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SOLUTION OF VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS WITH GENERALIZED MITTAG-LEFFLER FUNCTION IN THE KERNELS

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ABSTRACT. The present paper is intended for the investigation of the integro-differential equation of the form

$$(*)\quad (\mathcal{D}_{a+}^{\alpha}y)(x)=\lambda\int_a^x (x-t)^{\mu-1}E_{\rho,\mu}^{\gamma}[\omega(x-t)^{\rho}]y(t)\,dt+f(x),\\ a<x\leqq b,$$

with complex α,ρ,μ,γ and ω ($\mathrm{Re}\,(\alpha),\mathrm{Re}\,(\rho),\mathrm{Re}\,(\mu)>0$) in the space of summable functions $L(a,b)$ on a finite interval $[a,b]$ of the real axis. Here $\mathcal{D}_{a+}^{\alpha}$ is the operator of the Riemann-Liouville fractional derivative of complex order α ($\mathrm{Re}\,(\alpha)>0$) and $E_{\rho,\mu}^{\gamma}(z)$ is the function defined by

$$E_{\rho,\mu}^{\gamma}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\rho k+\mu)}\frac{z^k}{k!},$$

where, when $\gamma=1$, $E_{\rho,\mu}^{1}(z)$ coincides with the classical Mittag-Leffler function $E_{\rho,\mu}(z)$, and in particular $E_{1,1}(z)=e^z$. Thus, when $f(x)\equiv 0$, $a=0$, $\alpha=1$, $\mu=1$, $\gamma=0$, $\rho=1$, $\lambda=-i\pi g$, $\omega=i\nu$, g and ν are real numbers, the equation $(*)$ describes the unsaturated behavior of the free electron laser. The Cauchy-type problem for the above integro-differential equation is considered. It is proved that such a problem is equivalent to the Volterra integral equation of the second kind, and its solution in closed form is established. Special cases are investigated.

1. Introduction. It is well known that solutions of integro-differential equations of Volterra type can be obtained as solutions of

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the corresponding Volterra integral equations of second kind (see, for example, [**15**] and [**18**]). The solutions of the linear Volterra integral equations of second kind are given in terms of the resolvent kernels which are constructed as a series of repeated kernels. In general, these solutions have complicated forms and they are not suitable for the practical interest and for the numerical treatment. Therefore, it is important to construct solutions of Volterra integral equations in closed form by their representation via a finite number of quadratures or in terms of some special functions.

Our paper is accomplished in this direction and is devoted to the study of the integro-differential equation

$$(\mathcal{D}_{a+}^{\alpha}y)(x) = \lambda \int_a^x (x-t)^{\mu-1} E_{\rho,\mu}^{\gamma}[\omega(x-t)^{\rho}]y(t)\,dt + f(x), \quad a < x \leqq b, \tag{1.1}$$

on a finite interval $[a,b]$ of the real axis $\mathbf{R} = (-\infty, \infty)$ with $\alpha, \rho, \mu, \gamma, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu) > 0)$, where $\mathbf{C}$ is the set of complex numbers. Here $(\mathcal{D}_{a+}^{\alpha}y)(x)$ is the Riemann-Liouville fractional derivative of order $\alpha \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha) > 0)$ defined for $a < x \leqq b$ by [**22**, Sections 2.3 and 2.4]

$$(\mathcal{D}_{a+}^{\alpha}y)(x) = \left(\frac{d}{dx}\right)^n (\mathcal{I}_{a+}^{n-\alpha}y)(x), \quad n = [\mathrm{Re}\,(\alpha)] + 1 \tag{1.2}$$

in terms of the Riemann-Liouville fractional integral

$$(\mathcal{I}_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt \quad a < x \leqq b;\ \alpha \in \mathbf{C}\ (\mathrm{Re}\,(\alpha) > 0), \tag{1.3}$$

and $E_{\rho,\mu}^{\gamma}(z)$ is a special function of the form

$$E_{\rho,\mu}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\rho k + \mu)} \frac{z^k}{k!}, \quad \rho, \mu, \gamma \in \mathbf{C},\ \mathrm{Re}\,(\rho) > 0, \tag{1.4}$$

where $(\gamma)_k$ is the Pochhammer symbol [**13**, Section 2.1.1]

$$(\gamma)_0 = 1, \quad (\gamma)_k = \gamma(\gamma+1)\cdots(\gamma+k-1), \quad k = 1, 2, \dots . \tag{1.5}$$

The function $E^{\gamma}_{\rho,\mu}(z)$ was introduced by Prabhakar [**20**]. When $\gamma = 1$,

$$E^{1}_{\rho,\mu}(z) \equiv E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \quad \rho, \mu \in \mathbf{C}, \ \mathrm{Re}\,(\rho) > 0; \tag{1.6}$$

and, in particular, for $\gamma = \mu = 1$,

$$E^{1}_{\rho,1}(z) \equiv E_{\rho}(z) = \sum_{k=0}^{\infty} \frac{z^k}{(\rho k + 1)}, \quad \rho \in \mathbf{C}, \ \mathrm{Re}\,(\rho) > 0. \tag{1.7}$$

The functions in (1.6) and (1.7) are known as Mittag-Leffler functions (see [**14**, Section 18.1]). The classical results in the theory of these functions are given in [**14**, Section 18.1], and modern results in [**11**] and [**12**]. When $\rho = 1$, $E^{\gamma}_{1,\mu}(z)$ coincides with Kummer's confluent hypergeometric function $\Phi(\gamma, \mu; z)$ defined by the ${}_1F_1$-hypergeometric series (see [**13**, Section 6.1]) with the exactness to the constant multiplier $[\Gamma(\mu)]^{-1}$:

$$E^{\gamma}_{1,\mu}(z) = \frac{1}{\Gamma(\mu)} \Phi(\gamma, \mu; z), \tag{1.8}$$

$$\Phi(\gamma, \mu; z) \equiv {}_1F_1(\gamma; \mu; z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{(\mu)_k} \frac{z^k}{k!}, \quad \gamma, \mu \in \mathbf{C}. \tag{1.9}$$

When $\alpha = m \in \mathbf{N} = \{1, 2, \dots\}$, $(\mathcal{D}^{m}_{a+})(x) = y^{(m)}(x)$ and (1.1) is the ordinary integro-differential equation

$$y^{(m)}(x) = \lambda \int_a^x (x-t)^{\mu} E^{\gamma}_{\rho,\mu}[\omega(x-t)^{\rho}] y(t)\, dt + f(x), \quad a < x \leqq b. \tag{1.10}$$

When $a = 0$, such a simplest homogeneous equation in the form

$$y'(x) = -i\pi g \int_0^x (x-t) e^{i\nu(x-t)} y(t)\, dt \qquad 0 < x \leqq 1; \ g, \nu \in \mathbf{R}, \tag{1.11}$$

with the initial condition $y(0) = 1$ describes the unsaturated behavior of the free electron laser [**8**], [**10**] and its solution in closed form was constructed in [**9**]. The fractional analogue of this equation

$$(\mathcal{D}^{\alpha}_{0+} y)(x) = -i\pi g \int_0^x (x-t) e^{i\nu(x-t)} y(t)\, dt \qquad 0 < x \leqq 1; \ g, \nu \in \mathbf{R}, \tag{1.12}$$

with $\alpha > 0$ was considered in [**3**] and [**5**]. In [**3**] there was an attempt for solving this equation, but the obtained solution is not correct because the fractional derivative (1.2) is not invariant with respect to e^{-kx}, $k > 0$. In [**5**] the solution of the equation (1.12) was proved by using the methods of variation of parameters and of successive approximations, and the numerical result was also obtained by employing the algebraic system MAPLE V. We also mention that the multi-dimensional analogues of the equations (1.11) and (1.12) were considered in [**4**]. In this connection (see [**6**, Section IV]).

The simplest inhomogeneous equation (1.1) in the form

$$(\mathcal{D}_{0+}^{\alpha}y)(x) = \lambda \int_0^x (x-t)e^{i\nu(x-t)}y(t)\,dt + \beta e^{i\nu x}, \quad 0 < x \leqq 1;\ \lambda, \beta \in \mathbf{C};\ \nu \in \mathbf{R}, \tag{1.13}$$

was investigated in [**7**] where its solution in closed form was obtained in terms of the Kummer function (1.9) and the tau method of approximation [**19**] was used for the numerical treatment of this solution. Similar results for the fractional integro-differential equations

$$(\mathcal{D}_{0+}^{\alpha}y)(x) = \lambda \int_0^x (x-t)^{\delta}e^{i\nu(x-t)}y(t)\,dt + \beta e^{i\nu x}, \quad 0 < x \leqq 1 \tag{1.14}$$

and

$$(\mathcal{D}_{0+}^{\alpha}y)(x) = \frac{\lambda}{\Gamma(\delta+1)} \int_0^x (x-t)^{\delta}\Phi(b, \delta+1; i\nu(x-t))y(t)\,dt + \beta\Phi(c, 1; i\nu x), \quad 0 < x \leqq 1 \tag{1.15}$$

with $\lambda, \beta, b, c \in \mathbf{C}$, $\nu \in \mathbf{R}$ and $\delta > -1$ were established in [**2**] and [**1**], respectively. It should be noted that, in accordance with (1.7) and (1.9), the equations (1.13) and (1.15) are particular cases of the integro-differential equation (1.1) when $\gamma = \rho = \mu = 1$, $f(x) = \beta e^{i\nu x}$ and $\mu = \delta + 1$, $\rho = 1$, $f(x) = \beta\Phi(c, 1; i\nu x)$, respectively.

The present paper is devoted to the solution in closed form of the integro-differential equation (1.1) with the Cauchy-type initial conditions

$$\lim_{x \to +a} (\mathcal{D}_{a+}^{\alpha-k}y)(x) = b_k, \quad k = 1, 2, \dots, n = -[-\mathrm{Re}\,(\alpha)], \tag{1.16}$$

with $\mathcal{D}_{a+}^{\alpha-n}y$ being understood as $\mathcal{D}_{a+}^{\alpha-n}y = \mathcal{I}_{a+}^{n-\alpha}y$, in the space $L(a,b)$ of summable functions on the interval $[a,b]$. Properties of the generalized Mittag-Leffler function (1.4) and of the integral operators involving such a function in the kernels were investigated in our paper [**17**]. Here we apply the results in [**17**] to reduce the problem (1.1), (1.16) to the equivalent Volterra integral equation of second kind

$$
\begin{aligned}
y(x) = & \sum_{k-1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)}(x-a)^{\alpha-k} \\
& + \lambda \int_a^x (x-t)^{\mu+\alpha-1} E_{\rho,\mu+\alpha}^{\gamma}[\omega(x-t)^{\rho}]y(t)\,dt \\
& + \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt, \quad a < x \leqq b.
\end{aligned} \tag{1.17}
$$

The solution of this equation is established by the method of successive approximations, see, for example, [**21**].

The paper is organized as follows. Section 2 has preliminary character and contains the results concerning some properties of the fractional calculus operators (1.2), (1.3) and of the integral operators with the generalized Mittag-Leffler function (1.4) in the kernels. Section 3 deals with the solution in closed form of the integro-differential equation (1.1) with the initial condition (1.16). Solutions of special cases are studied in Sections 4 and 5.

2. Preliminaries. First we give some properties of the operators of the Riemann-Liouville fractional integrals $\mathcal{I}_{a+}^{\alpha}$ and the fractional derivatives $\mathcal{D}_{a+}^{\alpha}$ given in (1.3) and (1.2), respectively. The operator $\mathcal{I}_{a+}^{\alpha}$ is defined on the space $L(a,b)$ of Lebesgue measurable functions $g(x)$ on $[a,b]$:

$$
L(a,b) = \left\{ g : \|g\|_1 \equiv \int_a^b |g(t)|\,dt < +\infty \right\}. \tag{2.1}
$$

Lemma 1 [**22**]. *The Riemann-Liouville fractional integral operator* $\mathcal{I}_{a+}^{\alpha}$ *of order* $\alpha \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha) > 0)$ *is bounded in the space* $L(a,b)$ *and*

$$
\|\mathcal{I}_{a+}^{\alpha}\varphi\|_1 \leqq A\|\varphi\|_1, \quad A = \frac{(b-a)^{\mathrm{Re}\,(\alpha)}}{\mathrm{Re}\,(\alpha)|\Gamma(\alpha)|}. \tag{2.2}
$$

Lemma 2 [**22**]. *If* $\alpha, \beta \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\beta) > 0)$*, then the semigroup property*

$$\mathcal{I}^{\alpha}_{a+}\mathcal{I}^{\beta}_{a+}\varphi = \mathcal{I}^{\alpha+\beta}_{a+}\varphi \tag{2.3}$$

holds for any $\varphi \in L(a,b)$.

The Riemann-Liouville fractional derivative (1.2) is the inverse to the corresponding fractional integral (1.3) from the left.

Lemma 3 [**22**, Theorem 2.4]. *If* $\alpha \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha) > 0)$*, then the equality*

$$\mathcal{D}^{\alpha}_{a+}\mathcal{I}^{\alpha}_{a+}\,\varphi = \varphi \tag{2.4}$$

holds for any summable function $\varphi \in L(a,b)$.

We also need the assertion on the equivalence of the Cauchy-type problem for the fractional differential equation and the corresponding Volterra integral equation.

Lemma 4 [**16**]. *Let* $\alpha \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha) > 0)$*,* $n = -[-\mathrm{Re}\,(\alpha)]$*, and let the functions* $y(x)$ *on* $[a,b]$ *and* $f(x,y)$ *on* $[a,b] \times \mathbf{R}$ *be given such that* $y(x) \in L(a,b)$ *and* $f[x,y(x)] \in L(a,b)$*. Then the solution of the Cauchy-type problem*

$$(\mathcal{D}^{\alpha}_{a+}y)(x) = f[x,y(x)], \quad a < x \leqq b, \tag{2.5}$$

$$\lim_{x\to+a}(\mathcal{D}^{\alpha-k}_{a+}y)(x) = b_k, \quad k = 1,2,\dots,n, \tag{2.6}$$

in the space $L(a,b)$ *is equivalent to the solution of the Volterra integral equation of second kind*

$$y(x) = \sum_{k=1}^{n}\frac{b_k}{\Gamma(\alpha-k+1)}(x-a)^{\alpha-k} + \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f[t,y(t)]}{(x-t)^{1-\alpha}}\,dt, \quad a < x \leqq b. \tag{2.7}$$

There holds the following relation [**17**] for the generalized Mittag-Leffler function (1.4):

$$\int_0^x (x-t)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega[x-t]^{\rho})t^{\nu-1}E^{\sigma}_{\rho,\nu}(\omega t^{\rho})\,dt = x^{\mu+\nu-1}E^{\gamma+\sigma}_{\rho,\mu+\nu}(\omega x^{\rho}) \tag{2.8}$$

for $\rho, \mu, \nu, \gamma, \sigma, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu), \mathrm{Re}\,(\nu) > 0)$. Now we consider the integral operator

$$(2.9)\quad (\mathbf{E}^{\gamma}_{\rho,\mu,\omega;\alpha+}\,\varphi)(x) = \int_a^x (x-t)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(x-t)^{\rho}]\varphi(t)\,dt, \quad x > a$$

with the generalized Mittag-Leffler function (1.4) in the kernel. In particular, when $\gamma = 1$ and $\rho = 1$, we have the operators

(2.10)
$$(\mathbf{E}_{\rho,\mu,\omega;a+}\,\varphi)(x) = \int_a^x (x-t)^{\mu-1} E_{\rho,\mu}[\omega(x-t)^{\rho}]\varphi(t)\,dt, \quad x > a$$

and

(2.11)
$$(\Phi_{\gamma,\mu,\omega;a+}\,\varphi)(x) = \int_a^x (x-t)^{\mu-1} \Phi[\gamma,\mu,\omega(x-t)]\varphi(t)\,dt, \quad x > 0,$$

containing the Mittag-Leffler function (1.6) and the Kummer hypergeometric function (1.9) in the kernels, respectively. When $\gamma = 0$,

$$(2.12)\qquad E^{0}_{\rho,\mu}(z) = \frac{1}{\Gamma(\mu)},$$

and hence

$$(2.13)\qquad (\mathbf{E}^{0}_{\mu,\rho,\omega;a+}\,\varphi)(x) = (\mathcal{I}^{\mu}_{a+}\,\varphi)(x).$$

The operator $\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}$ is bounded in $L(a,b)$.

Theorem 1 [**17**, Theorem 4]. *Let* $\rho, \mu, \gamma, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu) > 0)$, *then the operator* $\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}$ *is bounded on* $L(a,b)$ *and*

$$(2.14)\qquad \|\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}\,\varphi\|_1 \leqq B\|\varphi\|_1,$$

where

(2.15)
$$B = (b-a)^{\mathrm{Re}\,(\mu)} \sum_{k=0}^{\infty} \frac{|(\gamma)_k|}{|\Gamma(\rho k+\mu)|[\mathrm{Re}\,(\rho)k + \mathrm{Re}\,(\mu)]}\, \frac{|\omega(b-a)^{\mathrm{Re}\,(\rho)}|^k}{k!}.$$

Corollary 1.1. *Let* $\rho, \mu, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu) > 0)$, *then the operator* $\mathbf{E}_{\rho,\mu,\omega;a+}$ *is bounded on* $L(a,b)$ *and*

$$\|\mathbf{E}_{\rho,\mu,\omega;a+}\,\varphi\|_1 \leqq B_1 \|\varphi\|_1, \tag{2.16}$$

where

$$B_1 = (b-a)^{\mathrm{Re}\,(\mu)} \sum_{k=0}^{\infty} \frac{|\omega(b-a)^{\mathrm{Re}\,(\rho)}|^k}{|\Gamma(\rho k+\mu)|[\mathrm{Re}\,(\rho)k + \mathrm{Re}\,(\mu)]}. \tag{2.17}$$

Corollary 1.2. *Let* $\mu, \gamma, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\mu) > 0)$, *then the operator* $\Phi_{\gamma,\mu,\omega;a+}$ *is bounded on* $L(a,b)$ *and*

$$\|\Phi_{\gamma,\mu,\omega;a+}\,\varphi\|_1 \leq B_2 \|\varphi\|_1, \tag{2.18}$$

where

$$B_2 = (b-a)^{\mathrm{Re}\,(\mu)} \sum_{k=0}^{\infty} \frac{|(\gamma)_k|}{|\Gamma(k+\mu)|[k+\mathrm{Re}\,(\mu)]}\,\frac{|\omega(b-a)^{\mathrm{Re}\,(\mu)}|^k}{k!}. \tag{2.19}$$

The integral transform $\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}$ of power function $(t-a)^{\beta-1}$ yields the same generalized Mittag-Leffler function.

Lemma 5 [**17**, Lemma 4]. *Let* $\rho, \mu, \gamma, \omega, \beta \in \mathbf{C}$ $(\mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu), \mathrm{Re}\,(\beta) > 0)$, *then*

$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}[(t-a)^{\beta-1}])(x) = \Gamma(\beta)(x-a)^{\mu+\beta-1} E^{\gamma}_{\rho,\mu+\beta}(\omega(x-a)^{\rho}). \tag{2.20}$$

In particular,

$$(\mathbf{E}_{\rho,\mu,\omega;a+}[(t-a)^{\beta-1}])(x) = \Gamma(\beta)(x-a)^{\mu+\beta-1} E_{\rho,\mu+\beta}(\omega(x-a)^{\rho}) \tag{2.21}$$

and

$$(\Phi_{\gamma,\mu,\omega;a+}[(t-a)^{\beta-1}])(x) = \frac{\Gamma(\mu)}{\Gamma(\mu+\beta)}\,(x-a)^{\mu+\beta-1} \Phi(\gamma, \mu+\beta; \omega(x-a)). \tag{2.22}$$

Compositions of the operator $\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}$ with the Riemann-Liouville fractional integration operator $\mathcal{I}^{\alpha}_{a+}$ are given by the following:

Theorem 2 [**17**, Theorem 6]. *Let* $\alpha, \rho, \mu, \gamma, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu) > 0)$, *then the relations*

$$\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}\mathcal{I}^{\alpha}_{a+}\,\varphi = \mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+}\,\varphi = \mathcal{I}^{\alpha}_{a+}\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}\,\varphi \tag{2.23}$$

hold for any summable function $\varphi \in L(a,b)$.

Corollary 2.1. *Let* $\alpha, \rho, \mu, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu) > 0)$, *then for any* $\varphi \in L(a,b)$

$$\mathbf{E}_{\rho,\mu,\omega;a+}\mathcal{I}^{\alpha}_{a+}\,\varphi = \mathbf{E}_{\rho,\mu+\alpha,\omega;a+}\,\varphi = \mathcal{I}^{\alpha}_{a+}\mathbf{E}_{\rho,\mu,\omega;a+}\,\varphi. \tag{2.24}$$

Corollary 2.2. *Let* $\alpha, \mu, \gamma, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\mu) > 0)$*, then for any* $\varphi \in L(a,b)$,

$$\Phi_{\gamma,\mu,\omega;a+}\mathcal{I}^{\alpha}_{a+}\,\varphi = \frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}\,\Phi_{\gamma,\mu+\alpha,\omega;a+}\,\varphi = \mathcal{I}^{\alpha}_{a+}\Phi_{\gamma,\mu,\omega;a+}\,\varphi. \tag{2.25}$$

The next results generalize the semigroup property (2.3) to the integral operators of the form (2.9) with the generalized Mittag-Leffler function in the kernel.

Theorem 3 [**17**, Theorem 8]. *Let* $\rho, \mu, \gamma, \nu, \sigma, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu), \mathrm{Re}\,(\nu) > 0)$, *then the relation*

$$\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}\mathbf{E}^{\sigma}_{\rho,\nu,\omega;a+}\,\varphi = \mathbf{E}^{\gamma+\sigma}_{\rho,\mu+\nu,\omega;a+}\,\varphi = \mathbf{E}^{\sigma}_{\rho,\nu,\omega;a+}\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}\,\varphi \tag{2.26}$$

is valid for any summable function $\varphi \in L(a,b)$*. In particular,*

$$\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}\mathbf{E}^{-\gamma}_{\rho,\nu,\omega;a+}\,\varphi = \mathcal{I}^{\mu+\nu}_{a+}\,\varphi = \mathbf{E}^{-\gamma}_{\rho,\nu,\omega;a+}\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}\,\varphi. \tag{2.27}$$

Corollary 3.1. *Let* $\rho, \mu, \nu, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\mu), \mathrm{Re}\,(\nu) > 0)$, *then for any* $\varphi \in L(a,b)$

$$\mathbf{E}_{\rho,\mu,\omega;a+}\mathbf{E}_{\rho,\nu,\omega;a+}\,\varphi = \mathbf{E}^{2}_{\rho,\mu+\nu,\omega;a+}\,\varphi = \mathbf{E}_{\rho,\nu,\omega;\alpha+}\mathbf{E}_{\rho,\mu,\omega;a+}\,\varphi. \tag{2.28}$$

Corollary 3.2. *Let* $\mu, \gamma, \nu, \sigma, \omega \in \mathbf{C}$ $(\mathrm{Re}\,(\mu), \mathrm{Re}\,(\nu) > 0)$, *then for any* $\varphi \in L(a,b)$

$$\Phi_{\gamma,\mu,\omega;a+}\Phi_{\sigma,\nu,\omega;a+}\,\varphi = \frac{\Gamma(\mu)}{\Gamma(\mu+\nu)}\,\Phi_{\gamma+\sigma,\mu+\nu,\omega;a+}\,\varphi = \Phi_{\sigma,\nu,\omega;a+}\Phi_{\gamma,\mu,\omega;a+}\,\varphi. \tag{2.29}$$

3. Solution of the Cauchy-type problem for the fractional integro-differential equation with generalized Mittag-Leffler function in the kernel. First we prove that a solution of the Cauchy-type problem of (1.1) with (1.16) is equivalent to a solution of the Volterra integral equation (1.17).

Lemma 6. *Let* $\alpha, \rho, \mu, \gamma, \omega, \lambda \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu) > 0)$. *If* $f(x) \in L(a,b)$, *then the solution of the Cauchy-type problem*

$$(\mathcal{D}^{\alpha}_{a+}y)(x) = \lambda \int_a^x (x-t)^{\mu-1}E^{\gamma}_{\rho,\mu}[\omega(x-t)^{\rho}]y(t)\,dt + f(x), \quad a<x\leqq b, \tag{3.1}$$

$$\lim_{x\to+a}(\mathcal{D}^{\alpha-k}_{a+}y)(x) = b_k, \quad k=1,2,\dots,n=-[-\mathrm{Re}\,(\alpha)], \tag{3.2}$$

in the space $L(a,b)$ *is equivalent to the solution of the Volterra integral equation*

$$\begin{aligned} y(x) = {} & \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)}(x-a)^{\alpha-k} \\ & + \lambda \int_a^x (x-t)^{\alpha+\mu-1}E^{\gamma}_{\rho,\mu+\alpha}[\omega(x-t)^{\rho}]y(t)\,dt \\ & + \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt, \quad a<x\leqq b. \end{aligned} \tag{3.3}$$

Proof. Using (2.9) we rewrite the equation (3.1) in the form

$$(\mathcal{D}^{\alpha}_{a+}y)(x) = \lambda(\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}y)(x) + f(x), \quad a<x\leqq b. \tag{3.4}$$

Since the solution y of the equation (3.4) belongs to $L(a,b)$, then, in view of Theorem 1, $\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}y$ also belongs to $L(a,b)$. By the conditions of the lemma, the righthand side of (3.4) belongs to $L(a,b)$. Thus by Lemma 4, the solution of the problem (3.1)–(3.2) in $L(a,b)$ is equivalent to the solution of the integral equation

$$y(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)}(x-a)^{\alpha-k} + \lambda(\mathcal{I}^{\alpha}_{a+}\mathbf{E}^{\gamma}_{\rho,\mu,\omega;a+}y)(x) + (\mathcal{I}^{\alpha}_{a+}f)(x), \quad a < x \leqq b. \tag{3.5}$$

By virtue of (2.23), the second term of the righthand side is $(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+}y)(x)$ and (3.5) is just (3.3).

Our next main result gives the solution in closed form of the Cauchy-type problem (3.1)–(3.2).

Theorem 4. *Let* $\alpha, \rho, \mu, \gamma, \omega, \lambda \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu) > 0)$. *If* $f \in L(a,b)$, *then the Cauchy-type problem* (3.1)–(3.2) *is solvable in the space* $L(a,b)$ *and its unique solution is given by*

$$y(x) = \sum_{k=1}^{n} b_k y_k(x) + \int_a^x K(x-t)f(t)\,dt, \tag{3.6}$$

where

$$y_k(x) = (x-a)^{\alpha-k}\sum_{j=0}^{\infty}\lambda^j (x-a)^{(\mu+\alpha)}E^{\gamma j}_{\rho,(\mu+\alpha)j+\alpha-k+1}[\omega(x-a)^{\rho}], \quad k = 1, 2, \dots, n \tag{3.7}$$

and

$$K(u) = \sum_{j=0}^{\infty}\lambda^j u^{(\mu+\alpha)j+\alpha-1}E^{\gamma j}_{\rho,(\mu+\alpha)j+\alpha}(\omega u^{\rho}). \tag{3.8}$$

Proof. By Lemma 6 it is sufficient to solve the Volterra integral equation (3.3) or the equation (3.5). By the theory of Volterra integral equations of the second kind (see, for example, [**19**]) such an integral equation has a unique solution $y(x) \in L(a,b)$. To find the exact solution we apply the method of successive approximation (see, for example [**21**]) and consider the sequence $y_m(x)$ defined by

$$y_0(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} (x-a)^{\alpha-k}, \tag{3.9}$$

$$\begin{aligned} y_m(x) &= y_0(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} y_{m-1})(x) \\ &\quad + (\mathcal{I}^{\alpha}_{a+} f)(x), \quad m = 1, 2, \dots . \end{aligned} \tag{3.10}$$

For $m = 1$,

$$y_1(x) = y_0(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} y_0)(x) + (\mathcal{I}^{\alpha}_{a+} f)(x). \tag{3.11}$$

$y_2(x)$ is

$$y_2(x) = y_0(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} y_1)(x) + (\mathcal{I}^{\alpha}_{a+} f)(x)$$

and, in accordance with (3.11),

$$\begin{aligned} y_2(x) &= y_0(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} y_0)(x) \\ &\quad + \lambda^2(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} \mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} y_0)(x) \\ &\quad + (\mathcal{I}^{\alpha}_{a+} f)(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} \mathcal{I}^{\alpha}_{a+} f)(x). \end{aligned}$$

Applying (2.26) and (2.23) to the last relation, we obtain

$$\begin{aligned} y_2(x) &= y_0(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} y_0)(x) \\ &\quad + \lambda^2(\mathbf{E}^{2\gamma}_{\rho,2(\mu+\alpha),\omega;a+} y_0)(x) \\ &\quad + (\mathcal{I}^{\alpha}_{a+} f)(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+2\alpha,\omega;a+} f)(x). \end{aligned} \tag{3.12}$$

Similarly, for $m = 3$, we have

(3.13)
$$\begin{aligned} y_3(x) &= y_0(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} y_2)(x) + (\mathcal{I}^{\alpha}_{a+} f)(x) \\ &= y_0(x) + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+\alpha,\omega;a+} y_0)(x) + \lambda^2(\mathbf{E}^{2\gamma}_{\rho,2(\mu+\alpha),\omega;a+} y_0)(x) \\ &\quad + \lambda^3(\mathbf{E}^{3\gamma}_{\rho,3(\mu+\alpha),\omega;a+} y_0)(x) + (\mathcal{I}^{\alpha}_{a+} f)(x) \\ &\quad + \lambda(\mathbf{E}^{\gamma}_{\rho,\mu+2\alpha,\omega;a+} f)(x) + \lambda^2(\mathbf{E}^{2\gamma}_{\rho,2(\mu+\alpha)+\alpha,\omega;a+} f)(x) \\ &= y_0(x) + \sum_{j=1}^{3} \lambda^j (\mathbf{E}^{\gamma j}_{\rho,j(\mu+\alpha),\omega;a+} y_0)(x) \\ &\quad + (\mathcal{I}^{\alpha}_{a+} f)(x) + \sum_{j=1}^{2} \lambda^j (\mathbf{E}^{\gamma j}_{\rho,j(\mu+\alpha)+\alpha,\omega;a+} f)(x). \end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned} y_m(x) &= y_0(x) + \sum_{j=1}^{m} \lambda^j (\mathbf{E}^{\gamma j}_{\rho,j(\mu+\alpha),\omega;a+} y_0)(x) \\ &\quad + (\mathcal{I}^{\alpha}_{a+} f)(x) + \sum_{j=1}^{m-1} \lambda^j (\mathbf{E}^{\gamma j}_{\rho,j(\mu+\alpha)+\alpha,\omega;a+} f)(x). \end{aligned} \tag{3.14}$$

By virtue of (2.12)

$$\begin{aligned} y_0(x) &= \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} (x-a)^{\alpha-k} \\ &= \sum_{k=1}^{n} b_k (x-a)^{\alpha-k} E^{0}_{\rho,\alpha-k+1}[\omega(x-a)^{\rho}] \end{aligned} \tag{3.15}$$

and by (2.20),

(3.16)
$$\begin{aligned} (\mathbf{E}^{\gamma j}_{\rho,j(\mu+\alpha),\omega;a+} y_0)(x) &= \sum_{k=1}^{n} \frac{b_k}{\Gamma(\alpha-k+1)} \\ &\quad \cdot \big(\mathbf{E}^{\gamma j}_{\rho,j(\mu+\alpha),\omega;a+} [(t-a)^{\alpha-k}]\big)(x) \\ &= \sum_{k=1}^{n} b_k (x-a)^{j(\mu+\alpha)+\alpha-k} E^{\gamma j}_{\rho,j(\mu+\alpha)+\alpha-k+1,\omega;a+} \\ &\quad \cdot [\omega(x-a)^{\rho}], \quad j = 1.\dots, m. \end{aligned}$$

In accordance with (2.13)

$$(\mathcal{I}_{a+}^{\alpha} f)(x) = (\mathbf{E}_{\alpha,\rho,\omega;a+}^{0} f)(x). \tag{3.17}$$

Using (3.15), (3.16) and (3.17), we rewrite (3.14) in the form

$$\begin{aligned} y_m(x) = \sum_{k=1}^{n} b_k (x-a)^{\alpha-k} \Bigg\{ \sum_{j=0}^{m} & \lambda^j (x-a)^{(\mu+\alpha)j} \\ & \cdot E_{\rho,(\mu+\alpha)j+\alpha-k+1}^{\gamma j}[\omega(x-a)^{\rho}] \Bigg\} \\ + \sum_{j=0}^{m-1} \lambda^j (\mathbf{E}_{\rho,j(\mu+\alpha)+\alpha,\omega;a+}^{\gamma j} f)(x). \end{aligned} \tag{3.18}$$

Passing to the limit as $m \to \infty$, we obtain the following representation for the solution $y(x)$:

$$\begin{aligned} y(x) = \sum_{k=1}^{n} b_k (x-a)^{\alpha-k} \Bigg\{ \sum_{j=0}^{\infty} & \lambda^j (x-a)^{(\mu+\alpha)j} \\ & \cdot E_{\rho,(\mu+\alpha)j+\alpha-k+1}^{\gamma j}[\omega(x-a)^{\rho}] \Bigg\} \\ + \sum_{j=0}^{\infty} \lambda^j (\mathbf{E}_{\rho,j(\mu+\alpha)+\alpha,\omega;a+}^{\gamma j} f)(x). \end{aligned} \tag{3.19}$$

If we take (3.7) and (3.8) into account, then (3.19) can be represented in the form (3.6). This completes the proof of the theorem.

4. Solution of the Cauchy-type problem for fractional integro-differential equations with the Mittag-Leffler function and the Kummar function in the kernels. The results in the previous section can be applied to solve the Cauchy-type problems

$$(\mathcal{D}_{a+}^{\alpha} y)(x) = \lambda \int_a^x (x-t)^{\alpha-1} E_{\rho,\mu}[\omega(x-t)^{\rho}] y(t)\, dt + f(x), \quad a < x \leqq b, \tag{4.1}$$

$$\lim_{x \to +a} (\mathcal{D}_{a+}^{\alpha-k} y)(x) = b_k, \quad k = 1, 2, \dots, n = -[-\mathrm{Re}\,(\alpha)], \tag{4.2}$$

and

(4.3)
$$(\mathcal{D}_{a+}^{\alpha}y)(x) = \frac{\lambda}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}\Phi(\gamma,\mu;\omega(x-t))y(t)\,dt + f(x),$$
$$a < x \leqq b,$$

(4.4) $$\lim_{x\to+a}(\mathcal{D}_{a+}^{\alpha-k}y)(x) = b_k, \quad k = 1,2,\ldots,n = -[-\mathrm{Re}\,(\alpha)],$$

containing the Mittag-Leffler function (1.4) and the Kummer function (1.9) in the kernels, respectively.

Theorem 4 and (1.6) imply

Theorem 5. *Let* $\alpha,\rho,\mu,\omega,\lambda \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha),\mathrm{Re}\,(\rho),\mathrm{Re}\,(\mu) > 0)$. *If* $f \in L(a,b)$*, then the Cauchy-type problem* (4.1)–(4.2) *is solvable in the space* $L(a,b)$ *and its unique solution is given by*

(4.5) $$y(x) = \sum_{k=1}^{n} b_k y_k(x) + \int_a^x K(x-t)f(t)\,dt,$$

where

(4.6) $$y_k(x) = (x-a)^{\alpha-k}\sum_{j=0}^{\infty}\lambda^j (x-a)^{(\mu+\alpha)j}E^j_{\rho,(\mu+\alpha)j+\alpha-k+1}\,[\omega((x-a)^{\rho})]$$

for $k = 1,2,\ldots,n$*, and*

(4.7) $$K(u) = \sum_{j=0}^{\infty}\lambda^j u^{(\mu+\alpha)j+\alpha-1}E^j_{\rho,(\mu+\alpha)j+\alpha}\,(\omega u^{\rho}).$$

Theorem 6. *Let* $\alpha,\mu,\gamma,\omega,\lambda \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha),\mathrm{Re}\,(\mu) > 0)$. *If* $f \in L(a,b)$*, then the Cauchy-type problem* (4.3)–(4.4) *is solvable in the space* $L(a,b)$ *and its unique solution is given by*

(4.8) $$y(x) = \sum_{k=1}^{n} b_k y_k(x) + \int_a^x K(x-t)f(t)\,dt,$$

where

$$y_k(x) = (x-a)^{\alpha-k} \sum_{j=0}^{\infty} \lambda^j (x-a)^{(\mu+\alpha)j} \cdot \frac{1}{\Gamma[j(\mu+\alpha)+\alpha-k+1]} \cdot \Phi(\gamma j, (\mu+\alpha)j+\alpha-k+1; \omega(x-a)) \tag{4.9}$$

for $k = 1, 2, \dots, n$, *and*

$$K(u) = \sum_{j=0}^{\infty} \lambda^j u^{(\mu+\alpha)j+\alpha-1} \frac{1}{\Gamma[j(\mu+\alpha)+\alpha]} \Phi(\gamma j, (\mu+\alpha)j+\alpha; \omega u). \tag{4.10}$$

5. Solutions of the Cauchy-type problems in special cases. Now we treat the solution of the Cauchy-type problem of the form (3.1)–(3.2) with $f(x) = p(x-a)^{\nu-1} E^{\sigma}_{\rho,\nu}[\omega(x-a)^\rho]$:

$$(\mathcal{D}^{\alpha}_{a+} y)(x) = \lambda \int_a^x (x-t)^{\mu-1} E^{\gamma}_{\rho,\mu}[\omega(x-t)^\rho] y(t)\, dt + p(x-a)^{\nu-1} E^{\sigma}_{\rho,\nu}[\omega(x-a)^\rho], \quad a < x \leqq b, \tag{5.1}$$

$$\lim_{x\to+a} (\mathcal{D}^{\alpha-k}_{a+} y)(x) = b_k, \quad k = 1, 2, \dots, n = -[-\mathrm{Re}\,(\alpha)]. \tag{5.2}$$

From Theorem 4 we deduce the following result.

Theorem 7. *Let* $\alpha, \rho, \mu, \nu, \gamma, \sigma, \omega, p, \lambda \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu), \mathrm{Re}\,(\nu) > 0)$. *Then the Cauchy-type problem* (5.1)–(5.2) *is solvable in the space* $L(a,b)$ *and its unique solution is given by*

$$y(x) = \sum_{k=1}^{n} b_k y_k(x) + p(x-a)^{\alpha+\nu-1} \sum_{j=0}^{\infty} \lambda^j (x-a)^{(\mu+\alpha)j} \cdot E^{\gamma j+\sigma}_{\rho,(\mu+\alpha)j+\alpha+\nu}[\omega(x-a)^\rho], \tag{5.3}$$

where $y_k(x)$, $k = 1, 2, \dots, n$, *are given by* (3.7).

Proof. We apply Theorem 4 with $f(x) = p(x-a)^{\nu-1}E^{\sigma}_{\rho,\nu}[\omega(x-a)^{\rho}]$. Since such an f belongs to $L(a,b)$, then by Theorem 4 the problem (5.1)–(5.2) is solvable in $L(a,b)$ and its unique solution has the form

$$y(x) = \sum_{k=1}^{n} b_k y_k(x) + p \sum_{j=0}^{\infty} \lambda^j K_j(x), \tag{5.4}$$

where

$$K_j(x) = \int_a^x (x-t)^{(\mu+\alpha)j+\alpha-1} E^{\gamma j}_{\rho,(\mu+\alpha)j+\alpha} \cdot [\omega(x-t)^{\rho}](t-a)^{\nu-1}E^{\sigma}_{\rho,\nu}[\omega(t-a)^{\rho}]\, dt \quad j = 0, 1, 2, \dots . \tag{5.5}$$

Making the change of variable $t - a = \tau$ and applying (2.8) with x being replaced by $x - a$, we find

$$K_j(x) = (x-a)^{(\mu+\alpha)j+\alpha+\nu-1} E^{\gamma j+\sigma}_{\rho,(\mu+\alpha)j+\alpha+\nu}[\omega(x-a)^{\rho}]. \tag{5.6}$$

Substituting (5.6) into (5.4), we obtain (5.3), and the theorem is proved.

Similarly from Theorems 5 and 6 we deduce the following statements which present solutions in closed form of the Cauchy-type problems (4.1)–(4.2) and (4.3)–(4.4) with the special function $f(x) = p(x-a)^{\nu-1}E_{\rho,\nu}[\omega(x-a)^{\rho}]$ and $f(x) = p(x-a)^{\nu-1}\Phi[\sigma,\nu;\omega(x-t)]/\Gamma(\nu)$, respectively.

Theorem 8. *Let* $\alpha, \rho, \mu, \nu, \omega, p, \lambda \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\rho), \mathrm{Re}\,(\mu), \mathrm{Re}\,(\nu) > 0)$. *Then the Cauchy-type problem*

$$(\mathcal{D}^{\alpha}_{a+}y)(x) = \lambda \int_a^x (x-t)^{\mu-1}E_{\rho,\mu}[\omega(x-t)^{\rho}]y(t)\, dt + p(x-a)^{\nu-1}E_{\rho,\nu}[\omega(x-a)^{\rho}], \quad a < x \leqq b, \tag{5.7}$$

$$\lim_{x\to+a} (\mathcal{D}^{\alpha-k}_{a+}y)(x) = b_k, \quad k = 1, 2, \dots, n = -[-\mathrm{Re}\,(\alpha)], \tag{5.8}$$

is solvable in the space $L(a,b)$ *and its unique solution is given by*

$$(5.9)\qquad \begin{aligned} y(x) = \sum_{k=1}^{n} b_k y_k(x) + p(x-a)^{\alpha+\nu-1} \sum_{j=0}^{\infty} \lambda^j (x-a)^{(\mu+\alpha)j} \\ \cdot E^{j+1}_{\rho,(\mu+\alpha)j+\alpha+\nu}[\omega(x-a)^{\rho}], \end{aligned}$$

where $y_k(x)$, $k = 1, 2, \dots, n$, *are given by* (4.6).

Theorem 9. *Let* $\alpha, \mu, \nu, \gamma, \sigma, \omega, p, \lambda \in \mathbf{C}$ $(\mathrm{Re}\,(\alpha), \mathrm{Re}\,(\mu), \mathrm{Re}\,(\nu) > 0)$. *Then the Cauchy-type problem*

$$(5.10)\qquad \begin{aligned} (\mathcal{D}^{\alpha}_{a+} y)(x) = \frac{\lambda}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} \Phi[\gamma, \mu; \omega(x-t)] y(t)\, dt \\ + \frac{p}{\Gamma(\nu)} (x-a)^{\nu-1} \Phi[\sigma, \nu; \omega(x-t)], \quad a < x \leqq b, \end{aligned}$$

$$(5.11)\qquad \lim_{x \to +a} (\mathcal{D}^{\alpha-k}_{a+} y)(x) = b_k, \quad k = 1, 2, \dots, n = -[-\mathrm{Re}\,(\alpha)],$$

is solvable in the space $L(a,b)$, *and its unique solution is given by*

$$(5.12)\qquad \begin{aligned} y(x) = \sum_{k=1}^{n} b_k y_k(x) + p(x-a)^{\alpha+\nu-1} \sum_{j=0}^{\infty} \frac{[\lambda (x-a)^{\mu+\alpha}]^j}{\Gamma[(\mu+\alpha)j + \alpha + \nu]} \\ \cdot \Phi[\gamma j + \sigma, (\mu+\alpha)j + \alpha + \nu; \omega(x-a)], \end{aligned}$$

where $y_k(x)$, $k = 1, 2, \dots, n$, *are given by* (4.9).

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