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ON INTEGRAL EQUATIONS ARISING IN THE FIRST-PASSAGE PROBLEM FOR BROWNIAN MOTION

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ABSTRACT. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion started at zero, let $g:(0,\infty)\to\mathbf{R}$ be a continuous function satisfying $g(0+)\geq 0$, let

$$\tau = \inf\{t>0 \mid B_t \geq g(t)\}$$

be the first-passage time of B over g, and let F denote the distribution function of τ. Then the following system of integral equations is satisfied:

$$t^{n/2}H_n\left(\frac{g(t)}{\sqrt{t}}\right) = \int_0^t (t-s)^{n/2}H_n\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right)F(ds)$$

for $t>0$ and $n=-1,0,1\ldots$, where $H_n(x)=\int_x^\infty H_{n-1}(z)\,dz$ for $n\geq 0$ and $H_{-1}(x)=\varphi(x)=(1/\sqrt{2\pi})e^{-x^2/2}$ is the standard normal density. These equations are derived from a single 'master equation' which may be viewed as a Chapman-Kolmogorov equation of Volterra type. The initial idea in the derivation of the master equation goes back to Schrödinger [**23**].

1. Introduction. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion started at zero, let $g:(0,\infty)\to\mathbf{R}$ be a continuous function satisfying $g(0+)\geq 0$, let

$$\tau = \inf\{t>0 \mid B_t \geq g(t)\} \tag{1.1}$$

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be the first-passage time of B over g, and let F denote the distribution function of τ.

The *first-passage problem* seeks to determine F when g is given. The *inverse first-passage problem* seeks to determine g when F is given. Both the process B and the boundary g in these formulations may be more general, and our choice of Brownian motion is primarily motivated by the tractability of the exposition. The facts to be presented below can be extended to more general Markov processes and boundaries (such as two-sided ones) and the time may also be discrete.

The first-passage problem has a long history and a large number of applications. Yet explicit solutions to the first-passage problem (for Brownian motion) are known only in a limited number of special cases including linear or quadratic g. The law of τ is also known for a square-root boundary g but only in the form of a Laplace transform (which appears intractable to inversion). The inverse problem seems even harder. For example, it is not known if there exists a boundary g for which τ is exponentially distributed, cf. [**20**].

One way to tackle the problem is to derive an equation which links g and F. Motivated by this fact many authors have studied integral equations in connection with the first-passage problem (see e.g. [**23, 26, 11, 24, 19, 9, 22, 6**]) under various hypotheses and levels of rigor. The main aim of this paper is to present a unifying approach to the integral equations arising in the first-passage problem that is done in a rigorous fashion and with minimal tools.

The approach naturally leads to a system of integral equations for g and F (Section 6) in which the first two equations contain the previously known ones (Section 5). These equations are derived from a single *master equation* (Section 3) that can be viewed as a *Chapman-Kolmogorov equation of Volterra type* (Section 2). The initial idea in the derivation of the master equation goes back to Schrödinger [**23**]. The master equation cannot be reduced to a partial differential equation of forward or backward type, cf. [**14**]. A key technical detail needed to connect the second equation of the system to known methods leads to a simple proof of the fact that F has a continuous density when g is continuously differentiable (Section 4). The problem of finding F when g is given is tackled using classic theory of linear integral equations (Section 7). The inverse problem is reduced to solving a system of *nonlinear* Volterra integral equations of the second kind (Section 8). General theory of such systems seems far from being complete at present.

2. Chapman-Kolmogorov equations of Volterra type. It will be convenient to divide our discussion into two parts depending on if the time set T of the Markov process $(X_t)_{t \in T}$ is either discrete (finite or countable) or continuous (uncountable). The state space S of the process may be assumed to be a subset of $\mathbf{R}$.

1. *Discrete time* (*and space*). Recall that $(X_n)_{n \geq 0}$ is a (time-homogeneous) Markov process if the following condition is satisfied:

$$E_x(Y \circ \theta_k \mid \mathcal{F}_k) = E_{X_k}(Y) \tag{2.1}$$

for all (bounded) measurable Y and all k and x. (Recall that $X_0 = x$ under P_x and that $X_n \circ \theta_k = X_{n+k}$.) Then the *Chapman-Kolmogorov equation*, cf. [**4, 14**], holds:

$$P_x(X_n = z) = \sum_y P_y(X_{n-k} = z)\, P_x(X_k = y) \tag{2.2}$$

for x, z in S and $1 < k < n$ given and fixed, which is seen as follows

$$\begin{aligned} P_x(X_n = z) &= \sum_y P_x(X_n = z, X_k = y) \\ &= \sum_y E_x(I(X_k = y) E_x(I(X_{n-k} = z) \circ \theta_k \mid \mathcal{F}_k)) \\ &= \sum_y E_x(I(X_k = y) E_{X_k}(I(X_{n-k} = z))) \\ &= \sum_y P_x(X_k = y) P_y(X_{n-k} = z) \end{aligned} \tag{2.3}$$

upon using (2.1) with $Y = I(X_{n-k} = z)$.

A geometric interpretation of the Chapman-Kolmogorov equation (2.2) is shown in Figure 1 (note that the vertical line passing through k is given and fixed). Although for (2.2) we only considered the time-homogeneous Markov property (2.1) for simplicity, it should be noted that a more general Markov process creates essentially the same picture.

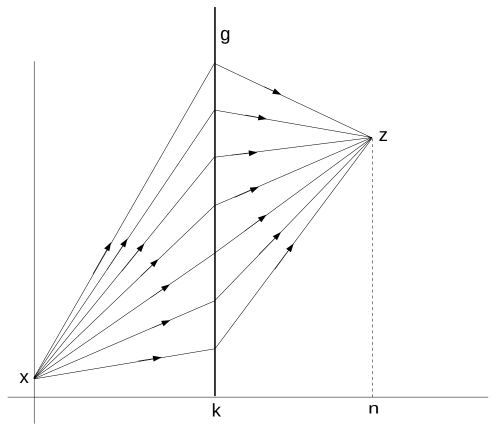

FIGURE 1. A symbolic drawing of the Chapman-Kolmogorov equation (2.2). The arrows indicate a time evolution of the sample paths of the process. The vertical line at k represents the state space of the process. The equations (2.11) have a similar interpretation.

Imagine now on Figure 1 that the vertical line passing through k begins to move continuously and eventually transforms into a new curve still separating x from z as shown in Figure 2. The question then arises naturally how the Chapman-Kolmogorov equation (2.2) extends to this case.

An evident answer to this question is stated in Theorem 2.1. This fact is then extended to the case of continuous time and space in Theorem 2.2 below.

Theorem 2.1. *Let* $(X_n)_{n\geq 0}$ *be a Markov process (taking values in a countable set* S*), let* x *and* z *be given and fixed in* S*, let* $g : \mathbf{N} \to S$ *be a function separating* x *and* z *relative to* X *(i.e. if* $X_0 = x$ *and* $X_n = z$ *for some* $n \geq 1$*, then there exists* $1 \leq k \leq n$ *such that* $X_k = g(k)$*) and let*

$$\tau = \inf\{k \geq 1 \mid X_k = g(k)\} \tag{2.4}$$

be the first-passage time of X *over* g*. Then the following sum equation holds*

$$P_x(X_n = z) = \sum_{k=1}^{n} P(X_n = z \mid X_k = g(k))\, P_x(\tau = k). \tag{2.5}$$

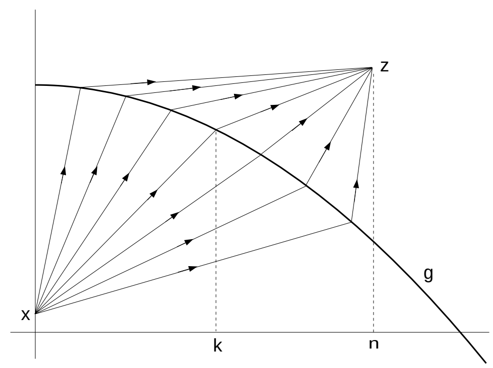

FIGURE 2. A symbolic drawing of the integral equation (2.5)–(2.6). The arrows indicate a time evolution of the sample paths of the process. The vertical line at k has been transformed into a time-dependent boundary g. The equations (2.16)–(2.17) have a similar interpretation.

Moreover, if the Markov process X is time-homogeneous, then (2.5) *reads as follows*

$$P_x(X_n = z) = \sum_{k=1}^{n} P_{g(k)}(X_{n-k} = z) P_x(\tau = k). \tag{2.6}$$

Proof. Since g separates x and z relative to X, we have

$$P_x(X_n = z) = \sum_{k=1}^{n} P_x(X_n = z, \tau = k). \tag{2.7}$$

On the other hand, by the Markov property

$$P_x(X_n = z \mid \mathcal{F}_k) = P_{X_k}(X_n = z) \tag{2.8}$$

and the fact that $\{\tau = k\} \in \mathcal{F}_k$, we easily find

$$P_x(X_n = z, \tau = k) = P(X_n = z \mid X_k = g(k))\, P_x(\tau = k). \tag{2.9}$$

Inserting this into (2.7) we obtain (2.5). The time-homogeneous simplification (2.6) follows then immediately, and the proof is complete. □

The equations (2.5) and (2.6) extend to the case when the state space S is uncountable. In this case the relation '$= z$' in (2.5) and (2.6) can be replaced by the relation '$\in G$' where G is any measurable set that is 'separated' from the initial point x relative to X in the sense described above. The extensions of (2.5) and (2.6) obtained in this way will be omitted.

2. *Continuous time* (*and space*). A passage from the discrete to the continuous case introduces some technical complications (regular conditional probabilities) which we set aside in the sequel.

Recall that $(X_t)_{t \geq 0}$ is a Markov process if the following condition is satisfied

$$P(X_t \in G \mid \mathcal{F}_s) = P(X_t \in G \mid X_s) \tag{2.10}$$

for all measurable G and all $s < t$. Then the Chapman-Kolmogorov equation, cf. [**4, 14**], holds:

$$P(t_1, x, t_3, G) = \int_S P(t_2, y, t_3, G) P(t_1, x, t_2, dy) \tag{2.11}$$

where $P(t_i, x, t_j, G) = P(X_{t_j} \in G \mid X_{t_i} = x)$ and $t_1 < t_2 < t_3$ are given and fixed.

Kolmogorov [**14**] calls (2.11) the 'fundamental equation,' notes that (under a desired Markovian interpretation) it is satisfied if the state space S is finite or countable (the 'total probability law'), and in the case when S is uncountable takes it as a 'new axiom.'

If X_{t_j} under $X_{t_i} = x$ has a density function f satisfying

$$P(t_i, x, t_j, G) = \int_G f(t_i, x, t_j, z)\, dz \tag{2.12}$$

for all measurable G, then the equations (2.11) reduce to

$$f(t_1, x, t_3, z) = \int_S f(t_1, x, t_2, y) f(t_2, y, t_3, z)\, dy \tag{2.13}$$

for x and z in S and $t_1 < t_2 < t_3$ given and fixed.

Kolmogorov [**15**] states that this integral equation was studied by Smoluchowski [**25**], recalls that he proved in [**14**] that under some additional conditions f satisfies certain differential equations of parabolic type (the *forward* and the *backward* equation), and in a footnote acknowledges that these differential equations were introduced by Fokker [**10**] and Planck [**21**] independently of the Smoluchowski integral equation. (The Smoluchowski integral equation [**25**] is a time homogeneous version of (2.13). The Bachelier-Einstein equation, cf. [**2, 8**]:

$$f(t+s,z) = \int_S f(t,z-x)f(s,x)\,dx \tag{2.14}$$

is a space-time homogeneous version of the Smoluchowski equation.)

Without going into further details on these facts, we will only note that the interpretation of the Chapman-Kolmogorov equation (2.2) described above by means of Figure 1 carries over to the general case of equation (2.11), and the same is true for the question raised above by means of Figure 2. The following theorem extends the result of Theorem 2.1 on this matter.

Theorem 2.2 (cf. Schrödinger [**23**] and Fortet [**11**, p. 217]). *Let $(X_t)_{t\geq 0}$ be a strong Markov process with continuous sample paths started at x, let $g : (0,\infty) \to \mathbf{R}$ be a continuous function satisfying $g(0+) \geq x$, let*

$$\tau = \inf\{t > 0 \mid X_t \geq g(t)\} \tag{2.15}$$

be the first-passage time of X over g, and let $F = F_x$ denote the distribution function of τ.

Then the following integral equation holds

$$P_x(X_t \in G) = \int_0^t P(X_t \in G \mid X_s = g(s))\,F(ds) \tag{2.16}$$

for each measurable set G contained in $[g(t),\infty)$.

Moreover, if the Markov process X is time-homogeneous, then (2.16) *reads as follows*

$$P_x(X_t \in G) = \int_0^t P_{g(s)}(X_{t-s} \in G)\,F(ds) \tag{2.17}$$

for each measurable set G contained in $[g(t),\infty)$.

Proof. The key argument in the proof is to apply a strong Markov property at time τ. This can be done informally (with $G \subseteq [g(t), \infty)$ given and fixed) as follows

$$\begin{aligned} P_x(X_t \in G) &= P_x(X_t \in G, \tau \le t) \\ &= E_x(I(\tau \le t) E_x(I(X_t \in G) \mid \tau)) \\ &= \int_0^t E_x(I(X_t \in G) \mid \tau = s)\, F(ds) \\ &= \int_0^t P(X_t \in G \mid X_s = g(s))\, F(ds) \end{aligned} \tag{2.18}$$

which is (2.16). In the last identity above we used that for $s \le t$ we have

$$E_x(I(X_t \in G) \mid \tau = s) = P(X_t \in G \mid X_s = g(s)), \tag{2.19}$$

which formally requires a precise argument. This is what we do in the rest of the proof.

For this, recall that $(Z_t)_{t \ge 0}$ is a strong Markov process if the following condition is satisfied

$$E_z(Y \circ \theta_\sigma \mid \mathcal{F}_\sigma) = E_{Z_\sigma}(Y) \tag{2.20}$$

for all (bounded) measurable Y and all stopping times σ. It turns out, however, that the process Z has to be chosen carefully. This is due to the fact that the time t on the left-hand side of (2.19) is deterministic. For example, by taking $Y = f(X_{t-\tau})$ on $\{\tau \le t\}$ we fail to achieve $Y \circ \theta_\tau = f(X_t)$, since $X_\beta \circ \theta_\sigma = X_{\sigma + \beta \circ \sigma}$ whenever $\sigma \le \beta$.

We thus choose $Z_t = (t, X_t)$ and define

$$\sigma = \inf\{t > 0 \mid Z_t \notin C\} \tag{2.21}$$

$$\beta = \inf\{t > 0 \mid Z_t \notin C \cup D\} \tag{2.22}$$

where $C = \{(s, y) \mid 0 < s < t,\ y < g(s)\}$ and $D = \{(s, y) \mid 0 < s < t,\ y \ge g(s)\}$ so that $C \cup D = \{(s, y) \mid 0 < s < t\}$. Thus $\beta = t$ under $P_{(0,x)}$, i.e., P_x and, moreover, $\beta = \sigma + \beta \circ \theta_\sigma$ since both σ and β are hitting times of the process Z to closed (open) sets, the second set being contained in the first one, so that $\sigma \le \beta$.

Setting $F(s,y) = 1_G(y)$ and $Y = F(Z_\beta)$, we thus see that $Y \circ \theta_\sigma = F(Z_\beta) \circ \theta_\sigma = F(Z_{\sigma+\beta\circ\sigma}) = F(Z_\beta) = Y$, which by means of (2.20) implies that

$$E_z(F(Z_\beta) \mid \mathcal{F}_\sigma) = E_{Z_\sigma}(F(Z_\beta)). \tag{2.23}$$

In the special case $z = (0,x)$, this reads

$$E_{(0,x)}(I(X_t \in G) \mid \mathcal{F}_\sigma) = E_{(\sigma,g(\sigma))}(I(X_t \in G)) \tag{2.24}$$

where $\mathcal{F}_\sigma$ on the left-hand side can be replaced by σ since the right-hand side defines a measurable function of σ. It then follows immediately from such modified (2.24) that

$$E_{(0,x)}(I(X_t \in G) \mid \sigma = s) = E_{(s,g(s))}(I(X_t \in G)) \tag{2.25}$$

and, since $\sigma = \tau \wedge t$, we see that (2.25) implies (2.19) for $s \le t$. Thus the final step in (2.18) is justified and therefore (2.16) is proved as well. The time-homogeneous simplification (2.17) is a direct consequence of (2.16), and the proof of the theorem is complete. □

The proof of Theorem 2.2 just presented is not the only possible one. The proof of Theorem 3.2 given below can easily be transformed into a proof of Theorem 2.2. Yet another quick proof can be given by applying the strong Markov property of the process (t, X_t) to establish (2.24) (multiplied by $I(\tau \le t)$ on both sides) with $\sigma = \tau \wedge t$ on the left-hand side and $\sigma = \tau$ on the right-hand side. The right-hand side then easily transforms to the right-hand side of (2.16), thus proving the latter.

In order to examine the scope of the equations (2.16) in a clearer manner, we will leave the realm of a general Markov process in the sequel and consider the case of a standard Brownian motion instead. The facts and methodology presented below extend to the case of more general Markov processes, or boundaries, although some of the formulas may be less explicit.

3. The master equation. The following notation will be used throughout

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \;\; \Phi(x) = \int_{-\infty}^{x} \varphi(z)\, dz, \;\; \Psi(x) = 1 - \Phi(x) \tag{3.1}$$

for $x \in \mathbf{R}$. We begin this section by recalling the result of Theorem 2.2. Thus, let $g : (0, \infty) \to \mathbf{R}$ be a continuous function satisfying $g(0+) \geq 0$, and let F denote the distribution function of τ from (2.15).

If specialized to the case of standard Brownian motion $(B_t)_{t \geq 0}$ started at zero, the equation (2.17) with $G = [g(t), \infty)$ reads as follows

$$\Psi\left(\frac{g(t)}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{g(t) - g(s)}{\sqrt{t-s}}\right) F(ds) \tag{3.2}$$

where the scaling property $B_t \sim \sqrt{t}\, B_1$ of B is used, as well as that $(z + B_t)_{t \geq 0}$ defines a standard Brownian motion started at z whenever $z \in \mathbf{R}$.

1. *Derivation.* It turns out that the equation (3.2) is just one in the sequence of equations that can be derived from a single master equation. This master equation can be obtained by taking $G = [z, \infty)$ in (2.17) with $z \geq g(t)$. We now present yet another proof of this derivation.

Theorem 3.1 (The master equation). *Let $(B_t)_{t \geq 0}$ be a standard Brownian motion started at zero, let $g : (0, \infty) \to \mathbf{R}$ be a continuous function satisfying $g(0+) \geq 0$, let*

$$\tau = \inf\{t > 0 \mid B_t \geq g(t)\} \tag{3.3}$$

be the first-passage time of B over g, and let F denote the distribution function of τ.

Then the following integral equation holds

$$\Psi\left(\frac{z}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{z - g(s)}{\sqrt{t-s}}\right) F(ds) \tag{3.4}$$

for all $z \geq g(t)$ where $t > 0$.

Proof. We will make use of the strong Markov property of the process $Z_t = (t, B_t)$ at time τ. This makes the present argument close to the argument used in the proof of Theorem 2.2.

For each $t > 0$, let $z(t)$ from $[g(t), \infty)$ be given and fixed. Setting $f(t, x) = I(x \geq z(t))$ and $Y = \int_0^\infty e^{-\lambda s} f(Z_s)\, ds$ by the strong Markov property (of the process Z) given in (2.20) with $\sigma = \tau$, and the scaling property of B, we find

$$
\begin{aligned}
\int_0^\infty & e^{-\lambda t} P_0(B_t \geq z(t))\, dt \\
&= E_0\Big(\int_0^\infty e^{-\lambda t} f(Z_t)\, dt \Big) \\
&= E_0\Big(E_0\Big(\int_\tau^\infty e^{-\lambda t} f(Z_t)\, dt \mid \mathcal{F}_\tau \Big)\Big) \\
&= E_0\Big(E_0\Big(\int_0^\infty e^{-\lambda(\tau+s)} f(Z_{\tau+s})\, ds \mid \mathcal{F}_\tau \Big)\Big) \\
&= E_0(e^{-\lambda\tau} E_0(Y \circ \theta_\tau \mid \mathcal{F}_\tau\Big)\Big) = E_0(e^{-\lambda\tau} E_{Z_\tau}(Y)) \\
&= \int_0^\infty e^{-\lambda t} E_{(t,g(t))}\Big(\int_0^\infty e^{-\lambda s} f(Z_s)\, ds \Big) F(dt) \qquad (3.5) \\
&= \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\lambda s} P_0(g(t) + B_s \geq z(t+s))\, ds F(dt) \\
&= \int_0^\infty e^{-\lambda t} \int_0^\infty e^{-\lambda s} \Psi\Big(\frac{z(t+s) - g(t)}{\sqrt{s}} \Big)\, ds F(dt) \\
&= \int_0^\infty e^{-\lambda t} \int_t^\infty e^{-\lambda(r-t)} \Psi\Big(\frac{z(r) - g(t)}{\sqrt{r-t}} \Big)\, dr F(dt) \\
&= \int_0^\infty e^{-\lambda r} \int_0^r \Psi\Big(\frac{z(r) - g(t)}{\sqrt{r-t}} \Big) F(dt)\, dr
\end{aligned}
$$

for all $\lambda > 0$. By the uniqueness theorem for Laplace transform, it follows that

$$P_0(B_t \geq z(t)) = \int_0^t \Psi\Big(\frac{z(t) - g(s)}{\sqrt{t-s}} \Big) F(ds) \qquad (3.6)$$

which is seen to be equivalent to (3.4) by the scaling property of B. The proof is complete. □

2. *Constant and linear boundaries.* It will be shown in Section 4 that when g is C^1 on $(0, \infty)$ then there exists a continuous density $f = F'$ of τ. The equation (3.2) then becomes

$$\Psi\Big(\frac{g(t)}{\sqrt{t}} \Big) = \int_0^t \Psi\Big(\frac{g(t) - g(s)}{\sqrt{t-s}} \Big) f(s)\, ds \qquad (3.7)$$

for $t > 0$. This is a linear *Volterra integral equation of the first kind* in f if g is known (it is a *nonlinear* equation in g if f is known). Its kernel

$$K(t,s) = \Psi\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right) \tag{3.8}$$

is *nonsingular* in the sense that the mapping $(s,t) \mapsto K(t,s)$ for $0 \le s < t$ is bounded.

If $g(t) \equiv c$ with $c \in \mathbf{R}$, then (3.2) or (3.7) reads as follows

$$P(\tau \le t) = 2P(B_t \ge c) \tag{3.9}$$

and this is the *reflection principle* of André [**1**], Bachelier [**2**, p. 64] and Lévy [**16**, p. 293].

If $g(t) = \alpha t + \beta$ with $\alpha \in \mathbf{R}$ and $\beta > 0$, then (3.7) reads as follows

$$\Psi\left(\frac{g(t)}{\sqrt{t}}\right) = \int_0^t \Psi(\alpha\sqrt{t-s}\,)f(s)\,ds \tag{3.10}$$

where we see that the kernel $K(t,s)$ is a function of the difference $t-s$ and thus of a convolution type. Standard Laplace transform techniques therefore can be applied to solve the equation (3.10) yielding the following explicit formula

$$f(t) = \frac{\beta}{t^{3/2}}\,\varphi\left(\frac{\alpha t + \beta}{\sqrt{t}}\right) \tag{3.11}$$

which is the well-known result of Doob [**5**, p. 397] and Malmquist [**17**, p. 526].

Closed form expressions for f in the case of more general boundaries g will be treated using classic theory of integral equations in Section 7 below.

3. *Numerical calculation.* The fact that the kernel (3.8) of the equation (3.7) is nonsingular in the sense explained above makes this equation especially attractive to numerical calculations of f if g is given. This can be done using the simple idea of Volterra (dating back to 1896).

Setting $t_j = jh$ for $j = 0, 1, \dots, n$ where $h = t/n$ and $n \geq 1$ is given and fixed, we see that the following approximation of the equation (3.7) is valid (when g is C^1 for instance):

$$\sum_{j=1}^{n} K(t, t_j) f(t_j) h = b(t) \tag{3.12}$$

where we set $b(t) = \Psi(g(t)/\sqrt{t})$. In particular, applying this to each $t = t_i$ yields

$$\sum_{j=1}^{i} K(t_i, t_j) f(t_j) h = b(t_i) \tag{3.13}$$

for $i = 1, 2, \dots, n$. Setting

$$a_{ij} = 2K(t_i, t_j), \quad x_j = f(t_j), \quad b_i = 2b(t_i)/h \tag{3.14}$$

we see that the system (3.13) reads as follows

$$\sum_{j=1}^{i} a_{ij} x_j = b_i, \quad i = 1, 2, \dots, n, \tag{3.15}$$

the simplicity of which is obvious, cf. [**19**]. We conjecture that this system constitutes an efficient method for numerical computation of f when g is given. Some examples of this computation are presented in [**28**] where references to other numerical methods can be found as well.

4. *Remarks.* It follows from (3.11) that for τ in (3.3) with $g(t) = \alpha t + \beta$ we have

$$P(\tau < \infty) = e^{-2\alpha\beta} \tag{3.16}$$

whenever $\alpha \geq 0$ and $\beta > 0$. This shows that F in (3.4) does not have to be a proper distribution function but generally satisfies $F(+\infty) \in (0, 1]$.

On the other hand, recall that *Blumenthal's* 0-1 *law* implies that $P(\tau = 0)$ is either 0 or 1 for τ in (3.3) and a continuous function $g : (0, \infty) \to \mathbf{R}$. If $P(\tau = 0) = 0$, then g is said to be an *upper function* for B, and if $P(\tau = 0) = 1$, then g is said to be a *lower function* for B.

Kolmogorov's test (see e.g. [**13**, pp. 33–35]) gives sufficient conditions on g to be an upper or lower function. It follows by Kolmogorov's test that $\sqrt{2t \log\log 1/t}$ is a lower function for B and $\sqrt{(2+\varepsilon)t \log\log 1/t}$ is an upper function for B for every $\varepsilon > 0$.

4. The existence of a continuous first-passage density. The equation (3.7) is a Volterra integral equation of the *first kind*. These equations are generally known to be difficult to deal with directly, and there are two standard ways of reducing them to Volterra integral equations of the *second kind*. The first method consists of differentiating both sides in (3.7) with respect to t, and the second method (Theorem 7.1) makes use of an integration by parts in (3.7) (see e.g. [**12**, pp. 40–41]). Our focus in this section is on the first method.

Being led by this objective we now present a simple proof of the fact that F is C^1 when g is C^1 (compare the arguments given below with those given in [**27**, p. 323] or [**9**, p. 322]).

Theorem 4.1. *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion started at zero, let $g : (0,\infty) \to \mathbf{R}$ be an upper function for B, and let τ in (3.3) be the first-passage time of B over g.*

If g is continuously differentiable on $(0,\infty)$ then τ has a continuous density f. Moreover, the following identity is satisfied

$$\frac{\partial}{\partial t}\Psi\left(\frac{g(t)}{\sqrt{t}}\right) = \frac{1}{2}f(t) + \int_0^t \frac{\partial}{\partial t}\Psi\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right) f(s)\,ds \tag{4.1}$$

for all $t > 0$.

Proof. 1. Setting $G(t) = \Psi(g(t)/\sqrt{t})$ and $K(t,s) = \Psi((g(t) - g(s))/\sqrt{t-s})$ for $0 \leq s < t$ we see that (3.2), i.e., (3.4) with $z = g(t)$, reads as follows

$$G(t) = \int_0^t K(t,s)F(ds) \tag{4.2}$$

for all $t > 0$. Note that $K(t,t-) = \psi(0) = 1/2$ for every $t > 0$ since $(g(t)-g(s))/\sqrt{t-s} \to 0$ as $s \uparrow t$ for g that is C^1 on $(0,\infty)$. Note also that

$$\frac{\partial}{\partial t} K(t,s) = \frac{1}{\sqrt{t-s}} \Big(\frac{1}{2} \frac{g(t)-g(s)}{t-s} - g'(t) \Big) \varphi \Big(\frac{g(t)-g(s)}{\sqrt{t-s}} \Big) \tag{4.3}$$

for $0 < s < t$. Hence we see that $(\partial K/\partial t)(t,t-)$ is not finite (whenever $g'(t) \neq 0$), and we thus proceed as follows.

2. Using (4.2) we find by Fubini's theorem that

$$\begin{aligned} &\lim_{\varepsilon \downarrow 0} \int_{t_1}^{t_2} \Big(\int_0^{t-\varepsilon} \frac{\partial}{\partial t} K(t,s) F(ds) \Big)\, dt \\ &\quad = \lim_{\varepsilon \downarrow 0} \Big(\int_0^{t_2-\varepsilon} K(t_2,s) F(ds) - \int_0^{t_1-\varepsilon} K(t_1,s) F(ds) \\ &\qquad - \int_{t_1-\varepsilon}^{t_2-\varepsilon} K(s+\varepsilon,s) F(ds) \Big) \\ &\quad = G(t_2) - G(t_1) - \frac{1}{2} \left(F(t_2) - F(t_1) \right) \end{aligned} \tag{4.4}$$

for $0 < t_1 \le t \le t_2 < \infty$. On the other hand, we see from (4.3) that

$$\Big| \int_0^{t-\varepsilon} \frac{\partial}{\partial t} K(t,s) F(ds) \Big| \le C \int_0^t \frac{F(ds)}{\sqrt{t-s}} \tag{4.5}$$

for all $t \in [t_1, t_2]$ and $\varepsilon > 0$ while, by Fubini's theorem, it is easily verified that

$$\int_{t_1}^{t_2} \Big(\int_0^t \frac{F(ds)}{\sqrt{t-s}} \Big)\, dt < \infty. \tag{4.6}$$

We may thus by the dominated convergence theorem (applied twice) interchange the first limit and the first integral in (4.4) yielding

$$\int_{t_1}^{t_2} \Big(\int_0^t \frac{\partial}{\partial t} K(t,s) F(ds) \Big)\, dt = G(t_2) - G(t_1) - \frac{1}{2} (F(t_2) - F(t_1)) \tag{4.7}$$

at least for those $t \in [t_1, t_2]$ for which

$$\int_0^t \frac{F(ds)}{\sqrt{t-s}} < \infty. \tag{4.8}$$

It follows from (4.6) that the set of all $t > 0$ for which (4.8) fails is of Lebesgue measure zero.

3. To verify (4.8) for all $t > 0$ we may note that a standard rule on the differentiation under an integral sign can be applied in (3.4), and this yields the following equation

$$\frac{1}{\sqrt{t}}\,\varphi\Big(\frac{z}{\sqrt{t}}\Big) = \int_0^t \frac{1}{\sqrt{t-s}}\,\varphi\Big(\frac{z-g(s)}{\sqrt{t-s}}\Big)F(ds) \tag{4.9}$$

for all $z > g(t)$ with $t > 0$ upon differentiating in (3.4) with respect to z. By Fatou's lemma, hence we get

$$\begin{aligned}\int_0^t \frac{1}{\sqrt{t-s}}\,\varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)F(ds)&\\ &= \int_0^t \liminf_{z\downarrow g(t)} \frac{1}{\sqrt{t-s}}\,\varphi\Big(\frac{z-g(s)}{\sqrt{t-s}}\Big)F(ds)\\ &\le \liminf_{z\downarrow g(t)} \int_0^t \frac{1}{\sqrt{t-s}}\,\varphi\Big(\frac{z-g(s)}{\sqrt{t-s}}\Big)F(ds)\\ &= \frac{1}{\sqrt{t}}\,\varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) < \infty\end{aligned} \tag{4.10}$$

for all $t > 0$. Now for $s < t$ close to t, we know that $\varphi((g(t)-g(s))/\sqrt{t-s}\,)$ in (4.10) is close to $1/\sqrt{2\pi} > 0$, and this easily establishes (4.8) for all $t > 0$.

4. Returning back to (4.7) it is easily seen using (4.3) that $t \mapsto \int_0^t (\partial K/\partial t)(t,s)F(ds)$ is right-continuous at $t \in (t_1, t_2)$ if we have

$$\int_t^{t_n} \frac{F(ds)}{\sqrt{t_n - s}} \longrightarrow 0 \tag{4.11}$$

for $t_n \downarrow t$ as $n \to \infty$. To check (4.11) we first note that by passing to the limit for $z \downarrow g(t)$ in (4.9), using (4.8) with the dominated convergence theorem, we obtain (5.1) below for all $t > 0$. Noting that $(s,t) \mapsto \varphi((g(t)-g(s))/\sqrt{t-s})$ attains its strictly positive minimum

$c > 0$ over $0 < t_1 \le t \le t_2$ and $0 \le s < t$, we may write

$$\begin{aligned} \int_t^{t_n} \frac{F(ds)}{\sqrt{t_n - s}} &\le \frac{1}{c} \int_t^{t_n} \frac{1}{\sqrt{t_n - s}}\, \varphi\Big(\frac{g(t_n) - g(s)}{\sqrt{t_n - s}}\Big) F(ds) \\ &= \frac{1}{c}\bigg(\frac{1}{\sqrt{t_n}}\, \varphi\Big(\frac{g(t_n)}{\sqrt{t_n}}\Big) \\ &\qquad - \int_0^t \frac{1}{\sqrt{t_n - s}}\, \varphi\Big(\frac{g(t_n) - g(s)}{\sqrt{t_n - s}}\Big) F(ds)\bigg) \end{aligned} \tag{4.12}$$

where the final expressions tends to zero as $n \to \infty$ by means of (5.1) below and using (4.8) with the dominated convergence theorem. Thus (4.11) holds and therefore $t \mapsto \int_0^t (\partial K/\partial t)(t,s) F(ds)$ is right-continuous. It can be similarly verified that this mapping is left-continuous at each $t \in (t_1, t_2)$ and thus continuous on $(0, \infty)$.

5. Dividing finally by $t_2 - t_1$ in (4.7) and then letting $t_2 - t_1 \to 0$, we obtain

$$F'(t) = 2\Big(G'(t) - \int_0^t \frac{\partial}{\partial t} K(t,s) F(ds)\Big) \tag{4.13}$$

for all $t > 0$. Since the right-hand side of (4.13) defines a continuous function of $t > 0$, it follows that $f = F'$ is continuous on $(0, \infty)$, and the proof is complete. □

5. Derivation of known equations. In the previous proof we saw that the master equation (3.4) can be once differentiated with respect to z implying the equation (4.9) and that in (4.9) one can pass to the limit for $z \downarrow g(t)$ obtaining the following equation

$$\frac{1}{\sqrt{t}}\, \varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = \int_0^t \frac{1}{\sqrt{t-s}}\, \varphi\Big(\frac{g(t) - g(s)}{\sqrt{t-s}}\Big) F(ds) \tag{5.1}$$

for all $t > 0$.

The purpose of this section is to show how the equations (4.1) and (5.1) yield some known equations studied previously by a number of authors.

1. We assume throughout that the hypotheses of Theorem 4.1 are fulfilled (and that $t > 0$ is given and fixed). Rewriting (4.1) more

explicitly by computing derivatives on both sides gives

(5.2)
$$\Big(\frac{1}{2}\,\frac{g(t)}{t^{3/2}} - \frac{g'(t)}{\sqrt{t}}\Big)\varphi\Big(\frac{g(t)}{\sqrt{t}}\Big)$$
$$= \frac{1}{2}\,f(t) + \int_0^t \Big(\frac{1}{2}\,\frac{g(t)-g(s)}{(t-s)^{3/2}} - \frac{g'(t)}{\sqrt{t-s}}\Big)\varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) f(s)\,ds.$$

Recognizing now the identity (5.1) multiplied by $g'(t)$ within (5.2) and multiplying the remaining part of the identity (5.2) by 2, we get

$$\frac{g(t)}{t^{3/2}}\,\varphi\Big(\frac{g(t)}{\sqrt{t}}\Big) = f(t) + \int_0^t \frac{g(t)-g(s)}{(t-s)^{3/2}}\,\varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) f(s)\,ds. \tag{5.3}$$

This equation has been derived and studied by Ricciardi et al. [**22**] using other means. Moreover, the same argument shows that the factor 1/2 can be removed from (5.2) yielding

(5.4)
$$\Big(\frac{g(t)}{t^{3/2}} - \frac{g'(t)}{\sqrt{t}}\Big)\varphi\Big(\frac{g(t)}{\sqrt{t}}\Big)$$
$$= f(t) + \int_0^t \Big(\frac{g(t)-g(s)}{(t-s)^{3/2}} - \frac{g'(t)}{\sqrt{t-s}}\Big)\varphi\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) f(s)\,ds.$$

This equation has been derived independently by Ferebee [**9**] and Durbin [**6**]. Ferebee's derivation is, set aside technical points, the same as the one presented here. Williams [**7**] presents yet another derivation of this equation (assuming that f exists). (Note also that multiplying both sides of (3.7) by $2r(t)$ and both sides of (5.1) by $2(k(t)+g'(t))$, and adding the resulting two equations to the equation (5.3), one obtains the equation (2.9)+(3.4) in Buonocore et al. [**3**] derived by other means.)

2. With a view to the inverse problem (of finding g if f is given) it is of interest to produce as many nonequivalent equations linking g to f as possible. (Recall that (3.7) is a *nonlinear* equation in g if f is known, and nonlinear equations are marked by a nonuniqueness of solutions.) For this reason it is tempting to derive additional equations to the one given in (5.1) starting with the master equation (3.4) and proceeding similarly to (4.9) above.

A standard rule on the differentiation under an integral sign can be inductively applied to (3.4), and this gives the following equations

$$\frac{1}{t^{n/2}}\,\varphi^{(n-1)}\Big(\frac{z}{\sqrt{t}}\Big) = \int_0^t \frac{1}{(t-s)^{n/2}}\,\varphi^{(n-1)}\Big(\frac{z-g(s)}{\sqrt{t-s}}\Big) F(ds) \tag{5.5}$$

for all $z > g(t)$ and all $n \geq 1$ where $t > 0$. Recall that

$$\varphi^{(n)}(x) = (-1)^n h_n(x)\varphi(x) \tag{5.6}$$

for $x \in \mathbf{R}$ and $n \geq 1$ where h_n is a Hermite polynomial of degree n for $n \geq 1$.

Noting that $\varphi'(x) = -x\varphi(x)$ and recalling (5.3) we see that a passage to the limit for $z \downarrow g(t)$ in (5.5) is not straightforward when $n \geq 2$ but complicated. For this reason we will not pursue it in further detail here.

3. The Chapman-Kolmogorov equation (2.11) is known to admit a reduction to the forward and backward equation, cf. [**14**], which are partial differential equations of parabolic type. No such derivation or reduction is generally possible in the entire-past dependent case of the equation (2.16) or (2.17), and the same is true for the master equation (3.4) in particular. We showed above how the differentiation with respect to z in the master equation (3.4) leads to the density equation (5.1), which together with the distribution equation (3.2) yields known equations (5.3) and (5.4). It was also indicated above that no further derivative with respect to z can be taken in the master equation (3.4) so that the passage to the limit for $z \downarrow g(t)$ in the resulting equation becomes straightforward.

6. Derivation of new equations.

1. Expanding on the previous facts a bit further, we now note that it is possible to proceed in a reverse order and integrate the master equation (3.4) with respect to z as many times as we please. This yields the whole spectrum of new nonequivalent equations which, taken together with (3.2) and (5.1), may play a fundamental role in the inverse problem (Section 8).

Theorem 6.1. *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion starting at zero, let $g : (0,\infty) \to \mathbf{R}$ be a continuous function satisfying $g(0+) \geq 0$, let τ in* (3.3) *be the first-passage time of B over g, and let F denote the distribution function of τ.*

Then the following system of integral equations is satisfied:

$$t^{n/2} H_n\left(\frac{g(t)}{\sqrt{t}}\right) = \int_0^t (t-s)^{n/2} H_n\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right) F(ds) \tag{6.1}$$

for $t > 0$ *and* $n = -1, 0, 1, \dots,$ *where we set*

$$H_n(x) = \int_x^\infty H_{n-1}(z)\, dz \tag{6.2}$$

with $H_{-1} = \varphi$ *being the standard normal density from* (3.1).

Remark. For $n = -1$, the equation (6.1) is the density equation (5.1). For $n = 0$, the equation (6.1) is the distribution equation (3.2). All equations in (6.1) for $n \neq -1$ are *nonsingular* (in the sense that their kernels are bounded over the set of all (s, t) satisfying $0 \leq s < t \leq T$).

Proof. Let $t > 0$ be given and fixed. Integrating (3.4) we get

$$\int_z^\infty \Psi\left(\frac{z'}{\sqrt{t}}\right) dz' = \int_0^t \int_z^\infty \Psi\left(\frac{z' - g(s)}{\sqrt{t - s}}\right) dz' F(ds) \tag{6.3}$$

for all $z \geq g(t)$ by means of Fubini's theorem. Substituting $u = z'/\sqrt{t}$ and $v = (z' - g(s))/\sqrt{t - s}$ we can rewrite (6.3) as follows

$$\sqrt{t} \int_{z/\sqrt{t}}^\infty \Psi(u)\, du = \int_0^t \sqrt{t - s} \int_{(z - g(s))/\sqrt{t - s}}^\infty \Psi(v)\, dv\, F(ds) \tag{6.4}$$

which is the same as the following identity

$$\sqrt{t}\, H_1\left(\frac{z}{\sqrt{t}}\right) = \int_0^t \sqrt{t - s}\, H_1\left(\frac{z - g(s)}{\sqrt{t - s}}\right) F(ds) \tag{6.5}$$

for all $z \geq g(t)$ upon using that H_1 is defined by (6.2) above with $n = 1$.

Integrating (6.5) as (3.4) prior to (6.3) above, and proceeding similarly by induction, we get

$$t^{n/2} H_n\left(\frac{z}{\sqrt{t}}\right) = \int_0^t (t - s)^{n/2} H_n\left(\frac{z - g(s)}{\sqrt{t - s}}\right) F(ds) \tag{6.6}$$

for all $z \geq g(t)$ and all $n \geq 1$. (This equation was also established earlier for $n = 0$ in (3.4) and for $n = -1$ in (4.9).) Setting $z = g(t)$ in (6.6) above we obtain (6.1) for all $n \geq 1$. (Using that $\Psi(x) \leq \sqrt{2/\pi}\, \varphi(x)$ for all $x > 0$, it is easily verified by induction that all integrals appearing in (6.1)–(6.6) are finite.) As the equation (6.1) was also proved earlier for $n=0$ in (3.2) and for $n=-1$ in (5.1) above, we see that the system (6.1) holds for all $n \geq -1$, and the proof of the theorem is complete. $\square$

2. In view of our considerations in subsection 1 of Section 5 above it is interesting to establish the analogues of the equations (5.3) and (5.4) in the case of other equations in (6.1).

For this, fix $n \geq 1$ and $t>0$ in the sequel, and note that taking a derivative with respect to t in (6.1) gives

(6.7)
$$\frac{n}{2}\, t^{n/2-1} H_n\Big(\frac{g(t)}{\sqrt{t}}\Big) + t^{n/2} H_n'\Big(\frac{g(t)}{\sqrt{t}}\Big)\Big(\frac{g'(t)}{\sqrt{t}} - \frac{g(t)}{2t^{3/2}}\Big)$$
$$= \int_0^t \bigg(\frac{n}{2}\,(t-s)^{n/2-1}\, H_n\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) + (t-s)^{n/2} H_n'\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\Big(\frac{g'(t)}{\sqrt{t-s}} - \frac{g(t)-g(s)}{2(t-s)^{3/2}}\Big)\bigg) F(ds).$$

Recognizing now the identity (6.1), with $n-1$ instead of n using that $H_n' = H_{n-1}$, multiplied by $g'(t)$ within (6.7), and multiplying the remaining part of the identity (6.7) by 2, we get

$$t^{n/2-1}\bigg(nH_n\Big(\frac{g(t)}{\sqrt{t}}\Big) - \frac{g(t)}{\sqrt{t}}\, H_{n-1}\Big(\frac{g(t)}{\sqrt{t}}\Big)\bigg)$$
$$= \int_0^t (t-s)^{n/2-1}\bigg(nH_n\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) - \frac{g(t)-g(s)}{\sqrt{t-s}}\, H_{n-1}\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\bigg) F(ds). \tag{6.8}$$

Moreover, the same argument shows that the factor $1/2$ can be removed from (6.7) yielding

$$t^{n/2}\bigg(\frac{n}{t}\, H_n\Big(\frac{g(t)}{\sqrt{t}}\Big) - \Big(\frac{g(t)}{t^{3/2}} - \frac{g'(t)}{\sqrt{t}}\Big) H_{n-1}\Big(\frac{g(t)}{\sqrt{t}}\Big)\bigg) \tag{6.9}$$
$$= \int_0^t (t-s)^{n/2}\bigg(\frac{n}{(t-s)}\, H_n\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big) - \Big(\frac{g(t)-g(s)}{(t-s)^{3/2}} - \frac{g'(t)}{\sqrt{t-s}}\Big) \cdot H_{n-1}\Big(\frac{g(t)-g(s)}{\sqrt{t-s}}\Big)\bigg) F(ds).$$

Each of the equations (6.8) and (6.9) is contained in the system (6.1). No equation of the system (6.1) is equivalent to another equation from the same system but itself.

7. A closed expression for the first-passage distribution. In this section we briefly tackle the problem of finding F when g is given using classic theory of linear integral equations (see e.g. [**12**]). The key tool in this approach is the *fixed-point theorem* for *contractive* mappings, which states that a mapping $T : X \to X$, where (X, d) is a complete metric space, satisfying

$$d(T(x), T(y)) \le \beta d(x, y) \tag{7.1}$$

for all $x, y \in X$ with some $\beta \in (0, 1)$ has a unique fixed point in X, i.e., there exists a unique point $x_0 \in X$ such that $T(x_0) = x_0$.

Using this principle and some of its ramifications developed within the theory of integral equations, the papers [**18**] and [**22**] present explicit expressions for F in terms of g in the case when X is taken to be a Hilbert space L^2. These results will here be complemented by describing a narrow class of boundaries g that allow X to be the Banach space $B(\mathbf{R}_+)$ of all bounded functions $h : \mathbf{R}_+ \to \mathbf{R}$ equipped with the sup-norm

$$\|h\|_\infty = \sup_{t \ge 0} |h(t)|. \tag{7.2}$$

While examples from this class range from a constant to a square-root boundary, the approach itself is marked by simplicity of the argument.

Theorem 7.1. *Let $(B_t)_{t \ge 0}$ be a standard Brownian motion started at zero, let $g : \mathbf{R}_+ \to \mathbf{R}$ be a continuous function satisfying $g(0) > 0$, let τ in (3.13) be the first-passage time of B over g, and let F denote the distribution function of τ.*

Assume, moreover, that g is C^1 on $(0, \infty)$, increasing, concave, and that it satisfies

$$g(t) \le g(0) + c\sqrt{t} \tag{7.3}$$

for all $t \ge 0$ with some $c > 0$. Then we have

$$F(t) = h(t) + \sum_{n=1}^{\infty} \left(\int_0^t K_n(t, s) h(s)\, ds \right) \tag{7.4}$$

where the series converges uniformly over all $t \geq 0$*, and we set*

$$h(t) = 2\Psi\left(\frac{g(t)}{\sqrt{t}}\right) \tag{7.5}$$

$$K_1(t,s) = \frac{1}{\sqrt{t-s}}\left(2g'(s) - \frac{g(t)-g(s)}{(t-s)}\right)\varphi\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right) \tag{7.6}$$

$$K_{n+1}(t,s) = \int_s^t K_1(t,r)K_n(r,s)\,dr \tag{7.7}$$

for $0 \leq s < t$ *and* $n \geq 1$.

Moreover, introducing the function

$$R(t,s) = \sum_{n=1}^{\infty} K_n(t,s) \tag{7.8}$$

for $0 \leq s < t$*, the following representation is valid*

$$F(t) = h(t) + \int_0^t R(t,s)h(s)\,ds \tag{7.9}$$

for all $t > 0$.

Proof. Setting $u = \Psi\big((g(t) - g(s))/\sqrt{t-s}\,\big)$ and $v = F(s)$ in the integral equation (3.2) and using the integration by parts formula, we obtain

$$\Psi\left(\frac{g(t)}{\sqrt{t}}\right) = \frac{1}{2}\,F(t) - \int_0^t \frac{\partial}{\partial s}\,\Psi\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right)F(s)\,ds \tag{7.10}$$

for each $t > 0$ that is given and fixed in the sequel. Using the notation of (7.5) and (7.6) above we can rewrite (7.10) as follows

$$F(t) - \int_0^t K_1(t,s)F(s)\,ds = h(t). \tag{7.11}$$

Introduce a mapping T on $B(\mathbf{R}_+)$ by setting

$$(T(G))(t) = h(t) + \int_0^t K_1(t,s)G(s)\,ds \tag{7.12}$$

for $G \in B(\mathbf{R}_+)$. Then (7.11) reads as follows

$$T(F) = F \tag{7.13}$$

and the problem reduces to solve (7.13) for F in $B(\mathbf{R}_+)$.

In view of the fixed-point theorem quoted above, we need to verify that T is a contraction from $B(\mathbf{R}_+)$ into itself with respect to the sup-norm (7.2). For this, note

$$\begin{aligned} \|T(G_1) - T(G_2)\|_\infty &= \sup_{t \geq 0} |(T(G_1 - G_2))(t)| \\ &= \sup_{t \geq 0} \left| \int_0^t K_1(t,s)(G_1(s) - G_2(s))\, ds \right| \\ &\leq \left(\sup_{t \geq 0} \int_0^t |K_1(t,s)|\, ds \right) \|G_1 - G_2\|_\infty . \end{aligned} \tag{7.14}$$

Since $s \to g(s)$ is concave and increasing, it is easily verified that $s \mapsto (g(t) - g(s))/\sqrt{t-s}$ is decreasing and thus $s \mapsto \Psi\big((g(t) - g(s))/\sqrt{t-s}\,\big)$ is increasing on $(0,t)$. It implies that

(7.15)

$$\begin{aligned} \beta := \sup_{t \geq 0} \int_0^t |K_1(t,s)|\, ds &= \sup_{t \geq 0} \int_0^t \left| 2 \frac{\partial}{\partial s} \Psi\left(\frac{g(t) - g(s)}{\sqrt{t-s}} \right) \right| ds \\ = \sup_{t \geq 0} \int_0^t 2 \frac{\partial}{\partial s} \Psi\left(\frac{g(t) - g(s)}{\sqrt{t-s}} \right) ds &= \sup_{t \geq 0} 2 \left(\frac{1}{2} - \Psi\left(\frac{g(t) - g(0)}{\sqrt{t}} \right) \right) \\ \leq 1 - 2\Psi(c) < 1 & \end{aligned}$$

using the hypothesis (7.3). This shows that T is a contraction from the Banach space $B(\mathbf{R}_+)$ into itself, and thus by the fixed-point theorem there exists a unique F_0 in $B(\mathbf{R}_+)$ satisfying (7.13). Since the distribution function F of τ belongs to $B(\mathbf{R}_+)$ and satisfies (7.13), it follows that F_0 must be equal to F.

Moreover, the representation (7.4) follows from (7.11) and the well-known formula for the resolvent of the integral operator $K = T - h$ associated with the kernel K_1:

$$(I - K)^{-1} = \sum_{n=0}^{\infty} K^n \tag{7.16}$$

upon using Fubini's theorem to justify that K_{n+1} in (7.7) is the kernel of the integral operator K^{n+1} for $n \geq 1$. Likewise, the final claim about (7.8) and (7.9) follows by the Fubini-Tonelli theorem since all kernels in (7.6) and (7.7) are nonnegative, and so are all maps $s \mapsto K_n(t,s)h(s)$ in (7.4) as well. This completes the proof. $\square$

Leaving aside the question on usefulness of the multiple-integral series representation (7.4), it is an interesting mathematical question to find a similar expression for F in terms of g that would not require additional hypotheses on g such as (7.3) for instance. In this regard especially those g satisfying $g(0+) = 0$ seem problematic as they lead to singular (or weakly singular) kernels generating the integral operators that turn out to be noncontractive.

8. The inverse problem. In this section we will reformulate the inverse problem of finding g when F is given using the result of Theorem 6.1. Recall from there that g and F solve

$$t^{n/2} H_n\left(\frac{g(t)}{\sqrt{t}}\right) = \int_0^t (t-s)^{n/2} H_n\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right) F(ds) \tag{8.1}$$

for $t > 0$ and $n \geq -1$ where $H_n(x) = \int_x^\infty H_{n-1}(z)\,dz$ with $H_{-1} = \varphi$. Then the inverse problem reduces to answer the following three questions:

Question 8.1. Does there exist a (continuous) solution $t \mapsto g(t)$ of the system (8.1)?

Question 8.2. Is this solution unique?

Question 8.3. Does the (unique) solution $t \mapsto g(t)$ solve the inverse first-passage problem, i.e., is the distribution function of τ from (3.3) equal to F?

It may be noted that each equation in g of the system (8.1) is a *nonlinear Volterra integral equation of the second kind*. Nonlinear equations are known to lead to nonunique solutions, so it is hoped that the totality of countably many equations could counterbalance this deficiency.

Perhaps the main example one should have in mind is when F has a continuous density f. Note that in this case $f(0+)$ can be strictly positive (and finite). Some information on possible behavior of g at zero for such f can be found in [**20**].

A numerical treatment of the inverse first-passage problem is given in a recent Ph.D. thesis by Zucca [**28**].

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