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ABSTRACT HYPERBOLIC VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT. This paper is devoted to the study of the problem of global solvability for the abstract hyperbolic Volterra integrodifferential equation

$$\text{(VIE)} \quad \begin{cases} u'(t) = A(t)u(t) + \int_0^t g(t,s,u(s))\,ds + f(t) & \text{for } t \geq 0 \\ S(t)u(t) \in D & \text{for } t \geq 0 \\ u(0) = \phi \end{cases}$$

in a general Banach space X. The result obtained here is applicable to semilinear hyperbolic integrodifferential equations with the so-called third kind boundary conditions in a space of continuous functions.

0. Introduction. The main object of this paper is the study of global solvability for the semilinear hyperbolic Volterra integrodifferential equation

$$\text{(VIE)} \quad \begin{cases} u'(t) = A(t)u(t) + \int_0^t g(t,s,u(s))\,ds + f(t) & \text{for } t \geq 0 \\ S(t)u(t) \in D & \text{for } t \geq 0 \\ u(0) = \phi \end{cases}$$

in a general Banach space X. Here $\{A(t) : t \geq 0\}$ is a given family of bounded linear operators on Y to X, where Y is another Banach space continuously imbedded in X, D is a closed linear subspace in Y, $\{S(t) : t \geq 0\}$ is a given family of isomorphisms of X onto X, $g(t,s,w)$ is an X-valued function of $(t,s) \in \Delta := \{(t,s) : 0 \leq s \leq t < \infty\}$ and $w \in Y$, and $f \in C^1([0,\infty) : X)$.

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This is the abstract version of the semilinear hyperbolic Volterra integrodifferential equation

$$u_{tt}(t,x) = a(t,x)u_{xx}(t,x) + \int_0^t b(t,s,x,u(s,x),u_x(s,x),u_{xx}(s,x))\,ds$$

with the so-called third kind boundary condition

$$u_x(t,0) - \alpha(t)u(t,0) = u_x(t,1) + \beta(t)u(t,1) = 0.$$

It should be noted here that the inclusion $S(t)u(t) \in D$ appearing in (VIE) is used to represent such boundary conditions. (See the final part of Section 3.)

To handle equations involving differential operators with time-dependent and nondense domains, we employ a family $\{S(t) : t \geq 0\}$ of isomorphisms of X onto X and generalize the notion of Hille-Yosida operators to the time-dependent case, where we mean by a *Hille-Yosida operator* that it satisfies the assumptions of the Hille-Yosida theorem characterizing the infinitesimal generators of semigroups of class (C_0) except for the density of their domains. It should be noted that the study of inhomogeneous abstract Cauchy problems for Hille-Yosida operators was initiated by Da Prato and Sinestrari [**1**] and their results have been extended to the case of quasi-linear evolution equations by Tanaka [**11**] and that the study of time-dependent initial value problems by using such a family $\{S(t) : t \geq 0\}$ is found in the paper [**6**] by Kato.

In Section 2 we study the problem of existence of local solutions to (VIE). The result obtained here is a generalization of the recent result by Nagel and Sinestrari [**10**] concerning the problem (VIE) with $S(t) = I$, the identity operator on X, for a Hille-Yosida operator, see also [**9**]. Our proof is different from theirs and based on the "generalized variation of constants" formula, see Theorem 1.4. This formula together with some properties of integral term appearing in (VIE), see Lemma 1.5, will enable us to study the problem of existence of local solutions by the usual contraction arguments.

Section 3 contains the study of global solvability for (VIE). In [**5**] Hrusa discussed by the energy method the problem of the global existence of solutions of semilinear integrodifferential equations in the L^2 framework. We are interested in the operator-theoretical approach to the problem of this kind. This is a motivation of our work in this section, and a similar investigation to ours was carried out by Heard [**3, 4**] in the special case of $S(t) = I$. Finally we give an application of our abstract theory to the concrete semilinear integrodifferential equations with the third kind boundary conditions.

In the rest of this section we list the notation used in this paper. We denote by $B(Y, X)$ the set of all bounded linear operators on Y to X with the associated operator norm $\|\cdot\|_{Y,X}$. We use subscript $_*$ to refer to the strong operator topology in $B(Y, X)$; namely $C_*([a, b] : B(Y, X))$ is the space of all strongly continuous operator functions on Y to X, while $C([a, b] : B(Y, X))$ is the space of all norm-continuous functions. We write for simplicity $M_B(\tau) = \sup\{\|B(t)\| : t \in [0, \tau]\}$ if $B(\cdot) \in C_*([0, \tau] : B(X))$. The symbol $\{\mathcal{A}(t) : t \in [0, T_0]\} \in S_\sharp(X, M, \omega)$ means that $(\omega, \infty) \subset \rho(\mathcal{A}(t))$ for $t \in [0, T_0]$, and

$$\left\| \prod_{j=1}^{k} (\lambda I - \mathcal{A}(t_j))^{-1} \right\| \leq M(\lambda - \omega)^{-k} \quad \text{for } \lambda > \omega$$

and every finite sequence $\{t_j\}_{j=1}^k$ with $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k \leq T_0$ and $k = 1, 2, \ldots$. Products containing $\{t_j\}$ will always be "time-ordered," namely a factor with a larger t_j stands to the left of ones with smaller t_j. We say that this condition is the *stability condition* and the pair $\{M, \omega\}$ is the *stability index*.

1. Basic hypotheses and preliminaries. Throughout this paper, let X be a Banach space with norm $\|\cdot\|$, Y another Banach space with norm $\|\cdot\|_Y$ which is continuously imbedded in X and D a closed linear subspace in Y. We begin by setting up basic hypotheses on the operators $A(t)$, $S(t)$ and $g(t, s, w)$ appearing in the equation (VIE).

The family $\{A(t) : t \geq 0\}$ in $B(Y, X)$ satisfies conditions (a_1) through (a_3) below.

(a_1) For each $\tau > 0$ there is a constant $c_A(\tau) \geq 1$ such that

$$\|y\|_Y \leq c_A(\tau)(\|y\| + \|A(t)y\|) \quad \text{for } (t, y) \in [0, \tau] \times Y. \tag{1.1}$$

Since $A(t) \in B(Y, X)$ and D is closed in Y, it follows from condition (a_1) that for each $t \geq 0$, $A(t)|_D$ is a closed linear operator in X. Here $A(t)|_D$ is the restriction of an operator $A(t)$ to D.

(a$_2$) For each $\tau > 0$ there are constants $M(\tau) \geq 1$ and $\omega(\tau) \geq 0$ such that

$$\{A(t)|_D : t \in [0, \tau]\} \in S_\sharp(X, M(\tau), \omega(\tau)).$$

(a$_3$) $A(\cdot) \in C^1_*([0, \infty) : B(Y, X))$.

The family $\{S(t) : t \geq 0\}$ of isomorphisms of X onto X satisfies the following conditions.

(s$_1$) $S(\cdot) \in C^2_*([0, \infty) : B(X))$.

(s$_2$) There is a family $\{B(t) : t \geq 0\}$ in $B(X)$ such that

$$S(t)A(t)S(t)^{-1} = A(t) + B(t) \quad \text{for } t \geq 0 \tag{1.2}$$

and that $B(\cdot) \in C^1_*([0, \infty) : B(X))$.

The X-valued function $g(t, s, w)$ defined for $(t, s) \in \Delta$ and $w \in Y$ satisfies two conditions below:

(g$_1$) For each $w \in Y$, $g(t, s, w)$ is continuous in X on Δ. For each $\tau, r > 0$ there exists $L_g(\tau, r) > 0$ such that

$$\|g(t, s, w) - g(t, s, z)\| \leq L_g(\tau, r)\|w - z\|_Y \tag{1.3}$$

for $(t, s) \in \Delta(\tau) := \{(t, s) : 0 \leq s \leq t \leq \tau\}$ and $w, z \in B_Y(r) := \{w \in Y : \|w\|_Y \leq r\}$.

(g$_2$) For each $(s, w) \in [0, \infty) \times Y$, $g(t, s, w)$ is differentiable in t with $(t, s) \in \Delta$. For each $w \in Y$, $(\partial/\partial t)g(t, s, w)$ is continuous in X on Δ. For each $\tau, r > 0$ there exists $\tilde{L}_g(\tau, r) > 0$ such that

$$\|(\partial/\partial t)g(t, s, w) - (\partial/\partial t)g(t, s, z)\| \leq \tilde{L}_g(\tau, r)\|w - z\|_Y \tag{1.4}$$

for $(t, s) \in \Delta(\tau)$ and $w, z \in B_Y(r)$.

We shall investigate some properties of the family $\{S(t) : t \geq 0\}$.

Lemma 1.1. *The following assertions hold.*

(i) $S(\cdot)^{-1} \in C^1_*([0, \infty) : B(X))$ *and* $(d/dt)S(t)^{-1}x = -S(t)^{-1}\partial S(t) \times S(t)^{-1}x$ *for* $x \in X$ *and* $t \geq 0$. *Here and subsequently* $\partial S(t)x$ *denotes the derivative of* $S(t)x$.

(ii) $S(t)(Y) = Y$ *for* $t \geq 0$.

(iii) *Both* $S(\cdot)$ *and* $S(\cdot)^{-1}$ *belong to* $C_*([0,\infty) : B(Y))$.

Proof. Assertion (i) is an elementary fact. Relation (1.2) implies $\{x \in X : S(t)^{-1}x \in Y\} = Y$, and so (ii) is true. We note that $S(t), S(t)^{-1} \in B(Y)$ by the closed graph theorem. To prove (iii), let $y \in Y$ and $t_0 \geq 0$. We have by (1.1)

$$\begin{aligned}\|S(t)y - S(t_0)y\|_Y \leq c_A(t_0+1)(&\|S(t)y - S(t_0)y\| \\ &+ \|A(t)S(t)y - A(t_0)S(t_0)y\| \\ &+ \|(A(t_0) - A(t))S(t_0)y\|)\end{aligned}$$

for $t \in [0, t_0+1]$. By (1.2) we have $A(t)S(t)y = S(t)A(t)y - B(t)S(t)y$ for $t \geq 0$, which shows that $A(t)S(t)y$ is continuous in $t \geq 0$. Assertion (iii) follows from this fact, (a_3) and (s_1). Similarly we have $S(\cdot)^{-1} \in C_*([0,\infty) : B(Y))$. $\square$

Lemma 1.2. *We define a family* $\{\mathcal{A}(t) : t \geq 0\}$ *by*

$$\mathcal{A}(t) = A(t)|_D + B(t) + \partial S(t)S(t)^{-1}$$

for $t \geq 0$. *Then the following assertions hold.*

(A1) *For each* $t \geq 0$, $\mathcal{A}(t)$ *is a closed linear operator in* X *with domain* D.

(A2) *For each* $\tau > 0$ *we have*

$$\|y\|_Y \leq c_{\mathcal{A}}(\tau)(\|y\| + \|\mathcal{A}(t)y\|) \quad \text{for } (t,y) \in [0,\tau] \times D,$$

where $c_{\mathcal{A}}(\tau) = c_A(\tau)(1 + \sup\{\|B(t) + \partial S(t)S(t)^{-1}\| : t \in [0,\tau]\})$.

(A3) *For each* $\tau > 0$, $\{\mathcal{A}(t) : t \in [0,\tau]\} \in S_\sharp(X, M(\tau), \beta(\tau))$, *where* $\beta(\tau) = \omega(\tau) + M(\tau)\sup\{\|B(t) + \partial S(t)S(t)^{-1}\| : t \in [0,\tau]\}$.

(A4) $\mathcal{A}(\cdot) \in C^1_*([0,\infty) : B(D,X))$.

Proof. (A1) and (A4) are obvious. (A2) follows from (1.1) by an easy computation. To prove (A3), let $\tau > 0$. The fact that $(\beta(\tau), \infty) \subset \rho(\mathcal{A}(t))$ for $t \in [0,\tau]$ is proved by the identity

$$\begin{aligned}(\lambda I - \mathcal{A}(t))^{-1} &= (\lambda I - A(t)|_D)^{-1} \\ &\quad \cdot \sum_{n=0}^{\infty}((B(t) + \partial S(t)S(t)^{-1})(\lambda I - A(t)|_D)^{-1})^n\end{aligned} \tag{1.5}$$

for $\lambda > \beta(\tau)$ and $t \in [0,\tau]$. It is known [**11**, Lemma 1.1] that there is a family $\{|\cdot|_t : t \in [0,\tau]\}$ of norms on X such that

(n_1) $\|x\| \le |x|_t \le |x|_s \le M(\tau)\|x\|$ for $x \in X$ and $(t,s) \in \Delta(\tau)$,

(n_2) $|(\lambda I - A(t)|_D)^{-1}x|_t \le (\lambda - \omega(\tau))^{-1}|x|_t$ for $x \in X, t \in [0,\tau]$ and $\lambda > \omega(\tau)$.

The stability condition is shown as follows. Let $\{t_i\}_{i=1}^k$ be a finite sequence such that $0 = t_0 \le t_1 \le \cdots \le t_k \le \tau$ and $\lambda > \beta(\tau)$. By (n_1) and (n_2) we have

$$\begin{aligned}
&|(B(t) + \partial S(t)S(t)^{-1})(\lambda I - A(t)|_D)^{-1}x|_t \\
&\qquad\qquad \le M(\tau)\|(B(t) + \partial S(t)S(t)^{-1})(\lambda I - A(t)|_D)^{-1}x\| \\
&\qquad\qquad \le M(\tau)\|B(t) + \partial S(t)S(t)^{-1}\|(\lambda - \omega(\tau))^{-1}|x|_t
\end{aligned}$$

for $x \in X$ and $t \in [0,\tau]$. By this fact we estimate (1.5) to give

$$|(\lambda I - \mathcal{A}(t))^{-1}x|_t \le (\lambda - \beta(\tau))^{-1}|x|_t \tag{1.6}$$

for $x \in X$ and $t \in [0,\tau]$. Set $a_l := |\prod_{i=1}^{l}(\lambda I - \mathcal{A}(t_i))^{-1}x|_{t_l}$ for $l = 0,1,\ldots,k$. By (1.6) we have $a_l \le (\lambda - \beta(\tau))^{-1}|\prod_{i=1}^{l-1}(\lambda I - \mathcal{A}(t_i))^{-1}x|_{t_l} \le (\lambda - \beta(\tau))^{-1}a_{l-1}$ for $l = 1,2,\ldots,k$. Here we have used property (n_1). Solving this inequality we find $a_k \le (\lambda - \beta(\tau))^{-k}a_0$. The stability condition is proved by using (n_1) again. □

The following is a direct consequence of [**11**, Theorem 1.5].

Theorem 1.3. *Let* $\{\mathcal{A}(t) : t \ge 0\}$ *be the family defined as in Lemma* 1.2*. Then the limit* $\mathcal{U}(t,s)x = \lim_{\lambda\downarrow 0}\mathcal{U}_\lambda(t,s)x$ *exists for* $x \in \overline{D}$ *and* $(t,s) \in \Delta$*. Here* $\mathcal{U}_\lambda(t,s) = \prod_{i=[s/\lambda]+1}^{[t/\lambda]}(I - \lambda\mathcal{A}(i\lambda))^{-1}$ *for* $(t,s) \in \Delta$*, and* $\overline{D}$ *denotes the closure of* D *in* X*. The family* $\{\mathcal{U}(t,s) : (t,s) \in \Delta\}$ *satisfies the following properties*:

(e_1) $\mathcal{U}(t,s) : \overline{D} \to \overline{D}$ *for* $(t,s) \in \Delta$;

(e_2) $\mathcal{U}(t,t)x = x$ *and* $\mathcal{U}(t,r)\mathcal{U}(r,s)x = \mathcal{U}(t,s)x$ *for* $x \in \overline{D}$ *and* $(t,r),(r,s) \in \Delta$;

(e_3) *the mapping* $(t,s) \to \mathcal{U}(t,s)x$ *is continuous on* Δ *for any* $x \in \overline{D}$;

(e$_4$) *for each* $\tau > 0$, $\|\mathcal{U}(t,s)x\| \le M(\tau)\exp(\beta(\tau)(t-s))\|x\|$ *for* $x \in \overline{D}$ *and* $(t,s) \in \Delta(\tau)$.

We say that the family $\{\mathcal{U}(t,s) : (t,s) \in \Delta\}$ is the *evolution operator on* $\overline{D}$ *generated by* $\{\mathcal{A}(t) : t \ge 0\}$.

Our argument in this paper is based on the following.

Theorem 1.4. *Let* $\{\mathcal{A}(t) : t \ge 0\}$ *be the family defined as in Lemma* 1.2. *Let* $\tau > 0$, $s \in [0,\tau)$ *and* $h \in C^1([s,\tau] : X)$. *If the compatibility condition* $x \in D$ *and* $\mathcal{A}(s)x + h(s) \in \overline{D}$ *is satisfied, then the problem*

$$\text{(CP;}(s,x),h) \qquad \begin{cases} u'(t) = \mathcal{A}(t)u(t) + h(t) & \text{for } t \in [s,\tau] \\ u(s) = x \end{cases}$$

has a unique solution $u \in C([s,\tau] : D) \cap C^1([s,\tau] : X)$ *given by*

$$u(t) = \mathcal{U}(t,s)x + \lim_{\lambda\downarrow 0} \int_s^t \mathcal{U}_\lambda(t,r)h(r)\,dr \tag{1.7}$$

for $s \le t \le \tau$. *Moreover, the solution* u *satisfies the equation*

$$\begin{aligned} \mathcal{A}(t)u(t) + h(t) &= \mathcal{U}(t,s)(\mathcal{A}(s)x + h(s)) \\ &\quad + \lim_{\lambda\downarrow 0} \int_s^t \mathcal{U}_\lambda(t,r)(\partial\mathcal{A}(r)u(r) + h'(r))\,dr \end{aligned} \tag{1.8}$$

for $s \le t \le \tau$. *Here* $\{\mathcal{U}(t,s) : (t,s) \in \Delta\}$ *is the evolution operator on* $\overline{D}$ *generated by* $\{\mathcal{A}(t) : t \ge 0\}$.

Proof. We define a family $\{\widetilde{\mathcal{A}}(t) : t \in [0,\tau]\}$ by

$$\widetilde{\mathcal{A}}(t) = \begin{pmatrix} \mathcal{A}(t) & h(s\vee t) \\ 0 & 0 \end{pmatrix}$$

for $t \in [0,\tau]$. We note that $\{\widetilde{\mathcal{A}}(t) : t \in [0,\tau]\}$ is stable by [**11**, Lemma 1.6] and that

$$\widetilde{\mathcal{A}} \in W_*^{1,1}\left(0,\tau : B\left(\begin{pmatrix} D \\ \mathbf{R} \end{pmatrix}, \begin{pmatrix} X \\ \mathbf{R} \end{pmatrix}\right)\right).$$

It follows from [**11**, Theorem 1.8] that the limit $\widetilde{\mathcal{U}}(t,s)\binom{x}{1} = \lim_{\lambda\downarrow 0}\widetilde{\mathcal{U}}_\lambda(t,s)\binom{x}{1}$ exists in $\binom{D}{\mathbf{R}}$ because of the compatibility condition, where $\widetilde{\mathcal{U}}_\lambda(t,r) = \prod_{i=[r/\lambda]+1}^{[t/\lambda]}(I-\lambda\widetilde{\mathcal{A}}(i\lambda))^{-1}$ for $(t,r)\in\Delta(\tau)$, and that the mapping $t\to\widetilde{\mathcal{U}}(t,s)\binom{x}{1}$ is continuously differentiable in $\binom{X}{\mathbf{R}}$ and $(d/dt)\widetilde{\mathcal{U}}(t,s)\binom{x}{1} = \widetilde{\mathcal{A}}(t)\widetilde{\mathcal{U}}(t,s)\binom{x}{1}$ for $t\in[s,\tau]$. According to the device due to Kato [**8**, Subsection 1.3], the solution u of (CP;$(s,x),h$) is given by the first component of $\widetilde{\mathcal{U}}(t,s)\binom{x}{1}$; hence (1.7) holds since we find by [**11**, (1.15)],

$$\widetilde{\mathcal{U}}_\lambda(t,s) = \begin{pmatrix} \mathcal{U}_\lambda(t,s) & \int_{[s/\lambda]\lambda}^{[t/\lambda]\lambda}\mathcal{U}_\lambda(t,r)h(([r/\lambda]+1)\lambda)\,dr \\ 0 & 1 \end{pmatrix}$$

for $t\in[s,\tau]$. The desired identity (1.8) is obtained by substituting $\widetilde{\mathcal{U}}(t,s)\binom{x}{1} = \binom{u(t)}{1}$ into the equality given by [**11**] that

$$\widetilde{\mathcal{A}}(t)\widetilde{\mathcal{U}}(t,s)\begin{pmatrix}x\\1\end{pmatrix} = \lim_{\lambda\downarrow 0}\left(\widetilde{\mathcal{U}}_\lambda(t,s)\widetilde{\mathcal{A}}(s)\begin{pmatrix}x\\1\end{pmatrix} + \int_s^t \widetilde{\mathcal{U}}_\lambda(t,r)\partial\widetilde{\mathcal{A}}(r)\widetilde{\mathcal{U}}(r,s)\begin{pmatrix}x\\1\end{pmatrix}dr\right)$$

for $s\le t\le\tau$. □

Lemma 1.5. *Let $t_0\ge 0$ and $T\in(t_0,\infty)$. For each $w\in C(J:Y)$, where $J=[t_0,T)$ or $[t_0,T]$, we define $F: C(J:Y)\to X$ by*

$$(Fw)(t) = S(t)\int_{t_0}^t g(t,s,S(s)^{-1}w(s))\,ds \tag{1.9}$$

for $t\in J$. Then the following assertions hold.

(F1) *For each τ, $r>0$ there exists $L_F(\tau,r)>0$ such that*

$$\|(Fw)(t)-(Fz)(t)\| \le L_F(\tau,r)\int_{t_0}^t \|w(\sigma)-z(\sigma)\|_Y\,d\sigma \tag{1.10}$$

for $w,z\in C(J:B_Y(r))$ and $t\in J\cap[0,\tau]$.

(F2) *For each τ, $r>0$ there exists $M_F(\tau,r)>0$ such that*

$$\begin{gathered}\|(Fw)(t)\| \le (t-t_0)M_F(\tau,r)\\ \text{for } w\in C(J:B_Y(r)) \quad\text{and}\quad t\in J\cap[0,\tau].\end{gathered}\tag{1.11}$$

(F3) *For each* $w \in C(J:Y)$, *we have* $(Fw)(\cdot) \in C^1(J:X)$. *For each* τ, $r > 0$ *there exists* $\tilde{L}_F(\tau, r) > 0$ *such that*

$$\int_{t_0}^{t} \|(d/d\sigma)(Fw)(\sigma) - (d/d\sigma)(Fz)(\sigma)\| \, d\sigma \leq \tilde{L}_F(\tau, r) \int_{t_0}^{t} \|w(\sigma) - z(\sigma)\|_Y \, d\sigma \tag{1.12}$$

for $w, z \in C(J : B_Y(r))$ *and* $t \in J \cap [0, \tau]$.

(F4) *For each* τ, $r > 0$ *there exists* $\tilde{M}_F(\tau, r) > 0$ *such that*

$$\|(d/dt)(Fw)(t)\| \leq \tilde{M}_F(\tau, r) \quad \text{for } w \in C(J : B_Y(r)) \quad \text{and} \quad t \in J \cap [0, \tau]. \tag{1.13}$$

Proof. By (iii) of Lemma 1.1 we have $c(\tau) := \sup\{\|S(t)^{-1}\|_Y : t \in [0, \tau]\} < \infty$ for each $\tau > 0$. To prove (1.10), let $w, z \in C(J : B_Y(r))$. We have by (1.3)

$$\|(Fw)(t) - (Fz)(t)\| \leq M_S(\tau) L_g(\tau, c(\tau)r) \int_{t_0}^{t} \|S(s)^{-1}w(s) - S(s)^{-1}z(s)\|_Y \, ds$$

for $t \in J \cap [0, \tau]$; hence (1.10) holds with $L_F(\tau, r) = M_S(\tau) L_g(\tau, c(\tau)r) c(\tau)$. We note here that for each τ, $r > 0$ there exist $M_g(\tau, r) > 0$ and $\tilde{M}_g(\tau, r) > 0$ such that

$$\|g(t, s, w)\| \leq M_g(\tau, r) \quad \text{for } (t, s) \in \Delta(\tau) \quad \text{and} \quad w \in B_Y(r), \tag{1.14}$$

$$\|(\partial/\partial t) g(t, s, w)\| \leq \tilde{M}_g(\tau, r) \quad \text{for } (t, s) \in \Delta(\tau) \quad \text{and} \quad w \in B_Y(r). \tag{1.15}$$

This fact follows readily from conditions (g_1) and (g_2). Equation (1.11) follows immediately from (1.14). For $w \in C(J : Y)$ we have

$(Fw)(\cdot) \in C^1(J : X)$ and

$$(1.16) \quad (d/dt)(Fw)(t) = \partial S(t) \int_{t_0}^{t} g(t,s,S(s)^{-1}w(s))\,ds + S(t)\Big(g(t,t,S(t)^{-1}w(t)) + \int_{t_0}^{t} (\partial/\partial t) g(t,s,S(s)^{-1}w(s))\,ds\Big)$$

for $t \in J$. Similarly to the argument above, we have (1.12) by (1.3) and (1.4). The desired inequality (1.13) follows easily from (1.14) and (1.15). □

2. Local solvability. This section is devoted to the local solvability for (VIE). We begin by showing the existence and uniqueness of solutions of the problem

$$(\mathrm{VIE}; t_0, u_0) \qquad \begin{cases} u'(t) = A(t)u(t) & \text{for } t \geq t_0 \\ \qquad + \int_{t_0}^{t} g(t,s,u(s))\,ds + f_0(t) & \\ S(t)u(t) \in D & \text{for } t \geq t_0 \\ u(t_0) = u_0, & \end{cases}$$

where $(t_0, u_0) \in [0,\infty) \times Y$ and $f_0 \in C^1([t_0,\infty) : X)$. Let J be an interval of the form $[t_0, t_0+\tau)$ or $[t_0, t_0+\tau]$ with τ such that $0 < \tau < \infty$. A function u in the class $C(J : Y) \cap C^1(J : X)$ is said to be a *solution to* (VIE; t_0, u_0) *on* J if $u(t_0) = u_0$, $S(t)u(t) \in D$ for $t \in J$, and u satisfies $u'(t) = A(t)u(t) + \int_{t_0}^{t} g(t,s,u(s))\,ds + f_0(t)$ for $t \in J$. Such a solution to (VIE; t_0, u_0) is called a *local solution to* (VIE; t_0, u_0).

Proposition 2.1 *The following assertions hold.*

(i) *The* (VIE; t_0, u_0) *has at most one solution on any closed interval* $[t_0, T]$.

(ii) *The* (VIE; t_0, u_0) *has a local solution if the compatibility condition* $u_0 \in Y$, $S(t_0)u_0 \in D$ *and* $\partial S(t_0)u_0 + S(t_0)(A(t_0)u_0 + f_0(t_0)) \in \overline{D}$ *is satisfied.*

Remark. If u is a solution to (VIE; t_0, u_0) on $[t_0, T]$ then

$$(d/dt)S(t)u(t) = \partial S(t)u(t) + S(t)\left(A(t)u(t) + \int_{t_0}^{t} g(t,s,u(s))\,ds + f_0(t)\right)$$

for $t \in [t_0, T]$. Since $S(t)u(t) \in D$ for $t \in [t_0, T]$, the righthand side belongs to the set $\overline{D}$; hence $\partial S(t_0)u_0 + S(t_0)(A(t_0)u_0 + f_0(t_0)) \in \overline{D}$.

Proof. Let u be a solution to (VIE; t_0, u_0) on $[t_0, T]$, and set $w(t) = S(t)u(t)$ for $t \in [t_0, T]$. Clearly $w(\cdot) \in C([t_0, T] : D) \cap C^1([t_0, T] : X)$ by (iii) of Lemma 1.1, and we have by (1.2)

$$\begin{cases} w'(t) = \mathcal{A}(t)w(t) + (Fw)(t) + S(t)f_0(t) & \text{for } t \in [t_0, T] \\ w(t_0) = S(t_0)u_0, \end{cases}$$

where $\{\mathcal{A}(t) : t \geq 0\}$ is the family defined as in Lemma 1.2 and F is defined by (1.9). Since $S(t_0)u_0 \in D$ and $\mathcal{A}(t_0)S(t_0)u_0 + S(t_0)f_0(t_0) = w'(t_0) \in \overline{D}$; namely the compatibility condition is satisfied, we have by Theorem 1.4

$$w(t) = \mathcal{U}(t, t_0)S(t_0)u_0 + \lim_{\lambda\downarrow 0}\int_{t_0}^{t} \mathcal{U}_\lambda(t,\sigma)((Fw)(\sigma) + S(\sigma)f_0(\sigma))\,d\sigma \tag{2.1}$$

and

$$\begin{aligned} \mathcal{A}(t)w(t) = {}& \mathcal{U}(t, t_0)(\mathcal{A}(t_0)S(t_0)u_0 + S(t_0)f_0(t_0)) \\ & + \lim_{\lambda\downarrow 0}\int_{t_0}^{t} \mathcal{U}_\lambda(t,\sigma)(\partial\mathcal{A}(\sigma)w(\sigma) \\ & \qquad + (d/d\sigma)((Fw)(\sigma) + S(\sigma)f_0(\sigma)))\,d\sigma \\ & - (Fw)(t) - S(t)f_0(t) \end{aligned} \tag{2.2}$$

for $t \in [t_0, T]$. Here $\{\mathcal{U}(t, s) : (t, s) \in \Delta\}$ is the evolution operator on $\overline{D}$ generated by $\{\mathcal{A}(t) : t \geq 0\}$. To prove (i), let v be another solution to (VIE; t_0, u_0) on $[t_0, T]$, and set $z(t) = S(t)v(t)$ for $t \in [t_0, T]$. We represent the difference between w and z by (2.1) and (2.2), and estimate it by (A3) of Lemma 1.2, (1.10) and (1.12). This yields

$$\begin{aligned}\|w(t)-z(t)\| &\le \lim_{\lambda\downarrow 0}\int_{t_0}^{t} M(T)(1-\lambda\beta(T))^{-([t/\lambda]-[\sigma/\lambda])}\\ &\qquad\cdot\left(L_F(T,R_0)\int_{t_0}^{\sigma}\|w(s)-z(s)\|_Y\,ds\right)d\sigma\\ &\le N(T)L_F(T,R_0)(T-t_0)\int_{t_0}^{t}\|w(s)-z(s)\|_Y\,ds\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{A}(t)(w(t)-z(t))\| &\le (N(T)(M_{\partial\mathcal{A}}(T)+\tilde{L}_F(T,R_0))\\ &\qquad + L_F(T,R_0))\int_{t_0}^{t}\|w(\sigma)-z(\sigma)\|_Y\,d\sigma\end{aligned}$$

for $t\in[t_0,T]$, where we set $R_0=\sup\{\|w(t)\|_Y\vee\|z(t)\|_Y : t\in[t_0,T]\}$. Here and subsequently we use the notation

$$\begin{gathered}N(\tau)=M(\tau)\exp(\beta(\tau)\tau),\\ M_{\partial\mathcal{A}}(\tau)=\sup\{\|\partial\mathcal{A}(t)y\| : y\in D, \|y\|_Y\le 1, t\in[0,\tau]\}.\end{gathered}$$

Adding these inequalities, and using (A2) we obtain the inequality of Gronwall type; hence $w=z$ on $[t_0,T]$. By the injectivity of $S(t)$ we have $u=v$ on $[t_0,T]$.

We shall prove assertion (ii) by the Picard-Banach fixed point theorem. To do so, let $T_0>t_0$ be fixed and choose $R>0$ so that $\|S(t_0)u_0\|_Y<R$. We now set $r_0=R-\|S(t_0)u_0\|_Y$, and define a set E by

$$E=\{w\in C([t_0,T]:D) : \|w(t)-S(t_0)u_0\|_Y\le r_0 \text{ for } t\in[t_0,T]\}.$$

Here $T\in(t_0,T_0]$ is yet to be determined. Clearly, E is a complete metric space with metric d defined by $d(w,z)=\sup\{\|w(t)-z(t)\|_Y : t\in[t_0,T]\}$. Let $w\in E$. By (F3) of Lemma 1.5, $(Fw)(\cdot)\in C^1([t_0,T]:X)$. Condition (s_1) implies $S(t)f_0(t)\in C^1([t_0,T]:X)$, since $f_0\in C^1([t_0,\infty):X)$. By (1.2) we have $\mathcal{A}(t_0)S(t_0)u_0+S(t_0)f_0(t_0)=S(t_0)A(t_0)u_0+\partial S(t_0)u_0+S(t_0)f_0(t_0)\in\overline{D}$. From Theorem 1.4 we deduce that the problem

$$\begin{cases}\tilde{w}'(t)=\mathcal{A}(t)\tilde{w}(t)+(Fw)(t)+S(t)f_0(t) & \text{for } t\in[t_0,T]\\ \tilde{w}(t_0)=S(t_0)u_0\end{cases}$$

has a unique solution $\tilde{w} \in C([t_0, T] : D) \cap C^1([t_0, T] : X)$. This fact enables us to define a mapping Ψ from E into $C([t_0, T] : D)$ by $\Psi w = \tilde{w}$. Since $(\Psi w)(\cdot)$ is a solution of (CP;$(t_0, S(t_0)u_0), (Fw)(\cdot) + S(\cdot)f_0(\cdot)$), we have by (1.7)

$$\begin{aligned}\|(\Psi w)(t) - S(t_0)u_0\| \le \sup\{\|\mathcal{U}(t, t_0)S(t_0)u_0 - S(t_0)u_0\| : t \in [t_0, T]\}\\ + \int_{t_0}^{t} N(T_0)\|(Fw)(\sigma) + S(\sigma)f_0(\sigma)\|\, d\sigma\end{aligned}$$

for $t \in [t_0, T]$. We note that $w(t) \in B_Y(R)$ for $t \in [t_0, T]$. By (1.11), the last term on the righthand side is bounded by

$$(T - t_0)N(T_0)((T_0 - t_0)M_F(T_0, R) + M_S(T_0)\sup\{\|f_0(t)\| : t \in [t_0, T_0]\})$$

for $t \in [t_0, T]$. We have by (1.8)

$$\begin{aligned}&\|\mathcal{A}(t)((\Psi w)(t) - S(t_0)u_0)\|\\ &\quad\le \sup\{\|(\mathcal{U}(t, t_0) - I)(\mathcal{A}(t_0)S(t_0)u_0 + S(t_0)f_0(t_0))\| : t \in [t_0, T]\}\\ &\qquad + \|(\mathcal{A}(t_0) - \mathcal{A}(t))S(t_0)u_0\| + \|S(t_0)f_0(t_0) - S(t)f_0(t)\|\\ &\qquad + \int_{t_0}^{t} N(T_0)\|\partial\mathcal{A}(\sigma)(\Psi w)(\sigma) + (d/d\sigma)((Fw)(\sigma) + S(\sigma)f_0(\sigma))\|\, d\sigma\\ &\qquad + \|(Fw)(t)\|\end{aligned}$$

for $t \in [t_0, T]$. By (1.13) the fourth term is estimated by

$$\begin{aligned}N(T_0)M_{\partial\mathcal{A}}(T_0)\int_{t_0}^{t} \|(\Psi w)(\sigma) - S(t_0)u_0\|_Y\, d\sigma\\ + (T - t_0)N(T_0)(M_{\partial\mathcal{A}}(T_0)R + \tilde{M}_F(T_0, R)\\ + \sup\{\|(d/dt)S(t)f_0(t)\| : t \in [t_0, T_0]\}).\end{aligned}$$

By (1.11) the last term is dominated by $(T - t_0)M_F(T_0, R)$. Combining these estimates and using (A2) we obtain the inequality

$$\|(\Psi w)(t) - S(t_0)u_0\|_Y \le \varepsilon(T) + C\int_{t_0}^{t} \|(\Psi w)(\sigma) - S(t_0)u_0\|_Y\, d\sigma$$

for $t \in [t_0, T]$, where C is a positive constant and $\{\varepsilon(T)\}$ is a positive sequence with $\lim_{T \downarrow t_0} \varepsilon(T) = 0$. Here we have used property (e_3) of

Theorem 1.3. We therefore find a $T \in (t_0, T_0]$ such that $\Psi(E) \subset E$. To show that Ψ is a contraction on E, let $w, z \in E$. Similarly to the proof of (i), we find by using (1.10) and (1.12) with $(\tau, r) = (T_0, R)$

$$\|(\Psi w)(t) - (\Psi z)(t)\| \leq N(T_0) L_F(T_0, R)(T_0 - t_0) \int_{t_0}^{t} \|w(\sigma) - z(\sigma)\|_Y \, d\sigma$$

and

$$\begin{aligned} &\|\mathcal{A}(t)((\Psi w)(t) - (\Psi z)(t))\| \\ &\qquad \leq N(T_0) \int_{t_0}^{t} (M_{\partial\mathcal{A}}(T_0) \|(\Psi w)(\sigma) - (\Psi z)(\sigma)\|_Y \\ &\qquad\qquad + \tilde{L}_F(T_0, R) \|w(\sigma) - z(\sigma)\|_Y) \, d\sigma \\ &\qquad + \int_{t_0}^{t} L_F(T_0, R) \|w(\sigma) - z(\sigma)\|_Y \, d\sigma \end{aligned}$$

for $t \in [t_0, T]$. Adding two inequalities, and using property (A2) we have

$$\begin{aligned} \|(\Psi w)(t) - (\Psi z)(t)\|_Y \leq\ & c(T_0)(T - t_0) d(\Psi w, \Psi z) \\ &+ c(T_0, R)(T - t_0) d(w, z) \end{aligned}$$

for $t \in [t_0, T]$, where we set $c(T_0) = c_{\mathcal{A}}(T_0) N(T_0) M_{\partial\mathcal{A}}(T_0)$ and

$$\begin{aligned} c(T_0, R) = c_{\mathcal{A}}(T_0)(&N(T_0) L_F(T_0, R)(T_0 - t_0) \\ &+ N(T_0) \tilde{L}_F(T_0, R) + L_F(T_0, R)); \end{aligned}$$

hence

$$d(\Psi w, \Psi z) \leq \frac{c(T_0, R)(T - t_0)}{1 - c(T_0)(T - t_0)} \, d(w, z).$$

This shows that Ψ is a contraction on E for a smaller $T \in (t_0, T_0]$. By the fixed point theorem there is a $w \in C([t_0, T] : D)$ satisfying the differential equation $w'(t) = \mathcal{A}(t) w(t) + (Fw)(t) + S(t) f_0(t)$ for $t \in [t_0, T]$, with initial condition $w(t_0) = S(t_0) u_0$. By using some properties of $\{S(t) : t \geq 0\}$ of Lemma 1.1 we see that the desired solution u is given by $u(t) = S(t)^{-1} w(t)$ for $t \in [t_0, T]$. $\square$

The main theorem in this section is provided by

Theorem 2.2. *If $\phi \in Y$, $S(0)\phi \in D$ and $\partial S(0)\phi + S(0)(A(0)\phi + f(0)) \in \overline{D}$, then there are a $t_{\max} \in (0,\infty]$ and a unique solution u to* (VIE) *on $[0, t_{\max})$ satisfying either*

(i) $t_{\max} = \infty$, *or*

(ii) $t_{\max} < \infty$ *and* $\limsup_{t\uparrow t_{\max}} \|u(t)\|_Y = \infty$.

Proof. By (ii) of Proposition 2.1, we define

$$t_{\max} = \sup\{T > 0 : \text{ the (VIE) has a solution on } [0,T]\}.$$

Clearly, $t_{\max} \in (0,\infty]$. Assertion (i) of Proposition 2.1 and the definition of $t_{\max}$ together imply that there is a unique solution u of (VIE) on $[0, t_{\max})$. If $t_{\max} = \infty$ then the proof is complete.

Now, assume $t_{\max} < \infty$. We have only to show $\limsup_{t\uparrow t_{\max}} \|u(t)\|_Y = \infty$. If this is false, then there is an $r_0 > 0$ such that $\|u(t)\|_Y \le r_0$ for $t \in [0, t_{\max})$. If we set $w(t) = S(t)u(t)$ for $t \in [0, t_{\max})$, then we have

$$\|w(t)\|_Y \le R := \sup\{\|S(t)\|_Y : t \in [0, t_{\max}]\} r_0, \quad \text{for } t \in [0, t_{\max}). \tag{2.3}$$

As in the proof of (i) of Proposition 2.1, we have

$$\begin{aligned} w(t) &= \mathcal{U}(t,s)S(s)u(s) \\ &\quad + \lim_{\lambda\downarrow 0} \int_s^t \mathcal{U}_\lambda(t,\sigma)((F_0 w)(\sigma) + S(\sigma)f(\sigma))\, d\sigma \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \mathcal{A}(t)w(t) &= \mathcal{U}(t,s)(\mathcal{A}(s)S(s)u(s) + S(s)f(s)) \\ &\quad + \lim_{\lambda\downarrow 0} \int_s^t \mathcal{U}_\lambda(t,\sigma)(\partial\mathcal{A}(\sigma)w(\sigma) \\ &\qquad + (d/d\sigma)((F_0 w)(\sigma) + S(\sigma)f(\sigma)))\, d\sigma \\ &\quad - (F_0 w)(t) - S(t)f(t) \end{aligned} \tag{2.5}$$

for $0 \le s \le t < t_{\max}$. Here $\{\mathcal{U}(t,s) : (t,s) \in \Delta\}$ is the evolution operator on $\overline{D}$ generated by $\{\mathcal{A}(t) : t \ge 0\}$, and $(F_0 w)(t) = S(t)\int_0^t g(t,\sigma,S(\sigma)^{-1}w(\sigma))\, d\sigma$ for $t \in [0, t_{\max})$. By (2.4) we have

$$\begin{aligned} \|w(t) - w(\hat{t})\| &\le \|(\mathcal{U}(t,s) - \mathcal{U}(\hat{t},s))S(s)u(s)\| \\ &\quad + \left(\int_s^t + \int_s^{\hat{t}}\right) N(t_{\max})(\|(F_0 w)(\sigma)\| + \|S(\sigma)f(\sigma)\|)\, d\sigma \end{aligned} \tag{2.6}$$

for $0 \leq s \leq t$, $\hat{t} < t_{\max}$. Since $S(s)u(s) \in D$, the first term on the righthand side tends to zero as $t, \hat{t} \uparrow t_{\max}$. Noting (2.3) we deduce from (1.11) that the last term is bounded by

$$\begin{aligned} &2(t_{\max} - s)N(t_{\max})(M_F(t_{\max}, R)t_{\max} \\ &\qquad + M_S(t_{\max}) \sup\{\|f(t)\| : t \in [0, t_{\max}]\}). \end{aligned}$$

Taking the limsup as $t, \hat{t} \uparrow t_{\max}$ in (2.6), and letting $s \uparrow t_{\max}$ we have

$$\lim_{t,\hat{t}\uparrow t_{\max}} \|w(t) - w(\hat{t})\| = 0. \tag{2.7}$$

We represent the difference between $\mathcal{A}(t)w(t)$ and $\mathcal{A}(\hat{t})w(\hat{t})$ by (2.5), and estimate it by (1.13). This yields

$$\begin{aligned} &\|\mathcal{A}(t)(w(t) - w(\hat{t}))\| \\ &\quad \leq \|(\mathcal{U}(t, s) - \mathcal{U}(\hat{t}, s))(\mathcal{A}(s)S(s)u(s) + S(s)f(s))\| \\ &\quad\quad + 2(t_{\max} - s)N(t_{\max})(M_{\partial\mathcal{A}}(t_{\max})R \\ &\quad\quad + \tilde{M}_F(t_{\max}, R) + C(t_{\max})) \\ &\quad\quad + \|(F_0 w)(t) - (F_0 w)(\hat{t})\| \\ &\quad\quad + \|S(t)f(t) - S(\hat{t})f(\hat{t})\| + \|(\mathcal{A}(t) - \mathcal{A}(\hat{t}))w(\hat{t})\| \end{aligned}$$

for $0 \leq s \leq t, \hat{t} < t_{\max}$, where we set $C(t_{\max}) = \sup\{\|(d/dt)S(t)f(t)\| : t \in [0, t_{\max}]\}$. The third term on the righthand side is equal to

$$\left\|(S(t) - S(\hat{t})) \int_0^t g(t, \sigma, u(\sigma))\, d\sigma + S(\hat{t}) \int_{\hat{t}}^t g(t, \sigma, u(\sigma))\, d\sigma\right\|,$$

which is estimated by $|t - \hat{t}|(M_{\partial S}(t_{\max})t_{\max} + M_S(t_{\max}))M_g(t_{\max}, r_0)$. Here we have used (1.14). The last term is dominated by $M_{\partial\mathcal{A}}(t_{\max})R|t-\hat{t}|$. From these estimates it follows that

$$\begin{aligned} &\limsup_{t,\hat{t}\uparrow t_{\max}} \|\mathcal{A}(t)(w(t) - w(\hat{t}))\| \\ &\qquad \leq 2(t_{\max} - s)N(t_{\max})(M_{\partial\mathcal{A}}(t_{\max})R \\ &\qquad\quad + \tilde{M}_F(t_{\max}, R) + C(t_{\max})) \end{aligned}$$

for $0 \leq s < t_{\max}$. The righthand side tends to zero as $s \uparrow t_{\max}$, and so this fact combined with (2.7) implies that the limit $w_* = \lim_{t\uparrow t_{\max}} w(t)$

exists in Y and $w_* \in D$ since D is a closed linear subspace in Y. If we set $u_* = S(t_{\max})^{-1}w_*$ and define the value of u at $t_{\max}$ by u_* then $u \in C([0, t_{\max}] : Y)$. We note that $u_* \in Y$, $S(t_{\max})u_* = w_* \in D$ and $\partial S(t_{\max})u_* + S(t_{\max})(A(t_{\max})u_* + \tilde{f}(t_{\max})) = \lim_{t \uparrow t_{\max}}(S(t)u(t))' \in \overline{D}$, where $\tilde{f}(t) := \int_0^{t_{\max}} g(t, \sigma, u(\sigma))\, d\sigma + f(t)$ for $t \in [t_{\max}, \infty)$. Since $\tilde{f}$ is continuously differentiable on $[t_{\max}, \infty)$, we deduce from (ii) of Proposition 2.1 that the problem

$$\begin{cases} \tilde{u}'(t) = A(t)\tilde{u}(t) + \int_{t_{\max}}^t g(t, s, \tilde{u}(s))\, ds + \tilde{f}(t) & \text{for } t \geq t_{\max} \\ S(t)\tilde{u}(t) \in D & \text{for } t \geq t_{\max} \\ \tilde{u}(t_{\max}) = u_* \end{cases}$$

has a solution $\tilde{u}$ on $[t_{\max}, t_{\max} + \delta]$ for some $\delta > 0$. The solution u on $[0, t_{\max})$ can be extended to the larger interval $[0, t_{\max} + \delta]$ by defining $u = \tilde{u}$ on $[t_{\max}, t_{\max} + \delta]$. This is a contradiction to the definition of $t_{\max}$. $\square$

3. Global solvability and its application. In this section we shall give a sufficient condition for the global solvability for (VIE).

Theorem 3.1. *For each $\phi \in Y$, $S(0)\phi \in D$ and $\partial S(0)\phi + S(0)(A(0)\phi + f(0)) \in \overline{D}$, the* (VIE) *has a global solution if the following condition is satisfied.*

For each $\tau > 0$ there are constants $K(\tau) > 0$ and $L(\tau) > 0$ such that

$$\|g(t, s, w)\| + \|(\partial/\partial t)g(t, s, w)\| \leq K(\tau)\|w\|_Y + L(\tau) \tag{3.1}$$

for $(t, s) \in \Delta(\tau)$ and $w \in Y$.

Proof. By Theorem 2.2 there is a $t_{\max} \in (0, \infty]$ and a unique solution u to (VIE) on $[0, t_{\max})$ satisfying either (i) or (ii) of Theorem 2.2. Assume to the contrary that $t_{\max} < \infty$. If we set $w(t) = S(t)u(t)$ for $t \in [0, t_{\max})$ then w is a solution of (CP;$(0, S(0)\phi), (Gw)(\cdot)$), where G is defined by

$$(Gw)(t) = S(t)\bigg(\int_0^t g(t, s, S(s)^{-1}w(s))\, ds + f(t)\bigg)$$

for $t \in [0, t_{\max})$. Using (1.16) we find by (3.1)

$$\|(Gw)(t)\| + \int_0^t \|(d/ds)(Gw)(s)\|\, ds \le c_1 + c_2 \int_0^t \|w(s)\|_Y\, ds \tag{3.2}$$

for $t \in [0, t_{\max})$, where c_i are positive constants. By Theorem 1.4 we have

$$\|w(t)\| \le N(t_{\max})\|S(0)\phi\| + \int_0^t N(t_{\max})\|(Gw)(s)\|\, ds$$

and

$$\begin{aligned}\|\mathcal{A}(t)w(t)\| \le{}& N(t_{\max})\|\mathcal{A}(0)S(0)\phi + S(0)f(0)\| \\ &+ \int_0^t N(t_{\max})(M_{\partial\mathcal{A}}(t_{\max})\|w(s)\|_Y \\ &+ \|(d/ds)(Gw)(s)\|)\, ds + \|(Gw)(t)\|\end{aligned}$$

for $t \in [0, t_{\max})$. The claim that $\|w(t)\|_Y$ is bounded on $[0, t_{\max})$ follows easily from the inequality of Gronwall type obtained by combining these inequalities and (3.2), and using property (A2). This contradicts assertion (ii) of Theorem 2.2. $\square$

Finally we shall give an application of our abstract results to the following semilinear integrodifferential equation with the third kind boundary condition:

(3.3)

$$\begin{cases} u_{tt}(t,x) = a(t,x)u_{xx}(t,x) \\ \qquad + \int_0^t b(t,s,x,u(s,x),u_x(s,x),u_{xx}(s,x))\, ds \\ \qquad\qquad\qquad \text{for } (t,x) \in [0,\infty) \times [0,1], \\ u_x(t,0) - \alpha(t)u(t,0) \\ \qquad = u_x(t,1) + \beta(t)u(t,1) = 0 \qquad \text{for } t \in [0,\infty), \\ u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x) \qquad \text{for } x \in [0,1]. \end{cases}$$

Here a is of class C^1 satisfying $a(t,x) \ge a_0 > 0$ for $(t,x) \in [0,\infty)\times[0,1]$, and α and β are of class C^2. The function b from $\Delta\times[0,1]\times\mathbf{R}\times\mathbf{R}\times\mathbf{R}$ to $\mathbf{R}$ satisfies the following properties:

(b_1) b is continuous on $\Delta \times [0,1] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$. For each $\tau, r > 0$ there exists $L_b(\tau, r) > 0$ such that

$$|b(t,s,x,\xi_1,\eta_1,\sigma_1) - b(t,s,x,\xi_2,\eta_2,\sigma_2)| \\ \leq L_b(\tau,r)(|\xi_1-\xi_2| + |\eta_1-\eta_2| + |\sigma_1-\sigma_2|)$$

for $(t,s) \in \Delta(\tau)$, $x \in [0,1]$ and $|\xi_i| + |\eta_i| + |\sigma_i| \leq r$.

(b_2) For each $(s,x,\xi,\eta,\sigma) \in [0,\infty) \times [0,1] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, b is differentiable in $t \geq s$, and $(\partial/\partial t)b(t,s,x,\xi,\eta,\sigma)$ is continuous on $\Delta \times [0,1] \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}$.

For each $\tau, r > 0$ there exists $\tilde{L}_b(\tau, r) > 0$ such that

$$|(\partial/\partial t)b(t,s,x,\xi_1,\eta_1,\sigma_1) - (\partial/\partial t)b(t,s,x,\xi_2,\eta_2,\sigma_2)| \\ \leq \tilde{L}_b(\tau,r)(|\xi_1-\xi_2| + |\eta_1-\eta_2| + |\sigma_1-\sigma_2|)$$

for $(t,s) \in \Delta(\tau)$, $x \in [0,1]$ and $|\xi_i| + |\eta_i| + |\sigma_i| \leq r$.

We are interested in getting a solution of (3.3) such that u_t, u_{tt}, u_x and u_{xx} are continuous on $[0,\infty) \times [0,1]$ and that (3.3) holds pointwise in $[0,\infty) \times [0,1]$.

Theorem 3.2. *Assume that $\varphi \in C^2[0,1]$ and $\psi \in C^1[0,1]$ satisfy the compatibility condition*

$$\begin{cases} \varphi'(0) - \alpha(0)\varphi(0) = \varphi'(1) + \beta(0)\varphi(1) = 0, \\ \psi'(0) - \alpha(0)\psi(0) - \alpha'(0)\varphi(0) \\ \quad = \psi'(1) + \beta(0)\psi(1) + \beta'(0)\varphi(1) = 0. \end{cases} \tag{3.4}$$

Then there exist a $t_{\max} \in (0,\infty]$ and a unique solution u of (3.3) *in the class $C([0,t_{\max}) : C^2[0,1]) \cap C^1([0,t_{\max}) : C^1[0,1]) \cap C^2([0,t_{\max}) : C[0,1])$. Moreover,* (3.3) *has a unique global solution if the following additional condition is satisfied.*

For each $\tau > 0$, there are constants $K(\tau) > 0$ and $L(\tau) > 0$ such that

$$|b(t,s,x,\xi,\eta,\sigma)| + |(\partial/\partial t)b(t,s,x,\xi,\eta,\sigma)| \\ \leq K(\tau)(|\xi| + |\eta| + |\sigma|) + L(\tau) \tag{3.5}$$

for all $(t,s) \in \Delta(\tau)$, $x \in [0,1]$ and ξ, η, $\sigma \in \mathbf{R}$.

Proof. Let $X = C^1[0,1] \times C[0,1]$ and $Y = C^2[0,1] \times C^1[0,1]$. Clearly Y is continuously imbedded in X, where the norms $\|\cdot\|$ and $\|\cdot\|_Y$ are defined by

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = \|u\|_{C^1[0,1]} \vee \|v\|_{C[0,1]} \quad \text{for } \begin{pmatrix} u \\ v \end{pmatrix} \in X$$

and

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_Y = \|u\|_{C^2[0,1]} \vee \|v\|_{C^1[0,1]} \quad \text{for } \begin{pmatrix} u \\ v \end{pmatrix} \in Y$$

respectively. We now define a family $\{A(t) : t \geq 0\}$ in $B(Y, X)$ and an X-valued function g by

$$\left(A(t) \begin{pmatrix} u \\ v \end{pmatrix} \right)(x) = \begin{pmatrix} 0 & 1 \\ a(t,x)\partial_x^2 & 0 \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \quad \text{for } \begin{pmatrix} u \\ v \end{pmatrix} \in Y$$

and

$$g\left(t, s, \begin{pmatrix} u \\ v \end{pmatrix} \right)(x) = \begin{pmatrix} 0 \\ b(t,s,x,u(x),u'(x),u''(x)) \end{pmatrix} \quad \text{for } \begin{pmatrix} u \\ v \end{pmatrix} \in Y,$$

respectively. It is seen that (3.3) is reduced to the abstract integrodifferential equation (VIE), by choosing the closed linear subspace $D = \{(u,v) \in C^2[0,1] \times C^1[0,1] : u'(0) = u'(1) = 0\}$ in Y and a family $\{S(t) : t \geq 0\}$ in $B(X)$ defined by

$$\left(S(t) \begin{pmatrix} u \\ v \end{pmatrix} \right)(x) = \begin{pmatrix} e^{s(t,x)} & 0 \\ 0 & e^{s(t,x)} \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \quad \text{for } \begin{pmatrix} u \\ v \end{pmatrix} \in X,$$

where $s(t,x) = -\alpha(t)x + (\alpha(t) + \beta(t))x^2/2$ for $(t,x) \in [0,\infty) \times [0,1]$.

We shall prove that the family $\{A(t) : t \geq 0\}$ defined as above satisfies conditions (a_1) through (a_3). An easy computation shows that (1.1) holds with $c_A(\tau) = 1 \vee (1/a_0)$. To prove condition (a_2), we use two families $\{A_0(t) : t \geq 0\}$ and $\{B_0(t) : t \geq 0\}$ of linear operators in $\mathfrak{X} = C[0,1] \times C[0,1]$ defined by

$$\begin{cases} (A_0(t)\tilde{v})(x) = \begin{pmatrix} \sqrt{a(t,x)} & 0 \\ 0 & -\sqrt{a(t,x)} \end{pmatrix} \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix}_x \\ \qquad\qquad\qquad\qquad\qquad \text{for } \tilde{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in D(A_0(t)) \\ D(A_0(t)) = \{\tilde{v} \in C^1[0,1] \times C^1[0,1] : \\ \qquad\qquad v_1(0) + v_2(0) = v_1(1) + v_2(1) = 0\} \end{cases}$$

and

$$(B_0(t)\tilde{v})(x) = -\begin{pmatrix} \sqrt{a(t,x)} & 0 \\ 0 & -\sqrt{a(t,x)} \end{pmatrix} q_x(t,x)q(t,x)^{-1}\tilde{v}(x)$$

for $\tilde{v} \in \mathfrak{X}$, where

$$q(t,x) = \begin{pmatrix} \sqrt{a(t,x)} & 1 \\ \sqrt{a(t,x)} & -1 \end{pmatrix}.$$

Similarly to the proof of [**2**, Theorem 2.1], we have $(0,\infty) \subset \rho(A_0(t))$ and $\|(\lambda I - A_0(t))^{-1}\|_{\mathfrak{X}} \le 1$ for $\lambda > 0$. Clearly, $B_0(t) \in B(\mathfrak{X})$ for $t \ge 0$, and $\omega(\tau) := \sup\{\|B_0(t)\|_{\mathfrak{X}} : t \in [0,\tau]\} < \infty$; hence

$$\{A_0(t) + B_0(t) : t \in [0,\tau]\} \in S_\sharp(\mathfrak{X}, 1, \omega(\tau)) \tag{3.6}$$

for each $\tau > 0$. We now turn to the check of condition (a_2). We first show that $(\omega(\tau),\infty) \subset \rho(A(t)|_D)$ for $t \in [0,\tau]$. To do so, let $\lambda > \omega(\tau)$ and $\binom{\xi}{\eta} \in X$. We want to solve the equation $(\lambda I - A(t)|_D)\binom{u}{v} = \binom{\xi}{\eta}$, namely

$$\begin{cases} \lambda u - v = \xi, \\ \lambda v - a(t,\cdot)u_{xx} = \eta, \\ u_x(0) = u_x(1) = 0. \end{cases} \tag{3.7}$$

We note that if $(\lambda I - A(t)|_D)\binom{u}{v} = \binom{\xi}{\eta}$, then

$$q(t,\cdot)\begin{pmatrix} u_x \\ v \end{pmatrix} = (\lambda I - (A_0(t) + B_0(t)))^{-1} q(t,\cdot)\begin{pmatrix} \xi_x \\ \eta \end{pmatrix}. \tag{3.8}$$

Indeed, if $\binom{u}{v}$ is a solution of (3.7) then we have

$$\begin{cases} \lambda \begin{pmatrix} u_x \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ a(t,\cdot) & 0 \end{pmatrix} \begin{pmatrix} u_x \\ v \end{pmatrix}_x = \begin{pmatrix} \xi_x \\ \eta \end{pmatrix}, \\ u_x(0) = u_x(1) = 0. \end{cases} \tag{3.9}$$

Since

$$q(t,x)\begin{pmatrix} 0 & 1 \\ a(t,x) & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{a(t,x)} & 0 \\ 0 & -\sqrt{a(t,x)} \end{pmatrix} q(t,x),$$

it follows easily that $\binom{w_1}{w_2} = q(t,\cdot)\binom{u_x}{v}$ is a solution of

$$(\lambda I - (A_0(t) + B_0(t)))\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = q(t,\cdot)\begin{pmatrix} \xi_x \\ \eta \end{pmatrix}.$$

On the basis of (3.8), we now put $\binom{z}{v} = q(t,\cdot)^{-1}(\lambda I - (A_0(t) + B_0(t)))^{-1}q(t,\cdot)\binom{\xi_x}{\eta}$. It is then seen that $\binom{z}{v} \in C^1[0,1] \times C^1[0,1]$ and (3.9) holds with u_x replaced by z. Defining $u = (v+\xi)/\lambda$ we have $u_x = z$ and see that $\binom{u}{v} \in D$ is a solution of (3.7). To prove the uniqueness of solutions of (3.7), set $\binom{\xi}{\eta} = \binom{0}{0}$ in (3.7). We have by (3.8), $\binom{u_x}{v} = \binom{0}{0}$ because of the injectivity of $q(t,\cdot)$, and then $u = 0$ by the first equation in (3.7). The proof of uniqueness is thus complete. Next, let $\lambda > \omega(\tau)$ and $\{t_i\}_{i=1}^k$ a finite sequence with $0 = t_0 \leq t_1 \leq \cdots \leq t_k \leq \tau$. For $\binom{\xi}{\eta} \in X$ we define

$$\begin{pmatrix} u_\lambda^l \\ v_\lambda^l \end{pmatrix} = \prod_{i=1}^{l}(\lambda I - A(t_i)|_D)^{-1}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\left(= (\lambda I - A(t_l)|_D)^{-1}\begin{pmatrix} u_\lambda^{l-1} \\ v_\lambda^{l-1} \end{pmatrix}\right)$$

for $1 \leq l \leq k$, where we set $\binom{u_\lambda^0}{v_\lambda^0} = \binom{\xi}{\eta}$. By (3.8) we find

$$q(t_l,\cdot)\begin{pmatrix} (u_\lambda^l)_x \\ v_\lambda^l \end{pmatrix} = (\lambda I - (A_0(t_l) + B_0(t_l)))^{-1}q(t_l,\cdot)\begin{pmatrix} (u_\lambda^{l-1})_x \\ v_\lambda^{l-1} \end{pmatrix},$$

and we have by (3.6)

$$\begin{aligned} q_\lambda^l &\leq (\lambda - \omega(\tau))^{-1}\left\| q(t_l,\cdot)\begin{pmatrix} (u_\lambda^{l-1})_x \\ v_\lambda^{l-1} \end{pmatrix}\right\|_{\mathfrak{X}} \\ &\leq (\lambda - \omega(\tau))^{-1}\exp(L_q(\tau)(t_l - t_{l-1}))q_\lambda^{l-1} \end{aligned}$$

for $1 \leq l \leq k$. Here we have used the notation

$$q_\lambda^l = \left\| q(t_l,\cdot)\begin{pmatrix} (u_\lambda^l)_x \\ v_\lambda^l \end{pmatrix}\right\|_{\mathfrak{X}}$$

and the fact

$$\left\| q(t,\cdot)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right\|_{\mathfrak{X}} \leq \exp(L_q(\tau)|t - \hat{t}|)\left\| q(\hat{t},\cdot)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}\right\|_{\mathfrak{X}}$$

for $t, \hat{t} \in [0, \tau]$ and $\binom{f_1}{f_2} \in \mathfrak{X}$. This is proved by using the fact that there exists $L_q(\tau) > 0$ such that $\|(q(t,\cdot) - q(\hat{t},\cdot))q(\hat{t},\cdot)^{-1}\binom{f_1}{f_2}\|_{\mathfrak{X}} \leq L_q(\tau)|t - \hat{t}|\|\binom{f_1}{f_2}\|_{\mathfrak{X}}$ for $t, \hat{t} \in [0, \tau]$ and $\binom{f_1}{f_2} \in \mathfrak{X}$ and the inequality $1 + r \leq e^r$ for $r \geq 0$. Solving the above inequality we have $q_\lambda^l \leq (\lambda - \omega(\tau))^{-l} \exp(L_q(\tau)(t_l - t_0))q_\lambda^0$, which implies

$$(\|(u_\lambda^l)_x\|_{C[0,1]} \vee \|v_\lambda^l\|_{C[0,1]}) \leq c_q(\tau)(\lambda - \omega(\tau))^{-l}(\|\xi_x\|_{C[0,1]} \vee \|\eta\|_{C[0,1]}) \tag{3.10}$$

for $0 \leq l \leq k$. By the definition of $\{u_\lambda^l\}$ we have $\lambda u_\lambda^l - v_\lambda^l = u_\lambda^{l-1}$ for $1 \leq l \leq k$; hence $u_\lambda^k = \sum_{i=1}^k (1/\lambda)^{k-i+1} v_\lambda^i + (1/\lambda)^k \xi$. We estimate it by (3.10). This yields

$$\|u_\lambda^k\|_{C[0,1]} \leq ((c_q(\tau)/\omega(\tau)) + 1)(\lambda - \omega(\tau))^{-k}(\|\xi\|_{C^1[0,1]} \vee \|\eta\|_{C[0,1]}).$$

Combining this and (3.10) we find

$$\{A(t)|_D : t \in [0, \tau]\} \in S_\sharp(X, M(\tau), \omega(\tau)),$$

where $M(\tau) = c_q(\tau) + (c_q(\tau)/\omega(\tau)) + 1$, and so condition (a_2) is satisfied. Condition (a_3) is clearly verified.

The family $\{S(t) : t \geq 0\}$ in $B(X)$ satisfies condition (s_1). Condition (s_2) is checked by taking a family $\{B(t) : t \geq 0\}$ in $B(X)$ defined by

$$\left(B(t)\begin{pmatrix} u \\ v \end{pmatrix}\right)(x) = \begin{pmatrix} 0 \\ -a(t,x)(s_{xx}(t,x)u(x) + 2s_x(t,x)u'(x) - s_x(t,x)^2 u(x)) \end{pmatrix}$$

for $\binom{u}{v} \in X$. One can easily check conditions (g_1) and (g_2) by virtue of (b_1) and (b_2). By (3.4) we have $S(0)\binom{\varphi}{\psi} \in D$ and $S(0)A(0)\binom{\varphi}{\psi} + \partial S(0)\binom{\varphi}{\psi} \in \overline{D}$; namely the compatibility condition is satisfied. It follows from Theorem 2.2 that there are a $t_{\max} \in (0, \infty]$ and a unique solution $\binom{u}{v}$ on $[0, t_{\max})$ of the Volterra integrodifferential equation (VIE) with initial data $\phi(x) := \binom{\varphi(x)}{\psi(x)}$, and it is therefore proved that the first component u is a unique solution of (3.3). If the function b satisfies (3.5) then (3.1) is easily verified, and so (3.3) has a unique global solution by Theorem 3.1. ◻

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