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A NEW INTEGRAL EQUATION FORMULATION FOR THE SCATTERING OF PLANE ELASTIC WAVES BY DIFFRACTION GRATINGS

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ABSTRACT. The scattering of plane elastic waves by a rigid periodic surface is considered. The Green's tensor for a half-space with a rigid surface is introduced and its properties, notably its asymptotic decay rate in horizontal layers above the plane, are analyzed. The Green's tensor is then used to define single and double layer potentials for a periodic surface, making use of the generalized stress tensor. Subsequently, a novel integral equation formulation for the scattering of plane waves by a diffraction grating is derived using a Brakhage/Werner type ansatz for the solution. Employing the Fredholm alternative, existence of solution is proved for all angles of incidence and all wave-numbers.

1. Introduction. It is the object of this paper to derive a new integral equation formulation for a certain scattering problem, namely the scattering of a time harmonic plane elastic wave by an unbounded periodic structure. The propagation of time harmonic waves with circular frequency ω in an elastic solid with Lamé constants μ, λ ($\mu, \lambda + \mu > 0$) and density ρ is governed by Hooke's law

$$\tau_{jk} = \lambda \operatorname{div} \mathbf{u} \delta_{jk} + \mu \left(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j} \right), \quad j, k = 1, 2, 3, \tag{1}$$

and by the equation of motion

$$\sum_{k=1}^{3} \frac{\partial \tau_{jk}}{\partial x_k} + \omega^2 \rho u_j = 0, \quad j = 1, 2, 3. \tag{2}$$

Here the vector field $\mathbf{u}$ denotes the displacements and τ the stress tensor. We will assume that the density ρ is constant throughout the

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medium, say $\rho \equiv 1$. Inserting the components of τ as given by (1) into the equations of motion (2) then yields the Navier equation

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \omega^2\mathbf{u} = 0. \tag{3}$$

All waves are assumed to be traveling in a half-space bounded by a diffraction grating, given as $x_2 = f(x_1)$ with f smooth and 2π-periodic, on which all displacements are assumed to vanish. As this special geometry is invariant in the x_3-direction, the system of equations (3) separates into two parts, one describing compressional and vertically polarized shear waves, the other describing horizontally polarized shear waves. The second part is in fact a scalar Helmholtz equation, and the solution of this problem has been studied in depth elsewhere, e.g., [**14, 21**] and the references contained therein. Here we will only consider the first part; the scattering problem is treated as a problem of plane strain.

The incident field is assumed to be a superposition of plane compressional and shear waves given by

$$\mathbf{u}^{inc}(x) = a_p\hat{\theta}e^{ik_p\mathbf{x}\cdot\hat{\theta}} + a_s\theta^{\perp}e^{ik_s\mathbf{x}\cdot\hat{\theta}}, \quad a_p, a_s \in \mathbf{C}. \tag{4}$$

Here $\theta \in (0,\pi)$ is the angle of incidence, $\hat{\theta} := (\cos\theta, -\sin\theta)^{\top}$, $\theta^{\perp} := (\sin\theta, \cos\theta)^{\top}$ and k_p, k_s denote the wave numbers for compressional and shear waves, respectively. Thus, denoting the scattered field by $\mathbf{u}$, it is the object to solve equation (3) in $\Omega := \{\mathbf{x} \in \mathbf{R}^2, x_2 > f(x_1)\}$ subject to the boundary condition

$$\mathbf{u}(\mathbf{x}) = -\mathbf{u}^{inc}(\mathbf{x}) \quad \text{on } \partial\Omega. \tag{5}$$

In both the mathematical and engineering literature, integral equation methods have been widely used to solve scattering problems for elastic waves, the books by Kupradze et al. [**16, 17**] and Constanda [**7**] giving a comprehensive mathematical introduction to the subject. Lately, work has focused on crack [**12, 13, 23**] and inverse problems [**10, 11, 15**]. Some investigations on elastic wave scattering by periodic free surfaces employing integral equation methods, including the formulation of a radiation condition, were made in [**9, 18**]. However, to the author's knowledge, [**1**] contains the first rigorous proof of uniqueness and existence of solution for the scattering of elastic waves by a diffraction grating.

Although [**1**] answers the question of uniqueness for the boundary value problem (3), (5) completely, existence of solution could only be proved for all but a discrete set of combinations of angle of incidence and wave numbers of the incident field. Similar problems arise when considering scattering of acoustic waves by diffraction gratings [**14**] but recent results by Chandler-Wilde/Ross [**4**] have shown how these difficulties can be avoided by employing the Green's function for a half-plane with an impedance boundary condition; see also Szemberg [**22**] for a related approach. The use of the simpler Dirichlet Green's function for a half-plane has recently been proposed by Zhang/Chandler-Wilde [**24**]. The advantage of these fundamental solutions are their faster asymptotic decay rates in vertically bounded strips. This ensures the existence of corresponding quasi-periodic Green's functions, which are used for diffraction grating problems, for all combinations of wave-number and angle of incidence and also makes extensions of these results to rough surfaces possible.

In the theory of elasticity, however, the question of fundamental solutions for half-space problems is a more delicate issue as, in general, no representations in terms of standard special functions exist. Since Lamb, in his classical paper [**19**], gave representations of Green's tensors for a number of free surface problems, much attention has been given to this issue. However, the author is not aware of any published work for the case of a rigid surface.

Thus, in Section 2, we introduce a Green's tensor for the elastic wave propagation problem in a half space with a rigid surface. Emphasis will be laid on a detailed analysis of its asymptotic decay rate in horizontal strips as this is the critical property for its successful use in solving the scattering problem for a diffraction grating for all wave-numbers and incident direction combinations. In Section 3, corresponding results for certain derivatives of the Green's tensor are established. The derivatives of interest are the generalized stresses as introduced by Kupradze [**16**] and also used in [**8, 20**].

In Section 4 definitions of quasi-periodic single- and double-layer potentials are given which will, in turn, by employed to prove existence of solution to the scattering problem for a diffraction grating in Section 5 through a Brakhage/Werner type ansatz for the solution [**2**]. As we make use of the generalized stress tensor, the operators in the equivalent integral equation formulation are compact so that the Fredholm alternative can be employed. However, the chosen ansatz limits the method to scattering surfaces with rigid boundaries, i.e., to Dirichlet boundary value problems. The final existence result holds for all combinations of angle of incidence and wave numbers.

This first section will close with some notations and general assumptions. Vectors and vector fields shall always be denoted in bold type. For $\mathbf{y} = (y_1, y_2)^\top$, we define $\mathbf{y}' := (y_1, -y_2)^\top$. For any set $\mathcal{M}$ and $m \in \mathbf{N}$, we define $BC^m(\mathcal{M})$ as the set of functions in $C^m(\mathcal{M})$ that, together with their derivatives up to order m, are bounded in $\mathcal{M}$. If $\mathcal{M}$ is compact, we denote by $C^{m,\alpha}(\mathcal{M})$, $0 < \alpha \le 1$, the usual set of Hölder continuous functions. For unbounded $\mathcal{M}$, we write $\phi \in C^{m,\alpha}(\mathcal{M})$ if $\phi \in C^{m,\alpha}(\mathcal{N})$ for every compact subset $\mathcal{N}$ of $\mathcal{M}$.

To abbreviate the differential operator in (3) we shall write

$$\Delta^* \mathbf{u} := \mu \Delta \mathbf{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \mathbf{u}.$$

The upper half plane shall be denoted by $\mathbf{R}^2_+ := \{\mathbf{x} \in \mathbf{R}^2 : x_2 > 0\}$. The Navier equation (3) is to be solved in the domain $\Omega := \{\mathbf{x} \in \mathbf{R}^2 : x_2 > f(x_1)\}$, where we assume $f \in BC^2(\mathbf{R})$ to be strictly positive. We finally set $S := \partial\Omega$ and, for $h > \max f$, call the sets

$$\mathcal{S}_h := \{\mathbf{x} \in \mathbf{R}^2_+ : f(x_1) < x_2 < h\}$$

horizontal layers (of height h) above S.

2. The Green's tensor. In the study of the Navier equation (3) one usually makes use of the matrix of fundamental solutions which is the Green's tensor for free-field conditions, given by

$$\begin{aligned} \Gamma(\mathbf{x}, \mathbf{y}) := {} & \frac{i}{4\mu} H_0^{(1)}(k_s|\mathbf{x} - \mathbf{y}|) \\ & + \frac{i}{4\omega^2} \nabla_x^\top \nabla_x \big(H_0^{(1)}(k_s|\mathbf{x} - \mathbf{y}|) - H_0^{(1)}(k_p|\mathbf{x} - \mathbf{y}|)\big), \end{aligned} \tag{6}$$

where $H_0^{(1)}(\cdot)$ denotes the Hankel function of the first kind and of order zero. Note the formulas

$$k_p^2 = \frac{\omega^2}{2\mu + \lambda}, \qquad k_s^2 = \frac{\omega^2}{\mu} \tag{7}$$

for the wave numbers.

With the help of the Bessel differential equation, it is easy to see that the components of this matrix can be written as

$$\Gamma_{jk}(\mathbf{x},\mathbf{y}) = \frac{i}{4\mu}\left\{\Phi_1(|\mathbf{x}-\mathbf{y}|)\delta_{jk} + \Phi_2(|\mathbf{x}-\mathbf{y}|)\frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^2}\right\}$$
$$j,k=1,2,$$

where, introducing the constant $\tau = k_p/k_s$,

$$\Phi_1(t) := H_0^{(1)}(k_s t) - \frac{1}{k_s t}(H_1^{(1)}(k_s t) - \tau H_1^{(1)}(k_p t)),$$
$$\Phi_2(t) := \frac{2}{k_s t}H_1^{(1)}(k_s t) - H_0^{(1)}(k_s t) - \frac{2\tau}{k_s t}H_1^{(1)}(k_p t) + \tau^2 H_0^{(1)}(k_p t).$$

As was pointed out in the introduction, the quasi-periodic Green's tensor corresponding to Γ that was employed in [**1**] does not exist for all combinations of angle of incidence and wave-numbers. To avoid this difficulty, the new integral equation formulation will make use of the Green's tensor for a half space with a rigid surface, i.e., Dirichlet boundary conditions. Theorems 2.1 and 2.5 below, proved by Fourier transform and other methods, show that this Green's tensor is given by

$$\Gamma_D(\mathbf{x},\mathbf{y}) := \Gamma(\mathbf{x},\mathbf{y}) - \Gamma(\mathbf{x},\mathbf{y}') + U(\mathbf{x},\mathbf{y}), \tag{8}$$

for $\mathbf{x},\mathbf{y}\in\mathbf{R}_+^2$, $\mathbf{x}\neq\mathbf{y}$, where

$$U(\mathbf{x},\mathbf{y}) := -\frac{i}{2\pi\omega^2}\int_{-\infty}^{\infty}(M_p(t,\gamma_p,\gamma_s;x_2,y_2) + M_s(t,\gamma_p,\gamma_s;x_2,y_2))e^{-iX_1 t}\,dt, \tag{9}$$

with

$$M_p(t,\gamma_p,\gamma_s;x_2,y_2) := \frac{e^{i\gamma_p(x_2+y_2)} - e^{i(\gamma_p x_2+\gamma_s y_2)}}{\gamma_p\gamma_s + t^2}\begin{pmatrix} -t^2\gamma_s & t^3 \\ t\gamma_p\gamma_s & -t^2\gamma_p \end{pmatrix}$$

$$M_s(t,\gamma_p,\gamma_s;x_2,y_2) := \frac{e^{i\gamma_s(x_2+y_2)} - e^{i(\gamma_s x_2+\gamma_p y_2)}}{\gamma_p\gamma_s + t^2}\begin{pmatrix} -t^2\gamma_s & -t\gamma_p\gamma_s \\ -t^3 & -t^2\gamma_p \end{pmatrix}.$$

Here $X_1 := x_1 - y_1$ and

$$\gamma_p := \begin{cases} \sqrt{k_p^2 - t^2}, & k_p^2 \ge t^2 \\ i\sqrt{t^2 - k_p^2}, & k_p^2 < t^2, \end{cases}$$
$$\gamma_s := \begin{cases} \sqrt{k_s^2 - t^2}, & k_s^2 \ge t^2 \\ i\sqrt{t^2 - k_s^2}, & k_s^2 < t^2. \end{cases}$$

The following theorem establishes that Γ_D possesses the properties of a fundamental solution and satisfies the boundary conditions.

Theorem 2.1. *For $\mathbf{y} \in \mathbf{R}_+^2$ fixed, $\Gamma_D(\cdot, \mathbf{y}) - \Gamma(\cdot, \mathbf{y}) \in C^2(\mathbf{R}_+^2) \cap C(\overline{\mathbf{R}_+^2})$ and its columns are solutions to the Navier equation* (3) *in $\mathbf{R}_+^2 \backslash \{\mathbf{y}\}$. Furthermore, $\Gamma_D(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{x} \in \partial\mathbf{R}_+^2$.*

Proof. From the definition of $U(\cdot, \cdot)$, it is clear that $\Gamma_D - \Gamma$ is infinitely often continuously differentiable in $\mathbf{R}_+^2$ with respect to both arguments. Furthermore, $\Gamma_D - \Gamma$ and all its derivatives can be continuously extended to $\overline{\mathbf{R}_+^2}$. Thus, $\Gamma_D - \Gamma$ has the required regularity.

Introducing the functions

$$\hat{\Phi}^{(1)}(x_2; y_2, t) := \frac{1}{\omega^2}(-t\gamma_s)\frac{e^{i\gamma_p(x_2+y_2)} - e^{i(\gamma_p x_2 + \gamma_s y_2)}}{\gamma_p\gamma_s + t^2},$$
$$\hat{\Psi}^{(1)}(x_2; y_2, t) := \frac{1}{\omega^2}t^2\frac{e^{i\gamma_s(x_2+y_2)} - e^{i(\gamma_s x_2 + \gamma_p y_2)}}{\gamma_p\gamma_s + t^2},$$
$$\hat{\Phi}^{(2)}(x_2; y_2, t) := \frac{1}{\omega^2}t^2\frac{e^{i\gamma_p(x_2+y_2)} - e^{i(\gamma_p x_2 + \gamma_s y_2)}}{\gamma_p\gamma_s + t^2},$$
$$\hat{\Psi}^{(2)}(x_2; y_2, t) := \frac{1}{\omega^2}t\gamma_p\frac{e^{i\gamma_s(x_2+y_2)} - e^{i(\gamma_s x_2 + \gamma_p y_2)}}{\gamma_p\gamma_s + t^2},$$

we further observe that the kth column of $U(\cdot, \cdot)$, $k = 1, 2$, can be written as

$$U_{\cdot k}(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{pmatrix} (-it)\hat{\Phi}^{(k)}(x_2; y_2, t) + \dfrac{\partial}{\partial x_2}\hat{\Psi}^{(k)}(x_2; y_2, t) \\ \dfrac{\partial}{\partial x_2}\hat{\Phi}^{(k)}(x_2; y_2, t) - (-it)\hat{\Psi}^{(k)}(x_2; y_2, t) \end{pmatrix} e^{-iX_1 t}\, dt.$$

$\hat{\Phi}^{(k)}$ and $\hat{\Psi}^{(k)}$ are also seen to satisfy the differential equations

$$\frac{\partial^2}{\partial x_2^2}\hat{\Phi}^{(k)} + (k_p^2 - t^2)\hat{\Phi}^{(k)} = 0,$$
$$\frac{\partial^2}{\partial x_2^2}\hat{\Psi}^{(k)} + (k_s^2 - t^2)\hat{\Psi}^{(k)} = 0.$$

This means that $\hat{\Phi}^{(k)}$ and $\hat{\Psi}^{(k)}$ are Fourier transforms of Lamé potentials with respect to X_1 and $U_{\cdot k}(\cdot, \mathbf{y})$ is a solution to (3).

It remains to show that $\Gamma_{D,\cdot k}(\mathbf{x}, \mathbf{y}) = 0$ for $x_2 = 0$. This is done in a straightforward way by computing the Fourier transform of the Hankel functions with respect to X_1. $\square$

The main advantage of using Γ_D over Γ is its faster asymptotic decay rate as $|x_1| \to \infty$ in horizontal layers above S for fixed $\mathbf{y}$. For the first two terms in its representation, this is shown in the following lemma.

Lemma 2.2. *For $\mathbf{x}, \mathbf{y} \in \mathbf{R}_+^2$, $|\mathbf{x} - \mathbf{y}| \geq 1$, the estimate*

$$\max_{j,k=1,2} |\Gamma_{jk}(\mathbf{x}, \mathbf{y}) - \Gamma_{jk}(\mathbf{x}, \mathbf{y}')| \leq \frac{h(x_2, y_2)}{|\mathbf{x} - \mathbf{y}|^{3/2}},$$

holds, where $h \in C(\mathbf{R}^2)$.

Proof. Using the notations $r = |\mathbf{x} - \mathbf{y}|$ and $r' = |\mathbf{x} - \mathbf{y}'|$, the following holds

$$\begin{aligned}\Gamma(\mathbf{x}, \mathbf{y}) - \Gamma(\mathbf{x}, \mathbf{y}') = \frac{i}{4\mu}\Bigg\{&(\Phi_1(r) - \Phi_1(r'))I \\ &+ \begin{pmatrix} 0 & -2y_2(x_1 - y_1) \\ -2y_2(x_1 - y_1) & 0 \end{pmatrix} \frac{\Phi_2(r)}{r^2} \\ &+ \begin{pmatrix} (x_1 - y_1)^2 & (x_1 - y_1)(x_2 + y_2) \\ (x_1 - y_1)(x_2 + y_2) & (x_2 + y_2)^2 \end{pmatrix} \\ &\cdot \left(\frac{\Phi_2(r)}{r^2} - \frac{\Phi_2(r')}{r'^2}\right)\Bigg\}.\end{aligned}$$

So it obviously suffices to show the estimate for the functions

$$\Phi_1(r) - \Phi_1(r'), \quad \frac{(x_1 - y_1)\Phi_2(r)}{r^2},$$

and

$$(x_1 - y_1)^2 \left(\frac{\Phi_2(r)}{r^2} - \frac{\Phi^2(r')}{r'^2} \right).$$

Using the mean value theorem yields

$$|\Phi_1(r) - \Phi_1(r')| \leq |r - r'| \max_{r \leq t \leq r'} |\Phi_1'(t)| = \frac{4x_2 y_2}{r + r'} \max_{r \leq t \leq r'} |\Phi_1'(t)|,$$

and thus the asymptotic decay rate of Hankel functions and their derivatives as, e.g., given in [**6**] yields the asserted estimate in the first case because of the assumption $|\mathbf{x} - \mathbf{y}| \geq 1$. In the second case, $(x_1 - y_1)/r$ is bounded and $(\Phi_2(r)/r)$ has the required decay rate. For the last function, we rewrite

$$\frac{\Phi_2(r)}{r^2} - \frac{\Phi_2(r')}{r'^2} = \frac{1}{r^2 + 4x_2 y_2} \left(\Phi_2(r) - \Phi_2(r') + \frac{4x_2 y_2}{r^2} \Phi_2(r) \right).$$

Now $(x_1 - y_1)^2/(r^2 + 4x_2 y_2)$ is bounded, $\Phi_2(r) - \Phi_2(r')$ can be estimated in the same way as $\Phi_1(r) - \Phi_1(r')$ above and $(\Phi_2(r)/r^2)$ decays even faster than required. $\square$

To prove a similar estimate for U, a more detailed analysis is required. To obtain alternative representations of the integrals used in the definition of U, γ_p and γ_s need to be extended to analytic functions in the complex plane. To this end, branch cuts from $\pm k_p$ and $\pm k_s$, respectively, to $\pm k_p \pm i\infty$ and $\pm k_s \pm i\infty$ are introduced. Further, note that the integrands in the definition of U do not have any singularities on the chosen branches of γ_p and γ_s. Restricting ourselves to the case $x_1 > y_1$ for the moment, we deform the path of integration into the lower half plane as illustrated in Figure 1.

It is easily seen that the integrals over the arcs vanish as their radius tends to infinity, so only the branch line integrals remain. Denoting the paths of integration along the branch cuts by $\mathcal{C}_1 \cup \mathcal{C}_2$ and $\mathcal{C}_3 \cup \mathcal{C}_4$, as indicated in Figure 1, and introducing the function

$$M(t, \gamma_p, \gamma_s) := -\frac{i}{2\pi\omega^2} (M_p(t, \gamma_p, \gamma_s; x_2, y_2) + M_s(t, \gamma_p, \gamma_s; x_2, y_2)),$$

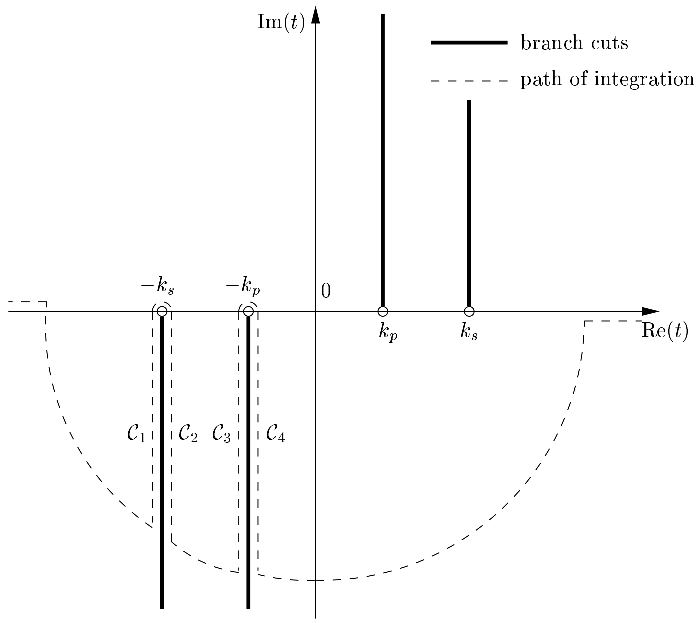

FIGURE 1. The path of integration.

we rewrite U as

$$\begin{aligned} U(\mathbf{x},\mathbf{y}) &= \int_{\mathcal{C}_1\cup\mathcal{C}_2} M(t,\gamma_p,\gamma_s)e^{-iX_1 t}\,dt \\ &\quad + \int_{\mathcal{C}_3\cup\mathcal{C}_4} M(t,\gamma_p,\gamma_s)e^{-iX_1 t}\,dt \\ &= (-i)\int_0^\infty \{M(-k_s - is, \gamma_p|_{\mathcal{C}_2}, \gamma_s|_{\mathcal{C}_2}) \\ &\qquad - M(-k_s - is, \gamma_p|_{\mathcal{C}_1}, \gamma_s|_{\mathcal{C}_1})\}e^{-X_1 s + iX_1 k_s}\,ds \\ &\quad + (-i)\int_0^\infty \{M(-k_p - is, \gamma_p|_{\mathcal{C}_4}, \gamma_s|_{\mathcal{C}_4}) \\ &\qquad - M(-k_p - is, \gamma_p|_{\mathcal{C}_3}, \gamma_s|_{\mathcal{C}_3})\}e^{-X_1 s + iX_1 k_p}\,ds. \end{aligned}$$

Note that $\gamma_s|_{\mathcal{C}_1} = -\overline{\gamma_s|_{\mathcal{C}_2}}$ and $\gamma_p|_{\mathcal{C}_1} = \gamma_p|_{\mathcal{C}_2}$; corresponding relations hold on $\mathcal{C}_3$ and $\mathcal{C}_4$. Using the mean-value theorem, we thus conclude

$$M(-k_s - is, \gamma_p|_{\mathcal{C}_2}, \gamma_s|_{\mathcal{C}_2}) - M(-k_s - is, \gamma_p|_{\mathcal{C}_1}, \gamma_s|_{\mathcal{C}_1}) \\ = 2\,\mathrm{Re}\,(\gamma_s|_{\mathcal{C}_2}) \frac{\partial M}{\partial \gamma_s}(-k_s - is, \gamma_p|_{\mathcal{C}_1}, \xi)$$

for some $\xi \in (\gamma_s|_{\mathcal{C}_1}, \gamma_s|_{\mathcal{C}_2})$. Now $(\partial M/\partial\gamma_s)(-k_s - is, \gamma_p|_{\mathcal{C}_1}, \xi)$ is seen to be continuously dependent on s in $[0,\infty)$, and for some constant C continuously dependent on x_2 and y_2, $|s^{-1/2}\,\mathrm{Re}\,(\gamma_s|_{\mathcal{C}_2})(\partial M/\partial\gamma_s)(-k_s - is, \gamma_p|_{\mathcal{C}_1}, \xi)| \le C$ holds for $s \in [0,1]$. Analogous results hold for the second integral. Therefore, we can estimate the asymptotic decay rate of U by employing the following lemma with $r = (1/2)$.

Lemma 2.3. *Assume $q \in C([0,\infty))$ so that $C_1 := \int_0^\infty |q(s)|e^{-s}\,ds$ exists. For $X > 1$, set*

$$I(X) := \int_0^\infty q(s) e^{-Xs}\,ds.$$

Further assume that for some $r > -1$ there exists $C_2 > 0$ with $|s^{-r} q(s)| \le C_2$ for all $s \in [0,1]$. Then for $X \ge 1 + (r+1)\log X$,

$$|I(X)| \le (C_1 + \Gamma(r+1)C_2)\frac{1}{X^{r+1}}.$$

Proof. We can estimate

$$\left|\int_0^1 q(s)e^{-Xs}\,ds\right| \le C_2 \int_0^1 s^r e^{-Xs}\,ds \le C_2 \frac{\Gamma(r+1)}{X^{r+1}}.$$

On the other hand, we have

$$\left|\int_1^\infty q(s)e^{-Xs}\,ds\right| \le e^{-(X-1)} \int_1^\infty |q(s)|e^{-s}\,ds \le \frac{C_1}{X^{r+1}}$$

for all $X \ge 1 + (r+1)\log X$. Adding these two estimates yields the assertion. $\square$

An identical analysis also yields this decay rate for $x_1 < y_1$. The only difference is that the path of integration has to be deformed into the upper half plane. Thus, also recalling Lemma 2.2, the following theorem is proved.

Theorem 2.4. *For $\mathbf{x}, \mathbf{y} \in \mathbf{R}^2_+$, $|x_1 - y_1| \geq e$, the estimate*

$$\max_{j,k=1,2} |\Gamma_{D,jk}(\mathbf{x}, \mathbf{y})| \leq \frac{h(x_2, y_2)}{|x_1 - y_1|^{3/2}}$$

holds, where $h \in C(\mathbf{R}^2)$.

The Green's tensor Γ_D also satisfies the radiation condition often used in elastic scattering problems, Kupradze's radiation condition. Let $\Gamma_D^{(p)} := -k_p^{-2} \operatorname{grad}_{\mathbf{x}} \Gamma_D$ denote the longitudinal and $\Gamma_D^{(s)} := \Gamma_D - \Gamma_D^{(p)}$ the transversal part of Γ_D. Then the following theorem holds.

Theorem 2.5. *Let $r := |\mathbf{x}|$. For $\mathbf{y} \in \mathbf{R}^2_+$ fixed,*

$$\begin{aligned}
&\Gamma_D^{(p)}(\mathbf{x}, \mathbf{y}) = O(r^{-1/2}), \\
&\frac{\partial \Gamma_D^{(p)}}{\partial r}(\mathbf{x}, \mathbf{y}) - ik_p \Gamma_D^{(p)}(\mathbf{x}, \mathbf{y}) = o(r^{-1/2}), \\
&\Gamma_D^{(s)}(\mathbf{x}, \mathbf{y}) = O(r^{-1/2}), \\
&\frac{\partial \Gamma_D^{(s)}}{\partial r}(\mathbf{x}, \mathbf{y}) - ik_s \Gamma_D^{(s)}(\mathbf{x}, \mathbf{y}) = o(r^{-1/2})
\end{aligned}$$

uniformly in $\mathbf{x}/r$ as $r \to \infty$.

Proof. As the assertion is a well-known fact for Γ, it suffices to show it for U. Observe that terms in (9) involving M_p represent the longitudinal and the term involving M_s the transversal part of U. For fixed $\mathbf{y}$, an entry $U_{jk}^{(p)}(\cdot, \mathbf{y})$, $j, k = 1, 2$, of the transversal part satisfies the scalar Helmholtz equation

$$\Delta_{\mathbf{x}} U_{jk}^{(p)}(\mathbf{x}, \mathbf{y}) + k_p^2 U_{jk}^{(p)}(\mathbf{x}, \mathbf{y}) = 0 \quad \text{in } \mathbf{R}^2_+$$

and the boundary condition

$$U_{jk}^{(p)}(\mathbf{x},\mathbf{y}) = \mathbf{g}(\mathbf{x}) := -\Gamma_{jk}(\mathbf{x},\mathbf{y}) + \Gamma_{jk}(\mathbf{x},\mathbf{y}') - U_{jk}^{(s)}(\mathbf{x},\mathbf{y}) \quad \text{on } \{x_2 = 0\}.$$

From (9) we see that $U_{jk}^{(p)}$ satisfies the *upward propagating radiation condition* of [**5**], see [**3**] for details. Reviewing the proof of Theorem 2.4, we see that $\mathbf{g}(\mathbf{x}) = O(|x_1|^{-3/2})$ as $|x_1| \to \infty$. We can thus use the argument presented in Section 5 of [**5**] to conclude

$$|U_{jk}^{(p)}(\mathbf{x},\mathbf{y})| \leq C(1+x_2)(1+r)^{-3/2}$$

and

$$\frac{\partial U_{jk}^{(p)}}{\partial r}(\mathbf{x},\mathbf{y}) - ik_p U^{(p)_{jk}}(\mathbf{x},\mathbf{y}) = o(r^{-1/2}).$$

The same argument can be applied to $U^{(s)}$. □

3. Derivatives of the Green's tensor. In this section we will investigate the properties of certain derivatives of Γ_D. Recalling Hooke's law (1), we follow Kupradze [**16**] in introducing a *generalized stress tensor* $\mathcal{P} = (\pi_{jk})$ by

$$\pi_{jk} := b \operatorname{div} \mathbf{u} \delta_{jk} + \mu \frac{\partial u_j}{\partial x_k} + a \frac{\partial u_k}{\partial x_j},$$

where a, b are real numbers satisfying $a + b = \lambda + \mu$. Given a curve $\Lambda \subset \mathbf{R}^2$ with a normal $\mathbf{n}$, the *generalized stress vector* is defined by

$$\mathbf{P}\mathbf{u} := \mathcal{P}\mathbf{n} = (\mu + a)\frac{\partial \mathbf{u}}{\partial \mathbf{n}} + b\mathbf{n}\operatorname{div}\mathbf{u} + a \begin{pmatrix} n_2 \left(\dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \right) \\ n_1 \left(\dfrac{\partial u_1}{\partial x_2} - \dfrac{\partial u_2}{\partial x_1} \right) \end{pmatrix}.$$

Where it is important to distinguish between derivatives taken with respect to $\mathbf{x}$ and $\mathbf{y}$, the notations $\mathbf{P}^{(\mathbf{x})}$ and $\mathbf{P}^{(\mathbf{y})}$ will be used. The choice of a and b is left open for the moment. In the case $a = \mu$ and $b = \lambda$, $\mathbf{P}\mathbf{u}$ reduces to the standard *stress vector* or *traction across* Λ. In the case $a = \mu(\lambda+\mu)/(\lambda+3\mu)$ and $b = (\lambda+2\mu)(\lambda+\mu)/(\lambda+3\mu)$,

$\mathbf{Pu}$ is called the *pseudostress vector* and will also be denoted by $\mathbf{Nu}$. The significance of this choice will become clear in Section 4.

By applying $\mathbf{P}\cdot$ to Γ_D, the matrix-functions $\Pi_D^{(1)}$ and $\Pi_D^{(2)}$ are introduced:

$$\Pi_{D,jk}^{(1)}(\mathbf{x},\mathbf{y}) := (\mathbf{P}^{(\mathbf{x})}(\Gamma_{D,\cdot k}(\mathbf{x},\mathbf{y})))_j,$$
$$\Pi_{D,jk}^{(2)}(\mathbf{x},\mathbf{y}) := (\mathbf{P}^{(\mathbf{y})}(\Gamma_{D,j\cdot}(\mathbf{x},\mathbf{y}))^\top)_k.$$

Theorem 3.1. *Theorems* 2.4 *and* 2.5 *hold with* Γ_D *replaced by* $\Pi_D^{(1)}$ *and* $\Pi_D^{(2)}$ *respectively.*

Proof. The theorem follows from Lemma 7.2 in [**1**]. □

A standard result is the third generalized Betti formula.

Lemma 3.2. *Let* $B \subseteq \mathbf{R}^2$ *be a bounded domain in which the divergence theorem holds. Then, for* $\mathbf{v}, \mathbf{w} \in C^2(\overline{B})$ *the third generalized Betti formula holds*

$$\int_B (\mathbf{v}\cdot\Delta^*\mathbf{w} - \mathbf{w}\cdot\Delta^*\mathbf{v})\,\mathbf{dx} = \int_{\partial B} (\mathbf{v}\cdot\mathbf{Pw} - \mathbf{w}\cdot\mathbf{Pv})\,ds. \tag{10}$$

Via an application of (10), it is possible to prove the following reciprocity relation for Γ_D.

Lemma 3.3. *Let* $\mathbf{x},\mathbf{y} \in \mathbf{R}_+^2$, $\mathbf{x} \neq \mathbf{y}$. *Then*

$$\Gamma_D(\mathbf{x},\mathbf{y}) = \Gamma_D(\mathbf{y},\mathbf{x})^\top.$$

Proof. Let $B_\varepsilon(\mathbf{z})$ denote the open ball with radius ε and center $\mathbf{z}$, and set $\Omega_{R,\varepsilon} := \{\mathbf{z} \in \mathbf{R}_+^2 : |\mathbf{z}| < R, \mathbf{z} \notin \overline{B_\varepsilon(\mathbf{x}) \cup B_\varepsilon(\mathbf{y})}\}$. Using (10), it then follows that

$$0 = \int_{\Omega_{R,\varepsilon}} \Gamma_{D,\cdot j}(\mathbf{z},\mathbf{x})\cdot\Delta_{\mathbf{z}}^*\Gamma_{D,\cdot k}(\mathbf{z},\mathbf{y}) - \Gamma_{D,\cdot k}(\mathbf{z},\mathbf{y})\cdot\Delta_{\mathbf{z}}^*\Gamma_{D,\cdot j}(\mathbf{z},\mathbf{x})\,d\mathbf{z}$$

$$= \int_{\partial\Omega_{R,\varepsilon}} \Gamma_{D,\cdot j}(\mathbf{z},\mathbf{x}) \cdot \Pi^{(1)}_{D,\cdot k}(\mathbf{z},\mathbf{y}) - \Gamma_{D,\cdot k}(\mathbf{z},\mathbf{y}) \cdot \Pi^{(1)}_{D,\cdot j}(\mathbf{z},\mathbf{x})\, ds(\mathbf{z})$$
$$\longrightarrow \Gamma_{D,kj}(\mathbf{y},\mathbf{x}) - \Gamma_{D,jk}(\mathbf{x},\mathbf{y}) \qquad \varepsilon \longrightarrow 0, R \longrightarrow \infty$$

as the integrals over $\{|\mathbf{z}| = R\}$ vanish for $R \to \infty$ by Theorems 2.5 and 3.1, and it can be seen by a standard argument from potential theory that the integrals over $\partial B_\varepsilon(\mathbf{x})$ and $\partial B_\varepsilon(\mathbf{y})$ converge to $-\Gamma_{D,jk}(\mathbf{x},\mathbf{y})$ and $\Gamma_{D,kj}(\mathbf{y},\mathbf{x})$, respectively. □

Theorem 3.4. (a) *For $\mathbf{y} \in \mathbf{R}^2_+$, the columns of $\Pi^{(2)}_D(\cdot,\mathbf{y})$ are solutions to the Navier equation* (3) *in $\mathbf{R}^2_+\backslash\{\mathbf{y}\}$.*

(b) *For $\mathbf{x} \in \mathbf{R}^2_+$, the rows of $\Pi^{(1)}_D(\mathbf{x},\cdot)$ are solutions to the Navier equation* (3) *in $\mathbf{R}^2_+\backslash\{\mathbf{x}\}$.*

(c) *For $\mathbf{x},\mathbf{y} \in \mathbf{R}^2_+$, $\mathbf{x} \neq \mathbf{y}$, the following holds:*

$$\Pi^{(2)}_D(\mathbf{x},\mathbf{y}) = \Pi^{(1)}_D(\mathbf{y},\mathbf{x})^\top.$$

(d) *Let $B \subseteq \mathbf{R}^2_+$ be a bounded domain in which the divergence theorem holds. Then any solution $\mathbf{u}$ to the Navier equation can be represented as*

$$\mathbf{u}(\mathbf{x}) = \int_{\partial B} \Gamma_D(\mathbf{x},\mathbf{y})\mathbf{P}\mathbf{u}(\mathbf{y}) - \Pi^2_D(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y})\, ds(\mathbf{y})$$

for all $\mathbf{x} \in B$.

Proof. Part (c) is an immediate consequence of Lemma 3.3. Part (a) follows from Theorem 2.1 and the definition of the generalized stress vector. Part (b) follows from parts (a) and (c). Part (d) holds because of the corresponding relation for Γ, see, e.g., [**7, 16**] together with Theorem 2.1 and (10). □

4. Single- and double-layer potentials. In the following, assume $\phi \in BC(S)$ to be a vector valued density. The normal $\mathbf{n}$ on S will be assumed to be pointing into Ω. We define a single-layer potential by

$$\mathbf{v}(\mathbf{x}) := \int_S \Gamma_D(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y}) \quad \text{for } \mathbf{x} \in \mathbf{R}^2_+\backslash S, \tag{11}$$

and a double-layer potential by

$$\mathbf{w}(\mathbf{x}) := \int_S \Pi_D^{(2)}(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y}) \quad \text{for } \mathbf{x} \in \mathbf{R}_+^2 \backslash S. \tag{12}$$

Remark 4.1. As a consequence of Theorems 2.4 and 3.1, for a vector field $\phi \in BC(S)$, the integrals

$$\int_S \Gamma_D(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y})$$

and

$$\int_S \Pi_D^{(j)}(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y}), \quad j = 1, 2,$$

exist as improper integrals for all $\mathbf{x} \in \overline{\mathbf{R}_+^2} \backslash S$.

From now on, we will assume that f is 2π-periodic (in the case of a different period, this can always be achieved through a simple change of variables). The following periodicity property, see also [**9, 14**], will be of some importance in the following.

Definition 4.2. A vector field $\mathbf{u} : D \to \mathbf{C}^2$, where either $D = \Omega$ or $D = S$, is called *quasi-periodic* with phase-shift $\alpha \neq 0$ if

$$\mathbf{u}(x_1 + 2\pi, x_2) = e^{i\alpha 2\pi}\mathbf{u}(x_1, x_2) \quad \text{for all } \mathbf{x} = (x_1, x_2) \in \Omega.$$

Set $\Lambda := \{\mathbf{x} \in S : 0 < x_1 < 2\pi\}$. The integrals over S in the definition of the single- and double-layer potentials can then be reduce to integrals over Λ.

Lemma 4.3. *Assume ϕ to be quasi-periodic with phase-shift α and $\mathcal{K}$ to be a compact subset of $\overline{\mathbf{R}_+^2} \times \overline{\mathbf{R}_+^2}$. Then, with $\mathbf{p} = (2\pi, 0)^\top$, the series*

$$\Gamma_{Dp}(\mathbf{x}, \mathbf{y}) := \sum_{n\in\mathbf{Z}} e^{-i\alpha 2\pi n}\Gamma_D(\mathbf{x} + n\mathbf{p}, \mathbf{y}),$$
$$\Pi_{Dp}^{(j)}(\mathbf{x}, \mathbf{y}) := \sum_{n\in\mathbf{Z}} e^{-i\alpha 2\pi n}\Pi_D^{(j)}(\mathbf{x} + n\mathbf{p}, \mathbf{y}), \quad j = 1, 2,$$

converge absolutely and uniformly on $\mathcal{K}\backslash\{(\mathbf{x},\mathbf{y}) : \mathbf{x}-\mathbf{y} \in \mathbf{pZ}\}$. *Furthermore,*

$$\int_S \Gamma_D(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y}) = \int_\Lambda \Gamma_{Dp}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y}),$$
$$\int_S \Pi_D^{(j)}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y}) = \int_\Lambda \Pi_{Dp}^{(j)}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y}),$$

$j=1,2$, *for* $\mathbf{x} \in \overline{\mathbf{R}_+^2}$.

Proof. The absolute and uniform convergence of the series is implied by Theorems 2.4 and 3.1. The second part of the lemma is a direct consequence of the dominated convergence theorem. □

From now on it will be assumed that ϕ is quasi-periodic. The following theorem then lists some standard results for the two layer potentials.

Theorem 4.4. (a) $\mathbf{v}$ *and* $\mathbf{w}$ *are quasi-periodic solutions to the Navier equation* (3) *in* Ω *and in* $\mathbf{R}_+^2\backslash\overline{\Omega}$.

(b) $\mathbf{v}$ *can be continuously extended to a function on* $\overline{\mathbf{R}_+^2}$ *and*

$$\mathbf{v}(\mathbf{x}) = \int_S \Gamma_D(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y})$$

for $\mathbf{x} \in S$ *where the integral exists as an improper integral.*

Proof. Part (a) of the theorem is obvious as it is possible to exchange differentiation and integration for $\mathbf{x} \notin S$. The quasi-periodicity of $\mathbf{v}$ and $\mathbf{w}$ is easily verified directly.

To prove part (b), set

$$\mathbf{v}_N(\mathbf{x}) := \sum_{n=-N}^{N} \int_\Lambda e^{-i\alpha 2\pi n}\Gamma_D(\mathbf{x}+n\mathbf{p},\mathbf{y})\phi(\mathbf{y})\,ds(\mathbf{y}).$$

For $\mathbf{v}_N$ with Γ_D replaced by Γ, the assertion is a well-known result [**7**]. However, from the properties of $\Gamma_D - \Gamma$, as listed in Theorem 2.1, it is easily seen that it also holds for $\mathbf{v}_N$.

Further, let $\mathcal{L}_h := \{\mathbf{x} \in \Omega : 0 \leq x_1 \leq 2\pi, x_2 < h\}$. From Lemma 4.3 it follows that $\|\mathbf{v} - \mathbf{v}_N\|_{\infty,\mathcal{L}_h} \to 0$. Thus (b) holds for $\mathbf{x}$ on Λ. But as $\mathbf{v}$ is quasi-periodic, the assertion follows for $\mathbf{x} \in S$. $\square$

At this stage it is advantageous to make a special choice for the two real numbers a and b in the definition of the generalized stress vector. In general, $\Pi^{(j)}(\mathbf{x}, \mathbf{y})$, $j = 1, 2$, has a strong singularity at $\mathbf{x} = \mathbf{y}$. However, when using the pseudo stress vector, i.e., $a = \mu(\lambda + \mu)/(\lambda + 3\mu)$ and $b = (\lambda + 2\mu)(\lambda + \mu)/(\lambda + 3\mu)$ all but the normal derivatives cancel [**16**] so that only weak singularities have to be dealt with.

Theorem 4.5. *Assume $a = \mu(\lambda+\mu)/(\lambda+3\mu)$ and $b = (\lambda+2\mu)(\lambda+\mu)/(\lambda+3\mu)$. Then the following statements hold.*

(a) *The vector field $\mathbf{w}_+ := \mathbf{w}|_\Omega$ can be extended continuously to $\overline{\Omega}$ and the relation*

$$\mathbf{w}_+(\mathbf{x}) = \frac{1}{2}\phi(\mathbf{x}) + \int_S \Pi_D^{(2)}(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y})$$

holds on S.

(b) *If $\phi \in C^{0,\alpha}(S)$, then $\mathbf{v}_+ := \mathbf{v}|_\Omega \in C^{1,\alpha}(\Omega)$ and $\mathbf{v}_- := \mathbf{v}|_{(\mathbf{R}_+^2 \setminus \overline{\Omega})} \in C^{1,\alpha}(\overline{\mathbf{R}_+^2} \setminus \Omega)$. The relations*

$$\mathbf{N}\mathbf{v}_+(\mathbf{x}) = -\frac{1}{2}\phi(\mathbf{x}) + \int_S \Pi_D^{(1)}(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y})$$

and

$$\mathbf{N}\mathbf{v}_-(\mathbf{x}) = \frac{1}{2}\phi(\mathbf{x}) + \int_S \Pi_D^{(1)}(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y})$$

hold on S.

(c) *For $\phi \in BC(S)$, the following holds*

$$\int_S \Gamma_D(\cdot, \mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y}), \qquad \int_S \Pi_D^{(j)}(\cdot, \mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y}) \quad \in C^{0,\alpha}(S).$$

Proof. The method of proof of (a) is essentially the same as for Theorem 4.4 (b).

For (b), we first note that the assertion holds for $\mathbf{v}_N$ as defined in the proof of Theorem 4.4. This can be shown in essentially the same way as in [**11, 16**] for the three-dimensional case. Furthermore, the sequence $(\mathbf{v}_N)$ is Cauchy in $C^{1,\alpha}(\mathcal{B}_\pm)$ where $\mathcal{B}_- := \{\mathbf{x} \in \mathbf{R}^2 : 0 \le x_1 \le 4\pi, 0 < x_2 < f(x_1)\}$ and $\mathcal{B}_+ := \{\mathbf{x} \in \mathbf{R}^2 : 0 \le x_1 \le 4\pi, f(x_1) < x_2 < h\}$ with $h > \max f$. This can easily be proven with an argument similar to that employed in the proof of Lemma 4.3 together with Lemma 7.2 in [**1**]. The assertion then follows for Ω and $\overline{\mathbf{R}^2_+}\backslash\Omega$ because of the quasi-periodicity of $\mathbf{v}$.

Part (c) is shown in the same way as part (b). $\square$

5. Scattering by diffraction gratings. We are now finally able to derive the new integral equation formulation for the problem of plane elastic wave scattering by a diffraction grating. Throughout this section it will be assumed that $a = \mu(\lambda+\mu)/(\lambda+3\mu)$ and $b = (\lambda+2\mu)(\lambda+\mu)/(\lambda+3\mu)$.

In the following, we will only consider plane incident waves of the form

$$\mathbf{u}^{inc}(\mathbf{x}) = \hat{\theta} e^{ik_p \mathbf{x}\cdot\hat{\theta}},$$

i.e., incident compressional waves. However, this is done only for technical simplicity and it is shown in [**1**] that all arguments can easily be generalized to hold for incident shear waves and incident plane waves of the general form (4).

We will thus consider the scattering problem:

$$\begin{aligned} \Delta^* \mathbf{u} + \omega^2 \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= -\mathbf{u}^{inc} \quad \text{on } S. \end{aligned} \tag{13}$$

To ensure uniqueness of solution to (13), a radiation condition has to be imposed. The condition we will use is similar to the one employed in solving diffraction grating problems for acoustic waves [**14**] and has been used successfully for elastic wave scattering in [**1, 9, 18**].

Definition 5.1. A bounded vector field $\mathbf{u} : \Omega \to \mathbf{C}^2$, quasi-periodic with phase-shift α, is called *radiating* if, for $x_2 > \max f$ it has an expansion of the form

$$\mathbf{u}(\mathbf{x}) = \sum_{n\in\mathbf{Z}} \left\{ u_{p,n} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} e^{i(\alpha_n x_1 + \beta_n x_2)} + u_{s,n} \begin{pmatrix} \gamma_n \\ -\alpha_n \end{pmatrix} e^{i(\alpha_n x_1 + \gamma_n x_2)} \right\},$$

where $u_{p,n},\ u_{s,n} \in \mathbf{C}$, $n \in \mathbf{Z}$, $\alpha_n := \alpha + n$,

$$\beta_n := \begin{cases} \sqrt{k_p^2 - \alpha_n^2}, & \alpha_n^2 \le k_p^2 \\ i\sqrt{\alpha_n^2 - k_p^2}, & \alpha_n^2 > k_p^2, \end{cases} \qquad \gamma_n := \begin{cases} \sqrt{k_s^2 - \alpha_n^2}, & \alpha_n^2 \le k_s^2 \\ i\sqrt{\alpha_n^2 - k_s^2}, & \alpha_n^2 > k_s^2. \end{cases}$$

The space of admissible solutions to the scattering problem will be denoted by

$$QR(\alpha) := \left\{ \mathbf{u} \in C^2(\Omega) \cap C(\overline{\Omega}) : \begin{array}{l} \mathbf{u} \text{ is quasi-periodic with phase-shift } \alpha, \\ \text{bounded, radiating and solves (3)} \end{array} \right\}.$$

Remark 5.2. It is shown in [**1**] that the scattering problem (13) has at most one solution in the set $QR(k_p\hat{\theta}_1)$.

Remark 5.3. Using Lemma 4.3 the methods employed in the proof of Theorem 7.1 in [**1**] can be used together with Theorems 2.1, 2.4, 2.5 and 3.1 to prove that the layer potentials $\mathbf{v}$ and $\mathbf{w}$ are elements of $QR(\alpha)$ provided the density ϕ is quasi-periodic with phase-shift α.

We now make an ansatz for the solution $\mathbf{u}$ to the scattering problem which was first proposed by Brakhage and Werner [**2**]:

$$\mathbf{u}(x) = \int_\Lambda \Pi^{(2)}_{Dp}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y}) - i\eta \int_\Lambda \Gamma_{Dp}(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, ds(\mathbf{y}) \tag{14}$$

where $\phi \in C(S)$ is a quasi-periodic density with phase-shift $k_p\hat{\theta}_1$ and η a complex number satisfying $\mathrm{Re}\,(\eta) \ne 0$. From Remark 5.3 it follows that $\mathbf{u} \in QR(k_p\hat{\theta}_1)$. From Theorems 4.4 and 4.5 we thus conclude that $\mathbf{u}$ is a solution to the scattering problem (13) if ϕ satisfies the integral equation

$$\frac{1}{2}\phi(\mathbf{x}) + \int_\Lambda (\Pi^{(2)}_{Dp}(\mathbf{x},\mathbf{y}) - i\eta\Gamma_{Dp}(\mathbf{x},\mathbf{y}))\phi(\mathbf{y})\, ds(\mathbf{y}) = -\mathbf{u}^{inc}(\mathbf{x}) \tag{15}$$

on Λ.

Theorem 5.4. *The integral equation* (15) *has a unique quasi-periodic solution* ϕ *with phase-shift* $k_p\hat{\theta}_1$.

Proof. The method of proof follows closely along the lines of the existence proof in [**1**]. Let $C_{2\pi}$ denote the space of 2π-periodic continuous functions. For $\psi \in C_{2\pi}$, we define the integral operators $S_{\hat{\theta}_1}, K_{\hat{\theta}_1}$ and $K'_{\hat{\theta}_1}$ as

$$\begin{aligned} S_{\hat{\theta}_1}\psi(t) &:= 2\int_0^{2\pi} e^{ik_p\hat{\theta}_1(s-t)}\Gamma_{Dp}(t, f(t), s, f(s)) \\ &\qquad\qquad \cdot \sqrt{1+f'(s)^2}\psi(s)\,ds, \\ K_{\hat{\theta}_1}\psi(t) &:= 2\int_0^{2\pi} e^{ik_p\hat{\theta}_1(s-t)}\Pi^{(2)}_{Dp}(t, f(t), s, f(s)) \\ &\qquad\qquad \cdot \sqrt{1+f'(s)^2}\psi(s)\,ds, \\ K'_{\hat{\theta}_1}\psi(t) &:= 2\int_0^{2\pi} e^{ik_p\hat{\theta}_1(s-t)}\Pi^{(1)}_{Dp}(t, f(t), s, f(s)) \\ &\qquad\qquad \cdot \sqrt{1+f'(s)^2}\psi(s)\,ds, \end{aligned}$$

with $t \in [0, 2\pi]$. As was remarked earlier, the kernels of these integral operators are weakly singular and the integral operators themselves are thus seen to be compact in $C_{2\pi}$. Furthermore, $S_{\hat{\theta}_1}$ and $S_{-\hat{\theta}_1}$ are adjoint with respect to the duality

$$\langle \phi, \psi \rangle := \int_0^{2\pi} \phi \cdot \psi \sqrt{1+f'(t)^2}\,dt,$$

as are $K_{\hat{\theta}_1}$ and $K'_{-\hat{\theta}_1}$.

By multiplying (15) by $e^{-ik_p\hat{\theta}_1 x_1}$ and setting $\psi(x_1) := e^{-ik_p\hat{\theta}_1 x_1}$ $\phi(x_1, f(x_1))$ (note that $\psi \in C_{2\pi}$), we obtain the equivalent equation

$$\begin{aligned} \psi(t) + K_{\hat{\theta}_1}\psi(t) - i\eta S_{\hat{\theta}_1}\psi(t) &= -2e^{-ik_p\hat{\theta}_1 t}\mathbf{u}^{inc}(t, f(t)), \\ & t \in [0, 2\pi]. \end{aligned} \tag{16}$$

The Fredholm alternative can be applied to (16). In fact, we will complete the proof by showing that the equation

$$\psi + K'_{-\hat{\theta}_1}\psi - i\eta S_{-\hat{\theta}_1}\psi = 0 \tag{17}$$

only admits the trivial solution.

Assume $\psi_0 \in C_{2\pi}$ to be a solution to (17). Setting $\phi_0(\mathbf{x}) := e^{-ik_p\hat{\theta}_1 x_1}\psi_0(x_1)$ and

$$\mathbf{v}(\mathbf{x}) := \int_\Lambda \Gamma_{Dp}(\mathbf{x},\mathbf{y})\phi_0(\mathbf{y})\,ds(\mathbf{y}), \quad \mathbf{x} \in \mathbf{R}^2_+ \backslash S,$$

we know from Theorem 4.4 and Remark 5.3 that $\mathbf{v}$ is continuous in $\overline{\mathbf{R}^2_+}$, a solution to the Navier equation in both Ω and $\mathbf{R}^2_+\backslash\overline{\Omega}$ and radiating in Ω. From Theorem 4.5 (c) and (17) we further conclude $\phi_0 \in C^{0,\alpha}(S)$. Thus, by Theorem 4.5 (b),

$$2e^{ik_p\hat{\theta}_1 x_1}\mathbf{N}\mathbf{v}_- = \psi_0 + K'_{-\hat{\theta}_1}\psi_0$$

and from (17) we conclude $\mathbf{N}\mathbf{v}_- - i\eta\mathbf{v}_- = 0$ on S. On the other hand, $\mathbf{v}_- = 0$ holds on $\{x_2 = 0\}$ because of the properties of the Green's tensor. Thus, setting $\Lambda_0 := \{\mathbf{x} \in \mathbf{R}^2 : 0 < x_1 < 2\pi, x_2 = 0\}$, we have

$$\begin{aligned}\operatorname{Re}(\eta)\int_\Lambda |\mathbf{v}_-|^2\,ds &= -\frac{1}{2i}\int_\Lambda \mathbf{v}_-\cdot\mathbf{N}\bar{\mathbf{v}}_- - \bar{\mathbf{v}}_-\cdot\mathbf{N}\mathbf{v}_-\,ds\\ &= \frac{1}{2i}\int_{\Lambda_0} \mathbf{v}_-\cdot\mathbf{N}\bar{\mathbf{v}}_- - \bar{\mathbf{v}}_-\cdot\mathbf{N}\mathbf{v}_-\,ds\\ &= 0\end{aligned}$$

by the third generalized Betti formula (10) as the integrals over $\{x_1 = 0\}$ and $\{x_1 = 2\pi\}$ cancel out because of the quasi-periodicity of $\mathbf{v}_-$. Thus $\mathbf{N}\mathbf{v}_- = \mathbf{v}_- = 0$ on Λ. Consequently, as $\mathbf{v}$ is continuous, $\mathbf{v}_+ = 0$ also holds on S. From this, recalling Remark 5.2, we conclude that $\mathbf{v}_+ \equiv 0$ in Ω. Thus, $\mathbf{N}\mathbf{v}_+ = 0$ on S; consequently, $\phi_0 = \mathbf{N}\mathbf{v}_- - \mathbf{N}\mathbf{v}_+ = 0$ and therefore also $\psi_0 = 0$. □

Thus the unique solvability of the scattering problem for all combinations of wave-numbers and angle of incidence is proved. Recalling that this result can easily be extended to general incident plane waves, we state our final theorem.

Theorem 5.5. *Given an incident plane wave of the form*

$$\mathbf{u}^{inc}(x) = a_p\hat{\theta}e^{ik_p\mathbf{x}\cdot\hat{\theta}} + a_s\theta^\perp e^{ik_s\mathbf{x}\cdot\hat{\theta}}, \quad a_p, a_s \in \mathbf{C},$$

the scattering problem

$$\mu\Delta\mathbf{u} + (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u} + \omega^2\mathbf{u} = 0 \quad \textit{in } \Omega,$$
$$\mathbf{u} + \mathbf{u}^{inc} = 0 \quad \textit{on } S$$

has a unique solution in the space $QR(k_p\hat{\theta}_1) + QR(k_s\hat{\theta}_1)$.

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