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THE COLLOCATION METHOD FOR SOLVING THE RADIOSITY EQUATION FOR UNOCCLUDED SURFACES

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This is dedicated to Professor Phil Anselone, a valued friend and colleague.

ABSTRACT. The *radiosity equation* occurs in computer graphics, and its solution leads to more realistic illumination for the display of surfaces. We consider the behavior of the radiosity integral operator, for smooth and piecewise smooth surfaces. A collocation method for solving the radiosity equation is proposed and analyzed. The method uses piecewise linear interpolation; and for one particular choice of such linear interpolation, it is shown that superconvergence results are obtained when solving on a smooth surface. Numerical results conclude the paper.

1. Introduction. The *radiosity equation* is a mathematical model for the brightness of a collection of one or more surfaces when their reflectivity and emissivity are given. The equation is

$$u(P) - \frac{\rho(P)}{\pi} \int_S u(Q)G(P,Q)V(P,Q)\, dS_Q = E(P), \qquad P \in S \tag{1}$$

with $u(P)$ the "brightness" or *radiosity* at P and $E(P)$ the *emissivity* at $P \in S$. The function $\rho(P)$ gives the *reflectivity* at $P \in S$, with $0 \le \rho(P) < 1$. In deriving this equation, reflections at every point are assumed to diffuse equally in all physically possible directions, that is the surface is a *Lambertian diffuse reflector*.

The function G is given by

$$\begin{aligned} G(P,Q) &= \frac{\cos\theta_P \cos\theta_Q}{|P-Q|^2} \\ &= \frac{[(Q-P)\cdot \mathbf{n}_P][(P-Q)\cdot \mathbf{n}_Q]}{|P-Q|^4}. \end{aligned} \tag{2}$$

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In this, $\mathbf{n}_P$ is the inner unit normal to S at P, and θ_P is the angle between $\mathbf{n}_P$ and $Q-P$; and $\mathbf{n}_Q$ and θ_Q are defined analogously. The function $V(P,Q)$ is a "line of sight" function. More precisely, if the points P and Q can "see each other" along a straight line segment which does not intersect S at any other point, then $V(P,Q)=1$; and otherwise, $V(P,Q)=0$. An *unoccluded* surface is one for which $V\equiv 1$ on S, and it is this case we investigate here. Note that S need not be connected, and it may be only piecewise smooth. The interior surface of a convex solid is unoccluded, but one can also be dealing with disconnected surfaces, as is illustrated with the two-piece surfaces used in some numerical examples in Section 5.

We often write (1) in the simpler form

$$u(P)-\int_S K(P,Q)u(Q)\,dS_Q = E(P), \quad P\in S \tag{3}$$

or in operator form as

$$(I-\mathcal{K})u = E. \tag{4}$$

In Section 2 we investigate some of the properties of G and $\mathcal{K}$, along with the solvability of (1). An introduction to the use of (1) in computer graphics is given in [**8**], along with methods for its numerical solution. The unoccluded case is of lesser importance in applications, but it is important to first understand it before proceeding with the case in which S is occluded.

In the numerical solution of (1), the Galerkin method has been the predominant form of numerical solution, with piecewise constant functions as the approximations. In this paper, we investigate collocation methods with approximations of all possible orders. In Section 3 we assume S is either a smooth surface or a finite collection of disconnected smooth surfaces. We give general results that are applicable to methods of arbitrary order and we investigate some *optimal* methods. In Section 4 we allow S to be only piecewise smooth, which is a more practical situation, and we also investigate the effect of using interpolation to approximate S. Numerical examples are given in Section 5.

2. Properties of the radiosity equation. The solvability theory for the radiosity equation (1) is relatively straightforward, being based on the geometric series theorem. We consider first the case that S is a smooth surface, and later we discuss the more difficult case of boundaries which are only piecewise smooth.

Assume S has a local representation at each $P_0 \in S$, i.e., there is a plane tangent to S at P_0 with the surface given locally by

$$\zeta = f(\xi, \eta), \quad (\xi, \eta) \text{ in a neighborhood about } P_0.$$

We assume that each such f is at least twice continuously differentiable, although this can be weakened somewhat. Over each such smooth surface S, the kernel function $G(P, Q)$ of (2) has a bounded singularity at $P = Q$ and otherwise it is a smooth function of P and Q. To see this, we first note that

$$|\cos \theta_P| \le c|P - Q| \tag{5}$$

with c independent of P and Q, e.g., see [**11**, p. 232]. (Note that, throughout this paper, we will use c to denote a generic constant.) Applying (5) to (2), we have

$$|G(P, Q)| \le c, \quad P, Q \in S, \ P \ne Q. \tag{6}$$

Using this boundedness, it is straightforward to show that when S is smooth, the integral operator $\mathcal{K}$ of (3) is compact as an operator on either $C(S)$ or $L^2(S)$ into itself, e.g., see [**10**, pp. 160–162]. We also note that the inequality (6) is still true when P and Q belong to a smooth sub-surface of a larger piecewise smooth surface.

As an aid in developing a solvability theory for (1), we must examine the norm of $\mathcal{K}$ when it is considered as an operator from $C(S)$ to $C(S)$. To do so, we use the following lemmas. We state the lemmas for surfaces S which need not be smooth, for application later with piecewise smooth surfaces.

Lemma 2.1. *Assume S is the boundary of a convex open set Ω, and assume S is a surface to which the divergence theorem can be applied. Let $P \in S$, and let S be smooth in an open neighborhood of P. Then $G(P, Q) \ge 0$ for $Q \in S$, and*

$$\int_S G(P, Q)\, dS_Q = \pi. \tag{7}$$

Proof. The positivity of $G(P,Q)$ follows from the inequalities

$$0 \le \theta_P, \theta_Q \le \frac{\pi}{2}$$

which follow, in turn, from the convexity of the region Ω.

Let $P \in S$, and let ε be a sufficiently small number. Exclude an ε-neighborhood of P from Ω, and denote the remaining set by Ω_ε:

$$\Omega_\varepsilon = \Omega \backslash \{Q : |Q - P| \le \varepsilon\}.$$

Let S_ε denote the boundary of Ω_ε, and let $\tilde{S}_\varepsilon$ denote the boundary of $\Omega \backslash \Omega_\varepsilon$, the ε-neighborhood of P that was excluded from Ω. Then

$$\int_S G(P,Q)\, dS_Q = \int_{S_\varepsilon} G(P,Q)\, ds_Q + \int_{\tilde{S}_\varepsilon} G(P,Q)\, ds_Q. \tag{8}$$

Note that the unit normal $\mathbf{n}_Q$ at $Q \in S$, S_ε, or $\tilde{S}_\varepsilon$ is directed into the interior of the region being bounded. Thus if $Q \in \tilde{S}_\varepsilon \cap S_\varepsilon$, then $\mathbf{n}_Q$ relative to S_ε is oriented opposite to that of $\mathbf{n}_Q$ relative to $\tilde{S}_\varepsilon$.

For a continuously differentiable vector function $\mathbf{v}(Q)$ defined Ω_ε, the divergence theorem says

$$\int_{S_\varepsilon} \mathbf{v}(Q) \cdot \mathbf{n}_Q\, ds_Q = -\int_{\Omega_\varepsilon} \nabla \cdot \mathbf{v}(Q)\, dQ.$$

We apply this with

$$\mathbf{v}(Q) = \frac{[(Q-P)\cdot \mathbf{n}_P]}{|P-Q|^4}(P-Q).$$

A straightforward computation shows

$$\nabla \cdot \mathbf{v}(Q) = 0, \quad Q \in \Omega_\varepsilon,$$

and, therefore,

$$\int_{S_\varepsilon} G(P,Q)\, dS_Q = \int_{S_\varepsilon} \mathbf{v}(Q) \cdot \mathbf{n}_Q\, ds_Q = 0.$$

Decompose $\tilde{S}_\varepsilon$ into two parts:

$$\tilde{S}_\varepsilon = T_\varepsilon \cup U_\varepsilon$$

with

$$\begin{aligned} T_\epsilon &= \{Q \in S \mid |Q - P| \le \varepsilon\} \\ U_\epsilon &= \{Q \in \Omega \mid |Q - P| = \varepsilon\}. \end{aligned}$$

Then

$$\int_{\tilde{S}_\varepsilon} G(P,Q)\, ds_Q = \int_{T_\varepsilon} G(P,Q)\, ds_Q + \int_{U_\varepsilon} G(P,Q)\, ds_Q, \tag{9}$$

and we examine separately each of these two righthand integrals.

Use (6) to write

$$\begin{aligned} 0 &\le \int_{T_\varepsilon} G(P,Q)\, ds_Q \\ &\le c \int_{T_\varepsilon} ds_Q \\ &= O(\varepsilon^2). \end{aligned} \tag{10}$$

Thus this integral goes to zero as $\varepsilon \to 0$.

For the last integral in (9), we can simplify $G(P,Q)$ and estimate the integral. For $Q \in U_\varepsilon$,

$$\mathbf{n}_Q = \frac{P-Q}{|P-Q|}, \qquad \mathbf{n}_Q \cdot \frac{P-Q}{|P-Q|} = 1$$

$$\begin{aligned} \int_{U_\varepsilon} G(P,Q)\, ds_Q &= \int_{U_\varepsilon} \frac{\mathbf{n}_P \cdot (Q-P)}{|P-Q|^3}\, dS_Q \\ &= \frac{1}{\varepsilon^3} \int_{U_\varepsilon} \mathbf{n}_P \cdot (Q-P)\, dS_Q. \end{aligned} \tag{11}$$

The set U_ε is approximately a hemisphere of radius ε. Change the variable of integration in the latter integral to $\mathbf{w}$, with $Q - P = \varepsilon \mathbf{w}$, so that $|\mathbf{w}| = 1$. In addition, reorient the set in such a manner that the unit normal $\mathbf{n}_P$ becomes the unit vector $\mathbf{k}$ directed along the positive w_3-axis in $\mathbf{R}^3$. Then the integral in (11) becomes

$$\int_{U_\varepsilon} G(P,Q)\,ds_Q = \int_{U_1} \mathbf{k}\cdot\mathbf{w}\,dS_w + o(\varepsilon)$$

with $U_1 = \{\mathbf{w} \in \mathbf{R}^3 \mid w_3 > 0\}$. In turn, this yields

$$\int_{U_\varepsilon} G(P,Q)\,ds_Q = \pi + o(\varepsilon). \tag{12}$$

Combining this with (8)–(10) and taking limits as $\varepsilon \to 0$, we have (7). □

Let S be a piecewise smooth unoccluded surface in $\mathbf{R}^3$. By this, we mean that S can be decomposed into a finite union,

$$S = S_1 \cup \cdots \cup S_J \tag{13}$$

with each S_j a smooth surface, i.e., there is a function

$$F_j : R_j \overset{1-1}{\underset{\text{onto}}{\to}} S_j \tag{14}$$

with R_j a closed simply-connected polygon in $\mathbf{R}^2$ and F_j a twice continuously differentiable function on R_j. We include the possibility that S may be disconnected.

Corollary 2.2. *Assume S is a piecewise smooth unoccluded surface in $\mathbf{R}^3$, and assume $S \subset \hat{S}$, with $\hat{S}$ the type of surface required in Lemma* 2.1. *Let $P \in S$ be a point at which S is smooth, and assume P does not lay on an edge or corner of S. Then*

$$\int_S G(P,Q)\,dS_Q \le \pi, \quad P \in S. \tag{15}$$

Proof. Apply the preceding Lemma 2.1 to $\hat{S}$, and then note that

$$\int_S G(P,Q)\,dS_Q \le \int_{\hat{S}} G(P,Q)\,dS_Q = \pi. \qquad \square$$

2.1. Solvability of the radiosity equation. Again, assume S is a smooth unoccluded surface, although it need not be connected. As an extension of the discussion following (1), we assume the reflectivity function $\rho(P)$ satisfies

$$\|\rho\|_\infty < 1. \tag{16}$$

Physically, this is a very sensible assumption, as real surfaces do not reflect 100 percent of all light that they receive. We also assume $\rho \in C(S)$.

With these assumptions, and with Lemma 2.1 and Corollary 2.2, we have $\mathcal{K}$ is a bounded compact operator on $C(S)$ to $C(S)$; and, moreover,

$$\|\mathcal{K}\| \le \|\rho\|_\infty < 1. \tag{17}$$

Using the geometric series theorem, the operator $I - \mathcal{K}$ is invertible on $C(S)$ to $C(S)$, with

$$\|(I - \mathcal{K})^{-1}\| \le \frac{1}{1 - \|\mathcal{K}\|}. \tag{18}$$

Thus the equation (1) is uniquely solvable for all emissivity functions $E \in C(S)$. In practice, the functions $E(P)$ and $\rho(P)$ are often discontinuous; and, later in the section, we discuss some appropriate modifications of the theoretical framework.

To talk about the regularity of solutions of (1), we need the following result.

Lemma 2.3. *Let $m \ge 0$ be an integer, and consider a surface S of the form in* (13). *Assume the parameterization functions of* (14) *are $m+2$ times continuously differentiable, and also assume the reflectivity function $\rho \in C^{m+1}(S)$. Then*

$$u \in C^m(S) \quad \Longrightarrow \quad \mathcal{K}u \in C^{m+1}(S). \tag{19}$$

Proof. For the case $m = 0$, the proof is relatively straightforward. Differentiate the kernel function $G(P,Q)$ with respect to P, to get

$$\frac{\partial G(P,Q)}{\partial P} = O\left(\frac{1}{|P-Q|}\right).$$

Then the associated function

$$\frac{\partial(\mathcal{K}u)}{\partial P}$$

is dominated by a 'single layer integral operator,' and, with the latter, it easily follows that $\mathcal{K}u \in C^1(S)$.

For the more general case, one needs to generalize the form of proof given in Günter [**9**, p. 49] for single and double layer potential integral operators. We omit it here. □

We summarize the solvability and regularity results in the following.

Theorem 2.4. *Let $m \geq 0$ be an integer. Let $\hat{S}$ be the boundary of a convex open set Ω, and assume $\hat{S}$ is a surface to which the divergence theorem can be applied. Assume S is a smooth (possibly disconnected) unoccluded surface $S \subset \hat{S}$, and assume it can be represented as in* (13) *with each parameterization function F_j being* $(m+2)$*-times continuously differentiable over its polygonal domain. Assume ρ, $E \in C^m(S)$. Then*:

(a) *the equation* (1) *is uniquely solvable, with the solution $u(P)$ satisfying*

$$\|u\|_\infty \leq \frac{\|E\|_\infty}{1-\|\mathcal{K}\|}.$$

(b) *The solution $u \in C^m(S)$.*

Proof. The proof of (a) is obvious from earlier remarks, and this also proves (b) for the case $m = 0$. For (b) with $m > 0$, write

$$u = E + \mathcal{K}u.$$

Use Lemma 2.3 to give an induction argument that $u \in C^m(S)$. □

2.2. Piecewise smooth surfaces. The majority of applications are likely to have surfaces S that are only piecewise smooth. As a simple example which illustrates the main mathematical difficulties of such surfaces, let S be the boundary of a rectangular solid, say that of the solid

$$\Omega = [0, a] \times [0, b] \times [0, c].$$

The function $G(P, Q)$ no longer is as well-behaved as for the smooth surface case, and it has singular behavior along all edges and corners; and, consequently, the integral operator $\mathcal{K}$ is also less well-behaved.

To illustrate the behavior of $G(P, Q)$, we use an even simpler surface S. Introduce

$$S_{xz} = \{(x, 0, z) \mid 0 \le x, z \le 1\}, \quad S_{xy} = \{(x, y, 0) \mid 0 \le x, y \le 1\},$$

the unit squares in the xz and xy-planes in $\mathbf{R}^3$, respectively. Define

$$S = S_{xz} \cup S_{xy} \tag{20}$$

which is not smooth along the edge $e_S \equiv \{(x, 0, 0) \mid 0 \le x \le 1\}$. Let $P = (x, y, z)$ and $Q = (\xi, \eta, \zeta)$. Then

$$G(P, Q) = \begin{cases} (y\zeta/[(x-\xi)^2 + y^2 + \zeta^2]^2), & P \in S_{xy},\, Q \in S_{xz} \\ (z\eta/[(x-\xi)^2 + \eta^2 + z^2]^2), & P \in S_{xz},\, Q \in S_{xy} \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

For $P, Q \in S_{xy}$, $G(P, Q) \equiv 0$. But for $P = ((1/2), 0, z)$ and $Q = ((1/2), \eta, 0)$,

$$G(P, Q) = \frac{z\eta}{\eta^2 + z^2} \cdot \frac{1}{\eta^2 + z^2}.$$

The first fraction is bounded, and, for $z = \eta$, it equals $1/2$ exactly. But the second fraction is unbounded as $\eta, z \to 0$. Thus $G(P, Q)$ is an unbounded function along the edge e_S common to the two smooth sub-surfaces S_{xy} and S_{xz}.

The results of Lemma 2.1 and Corollary 2.2 are still valid for the surface S of (20), but the function space needs to be changed to account for the discontinuity of $G(P, Q)$ for P or Q belonging to e_S. We use the Banach space $L^\infty(S)$, to allow for discontinuities along edges and corners of S, and this will also allow us to introduce emissivity $E(P)$

and reflectivity $\rho(P)$ which need not be continuous. The invertibility of $I - \mathcal{K}$ still follows as in (17)–(18). But the regularity result (19) of Lemma 2.3 is no longer valid. To further investigate the regularity of $\mathcal{K}u$, we compute $\mathcal{K}u(P)$ for the functions $u(Q) = 1$, $\xi - x$ and η, with $P \in S_{xz}$ and $Q \in S_{xy}$.

Letting $u \equiv 1$, and evaluating only at $P \in S_{xz}$ with $P = (x, 0, z)$,

$$\mathcal{K}u(P) = \int_0^1 \int_0^1 \frac{z\eta \, d\xi \, d\eta}{[(x-\xi)^2 + \eta^2 + z^2]^2}, \quad 0 < x, z < 1.$$

After some manipulation,

$$\begin{aligned} \mathcal{K}u(P) = {} & \frac{1}{2}\left\{\pi - \arctan\left(\frac{z}{x}\right) - \arctan\left(\frac{z}{1-x}\right)\right\} \\ & - \frac{z}{2\sqrt{1+z^2}}\left\{\arctan\left(\frac{x}{\sqrt{1+z^2}}\right) + \arctan\left(\frac{1-x}{\sqrt{1+z^2}}\right)\right\}. \end{aligned} \tag{22}$$

For $0 < z < 1$, this is a well-behaved function; but there are indications of problems near the edges at $x = 0$ and $x = 1$, particularly as $(x, 0, z) \to (0, 0, 0)$ or $(1, 0, 0)$.

Letting $u \equiv \xi - x$ for some $0 < x < 1$, and evaluating $\mathcal{K}u(P)$ for only $P \in S_{xz}$, we have

$$\mathcal{K}u(P) = \int_0^1 \int_0^1 \frac{(\xi - x)z\eta \, d\xi \, d\eta}{[(x-\xi)^2 + \eta^2 + z^2]^2}, \quad 0 < x, \ z < 1.$$

Then

$$\mathcal{K}u(P) = \frac{z}{4}\log\left\{\frac{[(1-x)^2 + z^2][x^2 + 1 + z^2]}{[(1-x)^2 + 1 + z^2][x^2 + z^2]}.\right\} \tag{23}$$

For $z \approx 0$,

$$\mathcal{K}u(P) \approx \frac{z}{4}\log\left\{\frac{[(1-x)^2][x^2+1]}{[(1-x)^2+1][x^2]}\right\},$$

which is well-defined for $0 < x < 1$, but has problems around $x = 0$ and $x = 1$.

Letting $u(\xi, \eta, 0) = \eta$ and $P = (x, 0, z)$,

$$\mathcal{K}u(P) = \int_0^1 \int_0^1 \frac{z\eta^2 \, d\xi \, d\eta}{[(x-\xi)^2 + \eta^2 + z^2]^2}, \quad 0 < x, \ z < 1.$$

Then

$$\begin{aligned}(24)\qquad \mathcal{K}u(P) &= \frac{-z}{2\sqrt{1+z^2}}\left\{\arctan\left(\frac{x}{\sqrt{1+z^2}}\right) + \arctan\left(\frac{1-x}{\sqrt{1+z^2}}\right)\right\}\\ &\quad + \frac{1}{2}\int_0^1\int_0^1 \frac{z\,d\xi\,d\eta}{(x-\xi)^2+\eta^2+z^2}.\end{aligned}$$

For the last integral in this formula, we break the integration region into two portions:

$$R_1 = \{(\xi,\eta,0)\mid (\xi-x)^2+\eta^2\le r^2,\ \eta\ge 0\},$$
$$R_2 = [0,1]\times[0,1]\backslash R_1$$

with some r chosen to be small enough that the semi-circle R_1 is located entirely within the unit square $[0,1]\times[0,1]$. The integral

$$(25)\qquad \iint_{R_2} \frac{z\,d\xi\,d\eta}{(x-\xi)^2+\eta^2+z^2}$$

is a smooth function of x and z as $z\to 0$, provided $0<x<1$. For the remaining integral, over R_1, change the integration variables to polar coordinates centered at $(\xi,\eta,0)=(x,0,0)$. Then

$$(26)\qquad \iint_{R_1} \frac{z\,d\xi\,d\eta}{(x-\xi)^2+\eta^2+z^2} = \frac{\pi z}{2}\log\left(r^2+z^2\right) - \pi z\log(z)$$

Thus for $u=\eta$, $\mathcal{K}u(P)$ is dominated by $-\pi z\log(z)$ as $z\to 0$. The function $\mathcal{K}u(P)$ does not belong to $C^1(S)$.

For more general density or brightness functions $u(Q)$, we can expand them about $(x,0,0)$ and then apply the above results to obtain more general regularity results. In Section 4, we return to some of these formulas, to investigate the effect on the behavior of our numerical schemes of S having edges and corners.

3. Collocation on smooth surfaces. To define numerical methods for solving (1), we follow closely the ideas used in defining boundary element methods for solving boundary integral equation reformulations of elliptic partial differential equations on regions in $\mathbf{R}^3$, e.g., see [**2**], [**4**, Chapters 5, 9], [**5, 12**]. In this section we develop numerical methods for the case that S is a smooth surface, although it need not be connected, and we follow the notation of (13)–(14) when considering S. In the following section, we extend the numerical theory to the cases where S is only piecewise smooth and where S is approximated by interpolation. The general idea of the numerical method is as follows. Begin by triangulating S and then approximate the unknown $u(P)$ by functions which are *piecewise polynomial* over the triangulation of S. The numerical solution is found by collocation, meaning that the approximate form of the solution is substituted into (1) and then the equation is forced to be true at the *collocation node points*, leading to a system of linear equations for determining the approximate solution.

We use the framework for collocation methods that is described in [**4**, Chapter 5] and [**5**], and only the most pertinent points are summarized here. An implementation of the numerical methods of this paper makes use of the boundary element package described in [**3**], to which the reader is referred for more detail.

We assume there is a sequence of triangulations of S, $\mathcal{T}_n = \{\Delta_{n,k} \mid 1 \le k \le n\}$, with some increasing sequence of integer values n converging to infinity. Usually in our codes, the values of n increase by a factor of 4. For example, if S is an ellipsoid, then we often subdivide S into a sequence of triangulations $\{\mathcal{T}_n \mid n = 8, 32, 128, ...\}$. There are assumptions made on the triangulations, most of which we leave to the cited references. Associated with most surfaces are parameterizations of the surface, as in (14). Consider only one such parameterization function, say

$$F_j : R_j \overset{1-1}{\underset{\text{onto}}{\rightarrow}} S_j$$

with R_j a polygonal region in the plane and some $1 \le j \le J$. Triangulate R_j, say as

$$\{\hat{\Delta}^j_{n,k} \mid k = 1, \dots, n_j\}. \tag{27}$$

Then triangulate the corresponding subsurface S_j using

$$\Delta^j_{n,k} = F_j(\hat{\Delta}^j_{n,k}), \quad k = 1, \dots, n_j. \tag{28}$$

For S as a whole, define

$$\mathcal{T}_n = \bigcup_{j=1}^{J} \{\Delta^j_{n,k} \mid k = 1, \dots, n_j\}.$$

Often we will dispense with the subscript n, although it is to be understood implicitly. The *mesh size* of this triangulation is defined by

$$h \equiv h_n = \max_{1 \le j \le J} \max_{1 \le k \le n_j} \text{diameter } (\hat{\Delta}^j_{n,k}).$$

For purposes of numerical integration and interpolation over the triangular elements in $\mathcal{T}_n$, we also need a parameterization over each $\Delta^j_{n,k}$ with respect to a standard reference triangle in the plane. Our reference triangle is the unit simplex,

$$\sigma = \{(s,t) \mid 0 \le s, t, s+t \le 1\}.$$

Let the vertices of $\hat{\Delta}^j_{n,k}$ be denoted by $\{v_1, v_2, v_3\}$, and define a parameterization function $m_k : \sigma \overset{1-1}{\underset{\text{onto}}{\to}} \Delta^j_{n,k}$ by

$$m_k(s,t) = F_j(uv_3 + tv_2 + sv_1), \quad (s,t) \in \sigma \tag{29}$$

with $u = 1 - s - t$. Using this, we can write

$$\int_{\Delta_k} f(Q)\, dS_Q = \int_\sigma f(m_k(s,t)) |(D_s m_k \times D_t m_k)(s,t)|\, d\sigma, \tag{30}$$

and this can be used to numerically evaluate the lefthand integral by using numerical integration formulas developed for the region σ.

Interpolation of functions over σ can be used to develop interpolatory approximations of functions defined on the triangular elements Δ_k. If

$$f(s,t) \approx \sum_{i=1}^{p} f(s_i, t_i) \ell_i(s,t) \tag{31}$$

is an interpolatory formula for functions $f \in C(\sigma)$, then define interpolation of functions $g \in C(\Delta_k)$ by

$$g(m_k(s,t)) \approx \sum_{i=1}^{p} f(m_k(s_i,t_i)) l_i(s,t), \quad (s,t) \in \sigma. \tag{32}$$

In the following, the formula (31) is used for interpolation of all possible degrees; and more detailed results are given for degrees 0, 1, and 2.

3.1. A piecewise linear collocation method. For the moment, let (31) denote a linear interpolation function. More precisely, let α be a given constant with $0 \leq \alpha < (1/3)$, and define interpolation nodes in σ by

$$\{q_1, q_2, q_3\} = \{(\alpha, \alpha), (\alpha, 1-2\alpha), (1-2\alpha, \alpha)\}. \tag{33}$$

If $\alpha = 0$, these are the three vertices of σ; otherwise, they are symmetrically placed points in the interior of σ. Define corresponding *Lagrange interpolation basis functions* by

$$l_1(s,t) = \frac{u-\alpha}{1-3\alpha}, \qquad l_2(s,t) = \frac{t-\alpha}{1-3\alpha}, \qquad l_3(s,t) = \frac{s-\alpha}{1-3\alpha}$$

for $(s,t) \in \sigma$ and $u = 1 - s - t$. The linear polynomial interpolating $f \in C(\sigma)$ is given by

$$f(s,t) \approx (\mathcal{L}_\sigma f)(s,t) \equiv \sum_{i=1}^{3} f(q_i) l_i(s,t). \tag{34}$$

For $g \in C(S)$, define

$$(\mathcal{P}_n g)(m_k(s,t)) = \sum_{i=1}^{3} g(m_k(q_i)) l_i(s,t), \quad (s,t) \in \sigma \tag{35}$$

for $k = 1, 2, \dots, n$. This interpolates $g(P)$ over each triangular element $\Delta_k \subset S$, with the interpolating function linear in the parameterization variables s and t. Let the interpolation nodes in Δ_k be denoted by

$$v_{k,i} = m_k(q_i), \quad i = 1, 2, 3; \quad k = 1, \dots, n.$$

Then (35) can be written

$$(\mathcal{P}_n g)(P) = \sum_{i=1}^{3} g(v_{k,i}) l_i(s,t), \qquad P = m_k(s,t) \in \Delta_k \tag{36}$$

for $k = 1, \dots, n$. Collectively, we refer to the interpolation nodes $\{v_{k,i}\}$ by $\{v_1, v_2, \dots, v_{3n}\}$, for $\alpha > 0$.

In the case $\alpha = 0$, the formula (35) defines a projection operator on $C(S)$, and, easily,

$$\|\mathcal{P}_n\| = 1, \quad \text{with} \quad \alpha = 0.$$

For $0 < \alpha < (1/3)$, the function $\mathcal{P}_n g$ is usually not continuous, and if the standard type of collocation error analysis is to be carried out in the context of function spaces, then $C(S)$ must be enlarged to include the piecewise linear approximants $\mathcal{P}_n g$. One way of doing this is by using the space $L^\infty(S)$. This is the set of all essentially bounded and Lebesgue measurable functions on S, and the norm is the essential supremum $\|\cdot\|_\infty$. This approach is fully explored in [**6**] and, with it, $\mathcal{P}_n$ can be extended to be a projection on $L^\infty(S)$. The reader is referred to [**6**] for details. For this case of α,

$$\|\mathcal{P}_n\| = \frac{1+\alpha}{1-3\alpha}, \quad \text{with} \quad 0 < \alpha < \frac{1}{3}. \tag{37}$$

A particularly important case is $\alpha = 1/6$, for which

$$\|\mathcal{P}_n\| = \frac{7}{3}, \quad \text{with} \quad \alpha = \frac{1}{6}.$$

Our collocation error analysis given below in Theorem 3.6 will use another approach, one using only the space $C(S)$ and not requiring $\mathcal{P}_n$ to be a projection operator.

It is clear that these definitions can be extended to interpolation with polynomials of any given degree. Particularly important cases are degree 0 and degree 2. For degree 0, define piecewise constant interpolation by

$$f(s,t) \approx f\left(\frac{1}{3}, \frac{1}{3}\right), \quad (s,t) \in \sigma,$$

for $f \in C(\sigma)$, with $(1/3, 1/3)$ the centroid of σ. Over $\Delta_k \subset S$, define $v_k = m_k(1/3, 1/3)$, which we call the *centroid* of Δ_k, $k = 1, \dots, n$. Define piecewise constant interpolation over S by

$$(\mathcal{P}_n g)(P) = g(v_k), \qquad P = m_k(s,t) \in \Delta_k \tag{38}$$

for $k = 1, \dots, n$ and $g \in C(S)$. The operator $\mathcal{P}_n$ can again be extended to be a projection on $L^\infty(S)$, with $\|\mathcal{P}_n\| = 1$. The case of quadratic interpolation is well-developed in the references for the case of interpolation at the vertices and midpoints of sides of Δ_k, e.g., see [**4**, Section 5.1], [**5**], and therefore the details of such interpolation are omitted here.

We define a collocation method with (36). Substitute

$$u_n(P) = \sum_{i=1}^{3} u_n(v_{k,i}) l_i(s,t), \qquad P = m_k(s,t) \in \Delta_k, \quad k = 1, \dots, n \tag{39}$$

into (1), with $V \equiv 1$ for an unoccluded surface. To determine the values $\{u_n(v_{k,i})\}$, force the equation resulting from the substitution to be true at the interpolation node points (which are now also called the *collocation* points). This leads to the linear system

$$u_n(v_i) - \frac{\rho(v_i)}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{3} u_n(v_{k,j}) \cdot \int_\sigma G(v_i, m_k(s,t))\, l_j(s,t) |(D_s m_k \times D_t m_k)(s,t)|\, d\sigma = E(v_i), \qquad i = 1, \dots, 3n, \tag{40}$$

which is of order $3n$. The system (40) contains integrals which must be evaluated numerically, and this is discussed in Section 5.

It is well known that (40) can be rewritten abstractly as

$$(I - \mathcal{P}_n \mathcal{K}) u_n = \mathcal{P}_n E, \tag{41}$$

which is to be compared to (4), the abstract formulation of (1). This is a standard form for an abstract error analysis of the collocation method; and for a general reference of such analyses for integral operators $\mathcal{K}$ which are compact on a Banach space $\mathcal{X}$ into itself, see [**1**, p. 54] or [**4**, Chapter 3]. We instead give an error analysis based on the *iterated collocation solution*, introduced below, as it is also needed in examining the question of *superconvergence* for some collocation solutions u_n at the collocation node points.

Given the collocation solution u_n for (41), introduce the *iterated collocation solution*

$$\hat{u}_n = E + \mathcal{K} u_n. \tag{42}$$

Then it is straightforward to show

$$\mathcal{P}_n \hat{u}_n = u_n. \tag{43}$$

This says that u_n and $\hat{u}_n$ agree at the collocation node points:

$$\hat{u}_n(v_i) = u_n(v_i), \quad i = 1, 2, \ldots, 3n. \tag{44}$$

Results for the convergence of $\hat{u}_n$ are in turn results for the convergence of u_n at the collocation node points. In addition,

$$\begin{aligned} u - u_n &= u - \mathcal{P}_n \hat{u}_n \\ &= (u - \mathcal{P}_n u) + \mathcal{P}_n (u - \hat{u}_n) \\ \|u - u_n\|_\infty &\le \|u - \mathcal{P}_n u\|_\infty + \|\mathcal{P}_n\| \|u - \hat{u}_n\|_\infty. \end{aligned} \tag{45}$$

Convergence results for $\hat{u}_n$ yield convergence results for u_n.

Substituting (43) into (42), we obtain

$$(I - \mathcal{K}\mathcal{P}_n)\hat{u}_n = E. \tag{46}$$

The operator

$$\mathcal{K}\mathcal{P}_n : C(S) \to C(S)$$

is a numerical integral operator based on *product integration*, for example, see [**1**, p. 106] or [**4**, Section 4.2]. An error analysis for (46) can be based on the general theory for such numerical integral operators.

Theorem 3.1. *Assume S is a smooth unoccluded surface in $\mathbf{R}^3$, and assume $S \subset \hat{S}$, with $\hat{S}$ the type of surface required in Lemma* 2.1. *Assume the surface S satisfies* (13)–(14) *with each $F_j \in C^3$. Assume the radiosity equation* (1) *is uniquely solvable for all emissivity functions $E \in C(S)$. Then for all sufficiently large n, say $n \ge n_0$, the operators $I - \mathcal{K}\mathcal{P}_n$ are invertible on $C(S)$ and have uniformly bounded inverses. Moreover, for the true solution u of* (1) *and the solution $\hat{u}_n$ of* (46),

$$\|u - \hat{u}_n\|_\infty \le \|(I - \mathcal{K}\mathcal{P}_n)^{-1}\| \, \|\mathcal{K}(u - \mathcal{P}_n u)\|_\infty, \quad n \ge n_0. \tag{47}$$

Furthermore, if the emissivity $E \in C^2(S)$, then

$$\|u - \hat{u}_n\|_\infty \le O(h^2), \quad n \ge n_0. \tag{48}$$

Proof. It is relatively straightforward to show that the family $\{\mathcal{K}\mathcal{P}_n \mid n \geq 1\}$ is collectively compact and pointwise convergent on $C(S)$ to $C(S)$. It then follows from the assumption of the existence of $(I-\mathcal{K})^{-1}$ and the theory of collectively compact operators, see [**1**, p. 96] or [**4**, Section 4.1.2] that the operators $I - \mathcal{K}\mathcal{P}_n$ are invertible on $C(S)$ and have uniformly bounded inverses for all sufficiently large n, say $n \geq n_0$. The bound (47) follows from the identity

$$u - \hat{u}_n = (I - \mathcal{K}\mathcal{P}_n)^{-1}\mathcal{K}(u - \mathcal{P}_n u). \tag{49}$$

The bound (48) follows from standard interpolation error bounds for linear interpolation. □

This theorem immediately generalizes to collocation based on interpolation of any given degree. For interpolation with polynomials of degree r, and with E sufficiently smooth, the error bound becomes

$$\|u - u_n\|_\infty \leq O(h^{r+1}), \quad n \geq n_0. \tag{50}$$

We omit the details, as they too are a straightforward consequence of existing theory.

We also note that the inverses for the collocation equation (41) and the iterated collocation equation (46) are related by the identities

$$\begin{aligned}(I - \mathcal{K}\mathcal{P}_n)^{-1} &= I + \mathcal{K}(I - \mathcal{P}_n\mathcal{K})^{-1}\mathcal{P}_n\\ (I - \mathcal{P}_n\mathcal{K})^{-1} &= I + \mathcal{P}_n(I - \mathcal{K}\mathcal{P}_n)^{-1}\mathcal{K}.\end{aligned}$$

See the discussion in [**4**, Section 3.4].

3.2. A superconvergent piecewise linear method. With the interpolation parameter $\alpha = 1/6$, we obtain a collocation method which converges more rapidly at the collocation node points. To show this, we must examine more carefully the term $\mathcal{K}(u - \mathcal{P}_n u)$ from (49).

By looking at the linear system associated with

$$(I - \mathcal{K}\mathcal{P}_n)(u - \hat{u}_n) = \mathcal{K}(u - \mathcal{P}_n u)$$

we can argue that

$$\max_{1\leq i\leq 3n} |u(v_i) - \hat{u}_n(v_i)| \leq c \max_{1\leq i\leq 3n} |\mathcal{K}(I - \mathcal{P}_n)u(v_i)|. \tag{51}$$

We omit the argument; but it is the same as that given in the discussion following formula (33) in [**5**]. Below we will look at the errors $\mathcal{K}(I-\mathcal{P}_n)u(v_i)$ occurring on the right side of (51). But first we need some preliminary results for the piecewise linear interpolation used in defining $\mathcal{P}_n$.

Consider the interpolation formula (34) for linear interpolation over σ. It also leads to a numerical integration formula

$$\int_\sigma f(s,t)\,d\sigma \approx \int_\sigma \mathcal{L}_\sigma f(s,t)\,d\sigma. \tag{52}$$

With any choice of the interpolation parameter α, if f is a linear, then the interpolation is exact, $\mathcal{L}_\sigma f = f$, and hence the integration formula (52) is also exact. However, if we choose $\alpha = (1/6)$, then the linear interpolation formula also satisfies

$$\int_\sigma f(s,t)\,d\sigma = \int_\sigma \mathcal{L}_\sigma f(s,t)\,d\sigma, \quad \deg(f) \le 2 \tag{53}$$

for f any polynomial in s,t of degree ≤ 2. The proof is a straightforward computation with the choices $f(s,t) = s^2, st, t^2$.

Integrating the right side of (52) yields the quadrature formula

$$\int_\sigma f(s,t)\,d\sigma \approx \mathcal{Q}_\sigma(f) \equiv \frac{1}{6}[f(\alpha,\alpha)+f(\alpha,1-2\alpha)+f(1-2\alpha,\alpha)] \tag{54}$$

for arbitrary $f \in C(\sigma)$ and $0 \le \alpha < 1/3$. This has degree of precision 1 for general α; and for $\alpha = 1/6$, it has degree of precision 2, based on (53). *For the remainder of this section, we assume* $\alpha = 1/6$.

We can extend this to an integration formula over the unit square $U \equiv [0,1]\times[0,1]$ by applying the same formula over both σ and its mirror image in U. Then we have a numerical integration formula over U which can be shown to have degree of precision 3:

$$\begin{aligned}\int_0^1\int_0^1 f(s,t)\,d\sigma \approx \frac{1}{6}[&f(\alpha,\alpha)+f(\alpha,1-2\alpha)+f(1-2\alpha,\alpha)\\ &+ f(1-\alpha,1-\alpha)+f(1-\alpha,2\alpha)+f(2\alpha,1-\alpha)].\end{aligned} \tag{55}$$

A proof based on using symmetry is straightforward, and we omit it.

Let $\tau \subset \mathbf{R}^2$ be a planar triangle with vertices $\{v_1, v_2, v_3\}$. The mapping

$$(x,y) \equiv \mu_\tau(s,t) = uv_3 + tv_2 + sv_1, \qquad u \equiv 1-s-t \tag{56}$$

is an affine one-to-one transformation of σ onto τ, and polynomials in s,t are transformed to polynomials in x,y of the same degree. For a function $g \in C(\tau)$, the function

$$\mathcal{L}_\tau g(x,y) = \sum_{i=1}^{3} g(\mu_\tau(q_i)) l_i(s,t), \qquad (x,y) = \mu_\tau(s,t)$$

in a linear polynomial which interpolates g at the points $\{\mu_\tau(q_1), \mu_\tau(q_2), \mu_\tau(q_3)\}$, with the latter symmetrically placed in τ. [Recall the definition of $\{q_i\}$ from (33).]

As earlier for integration over σ, define a numerical integration formula by

$$\int_\tau g(x,y)\, d\tau \approx \int_\tau \mathcal{L}_\tau g(x,y)\, d\tau. \tag{57}$$

Using the affine change of variables (56), and applying it to the earlier results over σ, we have that (57) can be written as

$$\int_\tau g(x,y)\, d\tau \approx \mathcal{Q}_\tau(g) \equiv \frac{\text{Area}(\tau)}{3} \sum_{i=1}^{3} g(\mu_\tau(q_i)). \tag{58}$$

This has degree of precision 2. Moreover, if τ_1 and τ_2 are triangles for which $\tau_1 \cup \tau_2$ is a parallelogram, then the formula

$$\int_{\tau_1 \cup \tau_2} g(x,y)\, d\tau \approx \mathcal{Q}_{\tau_1 \cup \tau_2}(g) \tag{59}$$

has degree of precision 3.

For differentiable functions f, introduce the notation

$$|D^k f(x,y)| = \max_{0 \le i \le k} \left| \frac{\partial^k f(x,y)}{\partial x^i \partial y^{k-i}} \right|.$$

Lemma 3.2. *Let τ be a planar right triangle, and assume the two sides which form the right angle have length h. Let $f \in C^3(\tau)$. Let $\Phi \in L^1(\tau)$ be differentiable with the first derivatives $D_x\Phi, D_y\Phi \in L^1(\tau)$. Assume $\alpha = 1/6$. Then*

$$\left|\int_\tau \Phi(x,y)(I-\mathcal{L}_\tau)f(x,y)\,d\tau\right| \le c\,h^3\left[\int_\tau (|\Phi| + |D\Phi|)\,d\tau\right]\max_\tau\{|D^2 f|, |D^3 f|\}. \tag{60}$$

In this and in the following proof, the letter c denotes a generic constant.

Proof. It is possible to find a linear polynomial $p_1(x,y)$ for which

$$\|f - p_1\|_\infty \le ch^2\|D^2 f\|_\infty, \quad f \in C^2(\tau) \tag{61}$$

for a suitable constant c. We can also find a quadratic polynomial $p_2(x,y)$ for which

$$\|f - p_2\|_\infty \le c\,h^3\|D^3 f\|_\infty, \quad f \in C^3(\tau) \tag{62}$$

Simply let p_1 and p_2 be Taylor polynomials of f. Similarly, we can find a constant ϕ_0 for which

$$\|\Phi - \phi_0\|_1 \le ch\|D^1\Phi\|_1 \tag{63}$$

with $\|\cdot\|_1$ denoting the norm on $L^1(\tau)$. As a general reference, see [**7**, Chapter 4].

To shorten the notation, let $\mathcal{L}'_\tau = I - \mathcal{L}_\tau$. To prove (60), write

$$\begin{aligned}\int_\tau \Phi\mathcal{L}'_\tau f\,d\tau &= \int_\tau \Phi\mathcal{L}'_\tau(f - p_2)\,d\tau \\ &\quad + \int_\tau (\Phi - \phi_0)\mathcal{L}'_\tau p_2\,d\tau + \int_\tau \phi_0\mathcal{L}'_\tau p_2\,d\tau.\end{aligned} \tag{64}$$

The first term on the right side is bounded using (62):

$$\begin{aligned}\left|\int_\tau \Phi\mathcal{L}'_\tau(f - p_2)\,d\tau\right| &\le \|\mathcal{L}'_\tau\|\|f - p_2\|_\infty \int_\tau |\Phi|\,d\tau \\ &\le ch^3\|D^3 f\|_\infty \int_\tau |\Phi|\,d\tau.\end{aligned}$$

The bound for $\|\mathcal{L}'_\tau\|$ comes from (37). The third term on the right side of (64) is zero, by using the fact that (57) has degree of precision 2.

For the second term on the right side of (64), note that

$$\mathcal{L}'_\tau p_2 = \mathcal{L}'_\tau(p_2 - p_1) = \mathcal{L}'_\tau([p_2 - f] - [p_1 - f])$$

since $\mathcal{L}_\tau p_1 = p_1$. Taking bounds and using (61)–(62),

$$\|\mathcal{L}'_\tau p_2\|_\infty \le c\, h^2 (h\|D^3 f\|_\infty + \|D^2 f\|_\infty).$$

Using this and (63),

$$\begin{aligned}\left| \int_\tau (\Phi - \phi_0)\mathcal{L}'_\tau p_2 \, d\tau \right| &\le \|\mathcal{L}'_\tau p_2\|_\infty \int_\tau |\Phi - \phi_0| \, d\tau \\ &\le ch^3 \int_\tau |D^1\Phi| \, d\tau \cdot \max_\tau \{\|D^2 f\|_\infty, \|D^3 f\|_\infty\}.\end{aligned}$$

Combining these results with (64) proves (60). □

Lemma 3.3. *Let τ_1 and τ_2 be planar right triangles that form a square R of length h on a side. Let $f \in C^4(R)$. Let $\Phi \in L^1(R)$ be twice differentiable with all first and second derivatives belonging to $L^1(R)$. Assume $\alpha = 1/6$. Then*

$$\begin{aligned}(65)\quad & \left| \int_R \Phi(x,y)(I - \mathcal{L}_\tau) f(x,y) \, d\tau \right| \\ & \qquad \le c\, h^4 \left[\int_R \sum_{i=0}^{2} |D^i \Phi| \, d\tau \right] \cdot \max_{\substack{R \\ i=2,3,4}} \{|D^i f|\}\end{aligned}$$

with $\mathcal{L}_\tau f(x,y) \equiv \mathcal{L}_{\tau_i} f(x,y)$ when $(x,y) \in \tau_i$, $i = 1,2$.

Proof. The proof is similar to the preceding one. Begin by letting $p_k(x,y)$ be a polynomial of degree k over R which satisfies

$$(66)\qquad \begin{aligned} \|f - p_k\|_\infty \le c\, h^{k+1} \|D^{k+1} f\|_\infty, \qquad & f \in C^{k+1}(R), \\ k = 1,2,3 & \end{aligned}$$

with $\|\cdot\|_\infty$ denoting the uniform norm on $C(R)$. From (66), we have

$$\|p_k - p_{k-1}\|_\infty \le ch^k(\|D^k f\|_\infty + h\|D^{k+1} f\|_\infty), \quad k = 2, 3. \tag{67}$$

In addition to (66), let $\phi_i(x, y)$ be a polynomial of degree i satisfying

$$\|\Phi - \phi_i\|_1 \le c\, h^{i+1} \|D^{i+1}\Phi\|_1, \quad i = 0, 1. \tag{68}$$

In this, $\|\cdot\|_1$ denotes the norm on $L^1(R)$.

In analogy with (64), consider the identity

$$\begin{aligned}\int_R \Phi \mathcal{L}'_\tau f \, d\tau = \int_R &\Phi \mathcal{L}'_\tau (f - p_3)\, d\tau \\ &+ \int_R (\Phi - \phi_0)\mathcal{L}'_\tau (p_3 - p_2)\, d\tau \\ &+ \int_R (\Phi - \phi_1)\mathcal{L}'_\tau (p_2 - p_1)\, d\tau \\ &+ \int_R (\phi_1 - \phi_0)\mathcal{L}'_\tau p_2 \, d\tau.\end{aligned} \tag{69}$$

This uses the identities $\mathcal{L}'_\tau p_1 = 0$ and

$$\int_R \phi_0 \mathcal{L}'_\tau p_3 \, d\tau = 0.$$

The first identity is immediate from the use of linear interpolation, and the second one follows from the fact that (59) has degree of precision 3.

Use the same type of arguments as in the proof of Lemma 3.2, together with the bounds (66)–(68). It follows easily that the first three terms on the right side of (69) are all $O(h^4)$, together with being multiplied by quantities of the form given on the right side of (65).

For the final term on the right side of (69), note first that

$$\mathcal{L}(\phi_i \mathcal{L} p_2) = \mathcal{L}(\phi_i p_2), \quad i = 0, 1. \tag{70}$$

To show this, we need show only that $\phi_i \mathcal{L} p_2$ and $\phi_i p_2$ agree at the node points for the interpolation operator $\mathcal{L}$. Call these node points

$\mu_j,\ j = 1, \ldots, 6$, and they are the analogues for R of the node points used in (55). Then

$$(\phi_i \mathcal{L} p_2)(\mu_j) = \phi_i(\mu_j) \cdot (\mathcal{L} p_2)(\mu_j) = \phi_i(\mu_j) \cdot p_2(\mu_j)$$

since $\mathcal{L} p_2$ interpolates p_2 at the nodes $\{\mu_j\}$. Next note that $(\phi_1 - \phi_0)\mathcal{L}'_\tau p_2$ is a polynomial of degree ≤ 3. Then from the fact that (59) has degree of precision 3,

$$\int_R (\phi_1 - \phi_0)\mathcal{L}'_\tau p_2 \, d\tau = \int_R \mathcal{L}_\tau[(\phi_1 - \phi_0)\mathcal{L}'_\tau p_2] \, d\tau.$$

It then follows from (70) that

$$\int_R (\phi_1 - \phi_0)\mathcal{L}'_\tau p_2 \, d\tau = \int_R \mathcal{L}'_\tau((\phi_1 - \phi_0)p_2) \, d\tau = 0. \tag{71}$$

The last step again uses the result that (59) has degree of precision 3.

This completes the proof of (65). □

The results in Lemmas 3.2 and 3.3 can be generalized to general triangles and parallelograms; but the derivatives of f and Φ will now involve the affine mapping μ_τ of (56). The bounds of (60) and (65) must now contain a term proportional to some power of

$$r(\tau) \equiv \frac{h(\tau)}{h^*(\tau)}. \tag{72}$$

In this fraction, $h(\tau)$ is the diameter of τ and $h^*(\tau)$ is equal to the radius of the circle inscribed in τ and tangent to its sides. We will only use triangulations for which the maximum of this ratio over the triangulation $\mathcal{T}_n = \{\widehat{\Delta}_{n,k}\}$ is uniformly bounded in n :

$$\sup_n \max_{\hat{\Delta}_k \in \mathcal{T}_n} r(\hat{\Delta}_k) < \infty. \tag{73}$$

This prevents the triangles $\hat{\Delta}_{n,k}$ from having angles which approach 0 as $n \to \infty$. We give the generalizations to arbitrary triangles in the following, and we omit the proof as it is basically straightforward.

Corollary 3.4. (a) *Let τ be a planar triangle of diameter h, let $f \in C^3(\tau)$ and let $\Phi \in L^1(\tau)$ with both first derivatives also in $L^1(\tau)$. Assume $\alpha = 1/6$. Then*

$$\left| \int_\tau \Phi(x,y)(I - \mathcal{L}_\tau) f(x,y)\, d\tau \right| \leq c(r(\tau))\, h^3 \left[\int_\tau (|\Phi| + |D\Phi|)\, d\tau \right] \max_\tau \{|D^2 f|, |D^3 f|\} \tag{74}$$

with $c(r(\tau))$ some multiple of a power of $r(\tau)$.

(b) *Let τ_1 and τ_2 be two planar triangles of diameter h, with $R \equiv \tau_1 \cup \tau_2$ a parallelogram. Let $f \in C^4(R)$, and let $\Phi \in L^1(R)$ have all of its second derivatives also belong to $L^1(R)$. Assume $\alpha = 1/6$. Then*

$$\left| \int_R \Phi(x,y)(I - \mathcal{L}_\tau) f(x,y)\, d\tau \right| \leq c(r(R))\, h^4 \left[\int_R \sum_{i=0}^{2} |D^i \Phi|\, d\tau \right] \max_{\substack{R \\ i=2,3,4}} \{|D^i f|\} \tag{75}$$

with $r(R) = \max_{i=1,2} r(\tau_i)$ and $c(r(R))$ some multiple of a power of $r(R)$.

The above results will be applied to the individual subintegrals in

$$\mathcal{K}u(v_i) = \frac{\rho(v_i)}{\pi} \cdot \sum_{k=1}^{n} \int_\sigma G(v_i, m_k(s,t)) u(m_k(s,t)) \|(D_s m_k \times D_t m_k)(s,t)|\, d\sigma \tag{76}$$

with the role of f played by $u(m_k(s,t))|(D_s m_k \times D_t m_k)(s,t)|$ and the role of Φ played by $G(v_i, m_k(s,t))$. Before doing so, we need to examine the growth of $G(P,Q)$ as $Q \to P$. We omit the proof as it is relatively straightforward.

Lemma 3.5. *Assume that S is a smooth C^2 surface. Then*

$$|D_Q^2 G(P,Q)| \leq \frac{c}{|P-Q|^2}, \quad P \neq Q \tag{77}$$

for all second order derivatives D_Q^2 with respect to Q.

Theorem 3.6. *Assume the hypotheses of Theorem* 3.1*, with each parameterization function $F_j \in C^4$. Let $\alpha = 1/6$, and assume the triangulation of S satisfies* (73)*. Assume $u \in C^3(S)$. Then*

$$\max_{1 \le i \le 3n} |u(v_i) - \hat{u}_n(v_i)| \le c\, h^3. \tag{78}$$

Proof. We omit the proof since it is very similar to that of the following theorem. □

We generally restrict our triangulations to be of a *symmetric* type, which refers to the method of carrying out the refinement process. When a parameterization triangle $\hat{\Delta}_k$ is refined, we divide it into four new triangles by connecting the midpoints of the three sides. With this, the number of triangles in a triangulation increases by a factor of 4 with each refinement. More importantly, most of the triangles can be grouped as parallelograms. More precisely, such a grouping will contain all but $O(\sqrt{n}) = O(h^{-1})$ of the triangles. More discussion of this method of refinement is given in [**3, 4**] and [**5**].

Theorem 3.7. *Assume the hypotheses of Theorem* 3.1*, with each parameterization function $F_j \in C^5$. Let $\alpha = 1/6$, and assume $u \in C^4(S)$. Assume the triangulation $\mathcal{T}_n$ of S satisfies* (73)*, and further assume it is a symmetric triangulation. For those integrals in* (76) *for which $\nu_i \in \Delta_k$, assume that all such integrals are evaluated with an error $O(h^4)$. Then*

$$\max_{1 \le i \le 3n} |u(v_i) - \hat{u}_n(v_i)| \le c\, h^4 |\log h|. \tag{79}$$

Proof. Following (51), we bound

$$\max_{1 \le i \le 3n} |\mathcal{K}(I - \mathcal{P}_n) u(v_i)|$$

to prove (79). For a given node point v_i, remove from $\mathcal{T}_n$ the triangle Δ^* which contains ν_i, calling the remaining triangulation $\mathcal{T}_n^*$. By assumption, the error in evaluating the integral of (76) over Δ^* will be $O(h^4)$.

Divide the triangles in $\mathcal{T}_n^*$ into two classes. Partition $\mathcal{T}_n^*$ into parallelograms (actually parallelograms in the parameterization plane) to the maximum extant possible. Let $\mathcal{T}_n^{(1)}$ denote the set of all triangles making up such parallelograms, and let $\mathcal{T}_n^{(2)}$ contain the remaining triangles. It can be shown that the number of triangles in $\mathcal{T}_n^{(1)}$ is $O(n)$, and the number of triangles in $\mathcal{T}_n^{(2)}$ is $O(\sqrt{n})$. Moreover, all but a finite number of the triangles in $\mathcal{T}_n^{(2)}$, bounded independent of n, will be at a minimum distance $\epsilon > 0$ from ν_i with ε independent of n and i. Based on the decomposition (76), consider the error $\mathcal{K}(I - \mathcal{P}_n)u(v_i)$ as composed of the errors over each of the triangles in $\mathcal{T}_n^*$.

Consider first the contributions to the error coming from triangles in $\mathcal{T}_n^{(2)}$. Applying Lemma 3.2 or Corollary 3.4(a), the error over each such triangle is $O(h^5\|D^3u\|_\infty)$, based on each such triangle having area proportional to $O(h^2)$. Since there are $O(\sqrt{n}) = O(h^{-1})$ such triangles in $\mathcal{T}_n^*$, the total error contributed from triangles in $\mathcal{T}_n^{(2)}$ is $O(h^4\|D^3u\|_\infty)$.

Consider next the contributions to the error coming from triangles in $\mathcal{T}_n^{(1)}$. We apply either Lemma 3.3 or Corollary 3.4(b). This yields an error of size $O(h^4)$ multiplied times the integral over each such parallelogram of the maximum of the second derivatives of $K(\nu_i, Q)$ with respect to Q. Combining these, we will have a bound

$$c\,h^4 \int_{S\setminus\Delta^*} |\nu_i - Q|^{-2}\, dS_Q.$$

Using a local representation of the surface, and then using polar coordinates to evaluate the integral, it can be shown to be of order $O(\log h)$. Thus the error arising from considering the triangles in $\mathcal{T}_n^{(1)}$ is $O(h^4 \log h)$.

Combining the errors arising from the integrals over Δ^* and the triangles in $\mathcal{T}_n^{(1)}$ and $\mathcal{T}_n^{(2)}$, we have (79). □

Some numerical examples to illustrate the error bound in (79) are given in Section 5.

4. Collocation on piecewise smooth surfaces. With S only piecewise smooth, there are additional problems, some of which were indicated following (21) in Section 2. On the practical side, there is difficulty when evaluating the unit normal $\mathbf{n}_P$ to the surface at points P located on an edge or at a corner of S. In part, for that reason, we consider only those collocation methods for which the collocation points are not on an edge or at a corner of S. With this caveat, we use the same basic numerical schemes as in Section 3. The difficulty of handling the evaluation of the unit normal $\mathbf{n}_P$ can be handled in other ways; but our assumption simplifies the overall implementation of the collocation method. The main result Theorem 3.1 is still valid for most cases, but the method of proof must change. With S smooth, the operator $\mathcal{K}$ is compact on $C(S)$, and this was used crucially in the proof of Theorem 3.1; but the lack of smoothness of $G(P,Q)$ shown in (21) implies $\mathcal{K}$ is no longer compact, nor is any power of it compact.

Use the same definitions for the triangulation of S and the definition of the collocation method as that used in Section 3. Note that, for the piecewise linear interpolation of (34), we now restrict the interpolation parameter α to satisfy $0 < \alpha < (1/3)$, to satisfy the restriction of the last paragraph on the location of the collocation points. When we prescribe that the function $f \in C^r(S)$, we mean the following: (1) $f \in C(S)$; (2) With respect to the decomposition of S in (13)–(14), the restriction of f to S_j belongs to $C^r(S_j)$, for each $j = 1, \dots, J$.

Theorem 4.1. *Assume S is a piecewise smooth unoccluded surface in $\mathbf{R}^3$, and assume $S \subset \hat{S}$, with $\hat{S}$ the type of surface required in Lemma* 2.1*. Assume the surface S satisfies* (13)–(14) *with each $F_j \in C^3$. For the interpolation method of* (34)*, assume*

$$\|\mathcal{P}_n\| \|\mathcal{K}\| \le \gamma < 1, \quad n \ge n_0 \tag{80}$$

for some constant γ and some $n_0 > 0$. The norm $\|\mathcal{P}_n\|$ is given in (37)*, and a bound on $\|\mathcal{K}\|$ is given in* (17)*. Then for all sufficiently large n, say $n \ge n_0$, the operators $I - \mathcal{P}_n\mathcal{K}$ are invertible on $\mathcal{X}$ and have uniformly bounded inverses. Moreover, for the true solution u of* (1) *and the solution u_n of* (41)*,*

$$\|u - u_n\|_\infty \le \|(I - \mathcal{P}_n\mathcal{K})^{-1}\| \|u - \mathcal{P}_n u\|_\infty, \quad n \ge n_0. \tag{81}$$

Furthermore, if the emissivity $E \in C^2(S)$*, then*

$$\|u - u_n\|_\infty \le O(h^2), \quad n \ge n_0. \tag{82}$$

Proof. The existence of $(I - \mathcal{P}_n\mathcal{K})^{-1}$ and the proof of stability comes immediately from the geometric series theorem and (80), and

$$\|(I - \mathcal{P}_n\mathcal{K})^{-1}\| \le \frac{1}{1 - \|\mathcal{P}_n\|\|\mathcal{K}\|} \le \frac{1}{1-\gamma}, \quad n \ge n_0.$$

The remainder of the proof is straightforward and standard, and we omit it. □

In Section 3 there was superconvergence in the case that $\alpha = 1/6$. This is not true here, because

$$\|\mathcal{K}(I - \mathcal{P}_n)u\|_\infty = O(h^2) \tag{83}$$

in general. To see this, we consider only the example surface $S = S_{xy} \cup S_{xz}$ of (20), and we use only special cases of u.

We first look at $\mathcal{K}u(P)$ for $u \equiv 1$; and we only look at a portion of the integral. More precisely, consider $P = (x, 0, z) = (ah, 0, bh)$ with $0 < a, b < 1$, and consider the integral over only the portion $[0, h] \times [0, h]$ of the integration region S_{xy}. Letting $Q = (\xi, \eta, 0)$, we have the integral

$$\begin{aligned} I_h(a,b) &\equiv \int_0^h \int_0^h \frac{z\eta \, d\xi \, d\eta}{[(x-\xi)^2 + \eta^2 + z^2]^2} \\ &= \int_0^h \int_0^h \frac{bh\eta \, d\xi \, d\eta}{[(ah-\xi)^2 + \eta^2 + (bh)^2]^2}. \end{aligned}$$

Change the variables of integration using

$$\xi = sh, \qquad \eta = th, \quad 0 \le s,\ t \le 1. \tag{84}$$

Then

$$I_h(a,b) = \int_0^1 \int_0^1 \frac{bt \, ds \, dt}{[(a-s)^2 + t^2 + b^2]^2}$$

and the value of this integral is given earlier in (22). Thus $I_h(a,b) \equiv I(a,b)$ is independent of h.

Now consider the portion of $\mathcal{K}(I-\mathcal{P}_n)u(P)$ consisting of the integral over $[0,h]\times[0,h]\subset S_{sy}$, and call it $e_h(P)$:

$$e_h(x,0,z) = \int_0^h \int_0^h \frac{[u(\xi,\eta,0)-(\mathcal{P}_n u)(\xi,\eta,0)]z\eta\, d\xi\, d\eta}{[(x-\xi)^2+\eta^2+z^2]^2}.$$

Using the change of (84),

$$e_h(ah,0,bh) = \int_0^1 \int_0^1 \frac{[u(sh,th,0)-(\mathcal{P}_n u)(sh,th,0)]bt\, ds\, dt}{[(a-s)^2+t^2+b^2]^2}. \tag{85}$$

Let $u(\xi,\eta,0)$ be a quadratic polynomial in ξ,η, say $u=\xi^2$. Assume $[0,h]\times[0,h]$ consists of two triangles (say, both containing the origin), named Δ_1 and Δ_2. Then we can show

$$u(sh,th,0)-(\mathcal{P}_n u)(sh,th,0) = h^2 q_i(s,t), \quad i=1,2$$

with each q_i quadratic in s,t. Then

$$e_h(ah,0,bh) = h^2 \int_0^1 \int_0^1 \frac{q(s,t)\, bt\, ds\, dt}{[(a-s)^2+t^2+b^2]^2} \tag{86}$$

with $q(s,t)=q_1$ or q_2, depending on which triangle contains (s,t). This effectively shows the asserted result (83), although an argument needs to be made regarding the remaining part of $\mathcal{K}(I-\mathcal{P}_n)u(P)$ that does not include $e_h(P)$, to show it too goes to zero like $O(h^2)$ or faster.

4.1. Approximation of the boundary. If the boundary S is curved rather than polyhedral, then it is convenient to approximate S by interpolation, obtaining an approximate boundary $\hat{S}$. This is then used in the approximate calculation of the collocation integrals of (40), using the interpolatory surface in the approximate calculation of the Jacobian $|(D_s m_k \times D_t m_k)(s,t)|$ and the approximate calculation of the unit normals $\mathbf{n}_P$ and $\mathbf{n}_Q$. This is also commonly done with boundary integral equations in potential theory, and an example can be found in [**5**].

Given a triangulation $\mathcal{T}_n = \{\Delta_{n,k}\}$, with $m_k : \sigma \to \Delta_{n,k}$, it is typical to approximate $\Delta_{n,k}$ by interpolating m_k at an evenly spaced grid on σ. For a discussion of this, see [**4**, Section 5.3]. Let $\tilde{m}_k$ be an interpolatory approximation of m_k of degree r. We let $\tilde{\Delta}_{n,k} = \tilde{m}_k(\sigma)$ and $\tilde{S}_n = \cup_1^n \tilde{\Delta}_{n,k}$. Generally, the interpolation is so chosen that adjoining triangles $\Delta_{n,k}$ and $\Delta_{n,l}$ will have interpolates which also join continuously.

It is straightforward that

$$\max_{(s,t)\in\sigma} |m_k(s,t) - \tilde{m}_k(s,t)| = O(h^{r+1}) \tag{87}$$

$$\max_{(s,t)\in\sigma} |Dm_k(s,t) - D\tilde{m}_k(s,t)| = O(h^{r}) \tag{88}$$

for $D = D_s$ and D_t. Let $\mathbf{n}_k(s,t)$ denote the unit normal to $\Delta_{n,k}$ at $m_k(s,t)$. It is easily computed from

$$\mathbf{n}_k(s,t) = \frac{D_s m_k(s,t) \times D_t m_k(s,t)}{|D_s m_k(s,t) \times D_t m_k(s,t)|}$$

and it is approximated by

$$\tilde{\mathbf{n}}_k(s,t) = \frac{D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)}{|D_s \tilde{m}_k(s,t) \times D_t \tilde{m}_k(s,t)|} \tag{89}$$

It is straightforward that

$$\max_{(s,t)\in\sigma} |\mathbf{n}_k(s,t) - \tilde{\mathbf{n}}_k(s,t)| = O(h^{r}). \tag{90}$$

The error bounds of (87)–(90) are all uniform with respect to k and n.

Using $\tilde{S}_n$, we approximate the linear system (40) by

$$\begin{aligned} \tilde{u}_n(v_i) - \frac{\rho(v_i)}{\pi} \sum_{k=1}^{n} \sum_{j=1}^{3} \tilde{u}_n(v_{k,j}) \int_\sigma G(v_i, \tilde{m}_k(s,t))\, l_j(s,t) \\ \cdot |(D_s \tilde{m}_k \times D_t \tilde{m}_k)(s,t)|\, d\sigma = E(v_i), \\ i = 1, \ldots, 3n. \end{aligned} \tag{91}$$

The kernel $G(v_i, \tilde{m}_k(s,t))$ is based on using $\tilde{\mathbf{n}}_k(s,t)$ in place of $\mathbf{n}_k(s,t)$ in the definition (2). An error analysis can be based on regarding (91) as a perturbation of (40). Doing so leads to the following convergence result when S is a smooth surface.

Theorem 4.2. *Assume the hypotheses of Theorem* 3.1*, with each parameterization function* $F_j \in C^5$*. Let* $\alpha = 1/6$*, and assume* $u \in C^4(S)$*. Assume the triangulation* $\mathcal{T}_n$ *of* S *satisfies* (73)*, and further assume it is a symmetric triangulation. For those integrals in* (91) *for which* $\nu_i \in \Delta_k$*, assume that all such integrals are evaluated with an error* $O(h^4)$*. Then there is a unique and stable solution* $\tilde{u}_n$ *to the system* (91)*, and for its error,*

$$\max_{1 \le i \le 3n} |u(v_i) - \tilde{u}_n(v_i)| \le c \max\{h^4 |\log h|, h^r\}. \tag{92}$$

The proof is similar to that given in [**5**, Theorem 3.5], and we omit it here. With smoother kernel functions which do not involve the normal $\mathbf{n}$ to the surface, it is known that the use of the approximate surface with quadratic interpolation ($r = 2$) will usually result in an error of $O(h^4)$ or better, e.g., see [**4**, Section 5.4]. But here, the presence of $\mathbf{n}_P$ in the kernel $G(P,Q)$ of (2) leads to part of the approximation error containing $|\mathbf{n}_P - \tilde{\mathbf{n}}_P| = O(h^r)$ with no possibility of cancellation due to integration over S. Thus to retain the error of $h^4|\log h|$ associated with using the exact surface, it is necessary to use interpolation of degree 4 when approximating S using interpolation.

For piecewise smooth surfaces, the generalization of Theorem 4.1 to using $\tilde{S}_n \approx S$ leads to an error bound

$$\max_{1 \le i \le 3n} |u(v_i) - \tilde{u}_n(v_i)| \le c\, h^{\min\{2,r\}}. \tag{93}$$

This implies we should use quadratic interpolation to preserve the order of convergence associated with solving on the exact surface. The programs in the package [**3**] use quadratic interpolation to approximate S , and thus this package will preserve the accuracy expected when using piecewise linear collocation to solve the radiosity equation on a piecewise smooth surface S.

4.2. A preference for using $\alpha \neq 0$. If one uses $\alpha = 0$ in defining the piecewise linear interpolation of Section 3 and Section 4, then the collocation solution u_n is continuous over S. However, if the surface S is approximated and the normals are approximated as in (89), then there is a problem in defining the normal at collocation points which are common to more than one triangular face Δ_k. Moreover, if the surface S is only piecewise continuous, then there is a necessity to define a normal at edges of S, regardless of whether the surface is approximated or not. All of these problems are avoided if we choose $0 < \alpha < 1/3$, so that collocation points are interior to each triangular face. This greatly simplifies the programming. If, later, we want to have the solution u_n evaluated at points on nearer to an edge or on an edge of a triangular face Δ_k, then the interpolation formula (39) can be used to obtain such values.

5. Numerical examples. For the examples with a smooth surface, we use a "two-piece surface." Define

$$\begin{aligned} S_1 &= \{(x, y, 0) \mid 0 \leq x, y \leq 1\} \\ S_2^{(p)} &= \{(x, y, z) \mid 0 \leq x, y \leq 1,\ z = 2 - x^p\} \end{aligned} \tag{94}$$

and let $S^{(p)} = S_1 \cup S_2^{(p)}$. We use $S^{(2)}$ and $S^{(3)}$. In line with the proof of Theorem 3.7 and the error bound (51),

$$\max_{1 \leq i \leq 3n} |u(v_i) - u_n(v_i)| \leq c \max_{1 \leq i \leq 3n} |\mathcal{K}(I - \mathcal{P}_n)u(v_i)|$$

we examine numerically the quantity

$$\mathcal{E}_n(u) \equiv \max_{1 \leq i \leq 3n} |\mathcal{K}(I - \mathcal{P}_n)u(v_i)|.$$

For the surface $S^{(2)}$, let

$$u(x, y, z) = \frac{1}{\sqrt{(x^2 + y^2 + (z - 0.5)^2}} \tag{95}$$

In Table 1 we give $\mathcal{E}_n(u)$ for both $\alpha = 1/6$ and $\alpha = 0.1$. The true value of $\mathcal{K}u(P)$ was obtained by an alternative numerical integration method.

TABLE 1. Discretization error $\mathcal{E}_n(U)$ on $S^{(2)}$.

	$\alpha = (1/6)$		$\alpha = 0.1$	
n	$\mathcal{E}_n(u)$	Ratio	$\mathcal{E}_n(u)$	Ratio
4	1.48E−3		4.81E−3	
16	1.17E−4	12.7	4.39E−4	11.0
64	1.06E−5	11.0	1.62E−4	2.7
256	7.18E−7	14.8	4.42E−5	3.7
1024	5.06E−8	14.2	1.13E−5	3.9

These results illustrate the superconvergence obtained when $\alpha = 1/6$ is used in defining the linear interpolation scheme. The results for $\alpha = 0.1$ are consistent with a convergence rate of $O(h^2)$, predicted by Theorem 3.1, and the results for $\alpha = 1/6$ appear to agree with a convergence rate of $O(h^4 \log h)$, predicted by Theorem 3.7. The slight slowdown for $n = 1024$ is due probably to integration errors in computing some values of $\mathcal{K}u(\nu_i)$ or $\mathcal{K}\mathcal{P}_n u(\nu_i)$, or it may represent erratic progress towards the eventual ratio of 16.

TABLE 2. Error in solving radiosity equation on $S^{(2)}$.

	$\alpha = \frac{1}{6}$		$\alpha = 0.1$	
n	$\|u - u_n\|_\infty$	Ratio	$\|u - u_n\|_\infty$	Ratio
4	1.39E−3		1.24E−2	
16	2.90E−4	4.8	3.48E−3	3.6
64	2.93E−5	9.9	9.20E−4	3.8
256	2.10E−6	13.9	2.38E−4	3.9

We solve the radiosity equation (1) with the emissivity $E(P)$ so chosen that the true solution is

$$u(x, y, z) = x^2 + y^2 + z^2. \tag{96}$$

The reflectivity $\rho(P) \equiv 1$, and for solvability of $(I - \mathcal{K})u = E$, this is okay since $\|\mathcal{K}\| < 1$ due to the surface not being closed. The numerical results are given in Table 2, again for two values of α. The reader should note that the collocation system (40) being solved has order $3n$; and it is for that reason that we did not go to a larger value of n. In the table, we use

$$\|u - u_n\|_\infty = \max_{1 \le i \le 3n} |u(v_i) - u_n(v_i)|.$$

Again, the numerical results are consistent with the theoretical results of Theorems 3.1 and 3.7.

We want to illustrate the convergence result (92) for the effect of using an approximate surface based on quadratic interpolation ($r = 2$). In Table 3, we present both $\mathcal{E}_n(u)$ and $\|u - \tilde{u}_n\|_\infty$ for the two-piece surface $S^{(3)}$, with $\alpha = (1/6)$ and quadratic interpolation of the surface in (91). We use the function u of (96). The results clearly show a convergence rate of $O(h^2)$, which is consistent with (92).

TABLE 3. Errors with an approximate surface for $S^{(3)}$.

n	$\mathcal{E}_n(u)$	Ratio	$\|u - \tilde{u}_n\|_\infty$	Ratio
4	2.17E−1		2.27E−1	
16	3.26E−2	6.7	3.26E−2	7.0
64	1.09E−2	3.0	1.09E−2	3.0
256	2.80E−3	3.9	2.80E−3	3.9
1024	6.87E−4	4.1		

As a very simple piecewise smooth surface, we use the unit cube,

$$S = [0, 1] \times [0, 1] \times [0, 1].$$

Again we use the function u of (96), and we choose $\alpha = 1/6$. In Table 4, we present both $\mathcal{E}_n(u)$ and $\|u - u_n\|_\infty$. The reflectivity function is $\rho \equiv 0.5$. It is expected that the ratios for $\|u - u_n\|_\infty$ will approach 4 as n increases, as is more clearly the case for $\mathcal{E}_n(u)$. This is consistent with a rate of convergence of $O(h^2)$, as predicted in Theorem 4.1.

TABLE 4. Errors with the unit cube surface.

n	$\mathcal{E}_n(u)$	Ratio	$\|u - u_n\|_\infty$	Ratio
12	4.29E−3		8.26E−3	
48	1.04E−4	4.1	1.58E−3	5.2
192	1.80E−4	5.8	2.46E−4	6.4
768	4.26E−5	4.2		

We also illustrate the convergence to be expected when the collocation method is based on the centroid rule of (38). The problem being solved is the same as when the surface is only piecewise smooth, and Table 5 contains the numerical results. Note that the linear system being solved has order n, in contrast with that based on linear interpolation and having order $3n$. The error is clearly $O(h)$, which is consistent with piecewise constant interpolation. On a smooth surface with a smooth unknown function u, one would expect $O(h^2)$, with a proof similar to that given for Theorem 3.7. This is of special interest since much of the literature on the radiosity equation (1) uses numerical methods based on piecewise constant approximations.

TABLE 5. Errors with the centroid rule.

n	$\|u - u_n\|_\infty$	Ratio
12	9.12E−2	
48	2.91E−2	3.1
192	1.50E−2	1.9
768	7.91E−3	1.9

5.1. Practical remarks. The numerical methods were implemented by using the boundary element package described in [**3**]. This required modifying some of the numerical integration subroutines, but the main schema remained the same. In particular, the method for calculating the collocation integrals

$$\int_\sigma G(v_i, m_k(s,t))\, l_j(s,t) |(D_s m_k \times D_t m_k)(s,t)|\, d\sigma$$

of (40) was the same. As the distance between the field point ν_i and the triangular element Δ_k decreases, the kernel function $G(\nu_i, Q)$ becomes more ill-behaved. To compensate for this, the complexity of the integration was increased as ν_i approached Δ_k, so as to obtain about equivalent accuracy for all such integrals. The details of this process can be found in [**3**]. There are additional practical improvements which can be made when implementing the linear collocation method of this paper, but these will be left to a future paper dealing with the practical implementation of our ideas.

6. Concluding remarks. In this paper we have attempted to give some intuition on the use of collocation methods for solving the radiosity equation. It has been done for the simplest of cases, that of unoccluded surfaces. We so restricted it in order to make clearer the behavior of the approximation methods being used. In addition, we also wanted to show the difference in behavior between using smooth surfaces and using piecewise smooth surfaces. These effects will also make themselves known when using emissivity functions which are only piecewise smooth, and we expect to handle this more formally in a future paper. It should be inferred from the discussion near the end of Section 2 that the unknown function u is likely to be somewhat ill-behaved in the vicinity of edges and corners of a piecewise smooth surface; and this may require some grading of the mesh to compensate for this ill-behavior in u. We also will discuss in a future paper the more important case of occluded surfaces, which is a much more interesting case for real graphics applications.

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