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EXISTENCE OF SOLUTIONS FOR A CLASS OF INTEGRODIFFERENTIAL EQUATIONS IN BANACH SPACES

FANGQI CHEN

ABSTRACT. An existence theorem of solutions for a class of nonlinear integrodifferential equations in Banach spaces is established. This is achieved by means of the Mönch fixed point theorem and an integration inequality for the measure of noncompactness.

1. Introduction. Let E be a real Banach space, $R^+ = \{t \in R^1 : t \geq 0\}$. Consider the IVP of nonlinear integrodifferential equations on the infinite interval R^+ in Banach space E,

$$(1) \qquad x'(t) = F\left(t, x(t), \int_0^t K(t,s,x(s))\, ds\right), \quad x(0) = 0, \ t \in R^+$$

where $F \in C[R^+ \times E \times E, E]$, $K \in C[R^+ \times R^+ \times E, E]$. In the case that IVP (1) is a scalar integrodifferential equation, the existence theorem of solutions has been obtained by means of the topological transversality arguments in [**1**]. But it is easy to see that the method used in [**1**] is not successful in the Banach space case. In this paper we shall use the Mönch fixed point theorem and an integration inequality for the measure of noncompactness to investigate the existence of solutions of IVP (1). An existence theorem is obtained.

2. Preliminaries. Throughout this paper, for $T > 0$, $C[[0,T], E]$ denotes the Banach space with supremum norm. For $D \subset C[[0,T], E]$, we write $D(t) = \{x(t) : x \in D\} \subset E$, $t \in [0,T]$. α denotes the Kuratowski measure of noncompactness.

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Lemma 1 [**5**]. *Let $D \subset C[[0,T],E]$ be bounded, and suppose that the elements of D are equicontinuous on $[0,T]$. Then*

$$\alpha(D) = \sup_{t\in[0,T]} \alpha(D(t)).$$

Lemma 2 [**4**]. *Let countable set $D = \{x_n\} \subset L^1([0,T],E)$, and let there exist $g \in L^1([0,T],R^+)$ such that, for any $x_n \in D$, $\|x_n(t)\| \le g(t)$ almost everywhere $t \in [0,T]$. Then*

$$\alpha\biggl(\biggl\{\int_0^t x_n(s)\,ds\biggr\}\biggr) \le 2\int_0^t \alpha(D(s))\,ds, \quad \forall\, t \in [0,T].$$

Lemma 3 [**6**] (Mönch fixed point theorem). *Let E be a Banach space, $D \subset E$ closed convex, and $F : D \to D$ continuous with the further property that, for some $x \in D$, we have*

$$C \subset D \text{ countable}, \quad \overline{C} = \overline{\text{co}}(\{x\} \cup F(C))$$

imply that C is a relatively compact set. Then F has a fixed point in D.

3. Main theorem. We are now in a position to prove our existence result. It is clear that $x \in C^1[R^+,E]$ is a solution of the IVP (1) if and only if $x \in C[R^+,E]$ is a solution of the following integral equation

$$x(t) = \int_0^t F\biggl(s, x(s), \int_0^s K(s,\tau,x(\tau))\,d\tau\biggr)\,ds. \tag{2}$$

Consider the operator A defined by

$$Ax(t) = \int_0^t F\biggl(s, x(s), \int_0^s K(s,\tau,x(\tau))\,d\tau\biggr)\,ds. \tag{3}$$

Let us list some conditions for convenience:

(H1) F is uniformly continuous on any bounded subset of $R^+ \times E \times E$, so is K on any bounded subset of $R^+ \times R^+ \times E$.

(H2) There exist $f, g, h \in C[R^+, R^+]$ and $H_i(t,s) \in C[R^+ \times R^+, R^+]$, $i = 1, 2$, such that

$$\|F(t,x,y)\| \le f(t)\|x\| + g(t)\|y\| + h(t), \quad (t,x,y) \in R^+ \times E \times E, \tag{4}$$

and

$$\|K(t,s,x)\| \le H_1(t,s)\|x\| + H_2(t,s), \quad (t,s,x) \in R^+ \times R^+ \times E. \tag{5}$$

(H3) There exist $G_i(t) \in C[R^+, R^+]$, $i = 1, 2$, and $L(t,s) \in C[R^+ \times R^+, R^+]$ such that

$$\begin{aligned} \alpha(F(t, D_1, D_2)) \le G_1(t)\alpha(D_1) + G_2(t)\alpha(D_2), \quad t \in R^+, \\ \text{bounded } D_i \subset E, \quad i = 1, 2, \end{aligned} \tag{6}$$

and

$$\alpha(K(t,s,D)) \le L(t,s)\alpha(D), \quad (t,s) \in R^+ \times R^+, \text{ bounded } D \subset E. \tag{7}$$

Theorem. *Let conditions* (H1)–(H3) *be satisfied. Then* IVP (1) *has a solution in* $C^1[R^+, E]$.

Proof. Our proof is divided into two parts.

(1) In this part, we prove that IVP (1) has a solution defined on $[0, 1]$. Evidently, we need only to prove that operator A has a fixed point in $C[[0,1], E]$. Our proof is based on the Mönch fixed point theorem and is divided into three steps again.

Step 1. It is easy to see by (3) and the hypotheses of the theorem that operator A maps $C[[0,1], E]$ into $C[[0,1], E]$. It is not difficult to verify that the uniform continuity of F on any bounded subset of $R^+ \times E \times E$ and that of K on any bounded subset of $R^+ \times R^+ \times E$ imply the continuity of operator A, i.e., A is a continuous operator from $C[[0,1], E]$ into $C[[0,1], E]$.

Step 2. Let

$$M_1 = \max\big\{ \max_{t\in[0,1]} f(t), \max_{t\in[0,1]} g(t), \max_{t\in[0,1]} h(t), \max_{(t,s)\in[0,1]\times[0,1]} H_i(t,s),\ i=1,2\big\}$$

and

$$M_2 = \max\big\{ \max_{t\in[0,1]} G_1(t), \max_{t\in[0,1]} G_2(t), \max_{(t,s)\in[0,1]\times[0,1]} L(t,s)\big\}.$$

Take constant $N > 0$ sufficiently large such that

$$\beta \equiv \frac{M_1}{N} + \frac{M_1^2}{N^2} < 1 \tag{8}$$

and

$$\gamma \equiv \frac{2M_2}{N} + \frac{4M_2^2}{N^2} < 1. \tag{9}$$

For any $x \in C[[0,1],E]$, let

$$\|x\|_0 = \max_{0\le t\le 1} \{e^{-Nt}\|x(t)\|\}.$$

Clearly, $C[[0,1],E]$ is a Banach space with norm $\|\cdot\|_0$ and two norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Therefore, operator A is also continuous with respect to norm $\|\cdot\|_0$.

In this step we shall prove that there exists a constant $R > 0$ sufficiently large such that $A : B_R \to B_R$ (where $B_R = \{x \in C[[0,1],E] : \|x\|_0 \le R\}$).

In fact, set

$$R \ge \frac{M_1(M_1+1)}{1-\beta}. \tag{10}$$

By virtue of (3)–(5), for any $x \in B_R$, $t \in [0,1]$, we have

$$\begin{aligned}
\|Ax(t)\| &\le \int_0^t \left\| F\left(s, x(s), \int_0^s K(s,\tau,x(\tau))\,d\tau\right)\right\| ds \\
&\le \int_0^t \left[f(s)\|x(s)\| + g(s)\int_0^s \|K(s,\tau,x(\tau))\|\,d\tau + h(s)\right] ds \\
&\le \int_0^t \left[M_1\|x(s)\| + M_1\left(\int_0^s M_1\|x(\tau)\|\,d\tau + M_1\right) + M_1\right] ds \\
&\le M_1\int_0^t e^{Ns}\|x\|_0\,ds + M_1^2\int_0^t\int_0^s e^{N\tau}\|x\|_0\,d\tau\,ds + M_1^2 + M_1 \\
&\le \left[\left(\frac{M_1}{N} + \frac{M_1^2}{N^2}\right)\|x\|_0 + (M_1^2 + M_1)\right] e^{Nt},
\end{aligned}$$

and so,

$$\begin{aligned}
\|Ax(t)e^{-Nt}\| &\le \beta\|x\|_0 + (M_1^2 + M_1) \\
&\le \beta R + (1-\beta)R = R, \\
&\forall\, t \in [0,1].
\end{aligned}$$

Consequently, we get that $\|Ax\|_0 \le R$, for any $x \in B_R$, i.e., $A : B_R \to B_R$.

Step 3. Let $D = \{x_n\} \subset B_R$ countable such that $\overline{D} = \overline{\mathrm{co}}(\{x\} \cup A(D))$ for some $x \in B_R$. It is not difficult to prove that the elements of $A(D)$ are equicontinuous on $[0,1]$, and so we have, by Lemma 1,

$$\alpha^*(A(D)) = \sup_{t\in[0,1]} \alpha^*((A(D))(t)), \tag{11}$$

where α^* denotes the Kuratowski measure of noncompactness in $C[[0,1], E]$ with norm $\|\cdot\|_0$.

Using (6), (7) and Lemma 2, for any $t \in [0,1]$, we find

$$\begin{aligned}\alpha((A(D))(t)) &= \alpha\bigg(\bigg\{\int_0^t F(s,x(s),\int_0^s K(s,\tau,x(\tau))\,d\tau)\,ds : x \in D\bigg\}\bigg)\\ &\le 2\int_0^t \bigg[G_1(s)\alpha(D(s))+2G_2(s)\int_0^s \alpha(K(s,\tau,D(\tau)))\,d\tau\bigg]\,ds\\ &\le 2M_2\int_0^t \bigg[\alpha(D(s))+2\int_0^s L(s,\tau)\alpha(D(\tau))\,d\tau\bigg]\,ds\\ &\le 2M_2\int_0^t \alpha(D(s))\,ds + 4M_2^2\int_0^t\int_0^s \alpha(D(\tau))\,d\tau\,ds\\ &= 2M_2\int_0^t e^{Ns}\alpha(e^{-Ns}D(s))\,ds\\ &\quad + 4M_2^2\int_0^t\int_0^s e^{N\tau}\alpha(e^{-N\tau}D(\tau))\,d\tau\,ds\\ &\le \frac{2M_2}{N}e^{Nt}\alpha^*(D)+\frac{4M_2^2}{N^2}e^{Nt}\alpha^*(D),\end{aligned}$$

and so,

$$\alpha(e^{-Nt}(A(D))(t)) \le \bigg(\frac{2M_2}{N}+\frac{4M_2^2}{N^2}\bigg)\alpha^*(D). \tag{12}$$

Finally, (11) and (12) imply

$$\alpha^*(A(D)) \le \gamma\alpha^*(D).$$

Therefore we deduce

$$\alpha^*(D) = \alpha^*(\overline{\mathrm{co}}(\{x\}\cup A(D))) = \alpha^*(A(D)) \le \gamma\alpha^*(D),$$

which, by $0 \le \gamma < 1$, implies $\alpha^*(D) = 0$, i.e., D is a relatively compact set in B_R. Hence, the Mönch fixed point theorem implies that A has a fixed point x^* in B_R. Consequently, IVP (1) has a solution x^* in $C[[0,1],E]$.

(2) Let

$$x_1^* = \lim_{t\to 1^-} x^*(t) = x^*(1). \tag{13}$$

We consider the following IVP of integrodifferential equation

$$\begin{aligned} y'(t) = F\bigg(&t+1, y(t)+x_1^*, \int_0^1 K(t+1,s,x^*(s))\,ds \\ &+ \int_0^t K(t+1,s+1,y(s)+x_1^*)\,ds\bigg), \\ &0 \le t \le 1, \quad y(0)=0. \end{aligned} \tag{14}$$

Clearly we have that conditions similar to (H1)–(H3) hold for the problem (14) so that a repetition of the above method yields a solution $y(t)$ of (14) on $[0,1]$.

We define a continuation of $x^*(t)$ on $[0,2]$ by

$$x^*(t) = \begin{cases} x^*(t), & 0 \le t \le 1, \\ y(t-1)+x_1^*, & 1 \le t \le 2, \end{cases}$$

and, by (13)–(14), we get that x^* is a solution to IVP (1) on $[0,2]$. Continuing in this way, we obtain a solution of IVP (1) on R^+. The proof is complete. □

Remark 1. If we consider IVP (1) with initial condition $x(0)=x_0$, we can avoid the apparent difficulty caused by the fact that x_0 might be different from the zero element by performing a transformation similar to the one made in the last part of the proof of our theorem.

Remark 2. We know that a class of nonlinear integral equations occurring in the mathematical theory of the infiltration of a fluid from a cylindrical reservoir into an isotropic homogeneous porous medium can be transformed into IVP (1) (in one-dimension form, see [**1, 7**]). Therefore, it seems to make sense to study IVP (1) in Banach spaces.

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Department of Mathematics, Shandong University, Jinan, Shandong 250100, P.R. China

Current address: Department of Mechanics, Tianjin University, Tianjin, 300072, People's Republic of China
E-mail address: `sunc@public1.tpt.tj.cn`