho(u^0))\|_{L^1(\Omega)}\\
&\quad+\frac{\varepsilon}{2}\sum_{i=1}^{m}\tau\|f_i\|_*^2+\frac{\varepsilon\lambda^2}{4\zeta}\sum_{i=1}^{m}\tau\|\delta\chi_i\|_*^2+\varepsilon\zeta\sum_{i=1}^{m}\tau\|u_i\|^2\\
&\quad+\varepsilon\tau\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\tau|k_j|^2\sum_{j=1}^{i}\tau\|u_j\|^2\right),\\
&\qquad\qquad\forall\, m=1,\dots,n,\quad\text{and}\quad\forall\,\zeta>0.
\end{aligned}$$

Second estimate. Testing (3.18) by $\tau\big(u_i - C_2\rho(u_i)\big)$, we can get

$$\begin{aligned}
&\tau^{1/4}\langle u_i-u_{i-1},u_i-C_2\rho(u_i)\rangle+\langle\rho(u_i)-\rho(u_{i-1}),u_i-C_2\rho(u_i)\rangle\\
&\quad+\lambda\langle\chi_i-\chi_{i-1},u_i-C_2\rho(u_i)\rangle+\tau(\!(u_i,u_i-C_2\rho(u_i))\!)\\
&\quad+\tau(\!((\underline{k}*_\tau\underline{u})_i,u_i-C_2\rho(u_i))\!)=\tau\langle f_i,u_i-C_2\rho(u_i)\rangle.
\end{aligned}$$

First, thanks to (2.4), the function $\upsilon := -1/\rho : \mathbf{R} \to [0, +\infty)$ is the subdifferential of a convex function $\widehat{\upsilon} : \mathbf{R} \to \mathbf{R}$, cf. Lemma 4.1, so that

(4.10)
$$\begin{aligned}
\tau^{1/4}\langle u_i-u_{i-1},u_i-C_2\rho(u_i)\rangle&=\tau^{1/4}\Big\langle u_i-u_{i-1},-\frac{C_1}{\rho(u_i)}\Big\rangle\\
&\ge C_1\tau^{1/4}\int_\Omega\widehat{\upsilon}(u_i)-C_1\tau^{1/4}\int_\Omega\widehat{\upsilon}(u_{i-1}).
\end{aligned}$$

Then, thanks to the fourth inequality in Remark 2.1, we have

$$\begin{aligned}
&\langle\rho(u_i)-\rho(u_{i-1}),u_i-C_2\rho(u_i)\rangle\\
&\qquad\ge -C_1\int_\Omega\ln(\rho(u_i))+C_1\int_\Omega\ln(\rho(u_{i-1})),
\end{aligned}\tag{4.11}$$

and, by the fifth inequality in Remark 2.1, we get

$$- \tau ((u_i, u_i - C_2 \rho(u_i))) \le \tau \, \gamma \, C_1 C_2 |\Gamma|. \tag{4.12}$$

Now, by (2.4), using the Schwarz inequality and (2.34), we can obtain

$$\begin{aligned} \tau \langle f_i, u_i - C_2 \rho(u_i) \rangle &\le \tau \|f_i\|_* \|u_i - C_2 \rho(u_i)\| \\ &\le \tau C_{\varepsilon,\zeta} \|f_i\|_*^2 + \tau \varepsilon \zeta \|u_i - C_2 \rho(u_i)\|^2 \\ &\le \tau C_{\varepsilon,\zeta} \|f_i\|_*^2 + 2\varepsilon\zeta\tau (\|u_i\|^2 + C_2^2 |\rho(0)|^2), \\ &\forall \, \zeta > 0, \quad \text{and for some } C_{\varepsilon,\zeta} > 0. \end{aligned} \tag{4.13}$$

Moreover, taking (2.4) into account and using (3.10) with the Schwarz inequality and (2.34), we have

$$\begin{aligned} &- \tau (((\underline{k} *_\tau \underline{u})_i, \, u_i - C_2 \rho(u_i))) \\ &\qquad \le 2\varepsilon\tau\zeta (\|u_i\|^2 + C_2^2 |\rho(0)|^2) + \frac{\tau C_\varepsilon}{4\zeta} \sum_{j=1}^{n} \tau |k_j|^2 \sum_{j=1}^{i} \tau \|u_j\|^2, \\ &\qquad \forall \, \zeta > 0 \quad \text{and for some } C_\varepsilon > 0. \end{aligned}$$

Now, thanks to (4.11)–(4.14), summing up (4.9) for $i = 1, \dots, m$, and applying Lemma 4.1, we get

$$\begin{aligned} &\lambda \sum_{i=1}^{m} \langle \chi_i - \chi_{i-1}, u_i - C_2 \rho(u_i) \rangle + C_1 \tau^{1/4} \|\widehat{v}(u_m)\|_{L^1(\Omega)} \\ &\quad + C_1 \| \ln(\rho(u_m)) \|_{L^1(\Omega)} - \varepsilon' \Big(\int_\Omega |\rho(u_m)|^2 + \tau^{1/4} \int_\Omega |u_m|^2 \Big) \\ &\le C(1 + \tau^{1/4}) + m\tau\gamma C_1 C_2 |\Gamma| + 4m C_2^2 |\rho(0)|^2 \tau\varepsilon\zeta + C_1 \tau^{1/4} \|\widehat{v}(u^0)\|_{L^1(\Omega)} \\ &\quad + C_1 \| \ln(\rho(u^0)) \|_{L^1(\Omega)} - \varepsilon' \Big(\int_\Omega |\rho(u^0)|^2 + \int_\Omega \tau^{1/4} |u^0|^2 \Big) \\ &\quad + C_{\varepsilon,\zeta} \sum_{i=1}^{m} \tau \|f_i\|_*^2 + 4\varepsilon\zeta \sum_{i=1}^{m} \tau \|u_i\|^2 + \frac{C_\varepsilon \tau}{4\zeta} \sum_{i=1}^{m} \Big(\sum_{j=1}^{n} \tau |k_j|^2 \sum_{j=1}^{i} \tau \|u_j\|^2 \Big), \\ &\qquad \forall \, m = 1, \dots, n, \quad \forall \, \zeta, \, \varepsilon' > 0, \quad \text{and for some } C_{\varepsilon,\zeta}, \, C_\varepsilon > 0. \end{aligned} \tag{4.15}$$

Third estimate. Multiplying equations (3.19) and (3.20) by C_1, testing the former by $\mathcal{N}(\chi_i - \chi_{i-1})$ (indeed, taking $v = 1$ in (3.19), we see that $(\chi_i - \chi_{i-1})$ satisfies the null-average condition) and the latter by $-(\chi_i - \chi_{i-1})$, summing up, using (2.10), and taking (A1) into account, we get

(4.16)
$$\begin{aligned} &C_1\tau\|\delta\chi_i\|_*^2 + \frac{C_1}{2}\int_\Omega |\nabla\chi_i - \nabla\chi_{i-1}|^2 + \frac{C_1}{2}\int_\Omega |\nabla\chi_i|^2 \\ &- \frac{C_1}{2}\int_\Omega |\nabla\chi_{i-1}|^2 - \lambda\int_\Omega u_i(\chi_i - \chi_{i-1}) + C_2\lambda\int_\Omega \rho(u_i)(\chi_i - \chi_{i-1}) \\ &+ C_1\|\widehat{\beta}_\tau(\chi_i)\|_{L^1(\Omega)} - C_1\|\widehat{\beta}_\tau(\chi_{i-1})\|_{L^1(\Omega)} \le -\tau\int_\Omega \sigma'(\chi_i)\delta\chi_i. \end{aligned}$$

Now, thanks to (A2), using the Schwarz inequality and (2.34), we can see that

(4.17)
$$\begin{aligned} -\tau\int_\Omega \sigma'(\chi_i)\delta\chi_i &\le \tau\,\frac{C_1}{4}\|\delta\chi_i\|_*^2 + \frac{\tau}{C_1}\|\sigma'(\chi_i)\|^2 \\ &\le \tau\,\frac{C_1}{4}\|\delta\chi_i\|_*^2 + 2\tau(|\Omega| + \gamma|\Gamma|)|\sigma'(0)|^2 + 2\tau\,\frac{C_{\sigma'}^2}{C_1}\|\chi_i\|^2. \end{aligned}$$

Summing up (4.16) for $i = 1, \ldots, m$, using (4.17), adding to both sides

$$\frac{C_1\gamma}{2}\|\chi_m\|_{L^2(\Gamma)}^2 = \frac{C_1\gamma}{2}\|\chi^0\|_{L^2(\Gamma)}^2 + C_1\gamma\tau\sum_{i=1}^m\int_\Gamma \delta\chi_i\,\chi_i$$

in order to recover the full V norm on the lefthand side, using also the Schwarz inequality and (2.34), we can get

$$\begin{aligned} &\frac{3C_1}{8}\sum_{i=1}^m \tau\|\delta\chi_i\|_*^2 + C\sum_{i=1}^m \|\chi_i - \chi_{i-1}\|^2 + \frac{C_1}{2}\|\chi_m\|^2 \\ &\quad - \lambda\sum_{i=1}^m\int_\Omega u_i(\chi_i - \chi_{i-1}) + C_2\lambda\sum_{i=1}^m\int_\Omega \rho(u_i)(\chi_i - \chi_{i-1}) \\ &\quad + C_1\|\widehat{\beta}_\tau(\chi_m)\|_{L^1(\Omega)} \\ &\le 2m(|\Omega| + \gamma|\Gamma|)\,|\sigma'(0)|^2\tau + C_1\|\widehat{\beta}_\tau(\chi^0)\|_{L^1(\Omega)} + C\|\chi^0\|^2 \\ &\quad + C'\tau\sum_{i=1}^m \|\chi_i\|^2, \quad \forall\, m = 1, \ldots, n, \end{aligned} \tag{4.18}$$

for some positive constant C, C'. Note that we can write here the V full norm of $\chi_i - \chi_{i-1}$, because it satisfies the null-average condition.

Summing up now the three estimates (4.8), (4.15), (4.18), and choosing $\zeta = 1/16$ and a suitable ε', thanks also to (A5), we can obtain

$$
\begin{aligned}
&\frac{3C_1}{8}\sum_{i=1}^m \tau\|\delta\chi_i\|_*^2 + C\sum_{i=1}^m \|\chi_i - \chi_{i-1}\|^2 + \frac{C_1}{2}\|\chi_m\|^2 \\
&+ C_1\|\widehat{\beta}_\tau(\chi_m)\|_{L^1(\Omega)} + \frac{\varepsilon\tau^{1/4}}{2}\sum_{i=1}^m |u_i - u_{i-1}|^2 + \frac{\varepsilon\tau^{1/4}}{2}|u_m|^2 \\
&+ C_1\tau^{1/4}\|\widehat{\upsilon}(u_m)\|_{L^1(\Omega)} + \frac{C_2\varepsilon}{2}\sum_{i=1}^m |\rho(u_i) - \rho(u_{i-1})|^2 \\
&+ C_2\varepsilon|\rho(u_m)|^2 + C_1(1+\varepsilon)\|\ln(\rho(u_m))\|_{L^1(\Omega)} + \frac{\varepsilon}{16}\sum_{i=1}^m \tau\|u_i\|^2 \\
\le{}& C(1+\varepsilon+\tau^{1/4}) + n\tau\Big[\gamma C_1 C_2|\Gamma| + \frac{\varepsilon C_2^2|\rho(0)|^2}{4} \\
&\qquad + 2(|\Omega| + \gamma|\Gamma|)|\sigma'(0)|^2\Big] + C_1\|\widehat{\beta}_\tau(\chi^0)\|_{L^1(\Omega)} \\
&+ C\|\chi^0\|^2 + \frac{C_2\varepsilon}{2}|\rho(u^0)|^2 + C_1(1+\varepsilon)\|\ln(\rho(u^0))\|_{L^1(\Omega)} \\
&+ \frac{\varepsilon\tau^{1/4}}{2}|u^0|^2 + C_1\tau^{1/4}\|\widehat{\upsilon}(u^0)\|_{L^1(\Omega)} \\
&+ \frac{\varepsilon}{2}\sum_{i=1}^m \tau\|f_i\|_*^2 + C_\varepsilon\sum_{i=1}^m \tau\|f_i\|_*^2 + 4\varepsilon\lambda^2\sum_{i=1}^m \tau\|\delta\chi_i\|_*^2 \\
&+ C'\tau\sum_{i=1}^m \|\chi_i\|^2 + C''_\varepsilon\tau\sum_{i=1}^m \tau\|u_i\|^2 + C'''_\varepsilon\tau\sum_{i=1}^m\sum_{j=1}^i \tau\|u_j\|^2, \\
&\forall\, m = 1,\dots,n, \quad \forall\,\varepsilon > 0, \text{ and for some } C''_\varepsilon,\, C'''_\varepsilon > 0.
\end{aligned}
$$

Now we can choose $\varepsilon = (C_1/16\lambda^2)$ and then

$$
\tau < \min\{C_1/2C',\, C_1/256\lambda^2 C''_\varepsilon,\, C_1/256\lambda^2 C'''_\varepsilon\},
$$

to obtain

$$
\begin{aligned}
& C \sum_{i=1}^{m} \tau \|\delta \chi_i\|_*^2 + C \sum_{i=1}^{m} \|\chi_i - \chi_{i-1}\|^2 + C\|\chi_m\|^2 \\
& \quad + C_1 \|\widehat{\beta}_\tau(\chi_m)\|_{L^1(\Omega)} + C\tau^{1/4} \sum_{i=1}^{m} |u_i - u_{i-1}|^2 \\
& \quad + C\tau^{1/4}|u_m|^2 + C\tau^{1/4}\|\widehat{v}(u_m)\|_{L^1(\Omega)} \\
& \quad + C \sum_{i=1}^{m} |\rho(u_i) - \rho(u_{i-1})|^2 + C|\rho(u_m)|^2 \\
& \quad + C\|\ln(\rho(u_m))\|_{L^1(\Omega)} + C \sum_{i=1}^{m} \tau \|u_i\|^2 \\
& \leq C + C_1 \|\widehat{\beta}_\tau(\chi^0)\|_{L^1(\Omega)} + C\|\chi^0\|^2 \\
& \quad + C|\rho(u^0)|^2 + C\|\ln(\rho(u^0)\|_{L^1(\Omega)} + C\tau^{1/4}|u^0|^2 \\
& \quad + C\|\widehat{v}(u^0)\|_{L^1(\Omega)} + C \sum_{i=1}^{m} \tau \|f_i\|_*^2 + C\tau \sum_{i=1}^{m-1} \sum_{j=1}^{i} \tau \|u_j\|^2 \\
& \quad + C\tau \sum_{i=1}^{m-1} \|\chi_i\|^2, \quad \forall\, m = 1, \ldots, n.
\end{aligned}
$$

Applying a discrete version of Gronwall's lemma (see e.g. the version reported in [**17**]) and owing to (3.16), we obtain from (4.19)

(4.20)

$$
\begin{aligned}
& \|\partial_t \chi_\tau\|_{L^2(0,T;V')}^2 + \tau \|\partial_t \chi_\tau\|_{L^2(0,T;V)}^2 + \tau^{5/4} \|\partial_t u_\tau\|_{L^2(0,T;H)}^2 \\
& + \|\overline{\chi}_\tau\|_{L^\infty(0,T;V)}^2 + \|\widehat{\beta}_\tau(\overline{\chi}_\tau)\|_{L^\infty(0,T;L^1(\Omega))} + \|\overline{u}_\tau\|_{L^2(0,T;V)}^2 \\
& + \tau^{1/4} \|\overline{u}_\tau\|_{L^\infty(0,T;H)}^2 + \tau \|\partial_t \rho(u_\tau)\|_{L^2(0,T;H)}^2 + \|\rho(\overline{u}_\tau)\|_{L^\infty(0,T;H)}^2 \leq C.
\end{aligned}
$$

Now it is straightforward to see that

$$
\|\rho(u_\tau) - \rho(\overline{u}_\tau)\|_{L^2(0,T;H)}^2 \leq \frac{\tau^2}{3} \|\partial_t \rho(u_\tau)\|_{L^2(0,T;H)}^2 \leq C\tau, \tag{4.21}
$$

$$
\|\chi_\tau - \overline{\chi}_\tau\|_{L^2(0,T;V)}^2 \leq \frac{\tau^2}{3} \|\partial_t \chi_\tau\|_{L^2(0,T;V)}^2 \leq C\tau, \tag{4.22}
$$

$$
\|u_\tau - \overline{u}_\tau\|_{L^2(0,T;H)}^2 \leq \frac{\tau^2}{3} \|\partial_t u_\tau\|_{L^2(0,T;H)}^2 \leq C\tau^{3/4}. \tag{4.23}
$$

Thanks to the estimates (4.20)–(4.22), well-known compactness results, and Lemma 3.1, cf. (A5), one can infer that there exists at least a subsequence of time steps, still denoted by τ, and some functions $\vartheta, u, \chi, \varphi$, such that

$$\rho(\overline{u}_\tau) \stackrel{*}{\longrightarrow} \vartheta \quad \text{in } L^\infty(0,T;H), \tag{4.24}$$

$$\overline{u}_\tau \rightharpoonup u \quad \text{in } L^2(0,T;V), \tag{4.25}$$

$$\chi_\tau \rightharpoonup \chi \quad \text{in } H^1(0,T;V'), \tag{4.26}$$

$$\overline{\chi}_\tau \stackrel{*}{\longrightarrow} \chi \quad \text{in } L^\infty(0,T;V), \tag{4.27}$$

$$\overline{(\underline{k} *_\tau \underline{u})}_\tau \rightharpoonup \varphi \quad \text{in } L^2(0,T;V), \tag{4.28}$$

as $\tau \searrow 0$. In addition, the generalized Ascoli theorem (see [**31**, Corollary 4]) ensures that, thanks to (4.26)–(4.27),

$$\overline{\chi}_\tau \to \chi \quad \text{in } \mathbf{C}^0([0,T];H), \quad \text{at least for a subsequence of} \quad \tau \searrow 0. \tag{4.29}$$

Now, taking (3.33) and (4.25)–(4.26), (2.11), (3.16), (2.4) into account, we may say that

$$\begin{aligned}&\|\tau^{1/4}\partial_t u_\tau + \partial_t \rho(u_\tau)\|_{L^2(0,T;V')}\\ &\qquad \le \lambda \|\partial_t \chi_\tau\|_{L^2(0,T;V')} + C\|\overline{u}_\tau\|_{L^2(0,T;V)} + \|\overline{f}_\tau\|_{L^2(0,T;V')} \le C\end{aligned}$$

and

$$\|\tau^{1/4}\overline{u}_\tau(s) + \rho(\overline{u}_\tau)(s)\|^2 \le C + C\|\overline{u}_\tau(s)\|^2, \quad \text{for a.e. } s \in (0,T).$$

Consequently, thanks to Ascoli's theorem and to (4.24)–(4.25), we may say that

$$\tau^{1/4}\overline{u}_\tau + \rho(\overline{u}_\tau) \to \vartheta \quad \text{in } C^0([0,T];H), \tag{4.30}$$

$$\rho(\overline{u}_\tau) \to \vartheta \quad \text{in } L^2(0,T;V) \quad \text{and so a.e. in } Q. \tag{4.31}$$

Now, to deduce that $\rho(u) = \vartheta$, we can use [**1**, p. 42], with the maximal monotonicity of α, and (4.31).

Moreover, Lemma 3.2, (4.25) and (2.30) lead to

$$\overline{(\underline{k} *_\tau \underline{u})}_\tau - k * \overline{u}_\tau \to 0 \quad \text{in } L^1(0,T;V), \tag{4.32}$$

and

$$k * \overline{u}_\tau - k * u \longrightarrow 0 \quad \text{in } L^2(0,T;V). \tag{4.33}$$

Thus, $\varphi = k * u$.

Fourth estimate. Now we need to estimate the L^2 norm of $\beta_\tau(\overline{\chi}_\tau)$, independently of τ. So, first we may observe that, from (3.19), it comes immediately

$$\langle \chi_i - m_0, 1\rangle = 0, \quad \forall\, i = 1, \dots, n, \quad \text{where} \quad m_0 := \int_\Omega \chi^0 \, dx.$$

Now, let us take

$$x_i := \frac{1}{|\Omega|}\langle \xi_i, 1\rangle,$$

and test (3.19) with $\tau\mathcal{N}(\xi_i - x_i)$ and (3.20) with $\tau(\xi_i - x_i)$. Then, subtracting the resulting equations and setting

$$F_i = -\mathcal{N}(\chi_i - \chi_{i-1}) - \tau\sigma'(\chi_i) + \frac{\tau\lambda}{C_1}\big(u_i - C_2\rho(u_i)\big),$$

and subtracting also $\langle x_i, \xi_i - x_i\rangle = 0$, we obtain the identity

$$\tau\big(\nabla\chi_i, \nabla(\xi_i - x_i)\big) + \tau|\xi_i - x_i|^2 = \tau\big(F_i - x_i, \xi_i - x_i\big) = \tau\big(F_i, \xi_i - x_i\big).$$

Since the first term on the lefthand side is nonnegative, due to the monotonicity of β_τ we deduce that

$$\tau|\xi_i - x_i| \le \tau|F_i|.$$

Then, summing it up for $i = 1, \dots, m$, and taking (4.20) into account, we get immediately

$$\|\overline{\xi}_\tau - x_\tau\|_{L^1(0,T;H)} \le C.$$

In the next step, we would like to derive an analogous estimate for $\beta_\tau(\overline{\chi}_\tau)$. To do that, we have to find an upper bound for the L^2-norm of x_τ.

Following exactly the argument reported for example in [**6**, Section 4], which follows closely the proof devised by Kenmochi, Niezgódka, and Pawlow in [**24**, Lemma 5.2], we can state that

$$\|\overline{\xi}_\tau\|_{L^1(0,T;H)} \le C. \tag{4.34}$$

Note that assumption (2.22) is used at this step.

Now, in order to derive an estimate of $\overline{w}_\tau$ in $L^2(0,T;V)$, we may observe that, thanks to (3.34), $\overline{w}_\tau - 1/|\Omega|\langle \overline{w}_\tau, 1\rangle$ is a solution of a problem like (2.8) with datum $\partial_t \chi_\tau \in L^2(0,T;V')$, thanks to (4.20). Hence, estimating the mean value of $\overline{w}_\tau$ with the help of (3.35) (choose $v = 1$) and using again (4.20), we can say that

$$\|\overline{w}_\tau\|_{L^2(0,T;V)} \le C. \tag{4.35}$$

Moreover, applying the same argument to (3.35), we obtain that $\overline{\chi}_\tau - m_0$ is the solution of a problem like (2.8) with datum in $L^1(0,T;H)$ and consequently we may obtain that

$$\|\overline{\chi}_\tau\|_{L^2(0,T;W)} \le C. \tag{4.36}$$

Thus, we can still take convergent subsequences by compactness as in (4.20), letting $\tau \searrow 0$. Finally, on account of (4.24)–(4.27), (4.33), (4.34)–(4.36) and (A2), passing to the limit in (3.33)–(3.35), we immediately recover (2.18)–(2.20) and the regularity (2.12)–(2.16).

By (4.29) and (4.31), we get also (2.21). Next, we note that $\{\beta_\tau(\overline{\chi}_\tau)\}_\tau$ and $\{\overline{\chi}_\tau\}_\tau$ converge to some ξ and χ weakly in $L^1(0,T;H)$, for instance, and we have to deduce (1.6). This can be done using [**1**, p. 42], and the strong convergence of $\{\overline{\chi}_\tau\}$, given in (4.29). This concludes the proof of Theorem 2.1.

Remark 4.1. Let we say that we could obtain the same existence result with less regularity on the kernel k, belonging to the intersection of $L^2(0,T)$ and suitable interpolation space between $L^1(0,T)$ and $BV(0,T)$, where $BV(0,T)$ denotes the space of the functions with bounded total variation (see e.g. [**2**]).

Indeed all the estimates can be repeated, taking as approximation for k the following one

$$k_i := \frac{1}{\tau}\int_{(i-1)\tau}^{i\tau} k(s)\,ds, \quad \text{for } i = 1, \dots, n.$$

In this case only Lemma 3.2 needs to be modified, but it is always possible to prove an analogous statement that allows us to say that

$$\overline{(\underline{k} *_\tau \underline{u})}_\tau - k * \overline{u}_\tau \longrightarrow 0 \quad \text{in } L^1(0,T;V), \quad \text{as } \tau \searrow 0$$

and so to pass to the limit into the discretized problem and get existence of solution to Problem (P).

5. Further regularity and uniqueness. This section is devoted to the proof of Theorem 2.2, so we now suppose that the assumptions (A1)–(A10) hold. Let now, in place of (3.15)

$$\overline{f}_\tau(t) = f(i\tau) \in V', \quad \text{for } t \in ((i-1)\tau, i\tau], \quad \forall\, i = 1, \dots, n,$$

so that we can say that

$$\|\delta\overline{f}_\tau\|_{L^1(0,T;V')} \le \mathrm{Var}_{[0,T];V'}[f],$$

and

$$\|f - \overline{f}_\tau\|_{L^1(0,T;V')} \le \tau \mathrm{Var}_{[0,T];V'}[f],$$

where $\mathrm{Var}_{[0,T];V'}[f]$ denotes the total variation on the interval $[0,T]$ in the space V' of the function f (see e.g. [**2**]).

Moreover, let $\chi_{-1} \in H$ be defined by

$$\left\langle \frac{\chi^0 - \chi_{-1}}{\tau}, v \right\rangle + \int_\Omega \nabla w_0 \nabla v = 0 \quad \forall\, v \in V, \text{ where} \tag{5.1}$$

$$\langle w_0, v\rangle := \int_\Omega \nabla\chi^0 \nabla v + \int_\Omega \Big(\xi_0 + \sigma'(\chi^0) - \frac{\lambda}{C_1}\big(u^0 - C_2\rho(u^0)\big)\Big)v, \quad \forall\, v \in V, \tag{5.2}$$

$$\xi_0 = \beta_\tau(\chi^0), \tag{5.3}$$

and, according to (3.5), be

$$\delta\chi_0 = \frac{\chi^0 - \chi_{-1}}{\tau}, \tag{5.4}$$

then, thanks to (5.1)–(5.4), (A2), (A10)–(A11), and (2.4), we have that a positive constant C exists such that

$$\|\delta\chi_0\|_* \le C, \tag{5.5}$$

and this estimate will be useful in the sequel.

Remark 5.1. We may now observe that, in substitution of (A11), the following condition

$$\left\| -\Delta\chi^0 + \beta_\tau(\chi^0) + \sigma'(\chi^0) - \frac{\lambda}{C_1}\big(u^0 - C_2\rho(u^0)\big)\right\| \le C \quad \forall\, \tau \in [0,1], \tag{5.6}$$

holding for some positive constant C, is sufficient to prove Theorem 2.2, together with (A1)–(A10) and (2.22). This is the most natural initial condition for this kind of problem, as it is possible to see looking at (5.1)–(5.3). In fact, using just (5.6), we may recover (5.5). Let us also note that obviously (A2) and (A10)–(A11), together with (2.4), imply condition (5.6).

Let us now perform some additional regularity estimates.

Fifth estimate. Multiplying (3.18) by $u_i - u_{i-1}$, we get

$$\begin{aligned} &\tau^{1/4}\int_\Omega \tau|\delta u_i|^2 + \int_\Omega \tau\delta\vartheta_i\delta u_i + \lambda\int_\Omega \delta\chi_i(u_i - u_{i-1}) \\ &\quad + \frac{1}{2}\|u_i - u_{i-1}\|^2 + \frac{1}{2}\|u_i\|^2 - \frac{1}{2}\|u_{i-1}\|^2 \\ &\le -(\!((\underline{k} *_\tau \underline{u})_i, u_i - u_{i-1})\!) + \langle f_i, u_i - u_{i-1}\rangle_{\cdot}, \end{aligned} \tag{5.7}$$

First, we can observe that, thanks the second equality in Remark 2.1,

$$\tau\int_\Omega \delta\vartheta_i\delta u_i \ge C_2\tau|\delta\vartheta_i|^2. \tag{5.8}$$

Then, we may sum up (5.7) for $i = 1, \dots, m$ and, using a discrete integration by parts (see (4.3)) we have

$$-\sum_{i=1}^{m} (\!((\underline{k} *_\tau \underline{u})_i, u_i - u_{i-1})\!) = -(\!((\underline{k} *_\tau \underline{u})_m, u_m)\!) + (\!((\underline{k} *_\tau \underline{u})_1, u^0)\!)$$
$$+ \sum_{i=1}^{m-1} (\!((\underline{k} *_\tau \underline{u})_i - (\underline{k} *_\tau \underline{u})_{i+1}, u_i)\!),$$
$$\forall\, m = 1, \dots, n.$$

Now, estimating the righthand side, we have

$$\sum_{i=1}^{m-1} (\!((\underline{k} *_\tau \underline{u})_i - (\underline{k} *_\tau \underline{u})_{i+1}, u_i)\!)$$
$$= \sum_{i=1}^{m-1} \tau (\!((\underline{\delta k} *_\tau \underline{u})_{i+1}, u_i)\!) + k_0 \sum_{i=1}^{m-1} \tau (\!(u_{i+1}, u_i)\!),$$

and

$$\|(\underline{k} *_\tau \underline{u})_m\|^2 \le \|k\|^2_{L^\infty(0,T)} \left(\sum_{j=1}^{m} \tau \|u_j\| \right)^2 \le \|k\|^2_{L^\infty(0,T)} m\tau^2 \sum_{j=1}^{m} \|u_j\|^2,$$
$$\forall\, m = 1, \dots, n,$$

whence,

$$-\sum_{i=1}^{m} (\!((\underline{k} *_\tau \underline{u})_i, u_i - u_{i-1})\!)$$
$$\le \frac{1}{4}\|u_m\|^2 + T\|k\|^2_{L^\infty(0,T)} \sum_{i=1}^{m} \tau \|u_i\|^2 + \frac{|k_1|^2}{2}\,\tau^2 \|u_1\|^2$$
$$+ \frac{1}{2}\|u^0\|^2 + \frac{1}{2}\sum_{i=1}^{m-1} \tau \Big(\|(\underline{\delta k} *_\tau u)_{i+1}\|^2 + \|u_i\|^2 \Big)$$
$$+ \frac{1}{2}|k_0| \sum_{i=1}^{m-1} \tau \Big(\|u_i\|^2 + \|u_{i+1}\|^2 \Big), \quad \forall\, m = 1, \dots, n.$$

Recalling (3.9) and (3.13), we finally deduce that

$$-\sum_{i=1}^{m}(((\underline{k} *_\tau \underline{u})_i, u_i - u_{i-1})) \\ \leq \frac{1}{4}\|u_m\|^2 + \frac{1}{2}\|u^0\|^2 + C\sum_{i=1}^{m}\tau\|u_i\|^2, \quad \forall\, m = 1, \dots, n. \tag{5.9}$$

Now, use again a discrete integration by parts (see (4.3)) in order to estimate the last term in (5.7). Indeed,

$$\sum_{i=1}^{m}\langle f_i, u_i - u_{i-1}\rangle = \langle f_m, u_m\rangle - \langle f_1, u^0\rangle + \sum_{i=1}^{m-1}\langle f_i - f_{i+1}, u_i\rangle \tag{5.10}$$

and

$$\sum_{i=1}^{m}\langle f_i, u_i - u_{i-1}\rangle \leq \frac{1}{8}\|u_m\|^2 + \frac{5}{2}\|\overline{f}_\tau\|^2_{L^\infty(0,T;V')} \\ + \frac{1}{2}\|u^0\|^2 + \sum_{i=1}^{m-1}\tau\|\delta f_{i+1}\|_*\|u_i\|. \tag{5.11}$$

Now, taking (5.7)–(5.11) into account, we have the following estimate

$$\tau^{1/4}\sum_{i=1}^{m}\tau|\delta u_i|^2 + C_2\sum_{i=1}^{m}\tau|\delta\vartheta_i|^2 + \lambda\sum_{i=1}^{m}\int_\Omega \delta\chi_i(u_i - u_{i-1}) \\ + \frac{1}{2}\sum_{i=1}^{m}\|u_i - u_{i-1}\|^2 + \frac{1}{8}\|u_m\|^2 \\ \leq \|u^0\|^2 + C'\tau\sum_{i=1}^{m}\|u_i\|^2 + \frac{5}{2}\|\overline{f}_\tau\|^2_{L^\infty(0,T;V')} + \sum_{i=1}^{m-1}\tau\|\delta f_{i+1}\|_*\|u_i\|, \\ \forall\, m = 1, \dots, n. \tag{5.12}$$

Sixth estimate. First, we can take the difference between (3.19) at i and (3.19) at $(i-1)$, then multiply it by C_1/τ, to get

$$C_1\left\langle \frac{\chi_i - \chi_{i-1}}{\tau^2} - \frac{\chi_{i-1} - \chi_{i-2}}{\tau^2}, v\right\rangle + C_1\int_\Omega \nabla\left(\frac{w_i - w_{i-1}}{\tau}\right)\nabla v = 0, \\ \forall\, v \in V. \tag{5.13}$$

Now we take the difference between (3.20) at i and (3.20) at $(i-1)$, then multiply it by C_1/τ, to get

$$\begin{aligned} C_1\Big\langle \frac{w_i - w_{i-1}}{\tau}, v\Big\rangle &= C_1 \int_\Omega \nabla\Big(\frac{\chi_i - \chi_{i-1}}{\tau}\Big)\nabla v + C_1 \int_\Omega \Big(\frac{\xi_i - \xi_{i-1}}{\tau}\Big) v \\ &\quad - \lambda \int_\Omega \Big(\frac{u_i - u_{i-1}}{\tau}\Big) v + \lambda C_2 \int_\Omega \Big(\frac{\vartheta_i - \vartheta_{i-1}}{\tau}\Big) v \\ &\quad + C_1 \int_\Omega \Big(\frac{\sigma'(\chi_i - \sigma'(\chi_{i-1})}{\tau}\Big) v, \quad \forall\, v \in V. \end{aligned} \tag{5.14}$$

Now multiply (5.13) by $\mathcal{N}(\chi_i - \chi_{i-1})$ and (5.14) by $\chi_i - \chi_{i-1}$.

Subtracting the two resulting equalities, we get

$$\begin{aligned} &\frac{C_1}{2}\|\delta\chi_i\|_*^2 + \frac{C_1}{2}\Big\|\frac{\chi_i - 2\chi_{i-1} + \chi_{i-2}}{\tau}\Big\|_*^2 \\ &- \frac{C_1}{2}\|\delta\chi_{i-1}\|_*^2 + \tau C_1 |\nabla(\delta\chi_i)|^2 - \lambda \int_\Omega (u_i - u_{i-1})\delta\chi_i \\ &+ C_1 \int_\Omega \big(\beta_\tau(\chi_i) - \beta_\tau(\chi_{i-1})\big)\delta\chi_i \\ \le &- C_1 \int_\Omega \big(\sigma'(\chi_i) - \sigma'(\chi_{i-1})\big)\delta\chi_i - \lambda C_2 \tau \int_\Omega \delta\vartheta_i \delta\chi_i. \end{aligned} \tag{5.15}$$

Now, in order to estimate the righthand side of (5.15), we can first observe that, thanks to (A1) and (3.17),

$$C_1 \int_\Omega \big(\beta_\tau(\chi_i) - \beta_\tau(\chi_{i-1})\big)\delta\chi_i \ge 0,$$

and (A2) lead to

$$-C_1 \int_\Omega \big(\sigma'(\chi_i) - \sigma'(\chi_{i-1})\big)\delta\chi_i \le \frac{\tau}{8}\|\delta\chi_i\|^2 + C\tau\|\delta\chi_i\|_*^2,$$

$$-\lambda C_2 \tau \int_\Omega \delta\vartheta_i\delta\chi_i \le \frac{C_2}{2}\tau|\delta\vartheta_i|^2 + \frac{\tau}{8}\|\delta\chi_i\|^2 + C\tau\|\delta\chi_i\|_*^2.$$

It follows

$$\begin{aligned} &\frac{C_1}{2}\|\delta\chi_i\|_*^2 + C_1\tau\|\delta\chi_i\|^2 + \frac{\tau^2 C_1}{2}\Big\|\frac{\chi_i - 2\chi_{i-1} + \chi_{i-2}}{\tau^2}\Big\|_*^2 \\ &- \frac{C_1}{2}\|\delta\chi_{i-1}\|_*^2 - \lambda \int_\Omega (u_i - u_{i-1})\delta\chi_i \\ \le &\frac{C_2}{2}\tau|\delta\vartheta_i|^2 + C\tau\|\delta\chi_i\|_*^2 + \frac{\tau}{4}\|\delta\chi_i\|^2, \end{aligned} \tag{5.15}$$

where we have written the V-norm of $\delta\chi_i$ instead of the H-norm of its gradient, because $\delta\chi_i$ is a zero mean value function.

Moreover, taking (5.1)–(5.4) and (A9) into account, summing up this estimate for $i = 1, \dots, m$ and then adding the result to (5.12), we have

(5.16)
$$\begin{aligned}
&\tau^{1/4}\sum_{i=1}^{m}\tau|\delta u_i|^2 + C_2\sum_{i=1}^{m}\tau|\delta\vartheta_i|^2 + \frac{1}{2}\sum_{i=1}^{m}\|u_i - u_{i-1}\|^2 + \frac{1}{4}\|u_m\|^2\\
&\quad + \frac{\tau^2}{2}\sum_{i=1}^{m}\|\delta(\delta\chi_i))\|_*^2 + \frac{1}{2}\|\delta\chi_m\|_*^2 + C\tau\sum_{i=1}^{m}\|\delta\chi_i\|^2 - \frac{1}{2}\|\delta\chi_0\|_*^2\\
&\le \frac{C_2}{2}\tau\sum_{i=1}^{m}|\delta\vartheta_i|^2 + C''\tau\sum_{i=1}^{m}\|\delta\chi_i\|_*^2 + C'\tau\sum_{i=1}^{m}\|u_i\|^2\\
&\quad + \|u^0\|^2 + \frac{5}{2}\|\overline{f}_\tau\|_{L^\infty(0,T;V')}^2 + \sum_{i=1}^{m-1}\tau\|\delta f_{i+1}\|_*\|u_i\|,
\end{aligned}$$

for all $m = 1, \dots, n$ and for some positive constants C' and C''.

Then we can choose $\tau < \min\{1/4C', 1/2C''\}$ in (5.16), so that we get, thanks also to (A9),

$$\begin{aligned}
&\tau^{1/4}\sum_{i=1}^{m}\tau|\delta u_i|^2 + \frac{C_2}{2}\sum_{i=1}^{m}\tau|\delta\vartheta_i|^2 + \frac{1}{2}\sum_{i=1}^{m}\tau^2\|\delta u_i\|^2 + C\|u_m\|^2\\
&\quad + \frac{\tau}{2}\sum_{i=1}^{m}\tau\|\delta(\delta\chi_i))\|_*^2 + \frac{1}{2}\|\delta\chi_m\|_*^2 + C\tau\sum_{i=1}^{m}\|\delta\chi_i\|^2\\
&\le C + \|u^0\|^2 + C\|\delta\chi_0\|_*^2 + C\tau\sum_{i=1}^{m-1}\|\delta\chi_i\|_*^2\\
&\quad + C\tau\sum_{i=1}^{m-1}\|u_i\|^2 + \sum_{i=1}^{m-1}\tau\|\delta f_{i+1}\|_*\|u_i\|, \quad \forall\, m = 1, \dots, n.
\end{aligned}$$

Now, we may use a discrete version of Gronwall's lemma to see that, thanks also to (5.5),

$$\begin{aligned}
&\tau\|\partial_t u_\tau\|_{L^2(0,T;V)}^2 + \tau^{1/4}\|\partial_t u_\tau\|_{L^2(0,T;H)}^2 + \|\overline{u}_\tau\|_{L^\infty(0,T;V)}\\
&\quad + \|\partial_t\vartheta_\tau\|_{L^2(0,T;H)}^2 + \|\partial_t\chi_\tau\|_{L^\infty(0,T;V')\cap L^2(0,T;V)}^2 \le C
\end{aligned} \tag{5.17}$$

and is straightforward to see that

$$\|u_\tau - \overline{u}_\tau\|^2_{L^2(0,T;V)} \le \frac{\tau^2}{3}\|\partial_t u_\tau\|^2_{L^2(0,T;V)} \le C\tau. \tag{5.18}$$

Now, by standard compactness argument, using also (5.18), at least for a subsequence of $\tau \searrow 0$, we have the following

$$\overline{u}_\tau \xrightarrow{*} u \quad \text{in } L^\infty(0,T;V), \tag{5.19}$$

$$\partial_t\vartheta_\tau \rightharpoonup \partial_t\vartheta \quad \text{in } L^2(0,T;H), \tag{5.20}$$

$$\partial_t\chi_\tau \rightharpoonup \partial_t\chi \quad \text{in } L^2(0,T;V), \tag{5.21}$$

$$\partial_t\chi_\tau \xrightarrow{*} \partial_t\chi \quad \text{in } L^\infty(0,T;V'). \tag{5.22}$$

Thus, (2.23)–(2.24) and the first two inclusions in (2.25) are satisfied. Moreover, let us observe that, proceeding exactly like in the fourth estimate, thanks to (5.17), we obtain

$$\|\overline{\xi}_\tau\|_{L^\infty(0,T;H)} + \|\overline{w}_\tau\|_{L^\infty(0,T;V)} + \|\overline{\chi}_\tau\|_{L^\infty(0,T;W)} \le C.$$

Now, by the standard compactness argument, at least for a subsequence of $\tau \searrow 0$, we have the following

$$\begin{aligned} \overline{\xi}_\tau &\xrightarrow{*} \xi \quad \text{in } L^\infty(0,T;H),\\ \overline{w}_\tau &\xrightarrow{*} w \quad \text{in } L^\infty(0,T;V),\\ \overline{\chi}_\tau &\xrightarrow{*} \chi \quad \text{in } L^\infty(0,T;W), \end{aligned}$$

and so we recover also regularity (2.25)–(2.27).

Now, let us come to the proof of Theorem 2.2. We use [**30**, Lemma 3.5] and state it for the reader's convenience.

Lemma 5.1. *Taking assumptions* (A4) *and* (2.1)–(2.4) *into account, there exists a constant* $d > 0$*, such that*

$$\frac{d|r_1 - r_2|}{1 + r_1^2 + r_2^2} \le |\rho(r_1) - \rho(r_2)|, \quad \forall\, r_1,\, r_2 \in \mathbf{R}.$$

Let now $(\vartheta_i, u_i, \chi_i, \xi_i)$ for $i = 1, 2$, be two quadruples fulfilling (2.12)–(2.21), that is, (ϑ_1, χ_1) and (ϑ_2, χ_2) are two solutions of problem (1.1)–(1.7), in the sense of Theorem 2.2.

First, we take the difference of equalities (2.18), then integrate from 0 to $s \in (0, T)$ and test it by $(u_1 - u_2)(s)$.

Thanks to (A3), integrating the result from 0 to $t \in (0, T)$, we get

$$\begin{aligned}&\int_0^t \int_\Omega \big(\rho(u_1) - \rho(u_2)\big)(s) u(s)\, ds + \frac{1}{2}\Big\| \int_0^t u(r)\, dr \Big\|^2 \\ &\le -\lambda \int_0^t \langle \chi(s), u(s)\rangle\, ds - \int_0^t \bigg(\!\!\bigg(\int_0^s (k*u)(r)\, dr, u(s) \bigg)\!\!\bigg)\, ds,\end{aligned} \tag{5.23}$$

where $u := u_1 - u_2$, and $\chi := \chi_1 - \chi_2$.

Now subtract the two equations (2.19), and test the result by $\mathcal{N}(\chi)$. Take the difference between the two equations (2.20), test it by $-\chi$ and then sum with the preceding expression. Using the monotonicity of β and integrating on $(0, t)$, with $t \in (0, T)$, we deduce that

$$\begin{aligned}&\frac{1}{2}\|\chi(t)\|_*^2 + \int_0^t \int_\Omega |\nabla \chi(s)|^2\, ds \\ &\le -\int_0^t \int_\Omega \big(\sigma'(\chi_1) - \sigma'(\chi_2)\big)(s) \chi(s)\, ds \\ &\quad + \frac{\lambda}{C_1} \int_0^t \int_\Omega \chi(s) \big[u(s) - C_2\big(\rho(u_1) - \rho(u_2)\big)\big](s)\, ds.\end{aligned} \tag{5.24}$$

Integrating by parts the latter term in (5.23), we obtain

$$\begin{aligned}&-\int_0^t \bigg(\!\!\bigg(\int_0^s (k*u)(\tau)\, d\tau, u(s) \bigg)\!\!\bigg)\, ds \\ &= -\bigg(\!\!\bigg(\int_0^t (k*u)(\tau)\, d\tau, \int_0^t u(s)\, ds \bigg)\!\!\bigg) \\ &\quad + \int_0^t \bigg(\!\!\bigg((k*u)(s), \int_0^s u(r)\, dr \bigg)\!\!\bigg)\, ds.\end{aligned} \tag{5.25}$$

Now, let us call

$$I_1 = -\bigg(\!\!\bigg(\int_0^t (k*u)(r)\, dr, \int_0^t u(s)\, ds \bigg)\!\!\bigg)$$

and

$$I_2 = \int_0^t \left(\!\!\left((k*u)(s), \int_0^s u(r)\,dr \right)\!\!\right) ds.$$

Using (2.28)–(2.30), (2.34), and (4.19), we can easily get

(5.26)

$$\begin{aligned} I_1 &\le \frac{1}{4}\Big\| \int_0^t u(s)\,ds \Big\|^2 + \|k*u\|^2_{L^1(0,T;V)} \\ &\le \frac{1}{4}\Big\| \int_0^t u(s)\,ds \Big\|^2 + 2T\big(|k(0)|^2 + \|k'\|^2_{L^1(0,T)}\big) \int_0^t \Big\| \int_0^s u(r)\,dr \Big\|^2 ds \end{aligned}$$

and

$$\begin{aligned} I_2 &\le \int_0^t \|(k*u)(s)\| \, \Big\| \int_0^s u(r)\,dr \Big\| \, ds \\ &\le 2T\big(|k(0)|^2 + \|k'\|^2_{L^1(0,T)}\big) \int_0^t \Big\| \int_0^s u(r)\,dr \Big\|^2 ds. \end{aligned}$$

Looking now at (5.24), we can observe that, thanks also to (A2) and (2.31),

$$\begin{aligned} &- \int_0^t \int_\Omega \big(\sigma'(\chi_1) - \sigma'(\chi_2)\big)(s)\chi(s)\,ds \\ &\le C \int_0^t \int_\Omega |\chi(s)|^2\,ds \le \zeta \|\chi\|^2_{L^2(0,T;V)} + C_\zeta \|\chi\|^2_{L^2(0,T;V')}, \\ &\qquad \forall\, \zeta > 0 \quad \text{and for some } C_\zeta > 0. \end{aligned}$$

Moreover, we can see that, thanks to (2.4),

$$\begin{aligned} &\frac{\lambda}{C_1} \int_0^t \int_\Omega \chi(s)\big[u(s) - C_2\big(\rho(u_1) - \rho(u_2)\big)\big](s)\,ds \\ &\le C \int_0^t \int_\Omega |u(s)||\chi(s)|\,ds \\ &\le C \int_0^t \int_\Omega \frac{|u(s)|}{\sqrt{1 + |u_1(s)|^2 + |u_2(s)|^2}} \sqrt{1 + |u_1(s)|^2 + |u_2(s)|^2 |\chi(s)|}\,ds \\ &\le \frac{1}{2} \int_0^t \int_\Omega \frac{d|u(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2}\,ds \\ &\quad + C \int_0^t \int_\Omega (1 + |u_1(s)|^2 + |u_2(s)|^2)|\chi(s)|^2\,ds, \end{aligned}$$

where d is the same constant of Lemma 5.1.

Now, thanks to (2.31), we have that

$$
\begin{aligned}
&C\int_0^t\int_\Omega (1+|u_1(s)|^2+|u_2(s)|^2)|\chi(s)|^2\,ds\\
&\le C\int_0^t |\chi(s)|^2\,ds + C\sum_{i=1}^2\int_0^t\int_\Omega |u_i(s)|^2||\chi(s)|^2\,ds\\
&\le \eta\|\chi\|^2_{L^2(0,T;V)} + C_\eta\|\chi\|^2_{L^2(0,T;V')}\\
&\quad + C\sum_{i=1}^2 \|u_i\|^2_{L^\infty(0,T;V)}\int_0^t \|\chi(s)\|^2_{L^4(\Omega)}\,ds,\\
&\qquad\qquad \forall\,\eta>0 \text{ and for some } C_\eta>0,
\end{aligned}
\tag{5.29}
$$

then, we can apply (2.31) to this last estimate and use the boundness of u_i in $L^\infty(0,T;V)$, given by (5.19), so that, choosing ζ,η suitably in (5.27) and (5.29), then Lemma 5.1, (5.24)–(5.29) and (A5) lead to

$$
\begin{aligned}
&\frac{1}{2}\int_0^t\int_\Omega \frac{d|u(s)|^2}{1+|u_1(s)|^2+|u_2(s)|^2}\,ds + \frac{1}{2}\Big\|\int_0^t u(s)\,ds\Big\|^2\\
&\quad + \frac{1}{2}\|\chi(t)\|_*^2 + \frac{1}{2}\int_0^t\int_\Omega |\nabla\chi(s)|^2\,ds\\
&\le C\|\chi\|^2_{L^2(0,T;V')} + C\int_0^t \Big\|\int_0^s u(r)\,dr\Big\|^2\,ds.
\end{aligned}
$$

Applying Gronwall's lemma (see [**2**, Lemmas A.4–A.5, pp. 156–157]), we finally have that $|u|=0$ almost everywhere, in Q and $\chi(t)=0$, for all $t\in[0,T]$.

So, on account of (2.17)–(2.21), this entails that the two pairs $(\vartheta_1,\,\chi_1)$ and $(\vartheta_2,\,\chi_2)$ must necessarily coincide and the proof of Theorem 2.2 is complete. $\square$

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