JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 11, Number 3, Fall 1999

OPERATOR NORMS OF POWERS OF THE VOLTERRA OPERATOR

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1. Introduction. The Volterra operator $V : L^2[0,1] \to L^2[0,1]$ will be defined by

$$Vf(x) = \int_0^x f(t)\,dt, \tag{1.1}$$

where f is real valued function.

Definition 1.1. The *operator norm*, $\|.\|$, is defined by

$$\|T\| = \sup_{\|f\|_2=1} \|Tf\|_2, \tag{1.2}$$

where

$$\|f\|_2 = \left[\int_0^1 |f(t)|^2\; dt\right]^{1/2}. \tag{1.3}$$

It is not difficult to show that the operator norm of V is $2/\pi$. In [**5**] N. Lao and R. Whitley give the numerical evidence which led them to the conjecture that

$$\lim_{m\to\infty} \|m!V^m\| = 1/2. \tag{1.4}$$

The purpose of this article is to verify that this is indeed the case. The analysis will be presented for a more general operator defined as follows.

Definition 1.2. The linear operator $A : L^2[0,1] \to L^2[0,1]$ is given by

$$Af(x) = \int_0^x a(x-t)f(t)\,dt, \tag{1.5}$$

Received by the editors in revised form on June 3, 1997.

where a is a nonnegative, nondecreasing L^2-integrable function on $[0, 1]$.

A is a Hilbert-Schmidt operator. It will be convenient to state some definitions and results concerning cones and u_0-positive operators. The general theory will be found in [**4**] from which the following are taken.

Definition 1.3. Let E be a real Banach space. A set $K \subset E$ is called a *cone* if the following conditions are satisfied:

(a) the set is closed,

(b) if $u, v \in K$ then $\alpha u + \beta v \in K$ for all nonnegative real numbers α, β,

(c) of each pair of vectors $f, -f$ at least one does not belong to K provided that $f \neq 0$.

We write $f \geq 0$ if $f \in K$, and $f \geq g$ if $f - g \in K$.

Definition 1.4. A cone is called *reproducing* if every element $f \in E$ can be represented in the form

$$f = u - v, \quad u, v \in K.$$

Example 1.1. The collection of nonnegative functions in C, the space of functions which are continuous on a bounded closed set, is a reproducing cone. In fact it is *solid*, that is to say, it contains interior points.

Example 1.2. Although $L^2[0, 1]$ does not contain a solid cone it does in fact contain the reproducing cone of functions which are positive almost everywhere, since every function $f \in L^2[0, 1]$ can be represented, as

$$f = f_+ - f_-,$$

where f_+ and f_- are nonnegative and belong to $L^2[0, 1]$.

Definition 1.5. The operator A defined on E is u_0-*positive* if there exists $u_0 \in K$ and a fixed positive integer p such that for each element

$f \in K$ there are positive numbers α and β, which depend on f, so that

$$\alpha\ u_0 \leq A^p f \leq \beta\ u_0.$$

Example 1.3. The Volterra operator V is not u_0-positive. For simplicity we take $p = 1$; the proof for a general value is similar. Suppose there did exist a nonnegative function u_0, and positive scalars α, β, such that

$$\alpha(f)\ u_0(x) \leq \int_0^x f(t)\,dt \leq \beta(f)\ u_0(x), \tag{1.6}$$

for all nonnegative functions f. Set $f_1(t) = 1$ on the right and $f_2(t) = t$ on the left to give, for almost all $x \in [0,1]$,

$$\frac{x}{\beta(f_1)} \leq u_0(x) \leq \frac{x^2}{2\alpha(f_2)},$$

which is clearly not true for all x.

Example 1.4. The operator G defined on $L^2[0,1]$ by

$$Gf(x) = (1-x)\int_0^x tf(t)\,dt + x\int_x^1 (1-t)f(t)\,dt, \tag{1.7}$$

is u_0-positive, with $p = 1$ and $u_0(x) = x(1-x)$.

Let $f \in L^2[0,1]$ and be positive almost everywhere; then

$$\begin{aligned}\frac{Gf(x)}{x(1-x)} &= \frac{1}{x}\int_0^x tf(t)\,dt + \frac{1}{(1-x)}\int_x^1 (1-t)f(t)\,dt \\ &\geq \min_{0\leq x\leq 1}\left[\frac{1}{x}\int_0^x tf(t)\,dt + \frac{1}{(1-x)}\int_x^1 (1-t)f(t)\,dt\right] \\ &= \alpha. \qquad (1.8)\end{aligned}$$

It follows that

$$Gf(x) \geq \alpha\ x(1-x). \tag{1.9}$$

Clearly $\alpha \geq 0$ since f is positive almost everywhere and not identically zero; in fact, it must be positive. For, suppose that $\alpha = 0$, in which case the lefthand side of (1.8) would vanish for some value of x, call this value x_0. If $0 < x_0 < 1$ this would imply that $Gf(x_0) = 0$. However, f is not identically zero and the integrands in (1.7) are positive; consequently, this cannot occur. On the other hand, if $x_0 = 0$ then

$$\lim_{x\to 0} \frac{Gf(x)}{x(1-x)} = \int_0^1 (1-t)f(t)\,dt,$$

which is not zero unless f is zero. The case of $x_0 = 1$ is treated in a similar fashion. We can take

$$\beta = \max_{0\leq x\leq 1}\left[\frac{1}{x}\int_0^x tf(t)\,dt + \frac{1}{(1-x)}\int_x^1 (1-t)f(t)\,dt\right]. \tag{1.10}$$

Finally we quote from [**4**] the following results. These will be found in the summary on pages 329–330.

Theorem 1.6 (Krasnosel'skiĭ). *Let K be a reproducing cone and T a u_0-positive linear operator. Then*

(a) *T has a unique eigenfunction which is in K,*

(b) *the corresponding eigenvalue, λ_0, is simple,*

(c) *if λ is any other eigenvalue, then*

$$|\lambda| < \lambda_0.$$

2. Equivalent formulation. The problem of finding the norm of A, defined by (1.5), is equivalent to that of finding the square root of the norm of A^*A, where A^* is the adjoint of A, given by

$$A^*f(x) = \int_x^1 a(t-x)f(t)\,dt. \tag{2.1}$$

We have to estimate the largest eigenvalue of A^*A.

Theorem 2.1. *Let the operator* $B : L^2[0,1] \to L^2[0,1]$ *be defined by*

$$Bf(x) = \int_0^{1-x} a(1-x-t)f(t)\,dt; \tag{2.2}$$

then

$$A^*A = B^2. \tag{2.3}$$

Proof. Let $f \in L^2[0,1]$; then

$$A^*Af(x) = \int_x^1 a(t-x)\int_0^t a(t-s)f(s)\,ds\,dt,$$

replace $t \mapsto 1-t$ to give

$$\begin{aligned} A^*Af(x) &= \int_0^{1-x} a(1-x-t)\int_0^{1-t} a(1-t-s)f(s)\,ds\,dt \\ &= B^2 f(x). \qquad \square \end{aligned}$$

We note in passing the more usual Fredholm form of the operators:

$$\begin{aligned} A^*Af(x) = B^2 f(x) = &\int_0^x f(s)\int_x^1 a(t-x)a(t-s)\,dt\,ds \\ &+ \int_x^1 f(s)\int_s^1 a(t-x)a(t-s)\,dt\,ds. \end{aligned} \tag{2.4}$$

Thus the problem of finding the spectral radius of A^*A can be replaced by that of finding the spectral radius of B^2. Denote it by λ_0^2 and the corresponding eigenfunction by ϕ_0. We shall show that ϕ_0 is of constant sign. This will enable us to estimate bounds for λ_0.

3. The u_0-positivity of B. We now show that the operator B defined by (2.2) is u_0-positive, with $p = 2$.

Lemma 3.1. *Let* $g(0) \neq 0$, $g(1) = 0$, $g'(t) \leq 0$, $a(t) \geq 0$, $a'(t) \geq 0$, $0 \leq t \leq 1$. *Then*

$$\max_{0\leq x_0\leq 1} g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt} \leq \frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} \leq g(0). \tag{3.1}$$

Proof. We note first that g decreases to zero.

Let x_0 satisfy $0 \leq x_0 \leq 1$; then

(a) $0 \leq x \leq x_0$, since g is a decreasing function, $g(t) \geq g(x_0)$, for $0 \leq t \leq x_0$, and so

$$\begin{aligned}\frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} &\geq g(x_0)\frac{\int_0^x a(x-t)\,dt}{\int_0^x a(t)\,dt}\\ &= g(x_0).\end{aligned} \tag{3.2}$$

(b) $x_0 \leq x \leq 1$, the integrand is positive and so

$$\begin{aligned}\int_0^x a(x-t)g(t)\,dt \geq \int_0^{x_0} a(x-t)g(t)\,dt\\ = \int_0^{x_0} [a(x-t) - a(x_0-t)]g(t)\,dt\\ + \int_0^{x_0} a(x_0-t)g(t)\,dt,\end{aligned}$$

which, since a' is nonnegative, gives

$$\int_0^x a(x-t)g(t)\,dt \geq \int_0^{x_0} a(x_0-t)g(t)\,dt.$$

It follows that

$$\int_0^x a(x-t)g(t)\,dt \geq \int_0^{x_0} a(x_0-t)g(t)\,dt \geq g(x_0)\int_0^{x_0} a(t)\,dt.$$

Now $\int_0^x a(t)\,dt \leq \int_0^1 a(t)\,dt$ and so

$$\frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} \geq g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt}. \tag{3.3}$$

The combination of (3.2) and (3.3) gives, for any x_0 in $[0,1]$,

$$\begin{aligned} \frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} &\geq \min\left[g(x_0), g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\ dt}\right] \\ &= g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt}. \end{aligned} \tag{3.4}$$

Since the lefthand side is independent of x_0, we have

$$\frac{\int_0^x a(x-t)g(t)\,dt}{\int_0^x a(t)\,dt} \geq \max_{0\leq x_0\leq 1} g(x_0)\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt}. \tag{3.5}$$

The upper bound in (3.1) follows from the fact that $g(t) \leq g(0)$, $0 \leq t \leq 1$. □

Theorem 3.2. *Let $f(t) \geq 0$, $0 \leq t \leq 1$. Then*

$$\alpha u_0 \leq B^2 f \leq \beta u_0, \tag{3.6}$$

where

$$u_0(x) = \int_0^{1-x} a(t)\,dt \tag{3.7}$$

$$\alpha = \max_{0\leq x_0\leq 1}\left[\frac{\int_0^{x_0} a(t)\,dt}{\int_0^1 a(t)\,dt}\int_0^{1-x_0} a(1-x_0-u)f(u)\,du\right] \tag{3.8}$$

$$\beta = \int_0^1 a(1-u)f(u)\,du. \tag{3.9}$$

Proof. In (3.1) replace $x \mapsto 1-x$ and set

$$g(t) = \int_0^{1-t} a(1-t-u)f(u)\,du. \tag{3.10}$$

This function will satisfy the conditions of Lemma 3.1, and the result follows. □

Hence, B is u_0-positive, and so by Theorem 1.6 the eigenvalue which gives the spectral radius of B is positive and the corresponding eigenfunction is nonnegative.

4. Mean value theorem. The proof of the next theorem is a generalization of one given by Collatz [**2**] for a finite dimensional operator, see also [**1**] and [**3**].

Theorem 4.1. *For any positive function $f \in C[0,1]$ the eigenvalue λ_0 which corresponds to an eigenfunction of constant sign satisfies*

$$\inf_{0<\tau<1} \left\{ \frac{\int_0^{1-\tau} a(1-\tau-x)u_0(x)f(x)\,dx}{u_0(\tau)f(\tau)} \right\} \leq \lambda_0 \leq \sup_{0<\tau<1} \left\{ \frac{\int_0^{1-\tau} a(1-\tau-x)u_0(x)f(x)\,dx}{u_0(\tau)f(\tau)} \right\}, \tag{4.1}$$

where

$$u_0(x) = \int_0^{1-x} a(t)\,dt, \quad 0 < x < 1. \tag{4.2}$$

Proof. Let ϕ_0 be the eigenvector which corresponds to λ_0, and as we have seen, $\phi_0 \in K$. Multiply (2.2) by $u_0 f$ and integrate to give

$$\lambda_0 \int_0^1 u_0(x)f(x)\phi_0(x)\,dx = \int_0^1 u_0(x)f(x) \int_0^{1-x} a(1-x-t)\phi_0(t)\,dt\,dx.$$

Interchange the order of integration, then

$$\lambda_0 \int_0^1 u_0(x)f(x)\phi_0(x)\,dx = \int_0^1 \phi_0(t) \int_0^{1-t} a(1-x-t)u_0(x)f(x)\,dx\,dt.$$

Hence

$$\lambda_0 \int_0^1 u_0(x) f(x) \phi_0(x)\, dx$$
$$= \int_0^1 u_0(t)\phi_0(t) f(t) \left[\frac{1}{u_0(t) f(t)} \int_0^{1-t} a(1-t-x) u_0(x) f(x)\, dx \right] dt.$$

The expression inside the square brackets is nonnegative, and so we can use the integral mean value theorem to give

(4.3)
$$\lambda_0 = \frac{\int_0^{1-\tau} a(1-\tau-x) u_0(x) f(x)\, dx}{u_0(\tau) f(\tau)}, \quad \text{for some } \tau, \quad 0 < \tau < 1,$$

from which the desired result follows. □

5. The Volterra operator. We now apply the results of the previous section to the problem of finding upper and lower bounds for $\|m! V^m\|$ where

$$(m-1)! V^m f(x) = \int_0^x (x-t)^{m-1} f(t)\, dt. \tag{5.1}$$

Theorem 5.1. *Let λ_0 be the largest eigenvalue of $(m-1)!V^m$. Then*

$$\frac{1}{2m} < \lambda_0 < \frac{1}{m}. \tag{5.2}$$

Proof. In this case $a(x) = x^{m-1}$ and the corresponding operator is u_0-positive, with

$$u_0(x) = \frac{(1-x)^m}{m}.$$

Hence the eigenfunction which corresponds to λ_0 is of constant sign, and so (4.3) becomes

$$\begin{aligned} \lambda_0 &= \frac{\int_0^{1-\tau} (1-\tau-x)^{m-1} (1-x)^m\, dx}{(1-\tau)^m} \\ &= \int_0^1 (1-x)^{m-1} (1-x+\tau x)^m\, dx. \end{aligned} \tag{5.3}$$

It follows that

$$\int_0^1 (1-x)^{2m-1}\,dx < \lambda_0 < \int_0^1 (1-x)^{m-1}\,dx,$$

which gives (5.2). □

The next corollary follows from the definition.

Corollary 5.2.

$$\frac{1}{2} < \|m!V^m\| < 1. \tag{5.4}$$

The upper bound in (5.2) can be improved by the use of the next result.

Theorem 5.3. *Let*

$$Af(x) = \int_0^x a(x-t)f(t)\,dt, \tag{5.5}$$

where $f \in L^2[0,1]$, *then*

$$\lambda_0^2 = \|A\|^2 \le \int_0^1 \int_0^t a^2(x)\,dx\,dt. \tag{5.6}$$

Proof. As we have seen

$$A^*A = B^2,$$

where

$$Bf(x) = \int_0^{1-x} a(1-x-t)f(t)\,dt.$$

Hence,

$$
\begin{aligned}
|\lambda_0 f(x)|^2 &= |Bf(x)|^2 \\
&\leq \int_0^{1-x} a^2(1-x-t)\,dt \int_0^{1-x} f^2(t)\,dt \\
&\leq \int_0^{1-x} a^2(1-x-t)\,dt \int_0^{1} f^2(t)\,dt.
\end{aligned}
\tag{5.7}
$$

Integrate this from 0 to 1 to give

$$
\begin{aligned}
\lambda_0^2 &\leq \int_0^1 \int_0^{1-x} a^2(1-x-t)\,dt\,dx \\
&= \int_0^1 \int_x^1 a^2(t-x)\,dt\,dx \\
&= \int_0^1 \int_0^t a^2(x)\,dx\,dt. \qquad \square
\end{aligned}
$$

In the present case

$$
Af(x) = (m-1)!V^m f(x) = \int_0^x (x-t)^{m-1} f(t)\,dt, \tag{5.8}
$$

and an easy calculation gives

$$
\|m!V^m\| \leq \left[\frac{m^2}{2m(2m-1)}\right]^{1/2} = \frac{1}{2}\left(1 - \frac{1}{2m}\right)^{-1/2}. \tag{5.9}
$$

This together with the lower bound in Corollary 5.2 gives the result stated in the introduction.

Theorem 5.4.

$$
\lim_{m\to\infty} \|m!V^m\| = \frac{1}{2}. \tag{5.10}
$$

Acknowledgments. The author thanks Norman Lao and Robert Whitley of the University of California who provided numerical evidence for the existence of the limit for the Volterra operator.

It is a pleasure to record his gratitude to his colleagues John Gilbert, who suggested the method of proof of Theorem 4.1 and Gordon Blower, who helped with the presentation. Thanks are also due to Graham Little of Manchester University for his invaluable comments.

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