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BOUNDARY INTEGRAL EQUATIONS FOR THE BIHARMONIC DIRICHLET PROBLEM ON NONSMOOTH DOMAINS

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ABSTRACT. In this paper we study the boundary reduction of the biharmonic interior and exterior Dirichlet problems in a plane domain with piecewise smooth boundary. The mapping properties of single and double layer biharmonic potentials, of biharmonic boundary integral operators, the Calderon projections and Poincaré-Steklov operators for domain with corners are analyzed. We derive existence and uniqueness results for direct boundary integral equations, which are equivalent to the variational formulation of the problems.

1. Introduction. The paper is devoted to the direct boundary integral method for solving the interior and exterior Dirichlet problems of the biharmonic equation

$$\begin{gathered}\Delta^2 u = 0 \quad \text{in } \Omega \subset \mathbf{R}^2,\\ u|_\Gamma = f_1, \qquad \partial_n u|_\Gamma = f_2.\end{gathered} \tag{1.1}$$

Here Ω is an interior or exterior domain bounded by a closed piecewise smooth curve Γ with corners, and the Dirichlet data $(f_1, f_2) = (v|_\Gamma, \partial_n v|_\Gamma)$ are the traces of a function v belonging on a neighborhood of Γ to the Sobolev space H^2. For the exterior problem, one has to impose additionally a special behavior of the solution at infinity.

The aim of the present paper is the study of direct boundary integral formulations which are equivalent to the variational solution of (1.1). As the main result, we derive different systems of integral equations on Γ and describe their solvability conditions. To do so we introduce certain boundary integral operators for the bi-Laplacian and study mapping properties in the corresponding trace spaces of H^2-functions.

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As a by-product, we are able to analyze the Steklov-Poincaré operators which map the Dirichlet data of biharmonic functions u to their Neumann data $(\Delta u|_\Gamma, \partial_n \Delta u|_\Gamma)$.

Among the different methods which exist for solving (1.1), integral equation methods play an important role, especially in connection with the boundary element method. For the interior problem and for sufficiently smooth boundary Γ such methods were investigated by several authors. Let us mention some results related to the contents of our paper.

In [**4**] and [**14**] a system of direct boundary integral equations was studied which is closely connected with the system (6.11) of our approach. In [**14**] Fuglede derived necessary and sufficient conditions for the equivalence of these equations to (1.1) if the Dirichlet data are sufficiently smooth. A general approach of direct first kind integral equations for (1.1) can be performed using the results of Costabel and Wendland, see [**6**] and [**13**]. Based on the theory of pseudodifferential operators, a complete description of the mapping properties of boundary integral operators, Calderon projections and Steklov-Poincaré operators can be obtained. This is mentioned in the paper of Costabel, Lusikka and Saranen [**9**], where approximation methods for solving the interior Dirichlet problem are studied which are based on three different boundary integral formulations. Besides the equations coinciding with our systems (6.13) and (6.11), the authors consider also an indirect method which goes back to Hsiao and MacCamy [**16**] and is based on a single layer representation. This approach was extended by Costabel, Stephan and Wendland studying in [**12**], to our knowledge for the first time, boundary integral equations for the bi-Laplacian on a nonsmooth curve. The authors consider the related boundary value problem $\operatorname{grad} u|_\Gamma = \mathbf{f}$ and obtain a system of two integral equations of the first kind with logarithmic principal part. Using Mellin techniques the continuity in Sobolev spaces and a Gårding inequality of the corresponding boundary integral operator are proven and the regularity of solutions is studied. Finally, we mention the paper [**2**] of Bourlard which proposes a direct Galerkin BEM for solving the interior Dirichlet problem on a polygonal domain and obtains optimal convergence rates for special graded meshes. Many of the stability results for the Galerkin method appear also in our approach, and we will comment on these results at the corresponding place.

The paper is organized as follows. In Section 2 we consider the space of Dirichlet data of H^2-functions and the space of Neumann data of H^2-functions u with $\Delta^2 u \in L^2$. The biharmonic potentials and their traces, the boundary integral operators, are introduced in Section 3. We investigate mapping properties of the boundary integral operators with respect to the trace spaces, the jump relations of the potentials and prove the Gårding inequality for the single layer potential operator. In Section 4 the behavior at infinity for solutions of the exterior Dirichlet problem is specified and we prove representation formulas for biharmonic functions. This allows the representation of Calderon projections via boundary integral operators. The special structure of these projections is used in Section 5 to analyze Steklov-Poincaré operators for the bi-Laplacian. After these preparations in the last section we deduce direct integral equations for solving the Dirichlet problem (1.1). For the interior problem we give three 2×2-systems of boundary integral equations and for the exterior problem two of those systems. In particular, the interior Dirichlet problem can be transformed to the system of integral equations on Γ

$$\tag{1.2} \begin{aligned} &\frac{\partial n_x}{4\pi} \int_\Gamma \eta_1(y) \partial_{n_y} |x-y|^2 \log|x-y| \, ds_y \\ &\qquad\qquad - \frac{\partial n_x}{4\pi} \int_\Gamma \eta_2(y) |x-y|^2 \log|x-y| \, ds_y \\ &\qquad = -\frac{\partial n_x}{\pi} \int_\Gamma f_1(y) \partial_{n_y} \log|x-y| \, ds_y \\ &\qquad\qquad + f_2(x) + \frac{\partial n_x}{\pi} \int_\Gamma f_2(y) \log|x-y| \, ds_y \\ &\eta_1(x) - \frac{1}{\pi} \int_\Gamma \eta_1(y) \partial_{n_y} \log|x-y| \, ds_y \\ &\qquad\qquad + \frac{1}{\pi} \int_\Gamma \eta_2(y) (\log|x-y| + 1) \, ds_y = 0. \end{aligned}$$

We study the solvability of the integral equations and prove their equivalence to the corresponding Dirichlet problem. For example, the equations (1.2) are uniquely solvable for any Dirichlet data (f_1, f_2) if the logarithmic capacity of the boundary cap $\Gamma \neq e^{-1}$ and the solution of (1.1) is given in Ω as the sum of potentials

$$\begin{aligned}u(x) = \frac{1}{2\pi}\int_\Gamma f_1(y)\partial_{n_y}\log|x-y|\,ds_y \\ - \frac{1}{2\pi}\int_\Gamma f_2(y)(\log|x-y|+1)\,ds_y \\ + \frac{1}{8\pi}\int_\Gamma \eta_1(y)\partial_{n_y}|x-y|^2\log|x-y|\,ds_y \\ - \frac{1}{8\pi}\int_\Gamma \eta_2(y)|x-y|^2\log|x-y|\,ds_y.\end{aligned}$$

To conclude the introduction we briefly comment on some topics not treated in this paper. We do not consider the approximate solution of the integral equations. If the boundary Γ is smooth, then well-known approximation results for pseudodifferential equations can be used to prove the convergences of different approximation methods, we refer to the papers [**9, 10**] and to Remark 6.1. For the case of piecewise smooth Γ the convergence of Galerkin and certain collocation methods for the strongly elliptic system (6.6) is rather clear, whereas the stability of approximation methods for solving the other systems seems to be open. To get error estimates one has to know the regularity of the corresponding solutions. We do not study this topic as well as the continuity of boundary integral operators in other than the energy norms because of the lack of space. Since we are dealing with direct methods, some regularity results can be derived from the known singularities of the solutions of the Dirichlet problem, see [**1**]. On the other hand, the calculus of Mellin operators provides a useful tool in this direction. A more interesting problem not treated is the analysis of direct integral methods for the biharmonic equation with other types of boundary conditions connected with this plate bending. The application of our methods to this problem will be considered in a forthcoming paper, see Remark 5.3.

2. Traces of H^2-functions on piecewise smooth boundaries. For the following let Γ be a simple closed curve in the plane (x_1, x_2) of the form

$$\Gamma = \bigcup_{i=1}^n \Gamma_i,$$

where Γ_i are of the class C^3 and adjacent arcs Γ_i form corners with angles different from 0 and 2π. The interior of Γ we denote by Ω_1, the exterior $\mathbf{R}^2\backslash\overline{\Omega}_1$ by Ω_2, and let the unit normal n on Γ be directed into Ω_2. The differentiation with respect to n is denoted by ∂_n. The starting point of our analysis is

Lemma 2.1 (Jakovlev [**17**]). *Let* $u \in H^2(\Omega_1)$. *Then*

$$u|_{\Gamma_i} \in H^{3/2}(\Gamma_i), \quad \partial_n u|_{\Gamma_i} \in H^{1/2}(\Gamma_i),$$
$$u|_\Gamma \in H^1(\Gamma), \quad \left.\frac{\partial u}{\partial x_1}\right|_\Gamma, \left.\frac{\partial u}{\partial x_2}\right|_\Gamma \in H^{1/2}(\Gamma),$$

and there exists a constant $c > 0$, *not dependent on* u, *such that*

$$\sum_{i=1}^n (\|u\|_{H^{3/2}(\Gamma_i)} + \|\partial_n u\|_{H^{1/2}(\Gamma_i)}) + \left\|\frac{\partial u}{\partial x_1}\right\|_{H^{1/2}(\Gamma)} + \left\|\frac{\partial u}{\partial x_2}\right\|_{H^{1/2}(\Gamma)} \le c\|u\|_{H^2(\Omega)}.$$

If the projections of the normal n onto the x_1- and x_2-axis are denoted by n_1 and n_2, respectively, then

$$\left.\frac{\partial u}{\partial x_1}\right|_\Gamma = n_1\partial_n u - n_2\partial_s u, \qquad \left.\frac{\partial u}{\partial x_2}\right|_\Gamma = n_2\partial_n u + n_1\partial_s u, \tag{2.1}$$

where ∂_s denotes the differentiation with respect to the arc length s. In the sequel we identify functions on Γ with periodic functions depending on s and write $\partial_s u = u'$. It is well known that, for $|t| \le 1$ the Sobolev spaces $H^t(\Gamma)$ can be identified with the corresponding periodic Sobolev spaces.

Note that the functions n_1 and n_2 as well as $\partial_n\varphi|_\Gamma$ and $\partial_s\varphi|_\Gamma$ for smooth $\varphi \in C_0^\infty(\mathbf{R}^2)$ belong piecewise to the class C^2 and C^3, respectively, having jumps at the corner points. Let us introduce the trace space

$$V(\Gamma) = \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} : u_1 \in H^1(\Gamma), n_1u_2 - n_2u_1' \in H^{1/2}(\Gamma), \right.$$
$$\left. n_2u_2 + n_1u_1' \in H^{1/2}(\Gamma) \right\},$$

equipped with the canonical norm and define the generalized trace

$$\gamma u := \begin{pmatrix} u|_\Gamma \\ \partial_n u|_\Gamma \end{pmatrix}.$$

Lemma 2.2 (Jakovlev [**17**]). *The linear mapping*

$$\gamma : H^2_{\mathrm{loc}}(\mathbf{R}^2) \longrightarrow V(\Gamma)$$

is continuous and has a continuous right inverse

$$\gamma^- : V(\Gamma) \longrightarrow H^2_{\mathrm{loc}}(\mathbf{R}^2).$$

In particular, γ maps $C_0^\infty(\mathbf{R}^2)$ onto a dense subspace of $V(\Gamma)$.

To describe the dual $V(\Gamma)'$ of the trace space, we introduce the duality form

$$\left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right] := -\langle v_1, u_1\rangle_\Gamma + \langle v_2, u_2\rangle_\Gamma, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle_\Gamma$ denotes the extension of the usual L^2-scalar product on Γ. Since the mapping

$$\begin{pmatrix} n_1 u_2 - n_2 u_1' \\ n_2 u_2 + n_1 u_1' \\ \int_\Gamma u_1\, ds \end{pmatrix} : V(\Gamma) \longrightarrow H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \times \mathbf{R}$$

is isomorphic we obtain

Lemma 2.3. *The vector $\binom{v_1}{v_2}$ belongs to $V(\Gamma)'$ if and only if there exist $z_1, z_2 \in H^{-1/2}(\Gamma)$ and a number $a \in \mathbf{R}$ such that, for any $\varphi \in C_0^\infty(\mathbf{R}^2)$, the equations*

$$\begin{aligned} \langle \varphi|_\Gamma, v_1\rangle_\Gamma &= \langle \partial_s\varphi|_\Gamma, n_2 z_1 - n_1 z_2\rangle_\Gamma + a\int_\Gamma \varphi\, ds, \\ \langle \varphi|_\Gamma, v_2\rangle_\Gamma &= \langle \varphi|_\Gamma, n_1 z_1 + n_2 z_2\rangle_\Gamma, \end{aligned}$$

are satisfied.

The trace $\gamma u \in V(\Gamma)$ will be called the *Dirichlet datum* of $u \in H^2_{\text{loc}}(\mathbf{R}^2)$ on Γ. Now we define the *Neumann datum*. We introduce the space

$$H^2(\Omega_1, \Delta^2) = \{u \in H^2(\Omega_1) : \Delta^2 u \in L^2(\Omega_1)\}$$

with the graph norm.

Lemma 2.4. *$C^\infty(\overline{\Omega}_1)$ is dense in $H^2(\Omega_1, \Delta^2)$.*

The proof is based on the same arguments as the proof for the case $H^1(\Omega_1, \Delta)$ given in the book of Grisvard [**15**].

Lemma 2.5. *Let $u \in H^2(\Omega_1, \Delta^2)$. Then the mapping*

$$\delta u : \psi \longrightarrow [\delta u, \psi] := \int_{\Omega_1} (\Delta u \Delta(\gamma^- \psi) - \gamma^- \psi \Delta^2 u)\, dx \tag{2.3}$$

is a continuous linear functional on $V(\Gamma)$ that coincides for sufficiently smooth u with the functional

$$\delta u := \begin{pmatrix} \partial_n \Delta u|_\Gamma \\ \Delta u|_\Gamma \end{pmatrix}. \tag{2.4}$$

Moreover, the mapping $\delta : H^2(\Omega_1, \Delta^2) \to V(\Gamma)'$ is continuous.

Proof. The first Green formula

$$\int_{\Omega_1} (\Delta u \Delta v - v \Delta^2 u)\, ds = \int_\Gamma (\Delta u \partial_n v - v \partial_n \Delta u)\, ds$$

is valid for all $u \in H^4(\Omega_1)$, $v \in H^2(\Omega_1)$, see [**5**]. Hence, for sufficiently smooth u,

$$\begin{aligned} |[\delta u, \psi]| &\le \|\Delta u\|_{L^2(\Omega_1)} \|\Delta(\gamma^- \psi)\|_{L^2(\Omega_1)} + \|\Delta^2 u\|_{L^2(\Omega_1)} \|\gamma^- \psi\|_{L^2(\Omega_1)} \\ &\le \|u\|_{H^2(\Omega_1, \Delta_2)} \|\gamma^- \psi\|_{H^2(\Omega_1)}. \end{aligned}$$

From Lemmas 2.2 and 2.4 the assertion follows by continuity. $\square$

Corollary 2.1. *For $u, v \in H^2(\Omega_1, \Delta^2)$ the second Green formula*

$$\int_{\Omega_1} (v\Delta^2 u - u\Delta^2 v)\, dx = [\delta v, \gamma u] - [\delta u, \gamma v]$$

holds. If $u \in H^2(\Omega_1)$ solves the biharmonic equation $\Delta^2 u = 0$, then $[\delta u, \gamma u] \geq 0$.

The construction of the Neumann data δu is standard, for second order equations we refer to [**15**] and [**7**], for the biharmonic equation a similar construction is given in [**2**]. We note that the definition of δu is based on the bilinear form

$$a(u, v) := \int_{\Omega_1} \Delta u \Delta v\, dx,$$

corresponding to the variational solution of the Dirichlet problem

$$\Delta^2 u = f \quad \text{in } \Omega_1, \quad \gamma u = \psi, \tag{2.5}$$

with $f \in L^2(\Omega_1)$, $\psi \in V(\Gamma)$. Since $a(u, u)^{1/2}$ defines a norm on $H_0^2(\Omega_1)$, see [**5**], Lemma 2.2 leads to

Lemma 2.6. *The Dirichlet problem* (2.5) *has for any $f \in L^2(\Omega_1)$, $\psi \in V(\Gamma)$ a unique solution $u \in H^2(\Omega_1, \Delta^2)$. The solution operator*

$$T : L^2(\Omega_1) \times V(\Gamma) \longrightarrow H^2(\Omega_1, \Delta^2) \tag{2.6}$$

is continuous.

Now we can prove

Lemma 2.7. *δ maps $C_0^\infty(\mathbf{R}^2)$ onto a dense subspace of $V(\Gamma)'$.*

Proof. Assume that there exists $\psi \in V(\Gamma)$ such that

$$[\delta\varphi, \psi] = 0 \quad \text{for all} \quad \varphi \in C_0^\infty(\mathbf{R}^2). \tag{2.7}$$

Due to Lemma 2.6, the boundary value ψ and an arbitrary $f \in L^2(\Omega_1)$ lead to solutions $T(0,\psi)$, $T(f,0) \in H^2(\Omega_1,\Delta^2)$ of the corresponding Dirichlet problems. Applying Corollary 2.1, we obtain

$$\begin{aligned}[\delta T(f,0),\psi] &= [\delta T(f,0),\gamma T(0,\psi)] - [\delta T(0,\psi),\gamma T(f,0)] \\ &= \int_{\Omega_1} (T(f,0)\Delta^2 T(0,\psi) - T(0,\psi)\Delta^2 T(f,0))\,dx \\ &= -\int_{\Omega_1} fT(0,\psi)\,dx.\end{aligned}$$

From Lemma 2.4, we conclude that (2.7) holds even for $\varphi = T(f,0) \in H^2(\Omega_1,\Delta^2)$, such that

$$\int_{\Omega_1} fT(0,\psi)\,dx = 0 \quad \text{for all} \quad f \in L^2(\Omega_1).$$

Thus $T(0,\psi) = 0$ and the relation $\psi = \gamma T(0,\psi) = 0$ shows that $\delta(C_0^\infty(\mathbf{R}^2))$ is dense in $V(\Gamma)'$. $\square$

In the sequel we consider also the Dirichlet problem in the exterior domain Ω_2. Besides the Dirichlet datum we have therefore to define the Neumann datum of functions given outside of Ω_1. Let $\tilde{\Omega}$ be a domain containing $\overline{\Omega}_1$, and let $u \in H^2(\tilde{\Omega}\backslash\Omega_1,\Delta^2)$. For $v \in H^2(\tilde{\Omega}\backslash\Omega_1)$, we define

$$[\delta u,\gamma v] = \int_{\tilde{\Omega}\backslash\Omega_1} (\varphi v \Delta^2 u - \Delta(\varphi v)\Delta u)\,dx,$$

where $\varphi \in C_0^\infty(\tilde{\Omega})$ with $\varphi \equiv 1$ on a neighborhood of $\overline{\Omega}_1$. It is clear that the definition of δ does not depend on φ. Moreover, it ensures that for $\varphi \in C_0^\infty(\mathbf{R}^2)$ there holds

$$\delta(\varphi|_{\Omega_2}) = \delta(\varphi|_{\Omega_1}).$$

In the following the pair of Dirichlet and Neumann data $(\gamma u,\delta u)$ will be called *Cauchy data* of u.

3. Boundary integral operators for the bi-Laplacian. Here we apply the approach developed in Costabel [**7**] to study boundary integral operators for second order equations on Lipschitz domains.

The boundary integral operators for the bi-Laplacian Δ^2 are based on the fundamental solution

$$G(x,y) := \frac{1}{8\pi}|x-y|^2 \log|x-y|, \quad x,y \in \mathbf{R}^2,$$

satisfying

$$\Delta_y^2 G(x,y) = \Delta_x^2 G(x,y) = \delta(x-y).$$

It is well known that the operator

$$\mathcal{G}u(x) := \langle G(x,\cdot), u\rangle_{\mathbf{R}^2}$$

is the two-sided inverse of Δ^2 on the space of compactly supported distributions on $\mathbf{R}^2$ and represents a pseudodifferential operator of order -4, i.e., the mapping

$$\mathcal{G} : H^s_{\mathrm{comp}}(\mathbf{R}^2) \longrightarrow H^{s+4}_{\mathrm{loc}}(\mathbf{R}^2), \quad s \in \mathbf{R}, \tag{3.1}$$

is continuous. Note that

$$\Delta_y G(x,y) = \Delta_x G(x,y) = \frac{1}{2\pi}\log|x-y| + \frac{1}{2\pi}. \tag{3.2}$$

We have the following representation formula.

Lemma 3.1. *Let $u \in L^2(\mathbf{R}^2)$ be a function with compact support such that the restrictions $u|_{\Omega_1} \in H^2(\Omega_1)$, $u|_{\Omega_2} \in H^2(\Omega_2)$ and $f = \Delta^2 u|_{\mathbf{R}^2\setminus\Gamma} \in L^2(\mathbf{R}^2)$. Then, for $x \in \mathbf{R}^2\setminus\Gamma$ the representation*

$$u(x) = \mathcal{G}f(x) - [\{\delta u\}, \gamma G(x,\cdot)] + [\delta G(x,\cdot), \{\gamma u\}],$$

holds, where

$$\{\gamma u\} := \gamma(u|_{\Omega_2}) - \gamma(u|_{\Omega_1}),\ \ \{\delta u\} := \delta(u|_{\Omega_2}) - \delta(u|\Omega_1),$$

denote the jumps across Γ.

The proof follows immediately from the second Green formula, Corollary 2.1, and the known representation formula for sufficiently smooth functions applied in a small ball enclosing the point x.

Next we define the biharmonic layer potentials for $x \in \mathbf{R}^2 \backslash \Gamma$ as

$$\begin{aligned} \mathcal{K}_0\chi(x) &:= [\chi, \gamma G(x,\cdot)], \quad \chi \in V(\Gamma)', \\ \mathcal{K}_1\psi(x) &:= [\delta G(x,\cdot)\psi], \quad \chi \in V(\Gamma), \end{aligned} \tag{3.3}$$

and the boundary integral operators

$$\begin{aligned} \mathcal{A}\chi &:= 2\gamma\mathcal{K}_0\chi, & \mathcal{B}\chi &:= 2\delta(\mathcal{K}_0\chi|_{\Omega_1}), \\ \mathcal{C}\psi &:= 2\gamma(\mathcal{K}_1\psi|_{\Omega_1}), & \mathcal{D}\psi &:= -2\delta(\mathcal{K}_1\psi|_{\Omega_1}). \end{aligned} \tag{3.4}$$

Lemma 3.2. *The mappings*

$$\mathcal{K}_0 : V(\Gamma)' \longrightarrow H^2_{\mathrm{loc}}(\mathbf{R}^2), \quad \mathcal{K}_1 : V(\Gamma) \to H^2(\Omega_1)$$
$$\mathcal{A} : V(\Gamma)' \longrightarrow V(\Gamma), \quad \mathcal{B} : V(\Gamma)' \longrightarrow V(\Gamma)', \quad \mathcal{C} : V(\Gamma) \longrightarrow V(T)$$

are continuous and $D\psi = 0$ for all $\psi \in V(\Gamma)$.

Proof. Because of

$$\mathcal{K}_0\chi(x) = \langle G(x,\cdot), \gamma'\chi\rangle_{\mathbf{R}^3}$$

we can write

$$\mathcal{K}_0\chi = \mathcal{G}\gamma'\chi. \tag{3.5}$$

The adjoint of the trace map $\gamma' : V(\Gamma)' \to H^{-2}_{\mathrm{comp}}(\mathbf{R}^2)$ is continuous, therefore the assertion for $\mathcal{K}_0$ follows from (3.1).

Due to Lemma 3.1 the solution $u = T(0,\psi)$ of the Dirichlet problem (2.5) can be represented in the form

$$T(0,\psi) = \mathcal{K}_0\delta T(0,\psi) - \mathcal{K}_1\psi,$$

such that from Lemmas 2.5 and 2.6, we derive

$$\|\mathcal{K}_1\psi\|_{H^2(\Omega_1)} \le c\|\psi\|_{V(\Gamma)}.$$

Now the mapping properties of $\mathcal{A}$ and $\mathcal{C}$ are a simple consequence of Lemma 2.1. The boundedness of $\mathcal{B}$ follows from Lemma 2.5 since $\Delta^2\mathcal{K}_0\chi = 0$ in Ω_1.

For $\psi = \binom{v_1}{v_2} \in V(\Gamma)$, we get from (2.2), (2.4) and (3.2) the representation

$$(3.6) \quad \mathcal{K}_1\psi(x) = -\frac{1}{2\pi}\int_\Gamma v_1(y)\partial_{n_y}\log|x-y|\,ds_y + \frac{1}{2\pi}\int_\Gamma v_2(y)(\log|x-y|+1)\,ds_y.$$

Hence, $\mathcal{K}_1\psi \in H^2(\Omega_1)$ is a harmonic function and, for any $\varphi \in V(\Gamma)$,

$$[\mathcal{D}\psi, \varphi] = 2\int_{\Omega_1}(\gamma^-\varphi\Delta^2\mathcal{K}_1\psi - \Delta(\gamma^-\varphi)\Delta\mathcal{K}_1\psi)\,dx = 0. \qquad \square$$

The layer potentials provide the following jump relations:

Lemma 3.3.

$$\{\gamma\mathcal{K}_0\chi\} = 0, \{\delta\mathcal{K}_0\chi\} = -\chi \quad \textit{for all } \chi \in V(\Gamma)',$$
$$\{\gamma\mathcal{K}_1\psi\} = \psi, \{\delta\mathcal{K}_1\psi\} = 0 \quad \textit{for all } \psi \in V(\Gamma).$$

Proof. Since $u = \mathcal{K}_0\chi \in H^2_{\mathrm{loc}}(\mathbf{R}^2)$, we have $\gamma(u|_{\Omega_1}) = \gamma(u|_{\Omega_2})$.

Further, from (3.5) we obtain $\Delta^2 u = \gamma'\chi$ in the distributional sense, i.e.,

$$\int_{\mathbf{R}^2} u\Delta^2\varphi\,dx = \langle\gamma'\chi, \varphi\rangle_{\mathbf{R}^2} = [\chi, \gamma\varphi]$$

for any $\varphi \in C_0^\infty(\mathbf{R}^2)$. On the other hand,

$$\begin{aligned}\int_{\Omega_1} u\Delta^2\varphi\,dx &= \int_{\Omega_1}\Delta u\Delta\varphi\,dx - [\delta\varphi, \gamma u]\\ &= [\delta(u|_{\Omega_1}), \gamma\varphi] - [\delta\varphi, \gamma u],\\ \int_{\Omega_2} u\Delta^2\varphi\,dx &= -[\delta(u|_{\Omega_2}), \gamma\varphi] + [\delta\varphi, \gamma u].\end{aligned}$$

Thus

$$[\chi, \gamma\varphi] = -[\delta(\mathcal{K}_0\chi|_{\Omega_2}) - \delta(\mathcal{K}_0\chi|_{\Omega_1}), \gamma\varphi], \quad \forall\,\varphi \in C_0^\infty(\mathbf{R}^2).$$

Now let $u = \mathcal{K}_1\psi$, $\psi = \binom{v_1}{v_2} \in V(\Gamma)$. From (3.6) and the jump relations of the harmonic potentials (proved for example in [**11**] for the more general case $(v_1, v_2) \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$) we obtain

$$u|_{\Omega_2} - u|_{\Omega_1} = v_1, \qquad \partial_n u|_{\Omega-2} - \partial_n u|_{\Omega_1} = v_2. \qquad \square$$

Now we consider the adjoints of the boundary integral operators (3.4) with respect to the duality form (2.2). Here and in the following, Id denotes the identity mapping in the spaces $V(\Gamma)$, $V(\Gamma)'$ or $V(\Gamma) \times V(\Gamma)'$.

Corollary 3.1. *The following hold:* $\mathcal{A} = \mathcal{A}'$ *and* $\mathcal{B}' = \mathcal{C} + 2\mathrm{Id}$.

Proof. The assertion follows immediately from the symmetry of the kernel function G and the jump relations, for example,

$$\begin{aligned}[\mathcal{B}\chi, \psi] &= [\delta(\mathcal{K}_0\chi|_{\Omega_1}) + \delta(\mathcal{K}_0\chi|_{\Omega_2}) + \chi, \psi] \\ &= \langle \mathcal{G}\gamma'\chi|_{\Omega_1} + \mathcal{G}\gamma'\chi|_{\Omega_2}, \delta'\psi\rangle_{\mathbf{R}^2} + [\chi, \psi],\end{aligned}$$

where $\delta'\psi$ denotes the compactly supported distribution on $\mathbf{R}^2$ defined by

$$\langle \varphi, \delta'\psi\rangle_{\mathbf{R}} = [\delta\varphi, \psi] \quad \text{for all } \varphi \in C_0^\infty(\mathbf{R}^2).$$

Since the jump relation yields, for $u = \mathcal{K}_1\psi$,

$$\int_{\mathbf{R}^2} u\Delta^2\varphi \, dx = [\delta\varphi, \psi] \quad \text{for all } \varphi \in C_0^\infty(\mathbf{R}^2)$$

we have $\Delta^2 u = \delta'\psi$ in the distributional sense, hence $\mathcal{K}_1\psi = \mathcal{G}\delta'\psi$ and

$$\begin{aligned}[\mathcal{B}\chi, \psi] &= \langle \gamma'\chi, \mathcal{G}\delta'\psi|_{\Omega_1} + \mathcal{G}\delta'\psi|_{\Omega_2}\rangle_{\mathbf{R}^2} + [\chi, \psi] \\ &= [\chi, \gamma(\mathcal{K}_1\psi|_{\Omega_1}) + \gamma(\mathcal{K}_1\psi|_{\Omega_2})] + [\chi, \psi] \\ &= [\chi, 2\gamma(\mathcal{K}_1\psi|_{\Omega_1}) + \psi] + [\chi, \psi] \\ &= [\chi, \mathcal{C}\psi] + 2[\chi, \psi]. \qquad \square\end{aligned}$$

Let us introduce the operator

$$\mathcal{W} := \mathrm{Id} + \mathcal{C}.$$

Then Corollary 3.1 implies that

$$\mathcal{B} = \mathrm{Id} + \mathcal{W}',$$

and from Lemma 3.3 we derive for $j = 1, 2$, the relations

$$\begin{aligned} \gamma(\mathcal{K}_1\psi|_{\Omega_j}) &= \frac{1}{2}(\mathcal{W} + (-1)^j \mathrm{Id})\psi, \\ \delta(\mathcal{K}_0\chi|_{\Omega_j}) &= \frac{1}{2}(\mathcal{W}' - (-1)^j \mathrm{Id})\chi. \end{aligned} \tag{3.7}$$

Therefore we call $\mathcal{W}$ the double layer potential operator of the bi-Laplacian on Γ. The corresponding single layer potential operator on Γ satisfies a Gårding inequality.

Lemma 3.4. *The operator $\mathcal{A}$ is strongly elliptic, i.e., there exist a compact operator $\mathcal{T} : V(\Gamma)' \to V(\Gamma)$ and a positive constant c such that*

$$|[\chi, (\mathcal{A} + \mathcal{T})\chi]| \geq c\|\chi\|^2_{V(\Gamma)'}, \quad \forall\, \chi \in V(\Gamma)'.$$

Proof. For $\chi \in V(\Gamma)'$ and $u = -\mathcal{K}_0\chi$ we have the relations

$$\gamma u|_{\Omega_1} = \gamma u|_{\Omega_2} = -\frac{1}{2}\mathcal{A}\chi, \quad \{\delta u\} = \chi.$$

We choose $\varphi \in C_0^\infty(\mathbf{R}^2)$ with $\varphi \equiv 1$ on a neighborhood of Ω_1 and set $u_1 = u|_{\Omega_1}$, $u_2 = \varphi u|_{\Omega_2}$. Then

$$\begin{aligned} \frac{1}{2}[\chi, \mathcal{A}\chi] &= [\delta u_1, \gamma u_1] - [\delta u_2, \gamma u_2] \\ &= \int_{\Omega_1} |\Delta u_1|^2\, dx + \int_{\Omega_2} |\Delta u_2|^2\, dx - \int_{\Omega_2} u_2 \Delta^2 u_2\, dx. \end{aligned}$$

Now we use the relation

$$\int_{\Omega_1} \left(\frac{\partial^2 u_1}{\partial x_1 \partial x_2}\right)^2 dx = \int_{\Omega_1} \frac{\partial^2 u_1}{\partial x_1^2}\frac{\partial^2 u_1}{\partial x_2^2}\, dx + \left\langle \frac{\partial u_1}{\partial x_2}, \partial_s \frac{\partial u_1}{\partial x_1}\right\rangle_\Gamma.$$

For smooth functions and any bounded domain this equality follows from Green's formula and (2.1). By Lemma 2.1, it holds therefore for any H^2-function. Hence, for $u_2 = \varphi u|_{\Omega_2}$, we have

$$\int_{\Omega_2} \left(\frac{\partial^2 u_2}{\partial x_1 \partial x_2} \right)^2 dx = \int_{\Omega_2} \frac{\partial^2 u_2}{\partial x_1^2} \frac{\partial^2 u_2}{\partial x_2^2} \, dx - \left\langle \frac{\partial u_2}{\partial x_2}, \partial_s \frac{\partial u_2}{\partial x_1} \right\rangle_\Gamma .$$

Because of $\gamma u_1 = \gamma u_2$, we get

$$\|\Delta u_1\|^2_{L^2(\Omega_1)} + \|\Delta u_2\|^2_{L^2(\Omega_2)} = |u_1|^2_{H^2(\Omega_1)} + |u_2|^2_{H^2(\Omega_2)};$$

here $|\cdot|_{H^2(\Omega_j)}$ denotes the usual seminorm, such that

$$\frac{1}{2}[\chi, \mathcal{A}\chi] = |u_1|^2_{H^2(\Omega_1)} + |u_2|^2_{H^2(\Omega_2)} - \int_{\Omega_2} u_2 \Delta^2 u_2 \, dx.$$

This leads, together with Lemma 2.5, to the inequality

$$\begin{aligned} \|\chi\|^2_{V(\Gamma)'} &= \|\delta u_1 - \delta u_2\|^2_{V(\Gamma)'} \\ &\le c(\|u_1\|^2_{H^2(\Omega_1)} + \|u_2\|^2_{H^2(\Omega_2)} + \|\Delta^2 u_2\|^2_{L^2(\Omega_2)}) \\ &\le \frac{c}{2}[\chi, \mathcal{A}\chi] + c\Bigg(\|u_1\|^2_{L^2(\Omega_1)} + \|u_2\|^2_{L^2(\Omega_2)} \\ &\qquad + \|\Delta^2 u_2\|^2_{L^2(\Omega_2)} + \int_{\Omega_2} u_2 \Delta^2 u_2 \, dx \Bigg). \end{aligned}$$

Since $\Delta^2 u_2$ is smooth and compactly supported in Ω_2, the term in the brackets is generated by a compact bilinear form of χ. $\square$

Corollary 3.2. *The operator $\mathcal{A} : V(\Gamma)' \to V(\Gamma)$ is Fredholm with index zero. If $\mathcal{A}\chi \in V(\Gamma)$, then $\chi \in V(\Gamma)'$.*

4. Calderon projections. Now we are in the position to define the Calderon projections which map onto the Cauchy data of functions biharmonic in Ω_1 or Ω_2. Here we follow a method developed in Costabel and Stephan [**11**] for second order equations.

We define the linear spaces

$$L_j := \{\mathcal{K}_0 \chi(x) - \mathcal{K}_1 \psi(x) : (\psi, \chi) \in V(\Gamma) \times V(\Gamma)', x \in \Omega_j\},$$

in which the solutions of the biharmonic equation are sought. From Lemmas 3.1 and 3.2 we conclude that L_1 is the set of all functions $u \in H^2(\Omega_1)$ satisfying $\Delta^2 u = 0$. Moreover, for $u \in L_1$, the representation formula

$$\mathcal{K}_0 \delta u(x) - \mathcal{K}_1 \gamma u(x) = \begin{cases} u(x) & x \in \Omega_1, \\ 0 & x \in \Omega_2, \end{cases} \tag{4.1}$$

holds.

The space L_2 consists of biharmonic functions $u \in H^2_{\mathrm{loc}}(\Omega_2)$ providing a special behavior at infinity, which we refer to as *radiation condition* and can be described as follows. We introduce the mappings

$$\begin{aligned} I_j \chi(x) &= [\chi, \gamma g_j(x, \cdot)], \quad \chi \in V(\Gamma)', \quad j = 1(1)4, \\ I_5 \psi &= [\delta g_3(x, \cdot), \psi], \quad \psi \in V(\Gamma), \end{aligned} \tag{4.2}$$

where the functions $g_j(x, y)$ are given by

$$\begin{aligned} g_1(x, y) &= 1, & g_2(x, y) &= \frac{x \cdot y}{|x|}, \\ g_3(x, y) &= |y|^2, & g_4(x, y) &= \frac{|y|^2}{2} + \frac{(x \cdot y)^2}{|x|^2}, \end{aligned}$$

(here $x \cdot y$ denotes the inner product of vectors $x, y \in \mathbf{R}^2$ and, as before, $|y|^2 = y \cdot y$). Note that I_1, I_3 and I_5 are linear functionals, while $I_2 \chi$ and $I_4 \chi$ are functions depending on the direction of x.

Lemma 4.1. *For given $(\psi, \chi) \in V(\Gamma) \times V(\Gamma)'$, the function*

$$u(x) = \mathcal{K}_1 \psi(x) - \mathcal{K}_0 \chi(x)$$

behaves for large $|x| = R$ as

$$\begin{aligned} u(x) = -\frac{1}{8\pi} &(I_1 \chi R^2 \log R - I_2 \chi(x)(2R \log R + R) \\ &+ (I_3 \chi - I_5 \psi) \log R + I_4 \chi(x) - I_5 \psi) \\ &+ O(R^{-1}). \end{aligned} \tag{4.3}$$

This expansion was proved in [**4**] for the case of ψ and χ having continuous components, such that from Lemmas 2.2 and 2.7 the assertion follows immediately.

A representation formula similar to (4.1) also holds for functions $u \in L_2$.

Lemma 4.2. *For $u \in L_2$ with Cauchy data $(\gamma u, \delta u)$, the following holds*

$$\mathcal{K}_1 \gamma u(x) - \mathcal{K}_0 \delta u(x) = \begin{cases} u(x) & x \in \Omega_2, \\ 0 & x \in \Omega_1. \end{cases} \tag{4.4}$$

Proof. We enclose Ω_1 by a ball B_R with radius $R > |x|$. Then the representation formula (4.1) is valid for the bounded domain $\Omega_2 \cap B_R$ yielding

$$\begin{aligned} u(x) = \mathcal{K}_1 \gamma u(x) - \mathcal{K}_0 \delta u(x) \\ + \int_{S_R} (u \partial_{n_z} \Delta G(x,z) - \Delta G(x,z) \partial_n u + \Delta u \partial_{n_z} G(x,z) \\ - G(x,z) \partial_n \Delta u) \, ds_z. \end{aligned}$$

Using the asymptotics (4.3) of $u(z)$ as $R = |z| \to \infty$ and the asymptotics of the fundamental solution given in [**4**],

$$\begin{aligned} G(x,z) = \frac{1}{8\pi} \Big(R^2 \log R - (x \cdot n_z)(2R \log R + R) \\ + |x|^2 \log R + \frac{|x|^2}{2} + (x \cdot n_z)^2 \Big) + O(R^{-1}), \end{aligned}$$

one obtains that the integrand permits the expansion

$$\begin{aligned} \frac{1}{64\pi^2} \Big[\Big((x \cdot n_z)\, I_1 \delta u - I_2 \delta u(z) \Big) \Big(3(\log R - 1) - 2(\log R)^2 \Big) \\ - \frac{2}{R} \Big((2(x \cdot n_z)^2 - |x|^2)\, I_1 \delta u + 2 I_3 \delta u - 2 I_4 \delta u(z) \Big) \Big] + O(R^{-2}). \end{aligned}$$

Obviously,

$$\int_{S_R} (x \cdot n_z) ds_z = \int_{S_R} I_2 \delta u(z) \, ds_z = 0,$$

such that the integral of the first term in the brackets vanishes. Further, denote by θ_z the angle between x and the integration point z. Then

$$2(x \cdot n_z)^2 - |x|^2 = |x|^2(2\cos^2\theta_z - 1) = |x|^2\cos^2\theta_z,$$

implying

$$\int_{S_R} (2(x \cdot n_z)^2 - |x|^2)\, I_1 \delta u \, ds_z = 0.$$

Finally, we have

$$I_4\delta u(z) - I_3\delta u = [\delta u, \gamma h(z,\cdot)]$$

with the function

$$h(z,y) = \frac{(z \cdot y)^2}{|z|^2} - \frac{|y|^2}{2} = \frac{|y|^2}{2}\cos 2\theta_z,$$

where now θ_z is the angle between y and z. Denoting by α the angle between y and n_y, we get

$$\begin{aligned}\partial_{n_y} h(z,y) &= 2(y \cdot n_z)(n_y \cdot n_z) - (y \cdot n_y)\\ &= |y|(2\cos\theta_z\cos(\theta_z - \alpha) - \cos\alpha)\\ &= |y|\cos(2\theta_z - \alpha).\end{aligned}$$

Hence,

$$\int_{S_R} (I_3\delta u - I_4\delta u(z))\, ds_z = 0,$$

such that

$$\begin{aligned}\int_{S_R} (u\partial_{n_z}\Delta G(x,z) - \Delta G(x,z)\partial_n u + \Delta u \partial_{n_z} G(x,z)&\\ - G(x,z)\partial_n \Delta u)\, ds_z = O(R^{-1}).&\end{aligned}$$ □

Now we introduce the linear operator

$$\mathfrak{A} := \begin{pmatrix} -\mathcal{W} & A \\ \mathcal{O} & \mathcal{W}' \end{pmatrix} : \begin{matrix} V(\Gamma) \\ \times \\ V(\Gamma)' \end{matrix} \longrightarrow \begin{matrix} V(\Gamma) \\ \times \\ V(\Gamma)' \end{matrix},$$

where $\mathcal{O}$ denotes the zero mapping, and define

$$\mathfrak{P}_j := \frac{1}{2}(\mathrm{Id} - (-1)^j \mathfrak{A}). \tag{4.5}$$

Theorem 4.1. *For $j = 1$ and 2, the operator $\mathfrak{P}_j$ is a bounded projection in $V(\Gamma)\times V(\Gamma)'$ mapping onto the set of Cauchy data $(\gamma u, \delta u)$ of all functions $u \in L_j$.*

Proof. The boundedness of $\mathfrak{P}_j$ follows from Lemma 3.2. Further, for any pair of densities $(\psi, \chi) \in V(\Gamma) \times V(\Gamma)'$, we have

$$u = (-1)^j(\mathcal{K}_1\psi - \mathcal{K}_0\chi) \in L_j$$

and, by Lemma 3.2 and (3.7), we derive

$$\begin{aligned}
\begin{pmatrix} \gamma u \\ \delta u \end{pmatrix} &= (-1)^j \begin{pmatrix} \gamma(\mathcal{K}_1\psi|_{\Omega_j}) - \gamma(\mathcal{K}_{0\chi}|_{\Omega_j}) \\ \delta(\mathcal{K}_1\psi|\Omega_j) - \delta(\mathcal{K}_0\chi|_{\Omega_j}) \end{pmatrix} \\
&= (-1)^j \begin{pmatrix} (\mathcal{W} + (-1)^j\mathrm{Id})\psi/2 - \mathcal{A}\chi/2 \\ -(\mathcal{W}' - (-1)^j\mathrm{Id})\chi/2 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \mathrm{Id} + (-1)^j\mathcal{W} & -(-1)^j\mathcal{A} \\ \mathcal{O} & \mathrm{Id} - (-1)^j\mathcal{W}' \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix} \\
&= \frac{1}{2}(\mathrm{Id} - (-1)^j\mathfrak{A}) \begin{pmatrix} \psi \\ \chi \end{pmatrix} \\
&= \mathfrak{P}_j \begin{pmatrix} \psi \\ \chi \end{pmatrix}.
\end{aligned}$$

Now let $u \in L_j$. Then the representation formulas (4.1) and (4.4) give

$$u(x) = (-1)^j(\mathcal{K}_1\gamma u(x) - \mathcal{K}_0\delta u(x)), \quad x \in \Omega_j,$$

thus the jump relations of Lemma 3.3 and (3.7) lead to

$$\begin{pmatrix} \gamma u \\ \delta u \end{pmatrix} = \mathfrak{P}_j \begin{pmatrix} \gamma u \\ \delta u \end{pmatrix}.$$

This shows that the mappings $\mathfrak{P}_j$ are projections and that the Cauchy data of all functions from L_j belong to the image of $\mathfrak{P}_j$. $\square$

Since the Calderon projections corresponding to the interior and the exterior problem are conjugate,

$$\mathfrak{P}_1 + \mathfrak{P}_2 = \mathrm{Id},$$

the space $V(\Gamma) \times V(\Gamma)'$ can be decomposed as the direct sum of closed subspaces

$$V(\Gamma) \times V(\Gamma)' = \{(\gamma u, \delta u) : u \in L_1\} \dot{+} \{(\gamma u, \delta u) : u \in L_2\}.$$

Further, since $\mathfrak{P}_j^2 = \mathfrak{P}_j$, we get

Corollary 4.1.

$$(\mathrm{Id} \pm \mathcal{W})^2/4 = (\mathrm{Id} \pm \mathcal{W})/2, \qquad \mathcal{W}\mathcal{A} = \mathcal{A}\mathcal{W}'. \tag{4.6}$$

5. Steklov-Poincaré operators. In this section we derive equations with the strongly elliptic single layer potential operator $\mathcal{A}$ for the solution of the interior and of the exterior Dirichlet problem

$$\begin{aligned} &\Delta^2 u = 0 \quad \text{in} \quad \Omega_j, \qquad \gamma u = \psi \in V(\Gamma), \\ &\text{if } j = 2 \text{ then } u \text{ satisfies the radiation condition (4.3),} \end{aligned} \tag{5.1}$$

and study the corresponding solution operators.

From Theorem 4.1 we know that any function $u \in L_j$ satisfies the relation

$$(\mathrm{Id} \ - \mathfrak{P}_j) \begin{pmatrix} \gamma u \\ \delta u \end{pmatrix} = 0, \tag{5.2}$$

the first line of this system yields in particular the equality

$$(\mathrm{Id} - (-1)^j \mathcal{W})\gamma u + (-1)^j A \delta u = 0.$$

Hence, if we consider the Dirichlet problem (5.1), then for given $\gamma u = \psi$ the unknown $\chi = \delta u$ has to solve the equation

$$\mathcal{A}\chi = (\mathcal{W} - (-1)^j \mathrm{Id})\psi. \tag{5.3}$$

In order to study the solvability of these equations, we make the assumption

Assumption A.1. *The exterior homogeneous Dirichlet problem* (5.1)*, i.e., $\psi = 0$, has only the trivial solution.*

Recently Costabel and Dauge proved in [**8**] that for any general curve Γ there exist between one and four values of the scaling factor $\rho > 0$ such that the scaled curve $\rho\Gamma = \{\rho x \in \mathbf{R}^2, x \in \Gamma\}$ violates assumption A.1.

Theorem 5.1. *Suppose* A.1. *The equations* (5.3) *are uniquely solvable for any $\psi \in V(\Gamma)$, and the weak solution $u \in L_j$ of the corresponding Dirichlet problem* (5.1) *is given by*

$$u(x) = (-1)^j(\mathcal{K}_1\psi(x) - \mathcal{K}_0\chi(x)), \quad x \in \Omega_j.$$

Proof. The unique solvability of the interior Dirichlet problem (Lemma 2.6) and the jump relations for the operator $\mathcal{K}_0$ (Lemma 3.3) imply that the equation

$$\mathcal{A}\chi = 0$$

has a nontrivial solution if and only if our assumption does not hold. Since by Corollary 3.2 $\mathcal{A}$ is a Fredholm with index zero, it is clear that $\mathcal{A} : V(\Gamma)' \to V(\Gamma)$ is bijective.

Remark 5.1. For a smooth boundary Γ and the interior Dirichlet problem this result was formulated in [**9**]. It follows from the general theory of strongly elliptic boundary integral operators developed in [**6**] and [**13**]. □

Now we analyze the solution operators of equations (5.3)

$$\mathcal{T}_j := \mathcal{A}^{-1}(\mathcal{W} - (-1)^j\mathrm{Id}) : V(\Gamma) \to V(\Gamma)', \tag{5.4}$$

which exist under assumption A.1 and map the Dirichlet data γu of a biharmonic function $u \in L_j$ to its Neumann data δu. The mappings $\mathcal{T}_j$ are called *Steklov-Poincaré operators* of the biharmonic equation.

Let us define the operators

$$\mathcal{P}_j := (\mathrm{Id} - (-1)^j \mathcal{W})/2 : V(\Gamma) \longrightarrow V(\Gamma), \tag{5.5}$$

which are bounded projections by Corollary 4.1. These projections are well studied. Indeed, by (3.6) we have, for $\psi = \binom{v_1}{v_2} \in V(\Gamma)$ and $x \in \mathbf{R}^2 \backslash \Gamma$

$$\begin{aligned} \mathcal{K}_1 \psi(x) = &-\frac{1}{2\pi} \int_\Gamma v_1(y) \partial_{n_y} \log|x-y| \, ds_y \\ &+ \frac{1}{2\pi} \int_\Gamma v_2(y)(\log|x-y|+1) \, ds_y, \end{aligned}$$

i.e., $\mathcal{K}_1 \psi$ is the sum of harmonic potentials and satisfies the radiation condition

$$a(\log|x|+1) + O(|x|^{-1}) \quad \text{for some } a \in \mathbf{R} \text{ as } |x| \to \infty. \tag{5.6}$$

Using the equality (3.7) for the double layer potential operator $\mathcal{W}$ and the well-known jump relations for harmonic potentials, it is easy to see that on Γ

$$\mathcal{W}\psi = \begin{pmatrix} D & -S \\ -H & -D' \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{5.7}$$

with the boundary integral operators

$$\begin{aligned} S\varphi(x) &:= -\frac{1}{\pi} \int_\Gamma \varphi(y)(\log|x-y|+1) \, ds_y, \\ D\varphi(x) &:= -\frac{1}{\pi} \int_\Gamma \varphi(y) \partial_{n_y} \log|x-y| \, ds_y, \\ D'\varphi(x) &:= -\frac{\partial n_x}{\pi} \int_\Gamma \varphi(y) \log|x-y| \, ds_y, \\ H\varphi(x) &:= \frac{\partial_{n_x}}{\pi} \int_\Gamma \varphi(y) \partial_{n_y} \log|x-y| \, ds_y. \end{aligned} \tag{5.8}$$

Here D' is the adjoint of the operator D with respect to the L^2-inner product on Γ. The mapping properties of matrices of the harmonic boundary integral operators corresponding to the fundamental solution,

$-\log|x-y|/2\pi$, were studied for example in [**11**]. From these results it is evident that the operators

$$\frac{1}{2}(\mathrm{Id} + (-1)^k \mathcal{W})$$

are bounded in $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ and project onto the boundary values of weak solutions of the Laplace equation in Ω_k, which are subjected to the radiation condition (5.6) if $k = 2$. By Lemma 3.2 the restrictions of these Calderon projections are bounded in $V(\Gamma)$. Therefore we obtain from (5.5)

Lemma 5.1. *The trace space* $V(\Gamma)$ *is the direct sum*

$$V(\Gamma) = V_1 \dot{+} V_2$$

of the closed subspaces

$$\begin{aligned} V_1 &:= \operatorname{im} \mathcal{P}_1 = \{\gamma u : u \in H^2_{\mathrm{loc}}(\Omega_2), \Delta u = 0, u \text{ satisfies } (5.6)\}, \\ V_2 &:= \operatorname{im} \mathcal{P}_2 = \{\gamma u : u \in H^2(\Omega_1), \Delta u = 0\}. \end{aligned}$$

Note that the mappings $\mathcal{P}_j$ appearing on the right-hand side of the boundary integral equation (5.3) for the interior, $j = 1$, and exterior, $j = 2$, Dirichlet problem project onto the traces of functions harmonic at the opposite domain.

The dual space $V(\Gamma)'$ is the direct sum of the corresponding polar sets

$$V(\Gamma)' = V_1^{\perp} \dot{+} V_2^{\perp},$$

which, in view of

$$V_j^{\perp} = (\operatorname{im} \mathcal{P}_j)^{\perp} = \ker \mathcal{P}_j' = \operatorname{im}\ (\mathrm{Id} - \mathcal{P}_j') \tag{5.9}$$

coincide with the image of the adjoint of the conjugate projection. The commutative relation (4.6) implies that

$$\mathcal{A}\mathcal{P}_j' = \mathcal{P}_j \mathcal{A} \mathcal{P}_j' = \mathcal{P}_j \mathcal{A},$$

yielding the equality

$$\mathcal{A} = \mathcal{P}_1 \mathcal{A} \mathcal{P}_1' + \mathcal{P}_2 \mathcal{A} \mathcal{P}_2'.$$

Using (5.9) and Theorem 5.1, we derive

Lemma 5.2. *The operator $\mathcal{A}$ is the direct sum of the mappings*

$$\mathcal{A} : V_1^{\perp} \longrightarrow V_2 \quad \textit{and} \quad \mathcal{A} : V_2^{\perp} \longrightarrow V_1,$$

which are bijective if the assumption A.1 *is satisfied.*

Now we show that $\mathcal{A}$ is a positive definite operator on a subspace of $V(\Gamma)'$. Let us denote by $\mathbf{P}_1$ the space of linear functions on $\mathbf{R}^2$ and set $l(\Gamma) := \gamma(\mathbf{P}_1)$.

Lemma 5.3. *There exists a constant $c > 0$ such that, for any $\chi \in l(\Gamma)^{\perp}$, the following holds*

$$[\chi, \mathcal{A}\chi] \geq c\|\chi\|^2_{V(\Gamma)'}.$$

Proof. We set $u = -\mathcal{K}_0\chi$, $u_1 = u|_{\Omega_1}$ and $u_2 = u|_{\Omega_2}$. For any ball B_R enclosing Ω_1 the first Green formula yields

$$\begin{aligned}
[\chi, \mathcal{A}\chi]/2 &= [\delta u_1, \gamma u_1] - [\delta u_2, \gamma u_2] \\
&= \int_{\Omega_1} |\Delta u_1|^2 \, dx + \int_{\Omega_2 \cap B_R} |\Delta u_2|^2 \, dx \\
&\quad - \int_{S_R} (\Delta u_2 \partial_n u_2 - u_2 \partial_n \Delta u_2) \, ds.
\end{aligned}$$

Because of $\chi \in l(\Gamma)^{\perp}$ and the definition (4.2), it is clear that $I_1\chi = I_2\chi(x) = 0$, leading to

$$u_2(x) = -\frac{1}{8\pi}(I_{3\chi} \log R + I_4\chi(x)) + O(R^{-1}) \quad \text{for } |x| = R.$$

Hence, $\Delta u_2 \in L^2(\Omega_2)$ and the integral over S_R converges to zero as $R \to \infty$ such that

$$[\chi, \mathcal{A}\chi] = 2\left(\int_{\Omega_1} |\Delta u_1|^2 \, dx + \int_{\Omega_2} |\Delta u_2|^2 \, dx \right) > 0$$

for $\chi \neq 0$. Since $\mathcal{A}$ is symmetric and strongly elliptic, the last inequality implies that $\mathcal{A}$ is even positive definite on $l(\Gamma)^{\perp}$. $\square$

Remark 5.2. The polar set $l(\Gamma)^{\perp}$ can be identified with the dual of the factor space $V(\Gamma)/l(\Gamma)$; therefore, the inequality

$$[\chi, \mathcal{A}\chi] \geq c\|\chi\|^2_{(V(\Gamma)/l(\Gamma))'}, \quad \forall\, \chi \in (V(\Gamma)/l(\Gamma))',$$

is valid with some constant $c > 0$. This was used by Bourlard in [**2**] to prove the existence of the solution $u \in L_1$ of (5.1) in the form

$$u(x) = \lfloor \mathcal{K}_0 \chi \rfloor (x) + p_1(x), \quad x \in \Omega_1,$$

where $\chi \in (V(\Gamma)/l(\Gamma))'$ solves

$$[\varphi, \mathcal{A}\chi] = 2[\varphi, \psi], \quad \forall\, \varphi \in (V(\Gamma)/l(\Gamma))',$$

$\lfloor \mathcal{K}_0 \chi \rfloor$ is an element of the corresponding factor class in $L_1/\mathbf{P}_1$ and $p_1 \in \mathbf{P}_1$ is the linear function satisfying

$$\gamma p_1 = \psi - \gamma \lfloor \mathcal{K}_0 \chi \rfloor. \qquad \square$$

Now we come to some consequences of the previous results.

Corollary 5.1. *The restriction of* $\mathcal{A}$ *on* $V_2^{\perp} \subset V(\Gamma)'$ *is a symmetric and positive definite operator between the dual spaces*

$$\mathcal{A} : V_2^{\perp} = im\, \mathcal{P}_1' \longrightarrow V_1 = im\, \mathcal{P}_1.$$

If the assumption A.1 *is violated, then* $\ker \mathcal{A} \subset V_1^{\perp}$ *and* $\ker \mathcal{A} \cap l(\Gamma)^{\perp} = \varnothing$.

Corollary 5.2. $\chi \in V(\Gamma)'$ *coincides with the Neumann data* δu *of a function* $u \in L_j$ *if and only if* $[\chi, \gamma v] = 0$ *for any harmonic function* $v \in H^2_{\text{loc}}(\Omega_j)$ *satisfying additionally the radiation condition* (5.6) *if* $j = 2$.

Corollary 5.3. *If $\chi \in V_j^{\perp}$, then the function $\mathcal{K}_0\chi \in H^2_{\mathrm{loc}}(\Omega_j)$ is harmonic in Ω_j and satisfies, in the case $j = 2$, the radiation condition* (5.6).

Proof. Corollary 5.2 states that, for any $\chi \in V_j^{\perp}$, there exist $u \in L_k$, $k = 3 - j$, such that $\chi = \delta u$. The representation formulas (4.1) and (4.4) imply that

$$\mathcal{K}_0\chi(x) = \begin{cases} \mathcal{K}_1\gamma u(x) - (-1)^k u(x) & x \in \Omega_k, \\ \mathcal{K}_1\gamma u(x) & x \in \Omega_j. \end{cases} \qquad \square$$

Now we are in a position to formulate some properties of the Steklov-Poincaré operators. By (5.4) and (5.5) we get

$$\mathcal{T}_j = 2\cdot(-1)^{j+1}\mathcal{A}^{-1}\mathcal{P}_j = 2\cdot(-1)^{j+1}\mathcal{P}_j'\mathcal{A}^{-1}\mathcal{P}_j$$

such that the following assertions hold.

Theorem 5.2. *The Steklov-Poincaré operator $\mathcal{T}_1$ which maps the Dirichlet data γu of a biharmonic function $u \in H^2(\Omega_1)$ to its Neumann data δU is continuous from $V(\Gamma)$ into $V(\Gamma)'$, symmetric with respect to the duality* (2.2), *and there exists $c > 0$ such that*

$$[\mathcal{T}_1\psi, \psi] \geq c\|\mathcal{P}_1\psi\|_{V(\Gamma)}, \quad \forall\, \psi \in V(\Gamma).$$

Moreover, the image $\operatorname{im}\mathcal{T}_1 \subset V(\Gamma)'$ is the closed subspace of elements which are orthogonal to the traces γv of all harmonic functions $v \in H^2(\Omega_1)$.

Theorem 5.3. *Suppose* A.1 *is true. Then the Steklov-Poincaré operator $\mathcal{T}_2$ which maps the Dirichlet data γu of a function $u \in L_2$ to its Neumann data δu is continuous from $V(\Gamma)$ into $V(\Gamma)'$, symmetric with respect to the duality* (2.2), *and there exists $c > 0$ such that*

$$-[\mathcal{T}_2\psi, \psi] \geq c\|\mathcal{P}_2\psi\|_{V(\Gamma)}, \quad \forall\, \psi \in \mathcal{A}(l(\Gamma)^{\perp}).$$

The image $\operatorname{im}\mathcal{T}_2 \subset V(\Gamma)'$ is the closed subspace of elements which are orthogonal to the traces γv of all harmonic functions $v \in H^2_{\mathrm{loc}}(\Omega_2)$ satisfying the radiation condition (5.6).

Remark 5.3. The previous results confirm the well-known fact that the Neumann problem

$$\Delta^2 u = 0 \quad \text{in } \Omega_j, \qquad \delta u = \chi \in V(\Gamma)'$$

is not elliptic. A variational approach to boundary conditions different from the Dirichlet one is based on the bilinear form

$$\int_{\Omega_1} \left(\Delta u \Delta v + (1-\sigma) \left(2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2} - \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} \right) \right) dx, \quad 0 < \sigma < 1, \tag{5.10}$$

which is closely connected with the plate equation. A detailed analysis of certain indirect integral equation methods on smooth boundaries for these problems is contained in the book [**3**] of Chen and Zhou. Up until now, the case of a nonsmooth curve Γ has not been analyzed in the literature. It is possible to modify our methods accordingly to the form (5.10) such that direct boundary integral equations for plate problems on domains with corners can be derived and analyzed. □

6. Boundary integral equations for Dirichlet problems. In this section we derive systems of integral equations for the interior and exterior Dirichlet problems and study the existence and uniqueness of corresponding solutions.

First we consider the concrete form of the mappings $\mathcal{A}$ and $\mathcal{P}_j$ which are 2×2 matrices of integral operators. In view of (2.2) and (3.4), the action of the operator $\mathcal{A}$ can be written as

$$\mathcal{A}\chi = \begin{pmatrix} A & -B \\ B' & C \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \qquad \chi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V(\Gamma)' \tag{6.1}$$

with the integral operators

$$\begin{aligned} A\varphi(x) &:= -2 \int_\Gamma G(x,y)\varphi(y)\, ds_y, \\ B\varphi(x) &:= -2 \int_\Gamma \partial_{n_y} G(x,y)\varphi(y)\, ds_y, \\ B'\varphi(x) &:= -2\partial_{n_x} \int_\Gamma G(x,y)\varphi(y)\, ds_y, \\ C\varphi(x) &:= 2\partial_{n_x} \int_\Gamma \partial_{n_y} G(x,y)\varphi(y)\, ds_y. \end{aligned}$$

By the duality (2.2) and (5.7) we have

$$\mathcal{W}' = \begin{pmatrix} D' & S \\ H & -D \end{pmatrix};$$

hence, the commutative relation (4.6) leads to the equalities between the integral operators

$$\begin{aligned} AD' - BS &= DA - SB', \\ B'H - CD &= HB - D'C, \\ AH + BD &= -DB - SC, \\ B'D' + CS &= -HA - D'B'. \end{aligned} \tag{6.2}$$

According to (5.5) the projections $\mathcal{P}_j$ have the form

$$\begin{aligned} \mathcal{P}_j &= \frac{1}{2}\begin{pmatrix} I - (-1)^j D & (-1)^j S \\ (-1)^j H & I + (-1)^j D' \end{pmatrix}, \\ \mathcal{P}_j' &= \frac{1}{2}\begin{pmatrix} I - (-1)^j D' & -(-1)^j H \\ -(-1)^j S & I + (-1)^j D \end{pmatrix}, \end{aligned}$$

which implies the well-known relations for harmonic boundary integral operators

$$\begin{aligned} D^2 + SH &= D'^2 + HS = I, \\ DS = SD, &\qquad HD = D'H. \end{aligned} \tag{6.3}$$

For the following we mention some other properties of these operators. It is well known that there exists a unique $\rho \in H^{-1/2}(\Gamma)$, the Robin potential, with $\langle \rho, 1 \rangle_\Gamma = 1$ such that the logarithmic potential

$$\int_\Gamma \rho(y) \log |x - y| \, ds_y$$

is constant, say $= \nu$, on Γ. The positive number

$$\operatorname{cap} \Gamma = e^\nu$$

is called the *logarithmic capacity* of Γ. We introduce the assumption

Assumption A.2. *The curve* Γ *is such that* $\operatorname{cap}\Gamma \neq e^{-1}$.

Clearly, A.2 means that the exterior Dirichlet problem for the Laplace equation

$$\Delta v = 0 \quad \text{in } \Omega_2, \qquad v|_\Gamma = 0, \quad v \text{ satisfies (5.6)},$$

has in $H^1_{\mathrm{loc}}(\Omega_2)$ only the trivial solution $v = 0$.

Let us note that, in the simplest case of a circle Γ both assumptions A.1 and A.2 coincide, they are valid if the radius of the circle $r \neq e^{-1}$.

Evidently, if A.2 holds, then the operator S has a trivial kernel. Moreover, S maps $H^{-1/2}(\Gamma)$ isomorphically onto $H^{1/2}(\Gamma)$ and the subspaces $V_j \subset V(\Gamma)$ can be characterized by the relation

$$\psi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_j \Longleftrightarrow v_2 = S^{-1}(D + (-1)^j I)v_1. \tag{6.4}$$

Turning to the duals and using (6.3) we obtain the characterization of $V_j^\perp \subset V(\Gamma)'$

$$\chi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_j^\perp \Longleftrightarrow v_1 = S^{-1}(D + (-1)^j I)v_2, \tag{6.5}$$

where, of course, the equality is understood in the weak sense, i.e.,

$$\langle v_1, \varphi|_\Gamma \rangle_\Gamma = \langle v_2, S^{-1}(D + (-1)^j I)\varphi|_\Gamma \rangle_\Gamma \quad \text{for all } \varphi \in C_0^\infty(\mathbf{R}^2).$$

Concerning the double layer potential D we note that the kernel of the operator $I - D$ is trivial, whereas the operators $I + D$ and $I + D'$ have one-dimensional kernels spanned by the constant function on Γ and by the Robin potential ρ, respectively.

The mentioned properties can be easily deduced from known results about harmonic potentials corresponding to the fundamental solution $-\log|x - y|/2\pi$, from Corollary 4.1 and the fact that the projections $\mathcal{P}_j$ are bounded in $V(\Gamma)$.

In Section 5 we have already studied the equations for solving the interior, $j = 1$, and exterior, $j = 2$, Dirichlet problem

$$\mathcal{A}\chi = 2 \cdot (-1)^{j+1}\mathcal{P}_j\psi,$$

which can be written as the system

$$
\begin{aligned}
Av_1 - Bv_2 &= (D-(-1)^j I)f_1 - Sf_2 \\
B'v_1 + Cv_2 &= -Hf_1 - (D' + (-1)^j I)f_2,
\end{aligned} \tag{6.6}
$$

where $\binom{f_1}{f_2} = \psi \in V(\Gamma)$ are the given Dirichlet data and $\binom{v_1}{v_2} = \delta u = \chi \in V(\Gamma)'$ are the unknowns. From the results of Section 5 it follows that, under assumption A.1, the unique solution χ belongs to the closed subspace

$$\chi \in \operatorname{im} \mathcal{P}'_j = \ker(I - \mathcal{P}'_j).$$

Consequently, we obtain, in view of (6.5),

Lemma 6.1. *Suppose Assumption* A.1 *holds, and let* $j = 1$ *or* 2. *The unique solution* $\binom{v_1}{v_2} \in V(\Gamma)'$ *of the system* (6.6) *solves the corresponding systems of boundary integral equations*

$$
\begin{aligned}
Av_1 - Bv_2 &= (D-(-1)^j I)f_1 - Sf_2 \\
Sv_1 - (D-(-1)^j I)v_2 &= 0,
\end{aligned} \tag{6.7}
$$

and

$$
\begin{aligned}
B'v_1 + Cv_2 &= -Hf_1 - (D' + (-1)^j I)f_2 \\
Sv_1 - (D-(-1)^j I)v_2 &= 0.
\end{aligned} \tag{6.8}
$$

For the opposite direction we have the following result.

Lemma 6.2. *Suppose Assumption* A.2 *holds, and let* $j = 1$ *or* 2. *Then any solution* $\binom{v_1}{v_2} \in V(\Gamma)'$ *of the system* (6.7) *satisfies the equations* (6.6).

Proof. Since the operator S is invertible, we get from (6.2) and (6.3) the equalities

$$
\begin{aligned}
&B'S^{-1}(D-(-1)^j I) + C \\
&\quad = (B'(D' - (-1)^j I) + CS)S^{-1} \\
&\quad = (-HA - D'B - (-1)^j B')S^{-1} \\
&\quad = S^{-1}((D^2 - I)A - (D + (-1)^j I)SB')S^{-1} \\
&\quad = S^{-1}(D + (-1)^j I)(-(-1)^j A + AD' - BS)S^{-1} \\
&\quad = S^{-1}(D + (-1)^j I)(AS^{-1}(D - (-1)^j I) - B)
\end{aligned} \tag{6.9}
$$

and

$$-Hf_1 - (D' + (-1)^j I)f_2 = S^{-1}(D + (-1)^j I)((D - (-1)^j I)f_1 - Sf_2). \tag{6.10}$$

Hence, for $v_1 = S^{-1}(D - (-1)^j I)v_2$ we derive

$$B'v_1 + Cv_2 = S^{-1}(D + (-1)^j I)(Av_1 - Bv_2),$$

showing that the second equation of (6.6) is a consequence of the system (6.7). □

Now we are able to prove solvability conditions for the boundary integral equations of the interior and exterior problem.

Theorem 6.1. *Suppose that* Γ *satisfies Assumption* A.2. *For any* $\psi = \binom{f_1}{f_2} \in V(\Gamma)$ *the systems of boundary integral equations*

$$\begin{aligned} Av_1 - Bv_2 &= (D + I)f_1 - Sf_2 \\ Sv_1 - (D + I)v_2 &= 0 \end{aligned} \tag{6.11}$$

and

$$\begin{aligned} B'v_1 + Cv_2 &= -Hf_1 - (D' - I)f_2 \\ Sv_1 - (D + I)v_2 &= 0 \end{aligned} \tag{6.12}$$

are uniquely solvable. The solution $\chi = \binom{v_1}{v_2} \in V(\Gamma)'$ *coincides with the Neumann data* δu *of* $u \in H^2(\Omega_1)$ *solving the interior Dirichlet problem*

$$\Delta^2 u = 0 \quad in\ \Omega_1, \qquad \gamma u = \psi.$$

Proof. Lemma 6.2 states that any solution of (6.11) solves the system

$$\begin{aligned} Av_1 - Bv_2 &= (D + I)f_1 - Sf_2 \\ B'v_1 + Cv_2 &= -Hf_1 - (D' - I)f_2. \end{aligned} \tag{6.13}$$

The same assertion is true for (6.12). Indeed, since the operator $I - D$ is invertible we get for $j = 1$ from (6.9) and (6.10)

$$\begin{aligned} Av_1 - Bv_2 &= (D-I)^{-1}S(B'v_1 + Cv_2) \\ &= (D-I)^{-1}S(-Hf_1 - (D'-I)f_2) \\ &= (D+I)f_1 - Sf_2. \end{aligned}$$

On the other hand, due to Lemma 6.1, both systems are solvable for any $\psi \in V(\Gamma)$ if Γ satisfies Assumption A.1. But the righthand side of (6.13) belongs to $V_1 = \operatorname{im}\mathcal{P}_1$ such that from Corollary 5.1 it follows that (6.11) and (6.12) are solvable for any $\psi \in V(\Gamma)$ even if A.1 is violated.

So it remains to show that the solution is unique under A.2. This follows immediately from Corollary 5.1, since Corollary 3.2 and (6.4) imply that $\chi \in V_2^{\perp}$. $\square$

Remark 6.1. The system of boundary integral equations (6.12), which coincides with (1.2) given in the introduction, seems to be a new equivalent integral formulation of the interior Dirichlet problem for the bi-Laplacian. This system has at least theoretically some advantages compared with (6.11) or (6.13) concerning the numerical solution of this problem. Let, for example, the boundary of Ω_1 be smooth. Then the integral operators are one-dimensional pseudodifferential operators on the closed curve Γ. In particular, the operator S with logarithmic kernel is a strongly elliptic pseudodifferential operator of order -1, whereas D is a smoothing operator, i.e., has the order $-\infty$. From

$$\begin{aligned} -\partial_{n_x} G(x,y) &= -(n_x, x-y)\left(\frac{1}{4\pi}\log|x-y| + \frac{1}{8\pi}\right) \\ \partial_{n_x}\partial_{n_y} G(x,y) &= -(n_x, n_y)\left(\frac{1}{4\pi}\log|x-y| + \frac{1}{8\pi}\right) \\ &\quad - \frac{(n_x, x-y)(n_y, x-y)}{4\pi|x-y|^2|} \end{aligned}$$

it can be easily seen that the mappings B' and $2C - S$ are pseudodifferential operators of order -3. Thus the system (6.12) corresponds to the bounded mapping

$$\mathcal{V} := \begin{pmatrix} C & B' \\ -I-D & S \end{pmatrix} : \begin{matrix} H^s(\Gamma) \\ \times \\ H^{s-1}(\Gamma) \end{matrix} \longrightarrow \begin{matrix} H^{s+1}(\Gamma) \\ \times \\ H^s(\Gamma) \end{matrix}, \quad s \in \mathbf{R},$$

which discretization provides a better conditioning than the systems (6.11) or (6.13). Moreover, $\mathcal{V}$ represents a compact perturbation of the simpler operator-matrix with lower triangular structure

$$\mathcal{V}_0 := \begin{pmatrix} S/2 & 0 \\ -I & S \end{pmatrix} : \begin{matrix} H^s(\Gamma) \\ \times \\ H^{s-1}(\Gamma) \end{matrix} \longrightarrow \begin{matrix} H^{s+1}(\Gamma) \\ \times \\ H^s(\Gamma) \end{matrix},$$

which is bijective for any $s \in \mathbf{R}$ if Γ satisfies assumption A.2. Thus, applying as in [**10**] well-established approximation results for pseudodifferential operators, it is easy to prove the convergence of Galerkin and certain collocation methods for solving (6.12). □

Now we consider the Dirichlet problem for the exterior domain.

Theorem 6.2. *Suppose that* Γ *satisfies Assumption* A.1. *Then, for any* $\psi = \binom{f_1}{f_2} \in V(\Gamma)$, *the systems of boundary integral equations*

$$\begin{aligned} Av_1 - Bv_2 &= (D-I)f_1 - Sf_2 \\ B'v_1 + Cv_2 &= -Hf_1 - (D'+I)f_2 \end{aligned} \tag{6.14}$$

and

$$\begin{aligned} Av_1 - Bv_2 &= (D-I)f_1 - Sf_2 \\ Sv_1 - (D-I)v_2 &= 0, \end{aligned} \tag{6.15}$$

are uniquely solvable. The solution $\chi = \binom{v_1}{v_2} \in V(\Gamma)'$ *coincides with the Neumann data* δu *of the function* $u \in L_2$ *solving the exterior Dirichlet problem* (5.1) *with* $\gamma u = \psi$. *If Assumption* A.1 *is violated, then systems (6.14) and (6.15) with vanishing righthand side possess nontrivial solutions.*

Proof. From Lemma 6.1 we know that, under A.1, the solution of (6.14) solves (6.15), too. Hence, in view of Theorem 5.1, it suffices to prove that (6.15) is uniquely solvable. To this end we show that the second equation of this system determines $V_1^\perp$ even if Assumption A.2 is violated.

Indeed, in this case the Robin potential ρ spans the kernel of the operator S and we have $\binom{0}{\rho} = \gamma w$ with a nontrivial solution w of the homogeneous Dirichlet problem for Laplace's equation in Ω_2 satisfying the radiation condition (5.6). Note that, in general, $\gamma w \notin V(\Gamma)$; this would require a smoother curve, say $\Gamma \in C^{1,\alpha}$. But the vector $\binom{\rho}{0} \in V(\Gamma)'$, and in view of Corollary 5.2 we obtain that $\binom{\rho}{0} \in V_1^\perp$ and consequently

$$\chi = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V_1^\perp \Longleftrightarrow Sv_1 - (D-I)v_2 = 0. \tag{6.16}$$

Thus the solution $\chi = \binom{v_1}{v_2}$ of the homogeneous system (6.15) belongs to $V_1^\perp$; from Corollary 5.3 we conclude that $K_0\chi \in H^2(\Omega_1)$ is harmonic. But the first equation of this system requires $\mathcal{K}_0\chi|_\Gamma = 0$ such that $\mathcal{K}_{0\chi} = 0$ in Ω_1 and $\mathcal{A}\chi = 0$. Hence, the homogeneous system (6.15) has a nontrivial solution only if Assumption A.1 is violated. □

The previous result gives a necessary and sufficient condition on Γ to derive equivalent boundary integral equations for the exterior Dirichlet problem, whereas Theorem 6.1 contains only a sufficient condition for the interior Dirichlet problem. We can prove that Assumption A.2 is also necessary for the unique solvability of the systems of integral equations if Γ is sufficiently smooth.

Theorem 6.3. *Let $\Gamma \in C^{1,\alpha}$, $0 < \alpha < 1$, $\operatorname{cap}\Gamma = e^{-1}$ and $f_1 = f_2 = 0$. Then the systems* (6.11) *and* (6.12) *possess nontrivial solutions.*

Proof. We construct the nontrivial solution of (6.11) following a method in Fuglede [**14**]. Since $\operatorname{cap}\Gamma = e^{-1}$ there exists a function w harmonic in Ω_2, satisfying the radiation condition (5.6) such that $w|_\Gamma = 0$ and $\gamma w \neq 0$. The condition $\Gamma \in C^{1,\alpha}$ ensures $\gamma w \in V(\Gamma)$ such that the solution u of the Dirichlet problem

$$\Delta^2 u = 0 \quad \text{in } \Omega_1, \qquad \gamma u = \gamma w,$$

provides $0 \neq \delta u \in V_2^\perp$ and $\mathcal{A}\delta u = 2\mathcal{P}_1\gamma w = 2\gamma w$. Hence $\delta u \in V(\Gamma)'$ solves the homogeneous system (6.11).

To get the nontrivial solution of (6.12), we start with $\mathcal{K}_0\binom{\rho}{0} \in H^2(\Omega_1)$, see the proof of Theorem 6.2, and denote $\mathcal{A}\binom{\rho}{0} = 2\gamma\mathcal{K}_0\binom{\rho}{0} =$

$2\binom{w_1}{w_2}$. Then we solve the Neumann problem for the Laplace equation

$$\Delta v = 0 \quad \text{in } \Omega_2, \qquad \partial_n v|_\Gamma = -w_2, \quad v \text{ satisfies (5.6).}$$

It is well known that $\Gamma \in C^{1,\alpha}$ implies $\gamma v \in V(\Gamma)$; hence, the solution of the Dirichlet problem

$$\Delta^2 u = 0 \quad \text{in } \Omega_1, \qquad \gamma u = \gamma v,$$

gives $\delta u \in V_2^\perp$ with $\mathcal{A}\delta u = 2\mathcal{P}_1 \gamma v = 2\gamma v$. So we derive

$$\mathcal{A}\left(\delta u + \begin{pmatrix} \rho \\ 0 \end{pmatrix}\right) = 2\begin{pmatrix} v|_\Gamma + w_1 \\ 0 \end{pmatrix}.$$

Now we use that A.2 is violated. Then $S\rho = 0$ and, for $\chi = \binom{v_1}{v_2} \in V(\Gamma)'$, we obviously obtain the relation

$$Sv_1 - (D+I)v_2 = 0 \Longleftrightarrow \chi \in V_2^\perp \dotplus \operatorname{span}\left\{\begin{pmatrix} \rho \\ 0 \end{pmatrix}\right\}. \tag{6.17}$$

Thus $\delta u + \binom{\rho}{0}$ is a nontrivial solution of the homogeneous system (6.12). □

Remark 6.2. The system (6.11) with the operator S replaced by the usual weakly singular operator

$$S_1\varphi(x) := -\frac{1}{\pi}\int_\Gamma \log|x-y|\varphi(y)\,ds_y$$

was introduced in [**4**] and analyzed in [**14**] for the case that the data satisfy the conditions $\Gamma \in C^{1,\alpha}$, $f_1 \in C^1(\Gamma)$, $f_2 \in C(\Gamma)$. It was proved that the corresponding integral equations are uniquely solvable and provide the solution of the interior Dirichlet problem if and only if the boundary is subjected to the assumption $\operatorname{cap}\Gamma \notin \{e^{-1}, 1\}$. It can be easily seen that, under this assumption, the assertions of Theorems 6.1 and 6.3 remain true for the systems (6.11) and (6.12) with the operator S_1 instead of S. □

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