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MULTI-LEVEL ITERATION METHODS FOR SOLVING INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. In this paper we develop multi-level iteration methods for solving Fredhom integral equations of the second kind based on the Galerkin method for which the Galerkin subspace has a multi-resolution decomposition. After expressing the equations using matrices of operators in accordance to the multi-resolution structure, we propose two iteration schemes for solving the equations that are analogues to the Jacobi and Gauss-Seidel iteration schemes for solving algebraic systems. We then discuss the two-grid nature of the schemes, compare them with the well-known two-grid schemes and a two-level scheme and prove their convergence. We also present our numerical implementation of these methods using piecewise linear polynomial wavelets for an integral equation with the logarithmic kernel.

1. Introduction. Numerical methods of solving integral equations often lead to solving very large linear algebraic systems that usually do not have convenient structures such as sparseness. Special efficient iteration methods (see, e.g., [**2, 3, 13, 14**]) are often used to solve such large systems by taking advantage of distinct properties from the integral equations and approximating subspaces. Recently more and more studies on using wavelet spaces for solving integral equations have emerged and shown promise ([**4, 6–9, 11, 12, 16, 17**]). The multi-resolution property of the wavelet subspaces makes such schemes most efficient in many applications, including solving integral equations. There have been existing iteration methods ([**5, 10**]) that utilize

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the multi-resolution structure of these approximating subspaces. The purpose of this paper is to develop some alternative iteration schemes that allow us to move several levels at once for Fredholm integral equations of the second kind. To illustrate the main idea using the Galerkin method, suppose an initial level k, usually not too large, has been chosen to compute an initial approximation for the solution of the integral equation under consideration. By using a multi-resolution decomposition for the approximating subspace, it is convenient to identify the operator at the $(k+l)$-th level with a matrix of operators at the k-th level. Using such an expression of the equations, we propose iteration schemes that are analogue to the Jacobi and Gauss-Seidel iteration methods for solving algebraic linear systems. Our derivation is based on the operator matrix reformulation of the Galerkin method on the multi-resolution structure of the approximating subspaces. These iteration methods turn out to be of two-grid nature if we properly choose residual corrections for the defect correction iteration. In the special case of $l = 1$, our two-level Gauss-Seidel iteration is the same as the well-known two-grid method for integral equations (see, e.g., [**2**]) with naturally chosen "prolongation" and "restriction" operators. As for the general case of $l > 1$, the Gauss-Seidel scheme does incorporate some of the multilevel information between the coarse grid and the fine grid, unlike most standard two-grid methods.

The rest of our presentation is organized as follows. We describe in Section 2 the multilevel iteration schemes and the ideas leading to these algorithms, and discuss their two-grid nature and relations to the existing two-grid methods and a two-level method from [**5**]. Section 3 is devoted to establishing the convergence results for these methods and discussing their rates of convergence. In Section 4 we present numerical implementations of these methods to demonstrate some of the theoretical results with numerical examples, and propose and test a stopping criterion for such iteration methods.

2. Galerkin based multilevel iterations. Let X be a Hilbert space and $\mathcal{K} : X \to X$ an operator such that $\mathcal{I} - \mathcal{K}$ is bijective on X. Consider the Fredholm integral equation of the second kind given by

$$(\mathcal{I} - \mathcal{K})u = f, \tag{2.1}$$

where $f \in X$ is given and $u \in X$ is the solution to be determined.

To solve equation (2.1) by the Galerkin method, let X_n be a nested sequence of finite dimensional subspaces of X:

$$X_n \subseteq X_{n+1}, \quad n = 0, 1, \ldots, \tag{2.2}$$

such that

$$\overline{\bigcup_{n=0}^{\infty} X_n} = X.$$

Let $\mathcal{P}_n : X \to X_n$ denote the sequence of orthogonal projections. Then

$$\|\mathcal{P}_n\| = 1, \quad X_n = \mathcal{P}_n X, \quad \mathcal{P}_n^* = \mathcal{P}_n, \quad \text{and} \quad \mathcal{P}_n \to \mathcal{I} \quad \text{pointwise in } X.$$

The Galerkin method is to find $u_n \in X_n$ satisfying the equation

$$(\mathcal{I} - \mathcal{P}_n \mathcal{K}) u_n = \mathcal{P}_n f, \tag{2.3}$$

or, equivalently,

$$(\mathcal{I} - \mathcal{P}_n \mathcal{K} \mathcal{P}_n) u_n = \mathcal{P}_n f. \tag{2.4}$$

We require that, for all n large enough, $\mathcal{I} - \mathcal{P}_n \mathcal{K}$ is invertible and its inverse is bounded by a constant C independent of n:

$$\|(\mathcal{I} - \mathcal{P}_n \mathcal{K})^{-1}\| \leq C. \tag{2.5}$$

This requirement is fulfilled when $\mathcal{K}$ is a compact operator on X. In this case, it follows from the pointwise convergence of $\mathcal{P}_n$ to $\mathcal{I}$ and the compactness of $\mathcal{K}$ and $\mathcal{K}^*$ that

$$\lim_{n \to \infty} \|\mathcal{K} - \mathcal{P}_n \mathcal{K}\| = \lim_{n \to \infty} \|\mathcal{K} - \mathcal{K} \mathcal{P}_n\| = 0,$$

which implies in particular condition (2.5). Consequently, equation (2.3) or (2.4) has a unique solution $u_n \in X_n$ which satisfies the estimates

$$\|u_n\| \leq c \|f\|, \quad \|u - u_n\| \leq c \inf_{v_n \in X_n} \|u - v_n\|,$$

where $u \in X$ is the exact solution of equation (1.1). For more details on the Galerkin method, see, e.g., [**2, 14**].

Next we develop our multilevel iteration methods. In the nestedness property of the approximating subspace sequence (2.2), each X_n represents a level of resolution in approximating X and solving $u_n \in X_n$ in (2.3) or (2.4) is seeking an approximation of up to the nth level of resolution to the exact solution $u \in X$. This nestedness property also implies that there exist subspaces W_n of X_{n+1} such that

$$X_{n+1} = X_n \oplus^{\perp} W_n, \quad n = 0, 1, \ldots . \tag{2.6}$$

Let

$$\mathcal{Q}_n = \mathcal{P}_{n+1} - \mathcal{P}_n.$$

It is then straightforward to show that $\mathcal{Q}_n$ is a projection onto W_n and

$$W_n = \mathcal{Q}_n X_{n+1}, \quad n = 0, 1, 2, \ldots .$$

Repeatedly using equation (2.6) produces, for $k \geq 0$ and $l \geq 1$, the following decomposition of the space X_n with $n = k + l$:

$$X_n = X_{k+l} = X_k \oplus^{\perp} W_k \oplus^{\perp} \cdots \oplus^{\perp} W_{k+l-1}. \tag{2.7}$$

We observe that the operators $\mathcal{P}_n$ and $\mathcal{Q}_n$ have the following orthogonal relations:

$$\mathcal{P}_n \mathcal{P}_m = \mathcal{P}_{\min(m,n)}, \quad \mathcal{Q}_n \mathcal{Q}_m = \delta_{nm} \mathcal{I},$$

and

$$\mathcal{P}_n \mathcal{Q}_m = \mathcal{Q}_m \mathcal{P}_n = \begin{cases} \mathcal{O} & \text{if } m \geq n, \\ \mathcal{Q}_m & \text{if } m < n. \end{cases}$$

It is convenient to develop our iteration schemes using matrices of operators. For this purpose, following **[5]** we identify the vector $[f_0, g_0, \ldots, g_{l-1}]^T$ in $X_k \times W_k \times \cdots \times W_{k+l-1}$ with the vector

$$f_0 + g_0 + \cdots + g_{l-1} \in X_k \oplus^{\perp} W_k \oplus^{\perp} \cdots \oplus^{\perp} W_{k+l-1}$$

for $f_0 \in X_k$ and $g_i \in W_{k+i}$, $k \geq 0$, $0 \leq i \leq l-1$. Accordingly, for $u_{k+l} \in X_{k+l}$, we write

$$u_{k+l} = u_{k,0} + v_{k,0} + \cdots + v_{k,l-1},$$

where $u_{k,0} \in X_k$ and $v_{k,i} \in W_{k+i}$ $(0 \le i \le l-1)$, and

$$\mathcal{P}_{k+l} = \mathcal{P}_k + \mathcal{Q}_k + \cdots + \mathcal{Q}_{k+l-1}.$$

For $n, m \ge 0$, define

$$\mathcal{K}_n = \mathcal{P}_n \mathcal{K} \mathcal{P}_n, \quad \mathcal{K}_{n,m} = \mathcal{Q}_n \mathcal{K} \mathcal{Q}_m,$$

$$\mathcal{B}_{n,m} = \mathcal{P}_n \mathcal{K} \mathcal{Q}_m \quad \text{for } n \le m, \quad \text{and} \quad \mathcal{C}_{n,m} = \mathcal{Q}_n \mathcal{K} \mathcal{P}_m \quad \text{for } n \ge m.$$

With these notations and convention, we can identify the operator $\mathcal{K}_{k+l}$ with the matrix of operators

$$\mathbf{A}_{k,l} = \begin{bmatrix} \mathcal{K}_k & \mathcal{B}_{k,k} & \cdots & \mathcal{B}_{k,k+l-1} \\ \mathcal{C}_{k,k} & \mathcal{K}_{k,k} & \cdots & \mathcal{K}_{k,k+l-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{C}_{k+l-1,k} & \mathcal{K}_{k+l-1,k} & \cdots & \mathcal{K}_{k+l-1,k+l-1} \end{bmatrix} \tag{2.8}$$

Thus, in analogy with the algebraic matrix operations, we can express operator equation (2.3) or (2.4) in the form

$$\mathbf{u}_{k,l} - \mathbf{A}_{k,l} \mathbf{u}_{k,l} = \mathbf{f}_{k,l} \tag{2.9}$$

where $\mathbf{u}_{k,l} = [u_{k,0}, v_{k,0}, \ldots, v_{k,l-1}]^T$ and $\mathbf{f}_{k,l} = [\mathcal{P}_k f, \mathcal{Q}_k f, \ldots, \mathcal{Q}_{k+l-1} f]^T$.

As in (2.3) or (2.4), we seek the solution to (2.9) in the approximating space X_{k+l}. Once the bases for X_k and W_{k+i} are chosen, the matrix representation of $\mathcal{K}_{k+l}$ has exactly the block structure as $\mathbf{A}_{k,l}$ in (2.8).

To develop multilevel iteration schemes for solving equation (2.9), we introduce two matrices of operators from $\mathbf{A}_{k,l}$: the strictly upper triangular matrix

$$\mathbf{U}_{k,l} = \begin{bmatrix} O & \mathcal{B}_{k,k} & \mathcal{B}_{k,k+1} & \cdots & \mathcal{B}_{k,k+l-1} \\ & O & \mathcal{K}_{k,k+1} & \cdots & \mathcal{K}_{k,k+l-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \mathcal{K}_{k+l-2,k+l-1} \\ & & & & O \end{bmatrix},$$

and the lower triangular matrix

$$\mathbf{L}_{k,l} = \begin{bmatrix} O & & & \\ \mathcal{C}_{k,k} & \mathcal{K}_{k,k} & & \\ \vdots & \vdots & \ddots & \\ \mathcal{C}_{k+l-1,k} & \mathcal{K}_{k+l-1,k} & \cdots & \mathcal{K}_{k+l-1,k+l-1} \end{bmatrix}.$$

In addition, we define

$$\mathbf{D}_{k,l} := \mathcal{I} - \mathbf{A}_{k,l} + \mathbf{U}_{k,l} + \mathbf{L}_{k,l}.$$

In terms of operators on X, this is in fact the operator $\mathcal{I} - \mathcal{K}_k$. When viewed as a matrix of operator on X_{k+l} (such as $\mathbf{U}_{k,l}$ and $\mathbf{L}_{k,l}$), $\mathbf{D}_{k,l}$ has the form

$$\mathbf{D}_{k,l} = \begin{bmatrix} \mathcal{I} - \mathcal{K}_k & & & \\ & \mathcal{I} & & \\ & & \ddots & \\ & & & \mathcal{I} \end{bmatrix}$$

where the $\mathcal{I}$s on the diagonal are understood as identity operators on the corresponding subspaces X_k, W_k, ..., W_{k+l-1}. Accordingly, equation (2.9) can be rewritten as

$$(\mathbf{D}_{k,l} - \mathbf{U}_{k,l} - \mathbf{L}_{k,l})\mathbf{u}_{k,l} = \mathbf{f}_{k,l}. \tag{2.10}$$

It is this matrix form that leads us to the following two multilevel iteration schemes for solving equation (2.3).

- Jacobi type iteration:

$$\mathbf{D}_{k,l}\mathbf{u}_{k,l}^{(m+1)} = (\mathbf{U}_{k,l} + \mathbf{L}_{k,l})\mathbf{u}_{k,l}^{(m)} + \mathbf{f}_{k,l}, \quad m = 0, 1, 2, \ldots \tag{2.11}$$

with any initial approximation $\mathbf{u}_{k,l}^{(0)}$. Except for the first component, all the other components of $\mathbf{u}_{k,l}^{(m+1)}$ are already expressed in terms of $\mathbf{u}_{k,l}^{(m)}$ since the diagonal blocks in $\mathbf{D}_{k,l}$ are identity operators. Only the first component needs to be solved by inverting $\mathcal{I} - \mathcal{K}_k$, which is also the difference between this iteration scheme and the usual algebraic Jacobi scheme.

- Gauss-Seidel type iteration:

$$(\mathbf{D}_{k,l} - \mathbf{U}_{k,l})\mathbf{u}_{k,l}^{(m+1)} = \mathbf{L}_{k,l}\mathbf{u}_{k,l}^{(m)} + \mathbf{f}_{k,l}, \quad m = 0, 1, 2, \ldots \tag{2.12}$$

with any initial approximation $\mathbf{u}_{k,l}^{(0)}$. Like the usual algebraic Gauss-Seidel iteration, (2.12) can be solved easily in the "backward substitutions" fashion for the components of $\mathbf{u}_{k,l}^{(m+1)}$, and the only inverse we need is for the first component, i.e., $(\mathcal{I} - \mathcal{K}_k)^{-1}$.

It is clear that the two methods are similar in the way of updating the components of $\mathbf{u}_{k,l}^{(m+1)}$; they differ only in that the Gauss-Seidel scheme uses newly available components within each iteration while the Jacobi scheme waits till an iteration is completed. Both methods require inverting the operator $\mathcal{I} - \mathcal{K}_k$ at the kth level. Hence, we have to choose k so that the inverse of $\mathcal{I} - \mathcal{K}_k$ exists and is relatively easy to find. In the meantime, we can take advantage of this to start the iteration with a good initial guess $\mathbf{u}_{k,l}^{(0)} := [(\mathcal{I} - \mathcal{K}_k)^{-1}\mathcal{P}_k f, 0, \ldots, 0]^T$.

Now we examine the two-grid nature of these schemes. Recall that a two-grid method for solving systems like (2.9) has the following form [**14**, Section 14.2]:

$$\mathbf{u}_{k,l}^{(m+1)} = [\mathbf{I} - \mathbf{B}_{k,l}(\mathbf{I} - \mathbf{A}_{k,l})]\mathbf{u}_{k,l}^{(m)} + \mathbf{B}_{k,l}\mathbf{f}_{k,l} \tag{2.13}$$

where $\mathbf{B}_{k,l}$ is an approximation of the inverse $(\mathbf{I} - \mathbf{A}_{k,l})^{-1}$ at a coarser grid. This approximation $\mathbf{B}_{k,l}$ is used in computing the correction from $\mathbf{u}_{k,l}^{(m)}$ to $\mathbf{u}_{k,l}^{(m+1)}$. It is easy to see that schemes (2.11) and (2.12) are results of (2.13) with the following choice of approximating matrix $\mathbf{B}_{k,l}$ at the coarse grid of the k-th level:

• For the Jacobi type scheme (2.11), choose $\mathbf{B}_{k,l} := \mathbf{D}_{k,l}^{-1}$ which has the matrix form on X_{k+l} as

$$\mathbf{B}_{k,l} = \begin{bmatrix} (\mathcal{I} - \mathcal{K}_k)^{-1} & & & \\ & \mathcal{I} & & \\ & & \ddots & \\ & & & \mathcal{I} \end{bmatrix}.$$

• For the Gauss-Seidel type scheme (2.12), choose $\mathbf{B}_{k,l} := (\mathbf{D}_{k,l} - \mathbf{U}_{k,l})^{-1}$, which on X_{k+l} has the matrix form

$$\mathbf{B}_{k,l} = \begin{bmatrix} (\mathcal{I} - \mathcal{K}_k)^{-1} & \mathcal{G}_{k,k} & \mathcal{G}_{k,k+1} & \cdots & \mathcal{G}_{k,k+l-1} \\ & \mathcal{I} & \mathcal{S}_{k,k+1} & \cdots & \mathcal{S}_{k,k+l-1} \\ & & \ddots & & \vdots \\ & & & \ddots & \mathcal{S}_{k+l-2,k+l-1} \\ & & & & \mathcal{I} \end{bmatrix}.$$

Here the operators $\mathcal{G}_{kj}$ have the form $\mathcal{P}_k\tilde{\mathcal{K}}\mathcal{Q}_j$ and $\mathcal{S}_{ij}$ are operators of the form $\mathcal{Q}_i\tilde{\mathcal{K}}\mathcal{Q}_j$ for some operators $\tilde{\mathcal{K}}$ on X.

Clearly, the Jacobi type is an obvious choice for the approximation of the inverse $(\mathbf{I} - \mathbf{A}_{k,l})^{-1}$ at the coarse grid of level k. In fact, it is a common choice for many two-grid methods. As for the Gauss-Seidel scheme, only in this matrix form can we see that this is also another natural choice for the approximation, which also utilizes that multi-level structure between the coarser grid k and the finer grid $n := k + l$.

To further compare with well-known two-grid methods ([**2, 13, 14**]) and the two-level method ([**5**]), let us consider the special case of $l = 1$ in the following. In this case equation (2.10) reduces to

$$\begin{bmatrix} \mathcal{I} - \mathcal{K}_k & -\mathcal{B}_{k,k} \\ -\mathcal{C}_{k,k} & \mathcal{I} - \mathcal{K}_{k,k} \end{bmatrix} \begin{bmatrix} u_{k,0} \\ v_{k,0} \end{bmatrix} = \begin{bmatrix} \mathcal{P}_k f \\ \mathcal{Q}_k f \end{bmatrix} \tag{2.14}$$

and our Gauss-Seidel type iteration (2.12) for solving (2.14) becomes

(2.15)

$$\begin{bmatrix} \mathcal{I} - \mathcal{K}_k & -\mathcal{B}_{k,k} \\ 0 & \mathcal{I} \end{bmatrix} \begin{bmatrix} u_{k,0}^{(m+1)} \\ v_{k,0}^{(m+1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathcal{C}_{k,k} & \mathcal{K}_{k,k} \end{bmatrix} \begin{bmatrix} u_{k,0}^{(m)} \\ v_{k,0}^{(m)} \end{bmatrix} + \begin{bmatrix} \mathcal{P}_k f \\ \mathcal{Q}_k f \end{bmatrix}$$

The two-grid method used in solving linear systems resulted from discretization of integral equations is often formulated through the introduction of the "restriction" and "prolongation" operators ([**2, 3**]). In our setting, these operators are naturally defined by the truncation $\mathbf{R}_{k,l} = [\mathcal{I}, O]$ (from X_{k+l} to X_k) and extension $\mathbf{P}_{k,l} = [\mathcal{I}, O]^T$ (from X_k to X_{k+l}). One of the formulations requires the following steps ([**2**, Section 6.3]) in each iteration from $[u_{k,0}^{(m)}, v_{k,0}^{(m)}]^T$ to $[u_{k,0}^{(m+1)}, v_{k,0}^{(m+1)}]^T$:

1. Smooth $[u_{k,0}^{(m)}, v_{k,0}^{(m)}]^T$ by the Picard iteration

$$\begin{bmatrix} \tilde{u}_{k,0}^{(m)} \\ \tilde{v}_{k,0}^{(m)} \end{bmatrix} = \begin{bmatrix} \mathcal{K}_k & \mathcal{B}_{k,k} \\ \mathcal{C}_{k,k} & \mathcal{K}_{k,k} \end{bmatrix} \begin{bmatrix} u_{k,0}^{(m)} \\ v_{k,0}^{(m)} \end{bmatrix} + \begin{bmatrix} \mathcal{P}_k f \\ \mathcal{Q}_k f \end{bmatrix}$$

2. Compute the defect of its restriction on the coarser grid

$$r_k^{(m)} = \mathcal{P}_k f - (\mathcal{I} - \mathcal{K}_k)\tilde{u}_{k,0}^{(m)} + \mathcal{B}_{k,k}\tilde{v}_{k,0}^{(m)}.$$

3. Compute a correction on the coarser grid

$$\delta u_{k,0}^{(m)} = (\mathcal{I} - \mathcal{K}_k)^{-1} r_k^{(m)}.$$

4. Prolongate the correction to the finer grid and update the approximation by

$$u_{k,0}^{(m+1)} = \tilde{u}_{k,0}^{(m)} + \delta u_{k,0}^{(m)} \quad \text{and} \quad v_{k,0}^{(m+1)} = \tilde{v}_{k,0}^{(m)}.$$

After combining the above steps into one iteration from $[u_{k,0}^{(m)}, v_{k,0}^{(m)}]^T$ to $[u_{k,0}^{(m+1)}, v_{k,0}^{(m+1)}]^T$, we can easily recover the Gauss-Seidel type scheme (2.15).

In [**5**], the following two-level method is introduced and analyzed for solving (2.14):

$$\begin{bmatrix} \mathcal{I} - \mathcal{K}_k & -\mathcal{B}_{k,k} \\ 0 & \mathcal{I} - \mathcal{K}_{k,k} \end{bmatrix} \begin{bmatrix} u_{k,0}^{(1)} \\ v_{k,0}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathcal{C}_{k,k} & 0 \end{bmatrix} \begin{bmatrix} u_k \\ 0 \end{bmatrix} + \begin{bmatrix} \mathcal{P}_k f \\ \mathcal{Q}_k f \end{bmatrix}, \tag{2.16}$$

where $u_k \in X_k$ solves exactly $(\mathcal{I} - \mathcal{K}_k)u_k = \mathcal{P}_k f$. In this scheme, although no iteration is required, we need to solve for the second component $v_{k,0}$ *exactly* from the equation

$$(\mathcal{I} - \mathcal{K}_{k,k})v_{k,0}^{(1)} = \mathcal{C}_{k,k}u_k + \mathcal{Q}_k f.$$

As for scheme (2.15), this component need not be solved exactly, but approximated through iterations which also involves approximations of the first component $u_{k,0}^{(m)}$:

$$v_{k,0}^{(m+1)} = \mathcal{C}_{k,k}u_{k,0}^{(m)} + \mathcal{K}_{k,k}v_{k,0}^{(m)} + \mathcal{Q}_k f.$$

Thus, we see that scheme (2.15) with finite steps of iterations and scheme (2.16) in fact provide different approximations for the exact solution to (2.14), and (2.16) requires solving an equation exactly without iterations while (2.15) needs iterations but not solving the equation exactly.

3. Convergence analysis. In this section, we provide convergence analysis for both Gauss-Seidel type and Jacobi type iteration schemes, (2.11) and (2.12), when the operator $\mathcal{K}$ is compact on X. In this case, the conditions we assume in Section 2 are satisfied. In particular, we have

$$\lim_{n\to\infty} \|(\mathcal{P}_{n+i} - \mathcal{P}_{n+j})\mathcal{K}\| = \lim_{n\to\infty} \|\mathcal{K}(\mathcal{P}_{n+i} - \mathcal{P}_{n+j})\| = 0. \tag{3.1}$$

Moreover, for n large enough, $\mathcal{I} - \mathcal{K}_n$ is invertible and the norm of the inverse operator is bounded uniformly for all n. In the following, we will establish convergence for both schemes (2.11) and (2.12) for k large enough, and discuss the rate of convergence.

To this end, we need to view the operators used to define these schemes as operators on X, and estimate their norms as operator norm on X.

From the definition of $\mathbf{U}_{k,l}$, we can see that $\mathbf{U}_{k,l}$ is in fact defined for all $u \in X$, and $\mathbf{U}_{k,l}$ acting on u is exactly the same as it acting on $(\mathcal{P}_{k+\ell} - \mathcal{P}_k)u$:

$$\mathbf{U}_{k,l}\, u = \mathcal{P}_k \mathcal{K}(\mathcal{P}_{k+l} - \mathcal{P}_k)u + \sum_{i=0}^{l-2} \mathcal{Q}_{k+i}\mathcal{K}(\mathcal{P}_{k+l} - \mathcal{P}_{k+i+1})u.$$

Hence, for each fixed $l \geq 1$, we can estimate its norm on X by

$$\|\mathbf{U}_{k,l}\| \leq \sum_{j=0}^{l-1} \|\mathcal{K}(\mathcal{P}_{k+l} - \mathcal{P}_{k+j})\| \to 0 \quad \text{as} \quad k \to \infty, \tag{3.2}$$

where we have used (3.1) and the fact that $\|\mathcal{Q}_j\| = 1$. Similarly, we have

$$\mathbf{L}_{k,l}\, u = (\mathcal{P}_{k+l} - \mathcal{P}_k)\mathcal{K}\mathcal{P}_k u + \sum_{i=0}^{l-1} (\mathcal{P}_{k+l} - \mathcal{P}_{k+i})\mathcal{K}\mathcal{Q}_{k+i}u.$$

and thus

$$\|\mathbf{L}_{k,l}\| \leq \sum_{i=0}^{l-1} \|(\mathcal{P}_{k+l} - \mathcal{P}_{k+i})\mathcal{K}\| \to 0 \quad \text{as} \quad k \to \infty.$$

Moreover, since $\mathbf{U}_{k,l} + \mathbf{L}_{k,l} = \mathcal{K}_{k+l} - \mathcal{K}_k$, we have that

$$\|\mathbf{U}_{k,l} + \mathbf{L}_{k,l}\| \leq \|(\mathcal{P}_{k+l} - \mathcal{P}_k)\mathcal{K}\| + \|\mathcal{K}(\mathcal{P}_{k+l} - \mathcal{P}_k)\| \to 0 \quad \text{as} \quad k \to \infty. \tag{3.3}$$

On the other hand, as an operator on X, $\mathbf{D}_{k,l} = \mathcal{I} - \mathcal{K}_k$. Hence

$$\mathbf{D}_{k,l}^{-1} = (\mathcal{I} - \mathcal{K}_k)^{-1}$$

and

$$(\mathbf{D}_{k,l} - \mathbf{U}_{k,l})^{-1} = (\mathcal{I} - (\mathcal{I} - \mathcal{K}_k)^{-1}\mathbf{U}_{k,l})^{-1}\,(\mathcal{I} - \mathcal{K}_k)^{-1}.$$

Therefore, we obtain the norm estimates for the two inverses:

$$\|\mathbf{D}_{k,l}^{-1}\| = \|(\mathcal{I} - \mathcal{K}_k)^{-1}\| \tag{3.4}$$

and

$$\|(\mathbf{D}_{k,l} - \mathbf{U}_{k,l})^{-1}\| \le \frac{\|(\mathcal{I} - \mathcal{K}_k)^{-1}\|}{1 - \|(\mathcal{I} - \mathcal{K}_k)^{-1}\|\|\mathbf{U}_{k,l}\|}.$$

Since $\|(\mathcal{I} - \mathcal{K}_k)^{-1}\|$ is uniformly bounded for large enough k, in view of (3.2), these two inverses are both bounded uniformly for large enough k.

Combining the above, we see that, for each fixed $l \ge 1$, the iteration operators for schemes (2.11) and (2.12) are both convergent to 0 in the operator norm:

$$\|\mathbf{D}_{k,l}^{-1}\,(\mathbf{L}_{k,l}+\mathbf{U}_{k,l})\| \to 0 \quad \text{and} \quad \|(\mathbf{D}_{k,l}-\mathbf{U}_{k,l})^{-1}\,\mathbf{L}_{k,l}\| \to 0 \quad \text{as } k \to \infty.$$

Hence they can be chosen less than 1 if k is large enough, which in turn yields the convergence of the iteration schemes (2.11) and (2.12).

Next we examine the rate of convergence of the iteration schemes. Let

$$q_{k,l} = \begin{cases} \|\mathbf{D}_{k,l}^{-1}(\mathbf{U}_{k,l} + \mathbf{L}_{k,l})\| & \text{for scheme (2.11),} \\ \|(\mathbf{D}_{k,l} - \mathbf{U}_{k,l})^{-1}\mathbf{L}_{k,l}\| & \text{for scheme (2.12).} \end{cases}$$

This is roughly the factor between two consecutive errors in the iteration. More precisely, for each (k, l), the quantity $q_{k,l}$ is the least upper bound for the ratios of two consecutive errors or the ratios of differences of two consecutive iterates:

$$\frac{\|\mathbf{u}_{k,l}^{(m)} - \mathbf{u}_{k,l}\|}{\|\mathbf{u}_{k,l}^{(m-1)} - \mathbf{u}_{k,l}\|} \le q_{k,l} \quad \text{and} \quad \frac{\|\mathbf{u}_{k,l}^{(m)} - \mathbf{u}_{k,l}^{(m-1)}\|}{\|\mathbf{u}_{k,l}^{(m-1)} - \mathbf{u}_{k,l}^{(m-2)}\|} \le q_{k,l}.$$

We have shown that, for fixed $l \ge 1$, $q_{k,l} \to 0$ as $k \to \infty$ and thus $q_{k,l} < 1$ for large enough k. Note that the argument depends on the *a priori* chosen l, the number of additional levels beyond the initial level k.

For the Jacobi type scheme (2.11), we can show that this rate is essentially independent of all large enough l. Indeed, from (3.3) and (3.4) above,

$$\begin{aligned} q_{k,l} &\le \|\mathbf{D}_{k,l}^{-1}\| \, \|(\mathbf{U}_{k,l} + \mathbf{L}_{k,l})\| \\ &\le \|(\mathcal{I} - \mathcal{K}_k)^{-1}\|(\|(\mathcal{P}_{k+l} - \mathcal{P}_k)\mathcal{K}\| + \|\mathcal{K}(\mathcal{P}_{k+l} - \mathcal{P}_k)\|), \end{aligned}$$

hence

$$\limsup_{l\to\infty} \; q_{k,l} \le \|(\mathcal{I} - \mathcal{K}_k)^{-1}\|(\|\mathcal{K} - \mathcal{P}_k\mathcal{K}\| + \|\mathcal{K} - \mathcal{K}\mathcal{P}_k\|) < 1 \tag{3.5}$$

if k is chosen large enough.

To derive an estimate for the rate of convergence of the Gauss-Seidel scheme (2.12), we further assume that

$$\sum_{i=0}^{\infty} \|(\mathcal{I} - \mathcal{P}_i)\mathcal{K}\| < \infty, \quad \sum_{i=0}^{\infty} \|\mathcal{K}(\mathcal{I} - \mathcal{P}_i)\| < \infty \tag{3.6}$$

and

$$\lim_{l\to\infty} l\|(\mathcal{P} - \mathcal{P}_l)\mathcal{K}\| = \lim_{l\to\infty} l\|\mathcal{K}(\mathcal{I} - \mathcal{P}_l)\| = 0. \tag{3.7}$$

For any positive integer k, we now introduce two quantities

$$r_k := \sum_{i=0}^{\infty} \|(\mathcal{I} - \mathcal{P}_{k+i})\mathcal{K}\| \quad \text{and} \quad r'_k := \sum_{i=0}^{\infty} \|\mathcal{K}(\mathcal{I} - \mathcal{P}_{k+i})\|.$$

Under assumption (3.6), we see that

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} r'_k = 0 \tag{3.8}$$

and hypothesis (3.7) ensures that

$$\limsup_{l\to\infty} \|\mathbf{L}_{k,l}\| \le r_k \quad \text{and} \quad \limsup_{l\to\infty} \|\mathbf{U}_{k,l}\| \le r'_k \tag{3.9}$$

Therefore, we conclude from (3.8) and (3.9) that for the Gauss-Seidel scheme

$$\begin{aligned} \limsup_{l\to\infty} q_{k,l} &\le \limsup_{l\to\infty} \frac{\|(\mathcal{I} - \mathcal{K}_k)^{-1}\|\|\mathbf{L}_{k,l}\|}{1 - \|(\mathcal{I} - \mathcal{K}_k)^{-1}\|\|\mathbf{U}_{k,l}\|} \\ &\le \frac{\|(\mathcal{I} - \mathcal{K}_k)^{-1}\| r_k}{1 - \|(\mathcal{I} - \mathcal{K}_k)^{-1}\| r'_k} < 1, \end{aligned} \tag{3.10}$$

if k is chosen large enough.

We summarize our discussions above in the following theorem.

Theorem 3.1. *Suppose that the operator $\mathcal{K}$ is compact on the Hilbert space X, $\mathcal{I} - \mathcal{K}$ is bijective on X, and the sequence of approximating subspaces possesses the nestedness property* (2.2) *and each has the decomposition* (2.7) *by orthogonal projections. Then the following statements hold.*

(i) *For each $l \geq 1$ and sufficiently large k, the iteration schemes* (2.11) *and* (2.12) *for solving the equation* (2.3) *or* (2.10) *are both convergent.*

(ii) *The rate of convergence for the Jacobi iteration scheme* (2.11) *is essentially independent of l for all sufficiently large k. If hypotheses* (3.6) *and* (3.7) *are satisfied, the rate of convergence for the Gauss-Seidel iteration scheme is also essentially independent of l for all sufficiently large k.*

(iii) *The rate of convergence for the Jacobi scheme can be made smaller (by increasing k) in the same order of approximation $\mathcal{P}_k\mathcal{K}$ to $\mathcal{K}$. Moreover, if hypotheses* (3.6) *and* (3.7) *are satisfied, the rate of convergence for the Gauss-Seidel scheme can be made smaller (by increasing k) in the same order of r_k going to zero.*

We remark that conditions (3.6) and (3.7) are fulfilled in many applications when X_n are chosen as piecewise polynomial spaces. This point will be further addressed in Section 4 by examples. As demonstrated in the previous sections, the iteration schemes (2.11) and (2.12) are two-grid in nature. Like other two-grid methods, it is possible to use them as efficiently as a multigrid method, since the overall number of iterations can be kept small and of equal order to that of multigrid method. More details on discussions and comparison of these two methods can be found in [**2, 3**].

4. Numerical examples. In this section, we carry out the numerical implementation of solving the following typical integral equation in

the setting outlined in the previous sections:

$$u(x) - \int_0^1 \ln(|x-y|)\; u(y)\, dy = f(x) \quad \text{for } x \in (0,1). \tag{4.1}$$

It is well known that the operator K defined by

$$(Ku)(x) := \int_0^1 \ln(|x-y|)\; u(y)\, dy, \quad x \in (0,1),$$

is compact from $L^2(0,1)$ to $L^2(0,1)$. In fact K is self-adjoint and maps $L^2(0,1)$ into the Sobolev space $H^{1-\varepsilon}(0,1)$ for any $\varepsilon > 0$.

First, we describe the construction of approximating subspaces $X_n \subset L^2(0,1)$. We will use piecewise linear polynomial wavelet spaces as described in [**1, 15**]. As mentioned in Section 2, the sequence of the subspaces X_n possesses the nestedness property (2.2), and each X_n has the orthogonal decomposition (2.7) or,

$$X_n = X_0 \oplus^\perp W_0 \oplus^\perp W_1 \oplus^\perp \cdots \oplus^\perp W_{n-1}.$$

We construct each of these subspaces by choosing its *orthonormal* basis:

$$X_0 = \text{span}\,\{w_{-1,1}, w_{-1,2}\} \quad \text{with} \begin{cases} w_{-1,1}(x) = 1, \\ w_{-1,2}(x) = 2\sqrt{3}\,(x - 1/2), \end{cases}$$

$$W_0 = \text{span}\,\{w_{01}, w_{02}\} \quad \text{with} \begin{cases} w_{01}(x) = 6(x-1/2) - 2\ \text{sgn}\,(x-1/2), \\ w_{02}(x) = 4\sqrt{3}\,|x-1/2| - \sqrt{3}. \end{cases}$$

These basis functions are defined for $x \in [0,1]$ by the expressions above and considered having zero extension to the rest of R for convenience. Starting with W_1, the W-spaces are defined successively as: if

$$W_{i-1} = \text{span}\,\{w_{i-1,j} :\ j = 1, \ldots, 2^i\},$$

then

$$W_i = \text{span}\,\{w_{ij} : j = 1, \ldots, 2^{i+1}\}$$

$$\text{with} \begin{cases} w_{ij}(x) = \sqrt{2}\, w_{i-1,j}(2x), \\ w_{i,2^i+j}(x) = \sqrt{2}\, w_{i-1,j}(2x-1), \end{cases} \quad 1 \le j \le 2^i.$$

Note that $\dim(X_n) = 2^{n+1}$, $\dim(W_i) = 2^{i+1}$, and the graph of each w_{ij} consists of two straight line segments on two neighboring intervals of width $1/2^{i+1}$.

The functions $\{w_{ij}\}$ form an *orthonormal* basis for X_n, arranged in the following order:

$$\{w_{-1,j}\}_{j=1}^{2},\ \{w_{0j}\}_{j=1}^{2},\ \{w_{1j}\}_{j=1}^{2^2}, \dots, \{w_{n-1,j}\}_{j=1}^{2^n}. \tag{4.2}$$

Thus, under this setting, solving the integral equation (4.1) in X_n by the Galerkin method leads to solving the following linear algebraic system, which is the matrix representation of the operator equation (2.9):

$$(I - A_n)v_n = f_n \tag{4.3}$$

where the unknown 2^{n+1}-vector v_n consists of the coefficients of the approximate solution u_n in X_n with the basis (4.2), and the components of f_n are the L^2-inner products of f with the basis functions w_{ij}. The coefficient matrices are of $2^{n+1} \times 2^{n+1}$: I is the identity matrix and A_n consists of double integrals of all possible pairs from the basis functions with the kernel K.

We employ the iteration schemes (2.11) and (2.12) to solve this linear system. In the matrix form, these schemes are block-matrix iterative schemes, where the block partition is based on the levels of resolutions. We begin with an initial level k, $1 \le k < n$, and let $l = n - k$ be the number of additional levels from this level to the final level n. Then A_n is partitioned into $(l+1) \times (l+1)$-block matrix:

$$A_n = A^{k,l} = [A_{ij}]_{i,j=k}^{k+l}:$$

$$A_{ij} \text{ is } 2^{\max(i,k+1)} \times 2^{\max(j,k+1)} \quad \text{and} \quad A_{kk} = A_k.$$

Accordingly, the three matrices of operators, $\mathbf{D}_{k,l}$, $\mathbf{U}_{k,l}$ and $\mathbf{L}_{k,l}$ introduced in Section 2, have their block-matrix representations as

follows.

$$D_{k,l} = \begin{bmatrix} I - A_k & & & \\ & I & & \\ & & \ddots & \\ & & & I \end{bmatrix},$$

$$U_{k,l} = \begin{bmatrix} O & A_{k,k+1} & \cdots & A_{k,k+l-1} \\ & O & \cdots & A_{k+1,k+l-1} \\ & & \ddots & \vdots \\ & & & O \end{bmatrix},$$

and

$$L_{k,l} = \begin{bmatrix} O & & & \\ A_{k+1,k} & A_{k+1,k+1} & & \\ \vdots & \vdots & \ddots & \\ A_{k+l-1,k} & A_{k+l-1,k+1} & \cdots & A_{k+l-1,k+l-1} \end{bmatrix}.$$

The iteration matrices for the two schemes are:

$$M_{k,l}^{\mathrm{GS}} = (D_{k,l} - U_{k,l})^{-1} L_{k,l} \quad \text{for the Gauss-Seidel scheme (2.12)},$$
$$M_{k,l}^{\mathrm{J}} = D_{k,l}^{-1}(U_{k,l} + L_{k,l}) \quad \text{for the Jacobi scheme (2.11)}.$$

We present some of our numerical results below. Our calculations are carried out by using Matlab. Note that the integrals needed for setting up the coefficient matrix and the righthand side vector can be computed exactly for this particular example, so we do not use numerical integrations in our calculations.

Example 4.1. The two-norm of the iteration matrices. First we calculate the 2-norm of the iteration matrix for each scheme, with various values of k and l. The purpose of this experiment is to verify the results proved on the convergence rate and its independence on the number l. We use the standard Matlab routines, `M=D\(U+L)` and `norm(M)`, to compute each iteration matrix and its matrix 2-norm. As shown in the tables, for each k, the norm clearly converges as l increases, and its variation is very small even from $l = 1$. From the last numbers in the columns (i.e., these with $k+l = 10$), we can also see that the norm

TABLE 4.1. The two-norm of the iteration matrices for the two schemes.

(a) $\|M_{k,l}^{\mathrm{J}}\|_2$ for the Jacobi scheme.

	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$l=1$	3.3888e−02	1.7038e−02	8.5434e−03	4.2779e−03	2.1406e−03
$l=2$	3.4015e−02	1.7101e−02	8.5746e−03	4.2934e−03	2.1483e−03
$l=3$	3.4213e−02	1.7198e−02	8.6228e−03	4.3174e−03	
$l=4$	3.4234e−02	1.7208e−02	8.6278e−03		
$l=5$	3.4236e−02	1.7209e−02			
$l=6$	3.4236e−02				

(b) $\|M_{k,l}^{\mathrm{GS}}\|_2$ for the Gauss-Seidel scheme.

	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$l=1$	3.0932e−02	1.5468e−02	7.7342e−03	3.8672e−03	1.9340e−03
$l=2$	3.0977e−02	1.5491e−02	7.7459e−03	3.8731e−03	1.9369e−03
$l=3$	3.1036e−02	1.5522e−02	7.7616e−03	3.8809e−03	
$l=4$	3.1041e−02	1.5525e−02	7.7630e−03		
$l=5$	3.1042e−02	1.5525e−02			
$l=6$	3.1042e−02				

decreases between two consecutive ks by the factor of almost exactly $1/2$. This is in fact the order of the approximation of $\mathcal{P}_k\mathcal{K}$ to $\mathcal{K}$:

$$\|\mathcal{K} - \mathcal{P}_k\mathcal{K}\|_2 = O(2^{-k(1-\varepsilon)}) \quad \text{for any} \quad \varepsilon > 0, \tag{4.4}$$

due to the fact that $\mathcal{K}u$ is actually in $H^{1-\varepsilon}(0,1)$ for $u \in L^2(0,1)$. For the Jacobi scheme, we have proved these results theoretically in the previous section (see (3.5)) and for the Gauss-Seidel scheme, we have also proved a similar result (see (3.10)). Table 4.1(a) and Table 4.1(b) confirm these theoretical results. In comparison, we can see that the convergence rate of the Gauss-Seidel scheme is an improvement of about 10% on that of the Jacobi scheme.

Example 4.2. Convergence rates for the Gauss-Seidel scheme. We employ the Gauss-Seidel iteration scheme to find approximate solutions to (4.3). Results using the Jacobi scheme are similar and thus are not presented here. We test on the example for which $u(x) = x^2$ is

the exact solution to (4.1); i.e., we set

$$f(x) = x^2 - \int_0^1 y^2 \ln|x-y|\, dy, \quad x \in [0,1].$$

As we have proved, the convergence of the iteration is global, and we can choose any initial guess for the iteration. In practice we usually choose the solution at the initial level as the initial guess: $u_{k,l}^{(0)} = (u_k^T, 0)^T$ where $u_k = u_{k,0}$ is the exact solution to (4.3) with $n = k$. This is in fact one of the reasons we are interested in iteration schemes of this type: a solution of the higher resolutions can be obtained faster from an existing solution of lower resolutions. We choose the initial level at $k = 5$ (A_5 is 64×64), and the final levels at $n = 7, 8, 9, 10$ (hence $l = 2, 3, 4, 5$, respectively). Recall that the system (4.3) has dimension of 2^{n+1}. For each n, we start the iteration (2.12) with $u_n^{(0)}$ as described above, and terminate the iteration when the difference of two successive iterates is smaller than $1.0\mathrm{e}-15$. At each iterate, the exact solution at the initial level $k = 5$ is solved by using the standard Matlab routine, `u=(I-A)\f`. Numerical results are tabulated in Table 4.2. The ratio of successive differences,

$$\nu_{k,l}^{(m)} = \frac{|u_{k,l}^{(m)} - u_{k,l}^{(m-1)}|_2}{|u_{k,l}^{(m-1)} - u_{k,l}^{(m-2)}|_2},$$

seems to converge as m increases, nearly independent of l, and is close to but slightly smaller than the 2-norm of the iteration matrix $M_{k,l}^{\mathrm{GS}}$ computed in the second column of Table 4.1(b). In the last column of Table 4.2, we also compute the L^2 error between $u_{k,l}^{(m)}$ (the function obtained by multiplying the vector $u_{k,l}^{(m)}$ and the basis functions) and the exact solution $u(x) = x^2$. We can see that the later iterates at each case do not improve the L^2 error from the exact solution u. The reason seems to be clear: it has reached the limit of accuracy determined by the approximation of X_n to X. We will return to this issue in Example 4.4.

TABLE 4.2. Iterations by Gauss-Seidel scheme (2.12) with $k = 5$.

(k,l)	$\dim = 2^{k+l+1}$	m	$\lvert u_{k,l}^{(m)} - u_{k,l}^{(m-1)}\rvert_2$	ratio $\nu_{k,l}^{(m)}$	$\lVert u_{k,l}^{(m)} - u\rVert_2$
		1	7.3733e-05		4.6795e-06
		3	1.7084e-08	1.5347e-02	4.5493e-96
(5,2)	256	5	4.0600e-12	1.5424e-02	4.5493e-06
		7	9.6675e-16	1.5433e-02	4.5493e-06
		1	7.3865e-05		1.5812e-06
		3	1.7137e-08	1.5364e-02	1.1373e-06
(5,3)	512	5	4.0740e-12	1.5424e-02	1.1373e-06
		7	9.6992e-16	1.5431e-02	1.1373e-06
		1	7.3874e-05		1.1349e-06
		3	1.714e-08	1.5365e-02	2.8454e-07
(5,4)	1024	5	4.0749e-12	1.5424e-02	2.8449e-07
		7	9.7014e-16	1.5431e-02	2.8459e-07
		1	7.3874e-05		1.1012e-06
		3	1.7142e-08	1.5365e-02	6.8082e-08
(5,5)	2048	5	4.0750e-12	1.5424e-02	6.9294e-08
		7	9.7014e-16	1.5431e-02	6.8893e-08

Example 4.3. Order of convergence rate for iteration in terms of k. In the following we test the order of convergence rates for the Gauss-Seidel iteration in terms of k, as predicted by (3.10). For this particular example, the order of $\|\mathcal{K} - \mathcal{P}_k\mathcal{K}\|$ is as given in (4.4). We use the Gauss-Seidel scheme to solve (4.3) when $n = 10$, with various choices of the initial level k. As in Example 4.2, for each k we stop the iteration when the successive difference is smaller than 1.0e−15, and use the last available ratio $\nu_{k,l}^{(m)}$ as the convergence rate $\nu_{k,l}$. Note that these ratios are comparable to the matrix norms computed in Table 4.1(b), and they are essentially independent of l. The order is then computed by

$$\text{order} = \log_2 \frac{\nu_{k-1,l+1}}{\nu_{k,l}}.$$

Numerical results are reported in Table 4.3. As indicated by (4.4), the order is nearly 1, and larger k requires fewer iterations to obtain the same accuracy in successive differences.

TABLE 4.3. Order of convergence rate for Gauss-Seidel scheme.

(k,l)	m	$\nu_{k,l}^{(m)} \approx \nu_{k,l}$	order
(4,6)	9	3.0795e−02	
(5,5)	7	1.5431e−02	0.9968
(6,4)	6	7.7243e−03	0.9987
(7,3)	5	3.8640e−03	0.9993
(8,2)	5	1.9322e−03	0.9999

Example 4.4. A stopping criterion for the iterations. The iteration schemes considered in this study are for solving linear system (4.3) whose solution $u_n \in X_n$ is an approximation to the exact solution $u \in X$. The accuracy of u_n to u is determined by the choice of the approximate subspaces, and for this particular example, it is well-known that the approximation is of second order:

$$\|u_n - u\|_2 = O(2^{-2n}). \tag{4.5}$$

Estimates of this type are available for commonly used approximate subspaces, such as piecewise polynomials (see, e.g., [**5**]). As seen in Example 4.2 above, the later iterations $u_{k,l}^{(m)}$ do not reduce error to the exact solution u, since the iterations are for finding $u_n \in X_n$ only. If our goal is simply to find an approximation to the exact solution $u \in X$, then $u_n \in X_n$ by the Galerkin method is one candidate with the accuracy stated in (4.5), and certainly we should also accept other approximations that have the same accuracy. This idea of finding an approximation of the Galerkin solution u_n with the same accuracy to the exact solution u is implemented and analyzed in [**5**] by a multi-level method. Here we propose a stopping criterion for our iteration schemes to obtain an iterate $u_{k,l}^{(m)}$ that falls into the neighborhood of the exact solution u as specified by (4.5). To do so we need to have an estimate for the "big-O" in (4.5), and one way to obtain this is to use $u_{k,1}^{(m_0)}$ as an estimate for the exact solution. Here $u_{k,1}^{(m_0)}$ is the m_0-th iterate at the $k+1$ level from the initial guess of u_k, the exact solution of (4.3) for $n = k$. Of course, the larger m_0 is, the better this estimate is, and we can in fact use iterations at another level $\ell > 1$ for the same purpose if so desired. In practice m_0 as small as 1 or 2 might work

TABLE 4.4. Results of using the stopping criterion (4.6) with $m_0 = 1$.

(k,l)	m	$\lvert u_{k,l}^{(m)} - u_{k,l}^{(m-1)}\rvert_2$	$\lVert u_{k,l}^{(m)} - u\rVert_2$	order	$\lVert u_{k+l}^{*} - u\rVert_2$	order
(5,2)	2	1.1132e−06	4.5493e−06		4.5493e−06	
(5,3)	2	1.1154e−06	1.1375e−06	1.9998	1.1373e−06	2.0001
(5,4)	3	1.7141e−08	2.8454e−07	1.9992	2.8459e−07	1.9986
(5,5)	3	1.7142e−08	6.8082e−08	2.0633	6.9094e−08	2.0422

well. Using such an approximation, we obtain the following stopping criterion: terminate the iteration at the m-th iterate if

$$|u_{k,l}^{(m)} - u_{k,l}^{(m-1)}|_2 < c_k 2^{-2l-1} \quad \text{with} \quad c_k = |u_{k,1}^{(m_0)} - u_k|_2. \tag{4.6}$$

In this way we save the unnecessary iterates yet yield an approximation of $u_{k,l}$ which has the same accuracy as the approximating subspace X_{k+l} would allow. As expected, when k is fixed, for larger l, it requires more iterates to obtain accurate approximations at the $k + l$ level. We tested this criterion for the Gauss-Seidel scheme with $m_0 = 1$ and obtain satisfactory results. When $k = 5$ and $m_0 = 1$, we have $c_k = 7.3874\text{e}{-5}$. We summarize results of this numerical experiment in Table 4.4. For comparison purposes, we use the Matlab routine, `|u=(I-A)\f|`, to obtain the exact solution $u_n^* = u_{k+l}^*$ to (4.3). As seen from the results, this criterion works very well for this example to save many unnecessary iterations. The approximations obtained are as close to the exact solution u in X as the exact solution u_n^* in X_n. The orders computed for both $u_{k,l}^{(m)}$ and u_n^* verify the order stated in (4.5). The savings of solving a larger system in this way can be clearly seen if we notice that, for example, instead of solving a full 2048×2048 system, we only need to solve a 64×64 system three times.

We remark that the simple stopping criterion tested in this example is also applicable for other iteration schemes provided that estimates similar to (4.5) are available.

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