JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 10, Number 2, Summer 1998

BPX PRECONDITIONER FOR HYPERSINGULAR INTEGRAL EQUATIONS

THANG CAO

ABSTRACT. In this paper we present the BPX (Bramble, Pasciak and Xu) preconditioner method for the Galerkin approximation of hypersingular integral equations on the interval $\Gamma = (-1, 1)$. The condition number of the resulting matrix with respect to the BPX preconditioner is shown to behave like $\mathcal{O}(h^{-\varepsilon})$ where ε is small and depends on the singularity of the exact solution at the end points of the open curve Γ. When Γ is closed, ε is reduced to zero, hence the condition number is independent of the mesh size. The implementations are based on the preconditioned conjugate gradient method using the BPX preconditioner. The numerical results are presented with a comparison between BPX preconditioner and HB (hierarchical basis) preconditioner.

1. Introduction. The discretization of partial differential equations and boundary integral equations leads to very large systems of linear equations, the direct solution of which can be very expensive in terms of storage and computational work. We now consider the BPX preconditioner method developed in the 1990s for finite element methods. Together with multigrid methods [**2**], domain decomposition methods [**9**], and hierarchical basis methods [**17**], the BPX preconditioner method is the fastest known method for solving large systems of linear equations arising from the discretization of partial differential equations. The theoretical foundation of the BPX preconditioner method started with Bramble et al. [**4**]. The BPX preconditioner method usually needs slightly more iteration steps than the multigrid methods, but the higher flexibility of these algorithms simplifies the use of parallel computing (the single subspace corrections are not applied in a sequential order but in parallel, see (3.8)). Another advantage is that it allows a simpler, more natural data structure and is therefore much better for non-uniformly refined grids. Consequently, it is possible to combine this method with adaptive methods. The combined adaptive

Received by the editors on February 14, 1997, and in revised form on September 3, 1997.
1991 AMS *Mathematics Subject Classification.* 65N55, 65N38.

additive multilevel method for the hypersingular integral equation is studied in [**7**] and [**8**].

In this paper we show that the BPX preconditioner method can be applied to the Galerkin boundary element method for the hypersingular integral equation. Our analysis follows closely the approach by Bramble et al. [**4, 3**] and Xu [**16**], whose applications were to finite element methods. The finite element theory does not carry over directly to boundary element methods because the stiffness matrix is dense and the boundary integral operators are non-local, and it is very difficult to deal with these problems directly via the boundary integral operators. However, the theoretical analysis for the hypersingular integral equation is based on the fractional Sobolev space $\tilde{H}^{1/2}(\Gamma)$. Therefore, we will use the interpolation properties, which make use of the well-known results for FEM (usually available for the Sobolev spaces $\tilde{H}^1(\Gamma)$ and $H^0(\Gamma)$), and the inverse inequalities, to overcome the above problems for our theoretical analysis. A limitation of our approach is that the mesh must be assumed to be quasi-uniform because we have to use inverse inequalities. It is much more complicated to apply these methods to the weakly singular integral equations, see [**1**].

The Galerkin discretization of the hypersingular integral equation on the interval $\Gamma = (-1, 1)$ leads to a system of linear equations

$$Au = f. \tag{1.1}$$

Here A is an $N \times N$ full matrix, so a Gauss solver requires $O(N^3)$ operations. For N large, $O(N^3)$ operations is too expensive; we need a good iterative method to approximate the Galerkin solution, keeping the error in the energy norm at the order of the Galerkin error (i.e., $O(h^{1/2-\varepsilon})$ for the hypersingular integral equation on open curves). The conjugate gradient method requires $CN^{1/2} \log N$ iterations, and therefore $CN^{5/2} \log N$ operations, to achieve this order of accuracy, see [**12**]. A multigrid method for equation (1.1) has been designed in [**12**] as a preconditioned conjugate gradient method to reduce the number of operations to CN^2. Recently, Tran and Stephan [**13**] applied the domain decomposition methods introduced by Dryja and Widlund [**9**] to equation (1.1). Their algorithm was the preconditioned conjugate gradient method, in which the condition number behaved like $O(h^{-\varepsilon})$. Their techniques of analysis were quite different from ours.

In this paper, we solve equation (1.1), using the BPX preconditioner introduced by Bramble, Pasciak and Xu [**2**] for FEM. The BPX preconditioner also yields a condition number that is $O(h^{-\varepsilon})$, and therefore required $O(h^{-\varepsilon}N^2)$ operations to obtain $O(h^{1/2-2\varepsilon})$ order of error.

The outline of this paper is as follows. In the next section we present the preliminary results of Sobolev spaces. In Section 3 the outline and the basic results for abstract additive multilevel methods are presented. In Section 4, we present the BPX preconditioner method and apply it to the hypersingular integral equation and show that the condition number is independent of the number of levels. Section 5 contains the matrix implementation of the BPX preconditioner. Numerical experiments that compare the performances of the BPX and the HB preconditioners are presented in Section 6.

2. Preliminaries.

2.1. *Sobolev spaces.* The definitions of the Sobolev spaces to be used throughout the paper are as follows. For $s \in \mathbf{R}$, the Sobolev space $H^s(\mathbf{R}^2)$ is defined as a space of temperate distributions $u \in S'$, such that

$$\|u\|^2_{H^s(\mathbf{R}^2)} = \int_{\mathbf{R}^2} |\hat{u}(\xi)|^2 (1+|\xi|^2)^s \, d\xi < \infty$$

where $\hat{u}$ is the Fourier transform of u. It is well known that

$$\|u\|^2_{H^0(\mathbf{R}^2)} = \|u\|^2_{L^2(\mathbf{R}^2)} = \int_{\mathbf{R}^2} |u(x)|^2 \, dx$$

and

$$\|u\|^2_{H^s(\mathbf{R}^2)} \sim \|u\|^2_{H^0(\mathbf{R}^2)} + \sum_{k=1}^{s} \|D^\alpha u\|^2_{H^0(\mathbf{R}^2)} \quad \text{for } s \in \mathbf{N},$$

where α are multi-indices and $|\alpha| = s$. Let Ω be a bounded domain with Lipschitz boundary $\tilde{\Gamma}$. Let $\Gamma \subseteq \tilde{\Gamma}$ be a closed or open curve such that $\Gamma = \tilde{\Gamma}$ if Γ is closed and $\Gamma \neq \tilde{\Gamma}$ if Γ is open. As in Lions and Magenes [**11**] and Triebel [**14**], we define

$$\begin{aligned} H^s(\tilde{\Gamma}) &= \{u|_{\tilde{\Gamma}} : u \in H^{s+1/2}(\mathbf{R}^2)\} \quad \text{for } s > 0 \\ H^0(\tilde{\Gamma}) &= L^2(\tilde{\Gamma}) \\ H^s(\tilde{\Gamma}) &= (H^{-s}(\tilde{\Gamma}))' \quad \text{for } s < 0. \end{aligned}$$

where $(.)'$ denotes the dual space with respect to L^2-inner product. Then, we define for all $s \in \mathbf{R}$

$$
\begin{aligned}
H^s(\Gamma) &= \{u|_\Gamma : u \in H^s(\tilde{\Gamma})\} \\
\tilde{H}^s(\Gamma) &= \{u \in H^s(\tilde{\Gamma}) : \operatorname{supp} u \subseteq \overline{\Gamma}\}.
\end{aligned} \tag{2.1}
$$

For $s = 1/2$, $\tilde{H}^s(\Gamma) = H^s_{00}(\Gamma)$ as defined in Lions and Magenes [**11**]. The duality properties are as follows (see Triebel [**14**])

$$
(H^s(\Gamma))' = \tilde{H}^{-s}(\Gamma) \quad \text{and} \quad (\tilde{H}^s(\Gamma))' = H^{-s}(\Gamma) \tag{2.2}
$$

For $s > 0$, the norms in $H^s(\tilde{\Gamma})$, $H^s(\Gamma)$ and $\tilde{H}^s(\Gamma)$ are defined by

$$
\begin{aligned}
\|u\|_{H^s(\tilde{\Gamma})} &= \inf\{\|v\|_{H^{s+1/2}(\mathbf{R}^2)} : v|_{\tilde{\Gamma}} = u\} \\
\|u\|_{H^s(\Gamma)} &= \inf\{\|v\|_{H^s(\tilde{\Gamma})} : v|_\Gamma = u\} \\
\|u\|_{\tilde{H}^s(\Gamma)} &= \|u\|_{H^s(\tilde{\Gamma})}.
\end{aligned}
$$

For $s < 0$, the norms are defined by duality. Their interpolation relationships for $0 < s < 1$ are as follows

$$
H^s(\Gamma) = [H^1(\Gamma), H^0(\Gamma)]_s \tag{2.3}
$$

$$
\tilde{H}^s(\Gamma) = [\tilde{H}^1(\Gamma), H^0(\Gamma)]_s \tag{2.4}
$$

where $[.,.]_s$ denotes complex interpolation spaces as defined in Lions and Magenes [**11**], Triebel [**14**] and (2.3) and (2.4) are from Theorems 2.10.1 and 2.10.4 of Triebel [**14**]. The duality theory for interpolation spaces Triebel [**14**], $[A_0, A_1]'_s = [A'_0, A'_1]_s$, and (2.2) imply that

$$
H^{-s}(\Gamma) = [\tilde{H}^1(\Gamma)', H^0(\Gamma)']_s = [H^{-1}(\Gamma), H^0(\Gamma)]_s \tag{2.5}
$$

$$
\tilde{H}^{-s}(\Gamma) = [H^1(\Gamma)', H^0(\Gamma)']_s = [\tilde{H}^{-1}(\Gamma), H^0(\Gamma)]_s. \tag{2.6}
$$

3. Abstract additive multilevel method. In this section we review recent developments in the abstract theory for the additive multilevel methods, in which the BPX preconditioner method is a prominent prototype. The basic references are [**4, 3, 16**]. The formulation of the problem is set in an abstract way such that it can

cover both the general discrete elliptic boundary value problems and the boundary integral equations.

Let $\mathcal{M}$ be a finite dimensional space, and $\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_J$ be subspaces of $\mathcal{M}$ such that

$$\mathcal{M} = \mathcal{W}_0 + \mathcal{W}_1 + \cdots + \mathcal{W}_J.$$

The symmetric bilinear form $a(u, v)$ is positive definite on $\mathcal{M} \times \mathcal{M}$, where $\mathcal{M}$ is a Hilbert space equipped with the inner product $(\cdot, \cdot)$. The energy norm $a(v, v)^{1/2}$ is denoted by $\|v\|_A$ for $v \in \mathcal{M}$. The symmetric positive definite operator $A : \mathcal{M} \to \mathcal{M}$ is defined by

$$(Au, v) = a(u, v) \quad \text{for all } v \in \mathcal{M}. \tag{3.1}$$

For $k = 0, \ldots, J$, we introduce the following operators:

1. The projection $P_k : \mathcal{M} \to \mathcal{W}_k$ is defined for $u \in \mathcal{M}$ by

$$a(P_k u, v) = a(u, v) \quad \text{for all } v \in \mathcal{W}_k$$

2. The projection $Q_k : \mathcal{M} \to \mathcal{W}_k$ is defined for $u \in \mathcal{M}$ by

$$(Q_k u, v) = (u, v) \quad \text{for all } v \in \mathcal{W}_k$$

3. The operator $A_k : \mathcal{W}_k \to \mathcal{W}_k$ is defined for $u \in \mathcal{W}_k$ by

$$(A_k u, v) = a(u, v) \quad \text{for all } v \in \mathcal{W}_k.$$

Since $a(u, v)$ is positive definite, we know that the following problem,

$$a(u, v) = (f, v) \quad \text{for all } v \in \mathcal{M}, \tag{3.2}$$

has a unique solution $u \in \mathcal{M}$.

Equation (3.2) is equivalent to the abstract linear equation

$$Au = f. \tag{3.3}$$

The additive multilevel methods belong to a class of fast solvers for the linear equation (3.3). Its idea is based on an iterative correction, with the correction term with respect to the subspace $\mathcal{W}_k$ being defined by

$$u_n = u_{n-1} + P_k(u - u_{n-1}), \tag{3.4}$$

where u and u_n are exact and approximate solutions of (3.3), respectively, and $P_k(u-u_{n-1})$ is the subspace correction term. The subspace correction term in (3.4) is considered to be ideal, because the resulting error $u-u_n$ is a-orthogonal to the subspaces $\mathcal{W}_k$. In practice, the subspace correction terms need to be expressed in terms of the righthand side f and the approximation u_n. We have

$$A_k P_k = Q_k A, \tag{3.5}$$

because from definition of the operators A, A_k, P_k and Q_k,

$$(A_k P_k v, w_k) = a(P_k v, w_k) = a(v, w_k) = (Av, w_k) = (Q_k Av, w_k),$$

for $v \in \mathcal{M}$ and $w_k \in \mathcal{W}_k$. By (3.3) and (3.5), the subspace correction term in (3.4) can be expressed as:

$$\begin{aligned} P_k(u-u_{n-1}) &= A_k^{-1} A_k P_k (u-u_{n-1}) \\ &= A_k^{-1} A_k P_k u - A_k^{-1} A_k P_k u_{n-1} \\ &= A_k^{-1} Q_k A u - A_k^{-1} Q_k A u_{n-1} \\ &= A_k^{-1} Q_k (f - A u_{n-1}). \end{aligned} \tag{3.6}$$

The computation of A_k^{-1} depends on the construction of the subspaces $\mathcal{W}_k$ and is normally far too expensive for a reasonable method. We replace A_k^{-1} by less expensive symmetric positive definite operators $R_k : \mathcal{W}_k \to \mathcal{W}_k$, which are called smoothing operators (note that these smoothing operators can be such as Jacobi, Gauss-Seidel, etc.). The iterative methods with the subspace corrections (3.4) then become

$$u_n = u_{n-1} + R_k Q_k (f - A u_{n-1}). \tag{3.7}$$

If the single subspace corrections are combined sequentially in the order $k = 0, 1, \ldots, J$, we have the multiplicative multilevel methods [**3**] or the multiplicative subspace correction methods [**16**], and these methods are actually classical multigrid methods. The additive multilevel methods are defined by

$$u_n = u_{n-1} + \sum_{k=0}^{J} R_k Q_k (f - A u_{n-1}). \tag{3.8}$$

Using the notation of Bramble et al. [**3**], we define the operator $B : \mathcal{M} \to \mathcal{M}$ by

$$B = \sum_{k=0}^{J} R_k Q_k, \tag{3.9}$$

the operators $T_k : \mathcal{W}_k \to \mathcal{W}_k$ by

$$T_k = R_k A_k P_k, \tag{3.10}$$

and the operator $T : \mathcal{M} \to \mathcal{M}$ by

$$\begin{aligned} T &= \sum_{k=0}^{J} T_k = \sum_{k=0}^{J} R_k A_k P_k \\ &\overset{(3.5)}{=} \sum_{k=0}^{J} R_k Q_k A = BA. \end{aligned} \tag{3.11}$$

Here, we think of B as an approximate inverse of A. The simple structure of the operator B offers many advantages as a preconditioner for the conjugate gradient method. On a parallel computer, each correction term can be assigned to a processor and can be computed in parallel. It is well known that the acceleration of the preconditioned conjugate gradient method depends on the condition number of the operator BA. This condition number can be estimated by obtaining the lower and upper bounds for the eigenvalues of the operator BA. The lower bound will depend on the stability of the following direct subspace splitting

$$\mathcal{M} = \mathcal{V}_0 \oplus \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_J, \tag{3.12}$$

where $\mathcal{V}_k$ is a subspace of $\mathcal{W}_k$ for $k = 0, 1, \ldots, J$. These subspaces $\mathcal{V}_k$ are only a tool for the theoretical analysis; they do not require practical computation, and the choice of these subspaces is quite flexible. The stability hypothesis is as follows:

Hypothesis 3.1. *There exists a positive constant K_1 such that*

$$\left\| \sum_{k=0}^{J} v_k \right\|_A^2 \geq K_1 \sum_{k=0}^{J} (R_k^{-1} v_k, v_k) \tag{3.13}$$

for all $v_k \in \mathcal{V}_k$.

The upper bound depends on the following hypothesis:

Hypothesis 3.2. *There exists a positive constant* K_2 *such that*

$$\left\| \sum_{k=0}^{J} w_k \right\|_A^2 \leq K_2 \sum_{k=0}^{J} (R_k^{-1} w_k, w_k) \tag{3.14}$$

for all $w_k \in \mathcal{W}_k$.

Hypothesis 3.2 is actually a consequence of the strengthened Cauchy-Schwarz inequality, as shown in the next section.

The following lemma is the main tool to obtain the lower bound.

Lemma 3.1. *Assume that Hypothesis* 3.1 *holds. Then we have*

$$\sum_{k=0}^{J} a(v_k, u_k) \leq \frac{1}{\sqrt{K_1}} \left\| \sum_{k=0}^{J} v_k \right\|_A \left(\sum_{k=0}^{J} a(T_k u_k, u_k) \right)^{1/2} \tag{3.15}$$

for all $v_k \in \mathcal{V}_k$, *and for all* $u_k \in \mathcal{M}$.

Proof. From the definition of A_k, P_k, R_k and T_k, we have

$$\begin{aligned}
\sum_{k=0}^{J} a(v_k, u_k) &= \sum_{k=0}^{J} a(v_k, P_k u_k) = \sum_{k=0}^{J} (v_k, A_k P_k u_k) \\
&= \sum_{k=0}^{J} (R_k^{-1/2} v_k, R_k^{1/2} A_k P_k u_k) \\
&\leq \left[\left(\sum_{k=0}^{J} (R_k^{-1/2} v_k, R_k^{-1/2} v_k) \right) \right. \\
&\qquad \left. \cdot \left(\sum_{k=0}^{J} (R_k^{1/2} A_k P_k u_k, R_k^{1/2} A_k P_k u_k) \right) \right]^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\sum_{k=0}^{J} (R_k^{-1} v_k, v_k) \Big)^{1/2} \Big(\sum_{k=0}^{J} (R_k A_k P_k u_k, A_k P_k u_k) \Big)^{1/2} \\
&= \Big(\sum_{k=0}^{J} (R_k^{-1} v_k, v_k) \Big)^{1/2} \Big(\sum_{k=0}^{J} (T_k u_k, A_k P_k u_k) \Big)^{1/2} \\
&= \Big(\sum_{k=0}^{J} (R_k^{-1} v_k, v_k) \Big)^{1/2} \Big(\sum_{k=0}^{J} a(T_k u_k, u_k) \Big)^{1/2}.
\end{aligned}
$$

Then (3.21) follows from Hypothesis 3.1. □

The main result in the theory of additive multilevel methods is as follows.

Theorem 3.1. *Assume that Hypotheses* 3.1–3.2 *hold. Then the operator B defined in* (3.9)*, is symmetric and positive definite, and the condition number of BA satisfies*

$$\kappa(BA) \le K_2/K_1, \tag{3.16}$$

where K_1 and K_2 are positive constants.

Proof. By the definition of the operators R_k and Q_k, we have

$$
\begin{aligned}
(Bv, u) &= \sum_{k=0}^{J} (R_k Q_k v, u) = \sum_{k=0}^{J} (R_k Q_k v, Q_k u) \\
&= \sum_{k=0}^{J} (Q_k v, R_k Q_k v) = \sum_{k=0}^{J} (v, R_k Q_k u) \\
&= (v, Bu),
\end{aligned} \tag{3.17}
$$

for $u, v \in \mathcal{M}$. Hence B is symmetric, and by the positive definiteness of R_k, $k = 0, \ldots, J$ and (3.17), it is obvious that B is positive definite.

To prove (3.16), we denote by $\lambda_{\min}(BA)$ and $\lambda_{\max}(BA)$ the minimum and the maximum eigenvalues of BA, respectively, and since

$$\kappa(BA) = \frac{\lambda_{\max}(BA)}{\lambda_{\min}(BA)},$$

we only need to show that

$$K_1 \le \lambda_{\min}(BA) \le \lambda_{\max}(BA) \le K_2. \tag{3.18}$$

In order to obtain the right hand inequality of (3.18), we use Hypothesis 3.2 as follows:

$$\begin{aligned} a(BAv, BAv) &\overset{(3.11)}{=} \Big\| \sum_{k=0}^{J} T_k v \Big\|_A^2 \\ &\overset{(3.14)}{\le} K_2 \sum_{k=0}^{J} (R_k^{-1} T_k v, T_k v) \\ &= K_2 \sum_{k=0}^{J} (A_k P_k v, T_k v) \\ &= K_2 \sum_{k=0}^{J} a(T_k v, v) \\ &= K_2 a(BAv, v), \end{aligned}$$

for $v \in \mathcal{M}$, and the last equality is from the identity $BA = \sum_{k=0}^{J} T_k$. By taking v to be an eigenvector of BA, we see that $\lambda_{\max}(BA) \le K_2$.

To show that the left inequality of (3.18) holds, we use Hypothesis 3.1 as follows:

Let $v \in \mathcal{M}$ have the decomposition (3.12), i.e., $v = \sum_{k=0}^{J} v_k$ with $v_k \in \mathcal{V}_k$. Then, it follows from Lemma 3.1 that

$$\begin{aligned} \|v\|_A^2 = a(v,v) &= \sum_{k=0}^{J} a(v_k, v) = \sum_{k=0}^{J} a(v_k, P_k v) \\ &\overset{(3.17)}{\le} \frac{1}{\sqrt{K_1}} \Big\| \sum_{k=0}^{J} v_k \Big\|_A \Big(\sum_{k=0}^{J} a(T_k P_k v, P_k v) \Big)^{1/2} \\ &= \frac{1}{\sqrt{K_1}} \|v\|_A \Big(\sum_{k=0}^{J} a(T_k v, v) \Big)^{1/2}. \end{aligned}$$

Hence,

$$K_1 a(v,v) \le \sum_{k=0}^{J} a(T_k v, v) = a(BAv, v),$$

and thus $K_1 \leq \lambda_{\min}(BA)$. □

4. BPX preconditioner. In this section, we study the BPX preconditioner for the hypersingular integral equation on the interval $\Gamma = (-1, 1)$. The hypersingular integral equation is written as

$$Wu(x) := \frac{1}{\pi} \frac{\partial}{\partial n_x} \int_\Gamma u(y) \frac{\partial}{\partial n_y} \log|x - y| \, ds_y = f(x) \quad \text{for } x \in \Gamma, \tag{4.1}$$

for $f \in H^{-1/2}(\Gamma)$. The spline spaces

$$\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_J = \mathcal{M}$$

are defined as in the last section, with the additional constraint that the functions vanish at both end points. For simplicity, the mesh size in τ_k, $k = 0, \ldots, J$, is assumed to be uniform and denoted by h_k; the results throughout the sequel can be extended easily to a quasi-uniform mesh. The Galerkin scheme is to find $v_h \in \mathcal{M}$ such that

$$\langle Wv_h, \chi \rangle = \langle f, \chi \rangle \quad \text{for all } \chi \in \mathcal{M} \, , \tag{4.2}$$

where $\langle ., . \rangle$ is the L^2-inner product. The stability and convergence of the scheme (4.2) were proved in [**15**]. For $u \in \tilde{H}^{1/2}(\Gamma)$, it is well known that there exist C_1 and C_2 independent of h such that

$$C_1 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq \langle Wu, u \rangle \leq C_2 \|u\|^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{4.3}$$

The operator $W : \tilde{H}^s(\Gamma) \to H^{s-1}(\Gamma)$ is continuous and bijective for $0 < s < 1$, see [**15**]. Thus for $0 < \varepsilon < 1$,

$$\|Wu\|_{H^{-\varepsilon}(\Gamma)} \leq C \|u\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \tag{4.4}$$

and

$$\|W^{-1}u\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \leq C \|u\|_{H^0(\Gamma)}. \tag{4.5}$$

We define

$$W(u, v) := \langle Wu, v \rangle \quad \text{and} \quad \|u\|^2_W := \langle Wu, u \rangle \sim \|u\|^2_{\tilde{H}^{1/2}(\Gamma)},$$

and introduce the following operators:

1. The projection $P_k : \mathcal{M} \to \mathcal{M}_k$ is defined for $u \in \mathcal{M}$ by
$$W(P_k u, v) = W(u, v) \quad \text{for all } v \in \mathcal{M}_k.$$
2. The operator $W_k : \mathcal{M}_k \to \mathcal{M}_k$ is defined for $u \in \mathcal{M}_k$ by
$$\langle W_k u, v\rangle = W(u, v) \quad \text{for all } v \in \mathcal{M}_k.$$

The behavior of the maximum eigenvalue of W_k is very important for the BPX preconditioner method. It was shown in [**7**] that

$$ch_k^{-1} \le \lambda_k \le Ch_k^{-1}, \tag{4.6}$$

where λ_k is the maximum eigenvalue of W_k.

The abstract inner product used in the last section becomes the L^2-inner product $\langle\cdot,\cdot\rangle$, and the symmetric positive definite bilinear form is $W(u, v)$. A prominent example of the additive multilevel methods is the BPX preconditioner, in which the correction subspaces are piecewise linear spaces $\mathcal{M}_k$. Therefore, the BPX preconditioner is also called the multilevel nodal basis preconditioner. The subspace decomposition (3.12) is based on the Q_k operator, which is as follows

$$\mathcal{V}_k = \{(Q_k - Q_{k-1})v \mid v \in \mathcal{M}\} \subseteq \mathcal{M}_k$$

for $k = 1, \ldots, J$, and $\mathcal{V}_0 = \mathcal{M}_0$. The BPX preconditioner B_{BPX} is defined by

$$B_{\mathrm{BPX}} = \sum_{k=0}^{J} R_k Q_k, \tag{4.7}$$

where the smoother $R_k : \mathcal{M}_k \to \mathcal{M}_k$ should be chosen to satisfy

$$C_1 \frac{\|v\|^2_{H^0(\Gamma)}}{\lambda_k} \le \langle R_k v, v\rangle \le C_2 \frac{\|v\|^2_{H^0(\Gamma)}}{\lambda_k} \tag{4.8}$$

for any $v \in \mathcal{M}_k$. Here, λ_k denotes the maximum eigenvalue of W_k; it was shown in [**7**] that $\lambda_k = \mathcal{O}(h_k^{-1})$. Since R_k is symmetric positive definite, $R_k^{-1/2}$ exists. Then replace v by $R_k^{-1/2}v$ in (4.8) to obtain

$$C_1\langle R_k^{-1}v, v\rangle \le \lambda_k\langle v, v\rangle \le C_2\langle R_k^{-1}v, v\rangle.$$

Hence,

$$C_2^{-1}\lambda_k\|v\|^2_{H^0(\Gamma)} \le \langle R_k^{-1}v, v\rangle \le C_1^{-1}\lambda_k\|v\|^2_{H^0(\Gamma)}. \tag{4.9}$$

The smoother R_k should also be chosen as simple as possible. The best choice is probably the Jacobi method which is defined by

$$R_k v = \sum_{x_i\in\mathcal{N}_k} \langle v, \psi_i^k\rangle \psi_i^k, \quad v \in \mathcal{M}_k \tag{4.10}$$

where $\psi_i^k \in \mathcal{M}_k$ are the nodal basis functions associated with the nodal points $x_i \in \mathcal{N}_k$. We will show that the smoother R_k given in (4.10), satisfies (4.8), as follows.

Any $v \in \mathcal{M}_k$ can be represented by

$$v = \sum_{x_i\in\mathcal{N}_k} v(x_i)\psi_i^k. \tag{4.11}$$

Let $\bar{v}$ be the vector with components $\bar{v}_i = v(x_i)$, $i = 1, \ldots, N_k$, and let G_k be the symmetric positive definite matrix with entries $(G_k)_{ij} = \langle \psi_i^k, \psi_j^k\rangle$, $i, j = 1, \ldots, N_k$. Then we have

$$\langle R_k v, v\rangle = \sum_{x_i\in\mathcal{N}_k} \langle v, \psi_i^k\rangle^2 = (G_k\bar{v}, G_k\bar{v}) \tag{4.12}$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product. For the uniform partition τ_k, it is easy to verify that

$$(G_k\bar{v}, \bar{v}) = \|v\|^2_{H^0(\Gamma)} \sim h_k \sum_{x_i\in\mathcal{N}_k} |v(x_i)|^2 = h_k(\bar{v}, \bar{v}), \tag{4.13}$$

where $\sim$ denotes equivalence of norms. Hence,

$$(G_k\bar{v}, G_k\bar{v}) \sim h_k^2(\bar{v}, \bar{v}) \sim h_k(G_k\bar{v}, \bar{v}) = h_k\|v\|^2_{H^0(\Gamma)}. \tag{4.14}$$

Since $\lambda_k = O(h_k^{-1})$, (4.8) is a consequence of (4.12) and (4.14).

The following theorem is the main result of this section. It is actually a special version of Theorem 3.1 applied to the hypersingular integral equation.

Theorem 4.1. *Let W be the hypersingular operator, and assume that the smoother R_k satisfies assumption* (4.8)*. Then the operator B_{BPX} defined in* (4.7) *is symmetric positive definite, and the condition number $\kappa(B_{\mathrm{BPX}}W)$ satisfies*

$$\kappa(B_{\mathrm{BPX}}W) \leq \frac{\tilde{K}_2}{\tilde{K}_1} h_J^{-2\varepsilon}. \tag{4.15}$$

The term $h_J^{-2\varepsilon}$ is associated with the singularity of the open curve problems, and since ε is small, it can be ignored in the general case.

In order to prove (4.15), we only need to show that

$$\tilde{K}_1 h_J^{2\varepsilon} \leq \lambda_{\min}(B_{\mathrm{BPX}}W) \leq \lambda_{\max}(B_{\mathrm{BPX}}W) \leq \tilde{K}_2. \tag{4.16}$$

The proof for the right inequality of (4.16) will depend on a version of the strengthened Cauchy-Schwarz inequality for the BPX method, whose proof relies on the following lemma

Lemma 4.1. *For $u \in \mathcal{M}_l$, $l \leq k$, we have*

$$\frac{\|W_k u\|_{H^0(\Gamma)}}{\lambda_k^{1/2}} \leq C\gamma^{k-l} W(u,u)^{1/2} \tag{4.17}$$

where $\gamma = (1/\sqrt{2})^{1-2\varepsilon}$ for some arbitrarily small $0 < \varepsilon < 1/2$.

Proof. We recall the following inverse inequality

$$\|u\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \leq C h_l^{-1/2+\varepsilon} \|u\|_{\tilde{H}^{1/2}(\Gamma)}, \quad u \in \mathcal{M}_l. \tag{4.18}$$

Hence

$$\begin{aligned}
\|W_k u\|^2_{H^0(\Gamma)} &= \langle W_k u, W_k u\rangle = W(W_k u, u) \\
&= \langle W_k u, W u\rangle \leq \|W_k u\|_{H^\varepsilon(\Gamma)} \|W u\|_{H^{-\varepsilon}(\Gamma)} \\
&\overset{(4.4)}{\leq} C \|W_k u\|_{H^\varepsilon(\Gamma)} \|u\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \\
&\overset{(4.18)}{\leq} C h_l^{-1/2+\varepsilon} h_k^{-\varepsilon} \|W_k u\|_{H^0(\Gamma)} W(u,u)^{1/2}
\end{aligned}$$

and

$$\frac{\|W_k u\|_{H^0(\Gamma)}}{\lambda_k^{1/2}} \le C h_l^{-1/2+\varepsilon} h_k^{-\varepsilon} \lambda_k^{-1/2} W(u,u)^{1/2}.$$

Using (4.6), we then have

$$\frac{\|W_k u\|_{H^0(\Gamma)}}{\lambda_k^{1/2}} \le C(h_k/h_l)^{1/2-\varepsilon} W(u,u)^{1/2} = C\gamma^{k-l} W(u,u)^{1/2}$$

with $\gamma = (1/\sqrt{2})^{1-2\varepsilon}$. □

The following lemma is the strengthened Cauchy-Schwarz inequality.

Lemma 4.2. *For $u \in \mathcal{M}_l$, $v \in \mathcal{M}_k$, $l \le k$, we have*

$$W(u,v) \le C\gamma^{k-l} \langle R_l^{-1} u, u\rangle^{1/2} \langle R_k^{-1} v, v\rangle^{1/2} \tag{4.19}$$

where $\gamma = (1/\sqrt{2})^{1-2\varepsilon}$ for some arbitrarily small $0 < \varepsilon < 1/2$.

Proof. Since $u \in \mathcal{M}_l$, $l \le k$, we use (4.17) in Lemma 4.1 to show that

$$\begin{aligned} W(u,v) = \langle W_k u, v\rangle &\le \|W_k u\|_{H^0(\Gamma)} \|v\|_{H^0(\Gamma)} \\ &= \frac{\|W_k u\|_{H^0(\Gamma)}}{\lambda_k^{1/2}} \lambda_k^{1/2} \|v\|_{H^0(\Gamma)} \\ &\overset{(4.17)}{\le} C\gamma^{k-l} \|u\|_{\tilde{H}^{1/2}(\Gamma)} \lambda_k^{1/2} \|v\|_{H^0(\Gamma)}. \end{aligned} \tag{4.20}$$

Using the inverse inequality, (4.9) and $\lambda_k = O(h_k^{-1})$, we have

$$\|u\|_{\tilde{H}^{1/2}(\Gamma)} \le C h_l^{-1/2} \|u\|_{H^0(\Gamma)} \le C\langle R_l^{-1} u, u\rangle^{1/2}, \tag{4.21}$$

then (4.19) obviously follows from (4.9) and (4.20)–(4.21). □

The proof for the left inequality of (4.6) will depend on the approximation and the stability properties of the operators Q_k, $k = 0, \ldots, J$, which are expressed in the following lemma.

Lemma 4.3. *Assume that $u \in \tilde{H}^1(\Gamma)$. Then we have*

$$\|(Q_k - Q_{k-1})u\|_{H^0(\Gamma)} \leq Ch_k^{1/2}\|u\|_{\tilde{H}^{1/2}(\Gamma)}, \quad k = 1, \ldots, J, \tag{4.22}$$

and

$$\|Q_k u\|_{\tilde{H}^{1/2}(\Gamma)} \leq C\|u\|_{\tilde{H}^{1/2}(\Gamma)}, \quad k = 0, \ldots, J. \tag{4.23}$$

Proof. It is well known that Q_k, $k = 0, \ldots, J$, have the following approximation property

$$\|(I - Q_k)u\|_{H^0(\Gamma)} \leq Ch_k\|u\|_{\tilde{H}^1(\Gamma)}, \quad u \in \tilde{H}^1(\Gamma), \tag{4.24}$$

and the following stability property

$$\|Q_k u\|_{\tilde{H}^r(\Gamma)} \leq C\|u\|_{\tilde{H}^r(\Gamma)}, \quad u \in \tilde{H}^r(\Gamma), \tag{4.25}$$

for $r = 0, 1$, see [**5**]. It is obvious that (4.23) follows from (4.25) and the interpolation property. To prove (4.22), we use the triangle inequality and (4.25) to get

$$\begin{aligned} \|(Q_k - Q_{k-1})u\|_{H^0(\Gamma)} &\leq \|Q_k u\|_{H^0(\Gamma)} + \|Q_{k-1}\|_{H^0(\Gamma)} \\ &\leq C\|u\|_{H^0(\Gamma)}. \end{aligned} \tag{4.26}$$

And, by triangle inequality, we have

$$\|(Q_k - Q_{k-1})u\|_{H^0(\Gamma)} \leq \|(I - Q_{k-1})u\|_{H^0(\Gamma)} + \|(I - Q_k)u\|_{H^0(\Gamma)}. \tag{4.27}$$

Since $h_{k-1} = 2h_k$, by interpolation, (4.46) obviously follows from (4.24), (4.26) and (4.27). □

The approximation property of Ritz-Galerkin operators P_i, $0, \ldots, J$, is expressed in the following lemma.

Lemma 4.4. *For $u \in \mathcal{M}$, let $\hat{u}_i = (P_i - P_{i-1})u$, $i = 1, \ldots, J$. Then we have*

$$\|\hat{u}_i\|_{H^0(\Gamma)} \leq Ch_i^{1/2-\varepsilon}\|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)}, \quad i = 1, \ldots, J. \tag{4.28}$$

Proof. Since P_i is the Ritz-Galerkin approximation, we can obtain the approximation property for $\hat{u}_i$, $i = 1, \ldots, J$, as follows.

First, find $v \in \tilde{H}^1(\Gamma)$ such that $\hat{u}_i = Wv$. Then for some $\chi \in \mathcal{M}_{i-1}$ we have

$$\begin{aligned} \|v - \chi\|_{\tilde{H}^{1/2}(\Gamma)} &\le C h_i^{1/2-\varepsilon} \|v\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \\ &= C h_i^{1/2-\varepsilon} \|W^{-1}\hat{u}_i\|_{\tilde{H}^{1-\varepsilon}(\Gamma)} \\ &\overset{(4.5)}{\le} C h_i^{1/2-\varepsilon} \|\hat{u}_i\|_{H^0(\Gamma)}. \end{aligned} \tag{4.29}$$

Using (4.29), we have

$$\begin{aligned} \|\hat{u}_i\|^2_{H^0(\Gamma)} &= \langle \hat{u}_i, \hat{u}_i \rangle = \langle \hat{u}_i, Wv \rangle = W(\hat{u}_i, v - \chi) \\ &\le C \|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)} \|v - \chi\|_{\tilde{H}^{1/2}(\Gamma)} \\ &\le C h_i^{1/2-\varepsilon} \|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)} \|\hat{u}_i\|_{H^0(\Gamma)}. \end{aligned}$$

Therefore, (4.28) obviously holds. □

Proof of Theorem 4.1. Since Theorem 4.1 is a special version of Theorem 3.1, we only need to show that Hypotheses 3.1–3.2 hold for the operator W, the subspaces $\mathcal{W}_k = \mathcal{M}_k$, $\mathcal{V}_k = (Q_k - Q_{k-1})\mathcal{M}$, and the smoother R_k defined by (4.10).

Hypothesis 3.2 can be proved by using the strengthened Cauchy-Schwarz inequality (4.19) as follows.

For any $w_k \in \mathcal{M}_k$, $k = 0, 1, \ldots, J$, we have

$$\begin{aligned} \|\sum_{k=0}^{J} w_k\|^2_W &= \sum_{i,j=0}^{J} W(w_i, w_j) \\ &\le C \sum_{i,j=0}^{J} \gamma^{|i-j|} \langle R_i^{-1} w_i, w_i \rangle^{1/2} \langle R_j^{-1} w_j, w_j \rangle^{1/2}. \end{aligned}$$

Let Θ be the symmetric matrix with entries $\Theta_{ij} = \gamma^{|i-j|}$, $i, j = 0, 1, \ldots, J$, and let $\lambda_{\max}$ be the maximum eigenvalue of Θ. Hence, as shown in Section 4.2, $\lambda_{\max}$ is bounded by

$$\lambda_{\max} \le 2 \sum_{k=0}^{\infty} \gamma^k \le \frac{2}{1-\gamma}.$$

Let $\vec{\eta}$ be the vector with components

$$\vec{\eta}_k = \langle R_k^{-1} w_k, w_k \rangle^{1/2} \quad \text{for } k = 0, 1, \ldots, J,$$

then we have

$$\left\| \sum_{k=0}^{J} w_k \right\|_W^2 \leq C(\Theta\vec{\eta}, \vec{\eta}) \leq \tilde{K}_2(\vec{\eta}, \vec{\eta}) = \tilde{K}_2 \sum_{k=0}^{J} \langle R_k^{-1} w_k, w_k \rangle.$$

The proof of the stability Hypothesis 3.1 will depend on the approximation and stability properties of the operators Q_k, $k = 0, \ldots, J$, and P_i, $i = 0, \ldots, J$, as follows.

For any $u_k \in \mathcal{V}_k$, $k = 0, 1, \ldots, J$, let $u = \sum_{k=0}^{J} u_k$; then $u \in \mathcal{M}$ and $u_k = (Q_k - Q_{k-1})u$, where $Q_{-1} := 0$. We also define $\hat{u}_i = (P_i - P_{i-1})u$, $i = 0, \ldots, J$, where $P_{-1} := 0$. Since $u = P_J u$, we have $u = \sum_{i=0}^{J} \hat{u}_i$. Therefore, it follows from the approximation property of P_i, $i = 0, \ldots, J$, (Lemma 4.4), and the approximation property of Q_k, $k = 0, \ldots, J$, (Lemma 4.3) that

$$\|(Q_k - Q_{k-1})\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)} \leq C h_i^{1/2-\varepsilon} h_k^{-1/2} \|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)}, \tag{4.30}$$

for $k, i = 0, \ldots, J$. Note that (4.30) holds for $k = 0$ or $i = 0$ because h_0 is fixed, i.e., it is a positive constant independent of h_J and J.

For $k - 1 \geq i$ or $k - 1 \geq j$, we observe that

$$\begin{aligned} W((Q_k - Q_{k-1})\hat{u}_i, (Q_k - Q_{k-1})\hat{u}_j) & \\ &= \langle W_k (Q_k - Q_{k-1})\hat{u}_i, (Q_k - Q_{k-1})\hat{u}_j \rangle \\ &= 0. \end{aligned} \tag{4.31}$$

Therefore

$$\begin{aligned}
\sum_{k=0}^{J}\|u_k\|_W^2 &= \sum_{k=0}^{J} W((Q_k-Q_{k-1})u,(Q_k-Q_{k-1})u)\\
&= \sum_{k=0}^{J}\sum_{i,j=0}^{J} W((Q_k-Q_{k-1})\hat{u}_i,(Q_k-Q_{k-1})\hat{u}_j)\\
&= \sum_{i,j=0}^{J}\sum_{k=0}^{\min(i,j)} W((Q_k-Q_{k-1})\hat{u}_i,(Q_k-Q_{k-1})\hat{u}_j)\\
&\le C\sum_{i,j=0}^{J}\sum_{k=0}^{\min(i,j)} \|(Q_k-Q_{k-1})\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)}\\
&\qquad\qquad \cdot\|(Q_k-Q_{k-1})\hat{u}_j\|_{\tilde{H}^{1/2}(\Gamma)}\\
&\overset{(4.30)}{\le} C\sum_{i,j=0}^{J}\sum_{k=0}^{\min(i,j)} h_k^{-1}h_i^{1/2-\varepsilon}h_j^{1/2-\varepsilon}\|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)}\|\hat{u}_j\|_{\tilde{H}^{1/2}(\Gamma)}.
\end{aligned}$$

For $j\ge i\ge k$, we have

$$\frac{h_i^{1/2}}{h_k^{1/2}} = \left(\frac{1}{\sqrt{2}}\right)^{i-k}$$

and

$$\frac{h_j^{1/2}}{h_k^{1/2}} = \left(\frac{1}{\sqrt{2}}\right)^{j-k} \le \left(\frac{1}{\sqrt{2}}\right)^{j-i}.$$

Therefore

$$\begin{aligned}
&\sum_{\substack{i=1\\ j\ge i}}^{J}\sum_{k=1}^{i} h_k^{-1}h_i^{1/2-\varepsilon}h_j^{1/2-\varepsilon}\|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)}\|\hat{u}_j\|_{\tilde{H}^{1/2}(\Gamma)}\\
&\qquad\le h_J^{-2\varepsilon}\sum_{\substack{i=1\\ j\ge i}}^{J}\left(\frac{1}{\sqrt{2}}i\right)^{j-i}\|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)}\|\hat{u}_j\|_{\tilde{H}^{1/2}(\Gamma)}\sum_{k=1}^{i}\left(\frac{1}{\sqrt{2}}\right)^{i-k}\\
&\qquad\le Ch_J^{-2\varepsilon}\sum_{\substack{i=1\\ j\ge i}}^{J}\left(\frac{1}{\sqrt{2}}\right)^{j-i}\|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)}\|\hat{u}_j\|_{\tilde{H}^{1/2}(\Gamma)},
\end{aligned}$$

and similarly we can show that

$$\sum_{\substack{i=1\\ j<i}}^{J}\sum_{k=1}^{j} h_k^{-1} h_i^{1/2-\varepsilon} h_j^{1/2-\varepsilon} \|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)} \|\hat{u}_j\|_{\tilde{H}^{1/2}(\Gamma)}$$

$$\leq C h_J^{-2\varepsilon} \sum_{\substack{i=1\\ j<i}}^{J} \left(\frac{1}{\sqrt{2}}\right)^{i-j} \|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)} \|\hat{u}_j\|_{\tilde{H}^{1/2}(\Gamma)}.$$

Using the same argument as in the proof of the right inequality, with $\Theta_{ij} = (1/\sqrt{2})^{|i-j|}$ and $\vec{\eta}_i = \|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)}$ for $i, j = 1, \ldots, J$, we have

$$\sum_{k=1}^{J} \|u_k\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C h_J^{-2\varepsilon} \sum_{i,j=1}^{J} \left(\frac{1}{\sqrt{2}}\right)^{|i-j|} \|\hat{u}_i\|_{\tilde{H}^{1/2}(\Gamma)} \|\hat{u}_j\|_{\tilde{H}^{1/2}(\Gamma)}$$

$$= C h_J^{-2\varepsilon} (\Theta\vec{\eta}, \vec{\eta}) \leq C h_J^{-2\varepsilon} \sum_{i=1}^{J} \|\hat{u}_i\|^2_{\tilde{H}^{1/2}(\Gamma)} \tag{4.32}$$

$$\leq C h_J^{-2\varepsilon} \sum_{i=1}^{J} W((P_i - P_{i-1})u, (P_i - P_{i-1})u).$$

Since $P_{i-1} \in \mathcal{M}_{i-1} \subset \mathcal{M}_i$, $i = 1, \ldots, J$, we have

$$W((P_i - P_{i-1})u, P_{i-1}u) = 0, \quad i = 1, \ldots, J.$$

Hence

$$\sum_{i=1}^{J} W((P_i - P_{i-1})u, (P_i - P_{i-1})u) \tag{4.33}$$

$$= \sum_{i=1}^{J} W((P_i - P_{i-1})u, u)$$

$$= W((P_J - P_0)u, u) = W((I - P_0)u, u)$$

$$= W((I - P_0)u, (I - P_0)u)$$

$$\leq C\|(I - P_0)u\|^2_{\tilde{H}^{1/2}(\Gamma)}$$

Therefore, it follows from (4.32)–(4.33) that

$$\sum_{k=0}^{J} \|u_k\|^2_{\tilde{H}^{1/2}(\Gamma)} \leq C h_J^{-2\varepsilon} (\|Q_0 u\|^2_{\tilde{H}^{1/2}(\Gamma)} + \|(I - P_0)u\|^2_{\tilde{H}^{1/2}(\Gamma)}).$$

The stability of the operator $Q_0 : \mathcal{M} \to \mathcal{M}_0$ (Lemma 4.3), and the well-known stability of the Ritz-Galerkin operator $P_0 : \mathcal{M} \to \mathcal{M}_0$ then ensure that

$$\sum_{k=0}^{J} \|u_k\|_W^2 \leq C h_J^{-2\varepsilon} \| \sum_{k=0}^{J} u_k \|_W^2 \tag{4.34}$$

for any $u_k \in \mathcal{V}_k$. We now only need to show that

$$\langle R_k^{-1} u_k, u_k \rangle \leq C \|u_k\|_W^2, \quad u_k \in \mathcal{V}_k. \tag{4.35}$$

We note that $Q_{k-1} u_k = 0$ for $u_k \in \mathcal{V}_k$, so $u_k = (I - Q_{k-1}) u_k$. Hence, using (4.9), (4.24), $\lambda_k = \mathcal{O}(h_k^{-1})$ and the inverse inequality,

$$\begin{aligned} \langle R_k^{-1} u_k, u_k \rangle &\overset{(4.9)}{\leq} C h_k^{-1} \|u_k\|_{H^0(\Gamma)}^2 \\ &= C h_k^{-1} \|(I - Q_{k-1}) u_k\|_{H^0(\Gamma)}^2 \\ &\overset{(4.24)}{\leq} C h_k \|u_k\|_{\tilde{H}^1(\Gamma)}^2 \\ &\overset{\text{inv.}}{\leq} C \|u_k\|_W^2, \end{aligned}$$

which proves (4.35). Now, (4.34)–(4.35) give

$$\left\| \sum_{k=0}^{J} u_k \right\|_W^2 \geq C h_J^{-2\varepsilon} \sum_{k=0}^{J} (R_k^{-1} u_k, u_k), \tag{4.36}$$

which is equivalent to (3.13) in Hypothesis 3.1 with $K_1 = O(h_J^{-2\varepsilon})$. □

5. Matrix implementation of the BPX preconditioner. The preconditioner B_{BPX} in (4.6) was defined as a functional operator, which is not easy to implement. It was shown in [**16**] or [**6**] that the BPX preconditioner can be expressed as a matrix $\overline{B}_{\mathrm{BPX}}$, where $\overline{B}_{\mathrm{BPX}}$ is defined by

$$\overline{B}_{\mathrm{BPX}} = \sum_{k=0}^{J} \Pi_k \Pi_k^T, \tag{5.1}$$

where Π_k is the matrix that represents the nodal basis in $\mathcal{M}_k$ in terms of the nodal basis in $\mathcal{M}$, with its entries $(\Pi_k)_{ij} = \psi_i^k(x_j)$ for $i = 1, \ldots, N_k$, and $j = 1, \ldots, N$, and Π_k^T is the transpose matrix of Π_k. BPX preconditioner is applied with the preconditioned conjugate gradient method, which requires the computation of $\overline{B}_{\mathrm{BPX}}\bar{v}$, $\bar{v} \in \mathbf{R}^N$, only. The implementation of the above computations was outlined as follows.

Let Π_{k-1}^k, $k = 1, \ldots, J$, be the matrix that represents the nodal basis in $\mathcal{M}_{k-1}$ in term of the nodal basis in $\mathcal{M}_k$, with its entries given by

$$(\Pi_{k-1}^k)_{ij} = \psi_i^{k-1}(x_j) \quad \text{for } x_j \in \mathcal{N}_k;\ i{=}1, \ldots, N_{k-1},\ j{=}1, \ldots, N_k.$$

Then each nodal basis function ψ_i^{k-1} in $\mathcal{M}_{k-1}$, $k = 1, \ldots, J$, can be expressed by

$$\psi_i^{k-1} = \sum_{j=1}^{N_k} \psi_i^{k-1}(x_j)\psi_j^k = \sum_{j=1}^{N_k} (\Pi_{k-1}^k)_{ij}\psi_j^k, \quad i = 1, \ldots, N_{k-1}. \tag{5.2}$$

Since the action of Π_{k-1}^k, $k = 1, \ldots, J$, is computed easily (see Algorithm 5.3), the action of Π_k will be implemented by the following factorization:

$$\Pi_k = \Pi_{J-1}^J \Pi_{J-2}^{J-1} \ldots \Pi_{k-1}^k. \tag{5.3}$$

Let Π_k^T be the transpose matrix of Π_k and Π_k^{k-1} the transpose matrix of Π_{k-1}^k, $k = 1, \ldots, J$. The actions of Π_{k-1}^k are also implemented easily (see Algorithm 5.2), and the actions of Π_k^T are implemented by the following factorization:

$$\Pi_k^T = \Pi_{k+1}^k \Pi_{k+2}^{k+1} \ldots \Pi_J^{J-1}. \tag{5.4}$$

We note that Π_{k-1}^k and Π_k^{k-1} are called the restriction and the prolongation, respectively, in multigrid terminology. We now present a high level algorithm for BPX preconditioner to compute $\overline{B}_{\mathrm{BPX}}\bar{v}$, that is based on (5.1), as follows.

Algorithm 5.1. (*Algorithm for computing* $\overline{B}_{\mathrm{BPX}}\bar{v}$, $\bar{v} \in \mathbf{R}^N$).

1. $\bar{v}^J = \bar{v}$
2. *for* $k = J$ *down to* 1, *do*
   $\bar{v}^{k-1} = \Pi_{k-1}^k \bar{v}^k$
   end
3. $\bar{w}^0 = \bar{v}^0$
4. *for* $k = 1$ *up to* J, *do*
   $\bar{w}^k = \bar{v}^k + \Pi_k^{k-1} \bar{w}^{k-1}$
   end
5. $\overline{B}_{\mathrm{BPX}} \bar{v} = \bar{w}^J$.

The algorithm to compute $\Pi_k^{k-1} \bar{v}$, $k = 1, \ldots, J$, is based on (5.3) as follows.

Algorithm 5.2. (*Algorithm for computing* $\Pi_k^{k-1} \bar{v}$, $\bar{v} \in \mathbf{R}^{N_{k-1}}$).
for $x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}$, *do*
$\bar{v}(i) \longleftarrow [\bar{v}(I1(i)) + \bar{v}(I2(i))]/2$
end.

where $I1(i), I2(i) \in \mathcal{N}_{k-1}$ are parent nodes of $x_i \in \mathcal{N}_k \backslash \mathcal{N}_{k-1}$. Similarly, the algorithm to compute $\Pi_{k-1}^k \bar{v}$, $k = J, \ldots, 1$, is based on (5.4), as follows.

Algorithm 5.3. (*Algorithm for computing* $\Pi_{k-1}^k \bar{v}$, $\bar{v} \in \mathbf{R}^{N_k}$).
for $x_i \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}$, *do*
$\bar{v}(I1(i)) \longleftarrow \bar{v}(I1(i)) + \bar{v}(i)/2$
$\bar{v}(I2(i)) \longleftarrow \bar{v}(I2(i)) + \bar{v}(i)/2$
end.

Let f be the number of operations to compute the action of $\overline{B}_{\mathrm{BPX}}$, and let f_k, $k = 0, \ldots, J$, be the number of operations to compute the actions of Π_k and Π_k^T. Since the number of operations to compute the

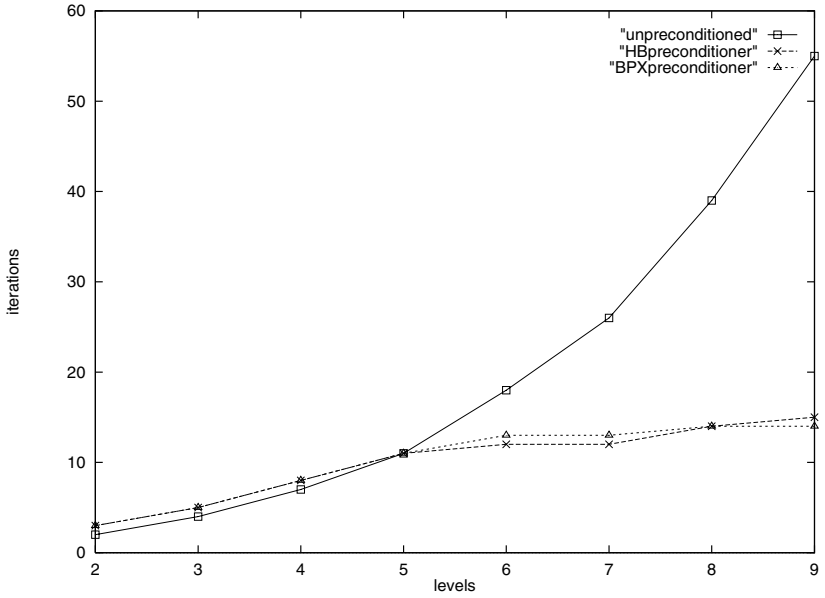

FIGURE 1. Iterations versus levels plot.

actions of Π_k^{k-1} and Π_{k-1}^k, $k = 1, \ldots, J$, is bounded by $C(N_k - N_{k-1})$, it is obvious that

$$f_k \leq C \sum_{l=1}^{k} (N_l - N_{l-1}) \leq C N_k, \quad k = 0, \ldots, J.$$

Since $N_k = \mathcal{O}(2^k)$, we conclude that

$$f \leq C \sum_{k=0}^{J} N_k \leq C \sum_{k=0}^{J} 2^k \leq C 2^J \leq CN.$$

This means that the action of $\overline{B}_{\mathrm{BPX}}$ can be carried out with $\mathcal{O}(N)$ operations, and therefore the BPX preconditioned conjugate gradient method for the hypersingular integral equation on an interval, would require $\mathcal{O}(N^2)$ operations to obtain the approximate solution, with the error in the energy norm of the Galerkin order.

6. Numerical experiments. In this section we will present numerical experiments that compare the performances of unpreconditioned, BPX preconditioned and HB preconditioned conjugate gradient methods. These two methods are denoted briefly as BPX PCG and HB PCG throughout this section, respectively. Our numerical example will be for the hypersingular integral equation (4.1) on $(-1, 1)$, with the right-hand side function $f(x) = 2$. In this special case, the exact solution is $u = 2\sqrt{1-x^2}$. The finite element spaces are piecewise linear spaces defined in Section 4. Therefore, the system of linear equations we need to solve, by applying the Galerkin method, is

$$\overline{W}\bar{u} = \bar{f}, \tag{6.1}$$

where $\bar{f}$ is the vector with components

$$\bar{f}_i = \langle 2, \psi_i \rangle = 2h_i, \quad i = 1, \ldots, N. \tag{6.2}$$

We will apply the PCG method, described in Golub and Van Loan [**10**], to solve (6.1) with $\overline{B}_{\mathrm{HB}}$ (HB preconditioner defined in matrix form [**7**]) and $\overline{B}_{\mathrm{BPX}}$ as the preconditioners. The relative error ε is chosen to be $\varepsilon = 10^{-8}$, which is below the order of the Galerkin error for up to nine levels of uniformly mesh refinement at least ($2^{-9} \simeq 2 \times 10^{-3}$). The condition numbers $\kappa(\overline{B}_{\mathrm{HB}}\overline{W})$, and $\kappa(\overline{B}_{\mathrm{BPX}}\overline{W})$ are calculated by

$$\kappa(\overline{B}_{\mathrm{HB}}\overline{W}) = \frac{\lambda_{\max}(\overline{B}_{\mathrm{HB}}\overline{W})}{\lambda_{\min}(\overline{B}_{\mathrm{HB}}\overline{W})}$$

and

$$\kappa(\overline{B}_{\mathrm{BPX}}\overline{W}) = \frac{\lambda_{\max}(\overline{B}_{\mathrm{BPX}}\overline{W})}{\lambda_{\min}(\overline{B}_{\mathrm{BPX}}\overline{W})}.$$

The extremal eigenvalues are calculated using the Lanczos algorithm to generate a symmetric tridiagonal matrix (see Golub and Van Loan [**10**, Section 10.2.6]), then using the symmetric QR algorithm in [**10**, pp. 419–425] to evaluate the extremal eigenvalues. The numerical experiments are performed on a Power Macintosh 6100/60 using the Metrowerk C++ compiler. The number of iterations, condition numbers, and CPU times for the HB and BPX PCG methods, and the unpreconditioned CG method, are given in Table 1. The comparisons of the iteration numbers between these three methods are given in Figure 1.

TABLE 1. CG: Unpreconditioned CG method, HB: HB preconditioned CG method, BPX: BPX preconditioned CG method.

		No. iterations			Condition Number			CPU times (sec)		
k	N_k	CG	HB	BPX	CG	HB	BPX	CG	HB	BPX
2	3	2	3	3	2.01	1.17	1.64	0	0	0
3	7	4	5	5	3.86	1.55	2.41	0	0	0
4	15	7	8	8	7.74	1.90	3.04	0	0	0
5	31	11	11	11	15.54	2.24	3.46	0	0	0
6	63	18	12	13	31.11	2.62	3.76	1	1	1
7	127	26	12	13	62.40	3.06	3.97	21	3	3
8	255	39	14	14	125.09	3.55	4.13	51	13	13
9	511	55	15	14	250.47	4.09	4.26	172	63	62

As shown by Figure 1 and Table 1, the performances of BPX PCG method is slightly better than HB PCG method, and they also show clearly the advantages of these two methods over the unpreconditioned CG method in term of CPU times, number of iterations and condition numbers. The numbers in Table 1 also show that the condition numbers $\kappa(\overline{B}_{\mathrm{BPX}}\overline{W})$, and the number of iterations for BPX PCG methods, behave like $\mathcal{O}(1)$, as shown by our theoretical results.

Acknowledgments. This work is a part of the author's PhD thesis at the School of Mathematics, The University of NSW. The author is supported by APRA and an ARC grant held by Professor I. Sloan and Dr. W. McLean. I would like to thank Professor I. Sloan and Dr. W. McLean for valuable advice and discussions. My thanks also go to Professor E.P. Stephan and Dr. Thanh Tran for many fruitful discussions.

REFERENCES

1. J.H. Bramble, Z. Leyk and J.E. Pasciak, *The analysis of multigrid algorithm for pseudo-differential operator of order minus one*, Math. Comp. **63** (1994), 461–478.

2. J.H. Bramble and J.E. Pasciak, *New convergence estimates for multigrid algorithms*, Math. Comp. **49** (1987), 311–329.

3. ———, *New estimates for multilevel algorithms including the V-cycles*, Math. Comp. **60** (1993), 447–471.

4. J.H. Bramble, J.E. Pasciak and J. Xu, *Parallel multilevel preconditioners*, Math. Comp. **55** (1990), 1–21.

5. J.H. Bramble and J. Xu, *Some Estimates for a weighted L^2 projection*, Math. Comp. **56** (1991), 463–476.

6. T. Cao, *Adaptive and additive multilevel methods for boundary integral equations*, PhD Thesis, Department of Applied Mathematics, The University of New South Wales, Sydney, Australia, 1996.

7. ———, *Hierarchical basis methods for hypersingular integral equations*, IMA J. Num. Anal. **17** (1997), 603–619.

8. ———, *Adaptive-additive multilevel methods for hypersingular integral equations*, submitted to Numer. Math. (1996).

9. M. Dryja and O. B. Widlund, *Multilevel additive methods for elliptic finite element problems*, in *Parallel algorithm for partial differential equations*, Proc. of the Sixth GAMM Seminar, Kiel, Germany, January 19–21, 1990 (Hackbusch, ed.), Vieweg, Braunschweig, 1991, 58–69.

10. G. Golub and C. Van Loan, *Matrix computations*, Johns Hopkins University Press, Baltimore, 1989.

11. J.L. Lions and E. Magenes, *Non-homogeneous boundary value problems and application* 1, Springer Verlag, Berlin, 1972.

12. T. von Petersdorff and E.P. Stephan, *On the convergence of the multigrid method for a hypersingular integral equation of the first kind*, Numer. Math. **57** (1990), 379–391.

13. T. Tran and E.P. Stephan, *Additive Schwarz method for the H-version boundary element method*, Appl. Anal. **60** (1996), 63–84.

14. H. Triebel, *Interpolation theory. Function spaces. Differential operator*, Berlin, Deutscher Verl. der Wissenschaften, 1978.

15. W.L. Wendland and E.P. Stephan, *A Hypersingular boundary integral method for two dimensional screen and crack problems*, Arch. Rational Mech. Anal. **112** (1990), 363–390.

16. J. Xu, *Iterative methods by space decomposition and subspace correction*, SIAM Rev. **34** (1992), 581–613.

17. H. Yserentant, *On the multi-level splitting of finite element spaces*, Numer. Math. **49** (1986), 379–412.

18. X. Zhang, *Multilevel Schwarz methods*, Numer. Math. **63** (1992), 521–539.

School of Mathematics, University of New South Wales, Kensington, NSW 2052, Australia
E-mail address: thang@maths.unsw.edu.au