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NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS— STABILITY AND NUMERICAL STABILITY OF θ-METHODS

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Dedicated to P.M. Anselone

ABSTRACT. In this work we consider equations of the form

$$(\dagger) \qquad y'(t) = -\int_0^t k(t-s)g(y(s))\,ds, \quad t \in \mathbf{R}^+,$$

and corresponding discretized equations of the form

$$(\ddagger) \qquad y_{n+1} - y_n = -h^2 \sum_{j=0}^{n+1} w_j^{(n+1)} k_{n+1-j} g(y_j), \quad j \in \mathbf{N}.$$

Levin and Nohel gave an analysis of the qualitative behavior of solutions to (†) by means of methods based on deriving a Lyapunov function for the solution. We analyze the qualitative behavior of solutions to (‡), basing our analysis on the earlier work by Levin and Nohel. We give a theorem on the qualitative behavior of solutions to (‡) and we are able to extend the analysis of both the continuous and discrete equations to a wider class of equations. We consider what conditions it would be natural to impose on the numerical method to guarantee that the qualitative behavior of solutions of (†) will be preserved in the solutions of the discrete scheme. We give a theorem in which we show that, under additional conditions on g and k, the qualitative behavior of solutions may be preserved in the discrete case, and we conclude with some numerical examples to illustrate our analytical results and demonstrate that a complete discrete analogue of the theory developed for (†) requires further investigation.

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1. Background. The stability and asymptotic stability of solutions to the types of problems we consider in this paper have been analyzed using two main approaches. Frequency domain methods were introduced by Popov, see [**9, 25, 23**]. These methods may be applied directly to analyze integral equations and integro-differential equations of Volterra type. Lyapunov's second method, see, for example, [**14, 21, 26**], is often used as a tool for the stability analysis and the prediction of qualitative behavior of solutions of differential equations. The method of Lyapunov may be applied to integro-differential equations [**2, 15, 23**]. Some authors, see, for example, [**9**], consider hybrid approaches that combine Lyapunov methods with frequency domain methods.

In this paper we have chosen to concentrate on Lyapunov approaches to the stability analysis. Frequency domain and hybrid methods will be considered in a sequel.

In their paper [**17**], Nohel and Levin applied a Lyapunov stability analysis to the Volterra integro-differential equation

$$y'(t) = -\int_0^t k(t-s)g(y(s))\,ds, \quad t \in \mathbf{R}^+. \tag{1.1}$$

The Lyapunov direct method has also been adapted for solutions of difference equations, see [**16**], and the work has recently been extended [**3, 11**] to include difference equations of Volterra type (and of unbounded order) such as, for example,

$$\begin{aligned} y_{n+1} - y_n &= -h^2 \sum_{j=0}^{n+1} w_j^{(n+1)} k_{n+1-j} g(y_j), \\ \sum_{j=0}^{n+1} w_j^{(n+1)} &= n+1, \quad n \in \mathbf{N}. \end{aligned} \tag{1.2}$$

Throughout our paper we use the notation $k_j = k(jh)$ where h is the (fixed) step length for our discrete scheme. We shall use the quadrature rules known as the θ methods. These methods include the implicit Euler formula for which

$$\{w_0^{(n)}, w_1^{(n)}, \dots, w_{n-1}^{(n)}, w_n^{(n)}\} = \{0, 1, \dots, 1\}, \tag{1.3}$$

the explicit Euler formula

$$\{w_0^{(n)}, w_1^{(n)}, \dots, w_{n-1}^{(n)}, w_n^{(n)}\} = \{1, \dots, 1, 0\} \tag{1.4}$$

and the trapezium rule for which

$$\{w_0^{(n)}, w_1^{(n)}, \dots, w_{n-1}^{(n)}, w_n^{(n)}\} = \{0.5, 1, \dots, 1, 0.5\}. \tag{1.5}$$

A general θ-rule has the form $\{w_0^{(n)}, w_1^{(n)}, \dots, w_{n-1}^{(n)}, w_n^{(n)}\} = \{\theta, 1, \dots, 1, 1-\theta\}$.

As one can easily observe, methods of this type have weights that are *almost constant*. This property was used in recent work by Vecchio [**29**] to enable the analysis of certain linear systems of Volterra integro-differential equations. We remark that the use of methods with repetition factor one can yield a similar analysis, see, for example, [**7**]. The methods we describe here have been widely studied in other contexts. We direct the interested reader to the references [**1, 10, 19, 22** and **28**].

Further recent analysis has considered some applications of the results in the paper [**3**] to numerical methods, see [**4**]. Our work, in this paper, is rather different from the other recent papers, since we aim to mimic existing analysis for the equation (1.1) rather than to develop an independent analysis of equation (1.2). A distinctive feature of our paper, when compared to the earlier work, is that we give concrete results for a class of nonlinear problems. An alternative approach to the work presented in this paper, based on the general method of Lyapunov functional construction, is described in the report [**8**].

Lyapunov-type methods are attractive, particularly for the analysis of nonlinear problems such as (1.1) and (1.2), because in principle they allow the investigation of stability concepts *directly* without the need to find an expression for the solution.

2. Lyapunov functions and a stability result for (1.1).

In [**17**] the stability of the equation (1.1) is discussed essentially using Lyapunov's second method. We shall describe their discussion here to set the scene for our investigation of the stability of (1.2).

Consider the equation

$$y'(t) = -\int_0^t k(t-s)g(y(s))\,ds,$$

subject to the following conditions:

H1. $k \in L^1[0, \infty)$ is completely monotone.

H2. $g(x) \in C(-\infty, \infty)$, $xg(x) > 0$, $x \neq 0$, and hence x and $g(x)$ always have the same sign and $g(0) = 0$.

H3. $G(x) := \int_0^x g(\xi)\, d\xi \to \infty$ as $|x| \to \infty$.

Levin and Nohel [**17**] give the following theorem.

Theorem 2.1 (Levin and Nohel). *Any solution $u(t)$ of* (1.1) *subject to* H1, H2, H3 *satisfies $u(t) \to 0$ as $t \to \infty$ providing the L^1-function $k(t)$ is non-null.*

Remarks. 1. If $k(t) \equiv 0$, then $u(t)$ is constant and therefore stable. If $k(t) \equiv k(0) \neq 0$, then the solution $u(t)$ may be stable but not asymptotically stable. For example, the equation $y'(t) = -\int_0^t y(s)\, ds$ is equivalent to the equation of simple harmonic motion $y''(t) = -y(t)$, whose solution is stable but not asymptotically stable.

2. The completely monotone kernel $k \in L^1$ has *fading* memory. It follows that $y'(t) = 0$ can only be satisfied as $t \to \infty$ if $\lim_{s\to\infty} g(y(s)) = 0$. This situation can arise only if $y(s) \to 0$ as $s \to \infty$. The ω-limit set of y thus consists of the single value zero.

3. While it may be convenient to assume that k is completely monotone, in fact the results we give can be shown to depend only on the assumption that $(-1)^j k^{(j)}(t) \geq 0$ for $j = 0, 1, 2$ and k is not constant. (This implies that k is a function of *strong positive type*, see [**24**].)

4. The assumption that k is differentiable guarantees that the evaluation (used later) of the sequence $k_n = k(nh)$ for fixed $h > 0$ is well-defined.

5. In this paper, stability is defined with respect to perturbations in the initial value $y(0)$. Theorem 2.1 implies that *every* solution to (1.1) is asymptotically stable and that perturbations in the initial value may be arbitrarily large. Our examples in the final section of this paper illustrate that this property is certainly not preserved in some of our discrete schemes.

6. The analysis of Levin and Nohel extends beyond the use of a

Lyapunov function. For the purposes of this paper, we will derive a discrete version of the following preliminary result, which is also contained in the paper [**17**].

Theorem 2.2 (Levin and Nohel [**17**]). *The zero solution of* (1.1) *subject to* H1, H2 *and* H3 *is asymptotically stable providing that* $k(t) \in L^1$ *is non-null.*

Proof. For comparison with the discrete version, we reproduce the main stages in the proof given by [**17**]. The proof is based around the derivation of a Lyapunov function V for a solution of (1.1) satisfying $V(t) \geq 0$, $V'(t) < 0$. The function

$$V(t) = G(u(t)) + \frac{1}{2}\int_0^t \int_0^t k(\tau + s) g(u(t-\tau)) g(u(t-s))\, d\tau\, ds,$$

is shown to be such a Lyapunov function. First we show that V is positive. This follows since, by condition (2) above, $G(u(t))$ is positive and $K(\tau, s) = k(\tau + s)$ is a kernel of positive type since k is completely monotone and a fortiori of positive type. The second term in the expression for V is therefore nonnegative.

To complete the proof, we show that dV/dt is negative.

We observe that an equivalent expression for V is

$$V(t) = G(u(t)) + \frac{1}{2}\int_0^t \int_0^t k(2t - \tau - s) g(u(\tau)) g(u(s))\, d\tau\, ds.$$

It follows from this observation that

$$\begin{aligned} V'(t) = G'(u(t))u'(t) &+ \int_0^t \int_0^t k'(2t - \tau - s) g(u(\tau)) g(u(s))\, d\tau\, ds \\ &+ \int_0^t k(t-s) g(u(t)) g(u(s))\, ds. \end{aligned}$$

Now

$$G'(u(t))u'(t) = -g(u(t)) \int_0^t k(t-s) g(u(s))\, ds \tag{2.1}$$

by (1.1). Hence, by substitution,

$$V'(t) = \int_0^t \int_0^t k'(\tau + s) g(u(t-\tau)) g(u(t-s))\, d\tau\, ds.$$

Complete monotonicity of k implies that the map $(\tau, s) \to -k'(\tau + s)$ is of positive type. Hence $V'(t) \le 0$ with equality only for $t = 0$. $\square$

The work considered so far has been concerned with the equation (1.1), which has no forcing term. Equations of the form

$$y'(t) + \int_0^t k(t-s) g(y(s))\, ds = f(t, y(t)), \quad t \in \mathbf{R}^+ \tag{2.2}$$

can also be analyzed. Indeed, Levin and Nohel give an analysis of equations of this form in the paper [**18**]. We remark that the analysis given in Theorem 2.2 can be extended simply to give a corresponding result for (2.2).

Corollary 2.3. *The conclusions of Theorem* 2.1 *are also valid for the equation* (2.2) *subject to the additional condition that* $f(t, \xi)$ *satisfies* $f(t, 0) = 0$ *and* $\xi f(t, \xi) \le 0$ *whenever* $\xi \ne 0$.

Proof. Corollary 2.3 follows by the same argument as in the proof of Theorem 2.2. We adapt the expression in (2.1)

$$G'(u(t)) u'(t) = g(u(t)) \left(f(t, u(t)) - \int_0^t k(t-s) g(u(s))\, ds \right). \tag{2.3}$$

It follows that the expression for $V'(t)$ includes the additional term $g(u(t)) f(t, u(t))$.

Since $\xi f(t, \xi) \le 0$ and x and $g(x)$ always have the same sign with $g(0) = 0$, we can conclude that $g(u(t)) f(t, u(t)) \le 0$. The final conclusion then follows as before.

3. Discretization of equations of the form (1.1). A natural approach to the numerical solution of equations of the form (1.1) would be the combination of a differential equation method with a quadrature rule for the integral. We consider a simple approach of this type. Further discussion for a variety of approaches can be found, for example, in [**1**].

With a θ-rule as a quadrature method, we analyze

$$y_{n+1} - y_n = -h^2 \sum_{j=0}^{n+1} w_j^{(n+1)} k_{n+1-j} g(y_j), \quad w_j^{(n)} \geq 0, \tag{3.1}$$

where $\{w_0^{(n)}, w_1^{(n)}, \ldots, w_{n-1}^{(n)}, w_n^{(n)}\} = \{\theta, 1, \ldots, 1, 1-\theta\}$, $y_0 = y(0)$.

Following on from Corollary 2.3, we remark here that our analysis can also be applied to equations of the form

$$y_{n+1} - y_n = -h^2 \sum_{j=0}^{n+1} w_j^{(n+1)} k_{n+1-j} g(y_j) + f(n, y_n). \tag{3.2}$$

4. Lyapunov functions and theorems on qualitative behavior for the discrete equation. The construction of a Lyapunov function for the discrete equation is less straightforward than one might hope. In particular, we need to be very careful that the function $V(n, \{y_j\})$ that we define does not depend on future values of the solution sequence $\{y_j\}$. In other words, we require that $V(n, \{y_j\})$ does not depend on any y_i with $i > n$. As we shall see, it proves to be impossible to give a complete discrete analogue of the continuous theory.

To begin with, we give a proof of asymptotic stability of the zero solution of equation (3.1) when only positive perturbations are permitted. This may seem rather restrictive, but nevertheless can be a useful result when the function y has a particular physical or biological meaning which implies that only nonnegative values of y_n are possible. (This situation covers, for example, models of population size or of concentration of a drug in the bloodstream.) Furthermore, it is our conjecture (supported by experimental evidence) that, for a wide class of kernels k and for suitable choice of initial value y_0 and $h > 0$, $y_n \geq 0$ for every n.

We require a discrete Lyapunov theorem of the type introduced in the recent papers [**3, 4**].

Theorem 4.1. *Let the sequence $\{y_n\}$ be a solution of a discrete Volterra equation, and let $V_i(y_0, y_1, \dots, y_i)$ be, for each natural number i, a scalar function continuous with respect to all its arguments, which satisfies*:

1. $V_0(0) = 0$,

2. $V_i(y_0, y_1, \dots, y_i) \geq \omega_i(\|y_i\|)$,

3. $\Delta V_i = V_{i+1}(y_0, y_1, \dots, y_i, y_{i+1}) - V_i(y_0, y_1, \dots, y_i) \leq 0$, *then the given solution of the equation is stable.*

4. *If, in addition, $\Delta V_i \leq -\omega_2(\|y_i\|)$, then the solution of the equation is asymptotically stable.*

Here the functions ω_i are assumed to be scalar increasing functions that satisfy $\omega_i(0) = 0$.

We can now give our first theorem on qualitative behavior of solutions to (3.1). The proof of the theorem is followed by an example and some remarks.

Theorem 4.2. *For the equation* (3.1), *we make the following assumptions*:

H4. *For each natural number n, the matrix $A(n) := (A(n)_{i,j} = k_{2n-i-j})$, of order n, is a positive definite matrix and that the matrix $A^{\ddagger}(n) := (A^{\ddagger}(n)_{i,j} = k_{2n+2-i-j} - k_{2n-i-j})$, of order n, is a negative semi-definite matrix.*

H5. *The function $g(u)$ satisfies conditions* H2 *and* H3 *of Theorem* 2.1 *and is also nondecreasing.*

H6. *The solution values satisfy $y_j \geq 0$ for each $j \geq 0$.*

H7. *The weights $w_j^{(n)}$ are given by a θ-method with $0 \leq \theta \leq 1/2$. Thus we insist that the θ-method is A-stable.*

Then, for every $\varepsilon > 0$ there is a corresponding $\delta_\varepsilon > 0$ and a natural number N_ε for which $|y_0| < \delta_\varepsilon$ implies $|y_n| < \varepsilon$ for each $n > N_\varepsilon$. If, in addition to H4–H7 *above, $A^{\ddagger}(n)$ is, for each n, a negative definite matrix, then $y_n \to 0$ as $n \to \infty$.*

In other words, the stability, respectively asymptotic stability, of the zero solution (*subject to* H6) *is preserved under discretization by a θ-*

method with $0 \le \theta \le 1/2$ *provided the admissible perturbations produce a solution satisfying* H6.

Proof. Following a similar approach to the one used in the proof of Theorem 2.2, we shall exhibit a Lyapunov function, this time for the sequence $\{y_n\}$ which satisfies the equation (3.1). The conclusions given in the statement of Theorem 4.2 then follow by Theorem 4.1.

Define

$$V(n,\{y_j\}_0^n) := \frac{h^2}{2}\sum_{j=0}^{n}\sum_{i=0}^{n} w_i^{(n)} w_j^{(n)} k_{2n-i-j}\, g(y_i) g(y_j) + G(n,\{y_j\})$$

where $G(n,\{y_j\}_0^n)(\ge 0)$, $G(n,0)=0$ will be defined later. As in the previous proof, we shall show that $V(n,\{y_j\})$ defined in this way has the properties required of a Lyapunov function for $\{y_j\}$.

Clearly, $V(n,0)=0$ and $V(n,\{y_j\})\ge 0$ because, by hypothesis, $A(n)$ is a positive definite matrix for each n and $G(n,\{y_j\})\ge 0$, $G(n,0)=0$.

Next we demonstrate that $V(n+1,\{y_j\}) - V(n,\{y_j\}) \le 0$.

$$\begin{aligned}
V(n+1,&\{y_j\}) - V(n,\{y_j\})\\
&= \frac{h^2}{2}\sum_{j=0}^{n}\sum_{i=0}^{n}\Big(w_i^{(n+1)} w_j^{(n+1)} k_{2n+2-i-j}\, g(y_i) g(y_j)\\
&\qquad - w_i^{(n)} w_j^{(n)} k_{2n-i-j}\, g(y_i) g(y_j)\Big)\\
&\quad + h^2(1-\theta) g(y_{n+1}) \sum_{j=0}^{n} w_j^{(n+1)} k_{n+1-j}\, g(y_j)\\
&\quad + \frac{h^2}{2}(1-\theta)^2 k_0\, g(y_{n+1})^2\\
&\quad + G(n+1,\{y_j\}) - G(n,\{y_j\}).
\end{aligned}$$

Now define

$$\begin{aligned}
G(n,\{y_j\}) &:= \sum_{j=1}^{n} w_j^{(n)} g(y_j)(y_j - y_{j-1}) + y_0 M,\\
G(0,\{y_j\}) &= y_0 M
\end{aligned}$$

where M is some positive constant chosen to make $G(n, \{y_j\}) \geq 0$. For example, with our hypotheses, we can choose $M = \max_{t\in[0,y_0]} g(t) = g(y_0)$. Now

$$\begin{aligned} G(n+1,\{y_j\}) - G(n,\{y_j\}) \\ &= (1-\theta)g(y_{n+1})(y_{n+1}-y_n) + \theta g(y_n)(y_n - y_{n-1}) \\ &= -h^2(1-\theta)g(y_{n+1})\sum_{j=0}^{n+1} w_j^{(n+1)} k_{n+1-j} g(y_j) \\ &\quad - h^2\theta g(y_n)\sum_{j=0}^{n} w_j^{(n)} k_{n-j} g(y_j). \end{aligned}$$

It follows, taking into account the change in the weight index from $w_i^{(n+1)}$ to $w_i^{(n)}$, that

$$\begin{aligned} V(n+1,\{y_j\}) - V(n,\{y_j\}) \\ &= \frac{h^2}{2}\bigg(\sum_{j=0}^{n}\sum_{i=0}^{n} w_i^{(n)} w_j^{(n)} (k_{2n+2-i-j} - k_{2n-i-j}) g(y_i) g(y_j)\bigg) \\ &\quad + h^2\theta g(y_n)\sum_{j=0}^{n} w_j^{(n)} (k_{n+2-j} - k_{n-j}) g(y_j) \\ &\quad + \frac{h^2}{2}\theta^2 k_2 g(y_n)^2 - \frac{h^2}{2}(1-\theta)^2 k_0 g(y_{n+1})^2. \end{aligned}$$

By hypothesis, the matrix of order $n+1$ with (i,j) entry $(k_{2n+2-i-j} - k_{2n-i-j})$ is negative semi-definite, and so the first two terms in the righthand side of this expression are less than or equal to 0. The condition $0 \leq \theta \leq 1/2$ combines with the observation that $k_2 - k_0 \leq 0$ and the fact that g is nondecreasing to yield the result that $V(n+1,\{y_j\}) - V(n,\{y_j\}) \leq 0$, as required. Moreover, if $(k_{2n+2-i-j} - k_{2n-i-j})$ is negative definite, then it follows that $V(n+1,\{y_j\}) - V(n,\{y_j\}) < (h^2/2)(k_2\theta^2 g(y_n)^2 - k_0(1-\theta)^2 g(y_{n+1})^2) < 0$. The conclusions of the theorem follow from Theorem 4.1 by choosing the function $\omega(s)$ to be an increasing function on the interval $[0, y_0]$ bounded above by $(h^2/2)(k_0 - k_2)\theta^2 g(s)^2$. $\square$

Example. In Section 5 we will consider the example equation

$$y'(t) = -\int_0^t e^{-\lambda(t-s)}[y(s)]^3\, ds$$

which satisfies conditions H1, H2 and H3, using θ-methods with $\theta = 0, 1/2, 1$. For $\theta = 0, 1/2$ the discrete equation satisfies H4, H5 and H7, and we will see that, for small enough $y_0 > 0$, H6 is also satisfied. For $\theta = 1$, the hypothesis H7 is violated and we are able to make some interesting observations about the numerical solution in this case.

Remarks. 1. It is possible to undertake a similar analysis and to reach a similar conclusion to that given in Theorem 4.2 by a direct argument and without recourse to a discrete Lyapunov function. The direct argument is based on showing that the sequence $\{y_n\}$ is a decreasing sequence of nonnegative values. However, it is in the spirit of this paper to proceed by Lyapunov methods.

2. The above analysis also applies to perturbations of the zero solution that are restricted to taking negative values. This follows since $xg(x) > 0$ for all nonzero x.

3. The sequence $\{k_j\}$ we have considered has fading memory, since it is in l^1 and is completely monotone. It follows from (3.1)that

(a) the only possible limit, λ, of the sequence $\{y_j\}$ must satisfy $g(\lambda) = 0$ (and so $\lambda = 0$),

(b) if $y_n \geq 0$ for $n \geq N$, then there is a $J \geq N$ for which $y_{j+1} - y_j \leq 0$ for every $j \geq J$,

(c) if $y_n \leq 0$ for $n \geq N$, then there is a $J \geq N$ for which $y_{j+1} - y_j \geq 0$ for every $j \geq J$.

4. In either of the last two cases, we can easily construct a Lyapunov function for $\{y_n\}$ as we did in our proof of Theorem 4.2. The only addition to the analysis we gave is that the constant M must be changed to ensure that $G > 0$.

5. The above remarks demonstrate that the zero solution of (3.1) is stable (asymptotically stable) under the conditions H4, H5 and H7 among the class of perturbed solutions that exhibit only finitely many changes of sign. In addition, we remark that the existence of a finite value of $M > 0$ for which $G > 0$ is sufficient (but not necessary) for the conclusions of Theorem 4.2.

We summarize these remarks in the following theorem, whose proof is identical to the proof of Theorem 4.2 apart from the choice of constant M.

Theorem 4.3. *For the equation* (3.1)*, assume* H4, H5, H7 *are satisfied as in Theorem* 4.2*. Let* y_0 *be given. Then either*

(i) *the sequence* y_n *exhibits infinitely many changes of sign, or*

(ii) *for every* $\varepsilon > 0$ *there is a corresponding* $\delta_\varepsilon > 0$ *and a natural number* N_ε *for which* $|y_0| < \delta_\varepsilon$ *implies* $|y_n| < \varepsilon$ *for each* $n > N_\varepsilon$.

If, in addition to H4, H5 *and* H7 *above,* $A^\ddagger(n)$ *is a negative definite matrix, then either* y_n *changes sign infinitely often or* $y_n \to 0$ *as* $n \to \infty$.

Note. All of the above analysis can be repeated with hardly any additional work for equations of the form (3.2), under the conditions $f(t, 0) = 0$, $\xi f(t, \xi) < 0$. We obtain the following corollary.

Corollary 4.4. *Under the additional hypothesis that the function* $f(t, \xi)$ *satisfies* $f(t, 0) = 0$ *and* $\xi f(t, \xi) \leq 0$ *for* $\xi \neq 0$*, the conclusions of Theorem* 4.3 *also apply to the equation* (3.2).

It remains to consider whether the possibility of persistent oscillatory solutions admitted by Theorem 4.3 can arise in practice. Based on experimental evidence we believe that, for each choice of θ in $[0, 1]$ and for suitable choice of kernel, k, and step length, h, one can choose y_0 sufficiently small that $y_n \geq 0$ for every $n \geq 1$. In other words, we believe that hypothesis H6 will apply in practical situations so long as we choose a sufficiently small value for y_0 (depending on the chosen step length h). The proof is technical and complicated, and we have only had success with a few simple examples. We make the following observation. Whatever choice of k and θ, too large a combination of $h > 0$ and y_0 results in spurious oscillations in the solution. However, we have observed oscillations that are *persistent* only in the case of the explicit rule ($\theta = 1$). In the next section we consider the behavior of solutions of difference equations arising from a continuous equation that satisfies the conditions of Theorem 2.1. We show that, for different choices of $h > 0$ and for different initial perturbations from zero we can produce solutions to the discrete equation exhibiting several types of behavior, including the possibility of infinitely many changes of sign. It remains to be considered whether precise conditions can be imposed that limit the size of the initial perturbation from zero for fixed $h > 0$ which guarantee that the solution does not exhibit oscillatory behavior. In this context, we remark that, for $k(t) \equiv k(0)$, an oscillatory solution of the continuous problem does arise.

5. Conclusions and numerical experiments. In this section we consider the particular integro-differential equation

$$y'(t) = -\int_0^t e^{-\lambda(t-s)}[y(s)]^3\, ds. \tag{5.1}$$

For λ real and positive, this equation satisfies the conditions of Theorem 2.1 and Theorem 2.2. We can therefore conclude that the zero solution of (5.2) is asymptotically stable and that every solution satisfies $y(t) \to 0$ as $t \to \infty$ whatever initial value $y(0)$ we choose. Further, from our analysis, we can predict that the numerical solution will either oscillate infinitely many times or will satisfy $y_n \to 0$. Our conjecture in the previous section suggests that, for sufficiently small starting value y_0, the solution will satisfy $y_n \geq 0$.

We consider three discrete equations ($\theta = 0, 1/2, 1$) and compare the long term solutions obtained for different initial values y_0.

First we consider the implicit Euler rule ($\theta = 0$) which provides an implicit scheme:

$$y_{n+1} - y_n = -h^2 \sum_{j=1}^{n+1} e^{-\lambda(n+1-j)h}[y_j]^3, \quad n \geq 1. \tag{5.2}$$

Previous experience with other types of problems lead us to expect a highly stable scheme to result. (Indeed, one can derive a definite result on qualitative behavior of the solution in this case by the method of stability by first approximation, see, for example, [**16**].)

Second, we consider the use of the trapezium rule ($\theta = 1/2$) to provide an alternative implicit scheme:

$$y_{n+1} - y_n = -h^2 \left(\sum_{j=1}^{n} e^{-\lambda(n+1-j)h}[y_j]^3 + \frac{y_{n+1}^3 + y_0^3}{2} \right), \quad n \geq 1. \tag{5.3}$$

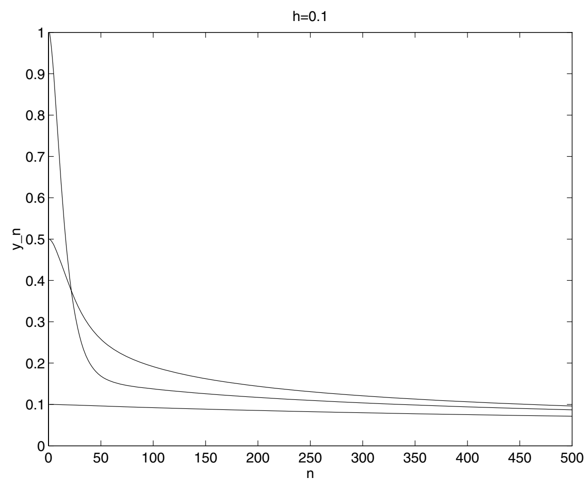

FIGURE 1. $\theta = 0$. With small initial value, the solution tends slowly to 0.

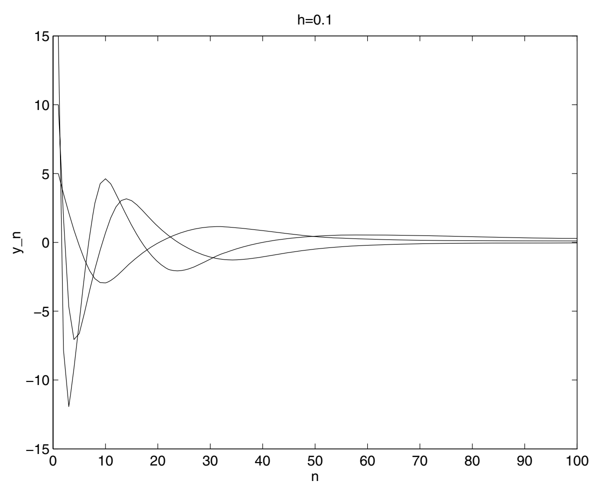

FIGURE 2. $\theta = 0$. With larger initial value, the solution tends to 0 after several oscillations.

Again we have reason to expect good stability behavior.

For our third scheme, we have used the explicit Euler rule ($\theta = 1$) for comparison:

$$y_{n+1} - y_n = -h^2 \sum_{j=0}^{n} e^{-\lambda(n-j)h}[y_j]^3, \quad n \geq 1. \tag{5.4}$$

Here we are using an explicit scheme for evaluating the sequence $\{y_n\}$. Previous experience leads us to suspect that the scheme may exhibit poor stability properties. Indeed, this scheme does not satisfy hypothesis H7.

Figures 1 and 2 show values of the solution y_n of (5.2) for fixed $h = 0.1$ when the initial value y_0 takes different values. We observe that, possibly after some initial oscillations, (according to the initial value y_0) $y_n \to 0$ as $n \to \infty$. The diagrams indicate that the zero solution to (5.2) is asymptotically stable for $h = 0.1$. In practice, when one has a priori knowledge that $y(t) \geq 0$, one would discard oscillatory solutions $\{y_n\}$ as unrealistic.

When we repeat the experiment for the discrete equations (5.3) we obtain similar results (Figures 3 and 4).

Finally, we give examples demonstrating where the use of an explicit rule leads to problems. For sufficiently small values of y_0, the solution of (5.4) satisfies $y_n \to 0$ as $n \to \infty$. However, we can observe that, for a particular choice of y_0, the numerical solution exhibits (spurious or unrealistic) persistent oscillations.

In our example, for each of the discrete schemes we have considered, when we choose our step length h sufficiently small, it appears from our calculations that, for sufficiently small perturbations of the initial value from zero, the solution tends to zero. For the two implicit schemes, we have seen no evidence that persistent oscillations do, in fact, arise. For the explicit scheme we considered, there is likely to be a relationship between the choice of h and the size of perturbation of the initial value from zero if persistent oscillations are to be avoided. However, more analysis is needed to predict a precise relationship between the numerical method, the step length and the stability of the zero solution for different choices of perturbation y_0.

We give this insight. We can seek (directly) solutions of equations (5.2), (5.3) and (5.4) that exhibit stable oscillations of period two. It

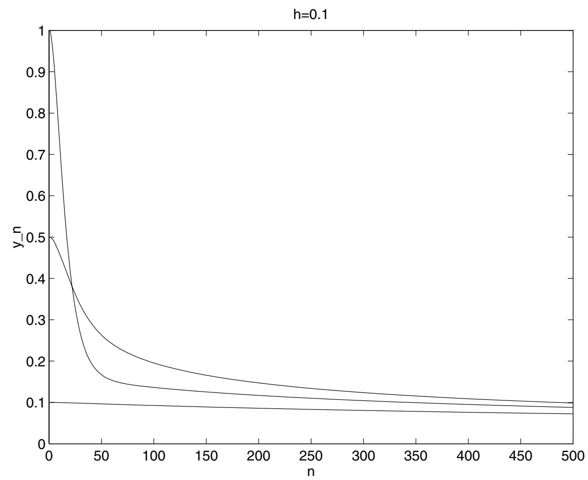

FIGURE 3. $\theta = 0.5$. With small initial value, the solution tends slowly to 0.

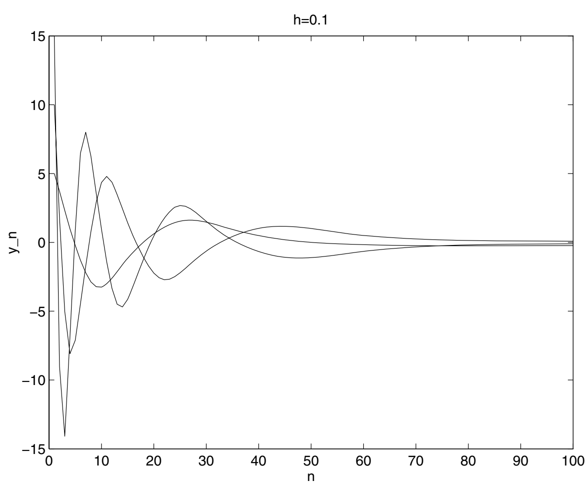

FIGURE 4. $\theta = 0.5$. With larger initial value, the solution tends to 0 after several oscillations.

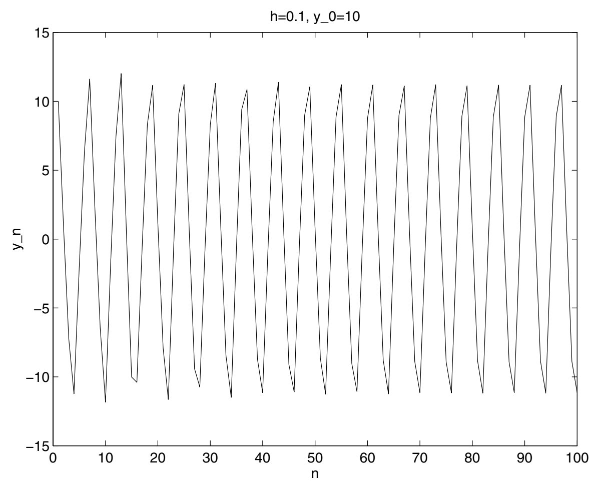

FIGURE 5. $\theta = 1$. A particular choice of y_0 yields persistent oscillations.

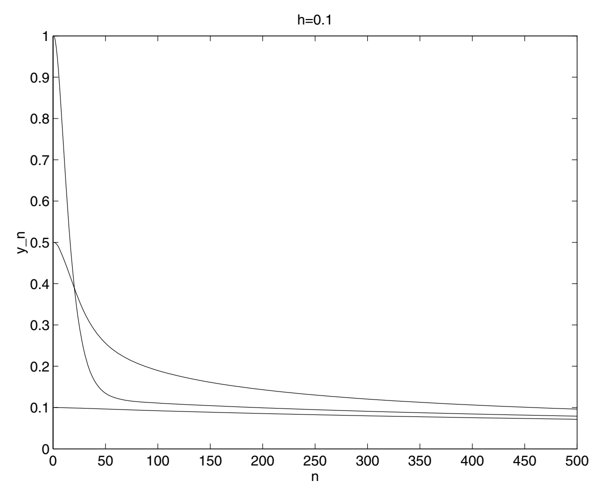

FIGURE 6. $\theta = 1$. Smaller choices of y_0 lead to solutions that tend to 0.

is easy to show that no solutions of this type arise for equations (5.2) and (5.3) whatever choice of initial value y_0 we make. However, for equation (5.4), solutions of this type do exist and the amplitude of the oscillations is

$$\frac{1}{h}(2+2e^{-\lambda h})^{1/2}.$$

For fixed h, λ, there is one possible amplitude of oscillatory solution of period 2. Persistent oscillations of period 2 can arise when the initial perturbation of the zero solution is precisely $(1/h)(2+2e^{-\lambda h})^{1/2}$. Further, as $h \to 0$, the size of the necessary perturbation approaches ∞. One can adopt a similar approach to calculate the amplitude of persistent oscillations of period greater than 2. The behavior we have observed is consistent with our expectation that, for sufficiently small perturbations of the zero solution (the size of perturbation depending on h) the solution will not exhibit persistent oscillations of this type even for the explicit scheme (5.4).

A further discussion of the relationship between the numerical method chosen, the step size $h > 0$ and the maximum size of the perturbation $y_0 > 0$ for which the solution satisfies $y_n \to 0$ as $n \to \infty$ will be the subject of a future report.

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