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A-POSTERIORI ESTIMATES AND ADAPTIVE SCHEMES FOR TRANSMISSION PROBLEMS

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ABSTRACT. The results presented here are directed to Galerkin schemes with respect to stable multiscale bases discretizations for boundary integral equations which describe transmission problems. We derive a posteriori estimates which are reliable and efficient with respect to any desirable tolerance. Moreover, the convergence of an adaptive scheme is investigated. The underlying ideas are applicable to a wide class of elliptic problems, cf. [**14**]. Here further details concerning decay estimates and appropriate index-sets for a system of boundary integral equations are presented.

1. Introduction. In [**14**] a posteriori estimates for a general class of elliptic problems were introduced, and their reliability and efficiency was shown. Moreover, convergence could be concluded for essentially symmetric problems, i.e., for problems which give bilinear forms that are, except for a sufficiently small perturbation, symmetric and elliptic. Here we apply and concretize this approach to a system of boundary integral equations.

To be more precise we consider transmission problems for the Helmholtz and Laplace equations where the transmission coefficient μ and the wave numbers k_1, k_2 are constant complex numbers. Scattering problems are included. Costabel and Stephan derived in [**12**] a system of boundary integral equations, such that the Cauchy data of the solutions of the transmission problems can be determined by the solutions of that system. An application of the representation formula yields solutions in the whole space. They prove a Gårding inequality and find conditions on μ and k_1, k_2 such that there exist unique

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solutions. The Gårding inequality and uniqueness imply for conforming Galerkin methods asymptotic quasi-optimal error estimates and for sufficiently fine discretization stability. We restrict our presentation to the polygonal plane case because of two reasons. First, there is no proof for a Gårding inequality for higher dimensional piecewise smooth geometries except for the case of Laplacians, where results by Costabel [**11**] can be applied. Second, the extension of the presented approach to higher dimensions would essentially apply the same ideas.

Generally, the understanding of a posteriori error estimates and adaptive schemes for boundary element methods appears to be far less developed than for finite element methods for partial differential operators, cf. [**1, 2, 3, 5, 22, 28**]. A discussion of rather general algorithms for boundary integral equations could be found in [**25**]. Some of the first rigorous results on a posteriori estimates for boundary element methods for strongly elliptic pseudodifferential operators on smooth boundaries were given by Wendland and Yu in [**30**]. They introduce a so-called influence index with respect to the underlying set of basis functions, which reflects in some sense the pseudo-local property of strongly elliptic pseudodifferential operators. Their adaptive method is most efficient if the influence index is much smaller than the number of basis functions. The involved local error indicators are solutions of local problems. They transfer their results also to those nodal collocation methods which can be interpreted as modified Galerkin methods. Faermann introduced in [**20**] the approach of local functionals developed by Bank and Rheinboldt for finite element methods. In this way she generalizes the results in [**30**] to some extent. Furthermore, we mention [**31**] where it is shown that the local error can be bounded by a sum of a local residual and some global terms. These results are improved in [**27**] where Saranen and Wendland considered pseudodifferential operators of general integer order on smooth closed Jordan curves in the plane. Recently Carstensen and Stephan presented in [**8, 9**] a posteriori estimates within another framework. Their approach leads to upper bounds for the global error consisting of terms which can be evaluated locally. Although they have no proof for the efficiency of their a posteriori estimates in general or the convergence of adaptive algorithms, their numerical examples show the desired behavior. For quasi-uniform meshes the efficiency of an a posteriori estimate is investigated in [**7**].

The objective of this note is to consider a posteriori estimates and adaptive schemes in the context of multiscale bases oriented methods. Multiscale bases seem to be an appropriate tool for taking systematic advantage of the pseudo-local property of boundary integral equations. Problems like preconditioning, compression and local a posteriori estimation can be handled in a nearly unified way, cf. [**16, 17, 14**]. Thus the multiscale approach offers a unified framework within which several problems can be handled. It differs from usual finite element techniques in that direct use of bases is made which span complements of successive trial spaces. Here we pose a posteriori error estimates which are reliable and efficient with respect to any prescribed accuracy and prove the convergence of an adaptive scheme induced by those a posteriori estimates at least for smooth boundaries.

The present investigations offer a variety of possibilities to fulfill assumptions made in [**14**] for obtaining an efficient and reliable a posteriori estimator. The estimates in [**14**] are based on representations of certain decay properties which allow proving the principal facts. Here we study those problem exemplarily in greater detail. We prepare moduli for deriving decay estimates and suitable index sets which are applicable for general integral equation problems. The presented results show a variety of possibilities for finally obtaining the desired estimates and shed light upon interdependencies of appearing constants. Especially, the results are in some sense preparatory for forthcoming numerical experiments. An analytic approach to the question of the asymptotic complexity has to be investigated in further notes. There the problem could be split into two parts: (i) The problem of adaptive approximation of the residual by biorthogonal multiscale bases; (ii) A compression step, taking into account the analytic properties of the underlying operators.

This note is organized as follows. In Section 2 we summarize known facts about the transmission problem. Galerkin schemes for the transmission problem with respect to multiscale bases discretizations are introduced in Section 3. There we also describe those properties of multiscale bases which seem important in considering a posteriori estimates. The a posteriori estimates are investigated in Section 4. The crucial idea is to use decay estimates for the definition of a posteriori estimators which are efficient and reliable up to a chosen tolerance. The same idea can also be used to obtain a posteriori estimates for a more general class of elliptic problems, cf. [**14**]. For the special case of a second order two point boundary value problem, this idea was introduced by Bertoluzza in [**4**]. Section 5 contains a result about the convergence of an adaptive scheme based on the a posteriori estimates.

For further comments about the underlying concept in a more general setting, we refer the reader to [**14**]. Finally we remark that the relation $a \lesssim b$ expresses that a can be bounded by some constant times b uniformly in the parameters on which a and b may depend. The symbol $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold.

2. A direct boundary integral equation for transmission problems.

2.1. *Formulation of the problem.* Let $\Omega_1 \subset \mathbf{R}^2$ be a bounded simply connected polygonal domain and $\Omega_2 = \mathbf{R}^2 \backslash \overline{\Omega}_1$ its complement with the common boundary $\Gamma := \partial\Omega_1 = \partial\Omega_2$. The normal derivative $\partial/\partial n$ on Γ is defined with respect to the normal pointing from Ω_1 to Ω_2.

For given functions $u_0 \in H^{1/2}(\Gamma)$, $v_0 \in H^{-1/2}(\Gamma)$ and $k_1, k_2, \mu \in \mathbf{C}$ we seek solutions (u_1, u_2) of the transmission problem

$$(\Delta + k_j^2) u_j = 0 \quad \text{in } \Omega_j, \quad j = 1, 2, \tag{2.1}$$

$$\begin{cases} u_1 - u_2 = u_0, \\ \mu(\partial u_1/\partial n) - (\partial u_2/\partial n) = v_0 \quad \text{on } \Gamma, \end{cases}$$

that satisfy certain conditions at infinity. For $k_2 \neq 0$ one has

$$u_2(x) = \mathcal{O}(|x|^{-1/2}), \quad |x| \to \infty,$$

and

$$\frac{\partial u_2(x)}{\partial |x|} - i k_2 u_2(x) = o(|x|^{-1/2}), \quad |x| \to \infty,$$

which is Sommerfeld's radiation condition. For $k_2 = 0$ there is a constant b such that

$$u_2(x) = \frac{b}{2\pi} \log |x| + o(1), \quad |x| \to \infty.$$

To formulate the so-called direct boundary integral equation problem which is studied in detail by Costabel and Stephan in [**12**], we introduce the fundamental solutions

$$g_j(x,y) := \begin{cases} 1/2\pi \log |x-y|, & k_j = 0, \\ -i/4 H_0^{(1)}(k_j |x-y|), & k_j \neq 0, \end{cases}$$

where $H_0^{(1)}$ is the modified Bessel function of the first kind satisfying the radiation conditions. For $j = 1, 2$ and $\varphi \in C^\infty(\Gamma)$ we define on Γ the integral operators

$$\begin{aligned}
V_j\varphi(x) &:= -2\int_\Gamma \varphi(y) g_j(x,y)\,ds_y,\\
K_j\varphi(x) &:= -2\int_\Gamma \varphi(y)\frac{\partial}{\partial n_y} g_j(x,y)\,ds_y,\\
K_j'\varphi(x) &:= -2\int_\Gamma \varphi(y)\frac{\partial}{\partial n_x} g_j(x,y)\,ds_y,\\
D_j\varphi(x) &:= -\frac{\partial}{\partial n_x} K_{\Omega_j}\varphi(x),
\end{aligned}$$

with $K_{\Omega_j}\varphi(x) := -2\int_\Gamma \varphi(y)(\partial/\partial n_y) g_j(x,y)\,ds_y$ for $x \in \Omega_j$.

These integral operators can be extended to continuous operators

$$\begin{aligned}
V_j &: H^{-1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma),\\
K_j &: H^{1/2+s}(\Gamma) \longrightarrow H^{1/2+s}(\Gamma),\\
K_j' &: H^{-1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma),\\
D_j &: H^{1/2+s}(\Gamma) \longrightarrow H^{-1/2+s}(\Gamma),
\end{aligned}$$

for $s \in [-1/2, 1/2]$, cf. [**11**], where

$$\begin{aligned}
&H^s(\Gamma) := \{u|_\Gamma \mid u \in H^{s+1/2}(\mathbf{R}^2)\}, \quad s > 0,\\
&H^0(\Gamma) := L^2(\Gamma),\\
&H^s(\Gamma) := (H^{-s}(\Gamma))', \quad \text{(dual spaces)}, \quad s < 0.
\end{aligned}$$

The norms in $H^s(\Gamma)$ are denoted by $\|\cdot\|_s$, $s \in \mathbf{R}$. On the product spaces $H^s(\Gamma) \times H^t(\Gamma)$, $s, t \in \mathbf{R}$, we introduce

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{s,t} := (\|u\|_s^2 + \|v\|_t^2)^{1/2}, \quad \begin{pmatrix} u \\ v \end{pmatrix} \in H^s(\Gamma) \times H^t(\Gamma).$$

If we consider restrictions of functions to subsets of Γ we note those subsets as additional indices.

The integral operators satisfy symmetry relations, i.e., one has

$$\begin{aligned}
\langle V_j v, z\rangle &= \langle v, V_j z\rangle,\\
\langle K_j u, z\rangle &= \langle u, K_j' z\rangle,\\
\langle D_j u, w\rangle &= \langle u, D_j w\rangle,
\end{aligned} \tag{2.2}$$

for $u, w \in H^{1/2}(\Gamma)$, $v, z \in H^{-1/2}(\Gamma)$ where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-s}(\Gamma)$ and $H^s(\Gamma)$ such that $\langle f, g \rangle = \int_\Gamma f(x)g(x)\,dx$ for sufficiently smooth functions f and g.

With $D := 1/2(D_1 + 1/\mu D_2)$, $K := 1/2(K_1 + K_2)$, $K' := 1/2(K_1' + K_2')$ and $V := 1/2(V_1 + \mu V_2)$ we define the matrix operator

$$H := \begin{pmatrix} D & K' \\ -K & V \end{pmatrix}. \tag{2.3}$$

Furthermore, we set

$$A_j := \begin{pmatrix} -K_j & V_j \\ D_j & K_j' \end{pmatrix} \quad \text{and} \quad M := \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}.$$

Then the direct integral equation formulation of the transmission problem reads

$$H \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \tag{2.4}$$

with

$$\begin{pmatrix} g_2 \\ g_1 \end{pmatrix} := 1/2 M^{-1}(I + A_2) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}.$$

If $\binom{u}{v} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ denotes a solution of (2.4), then $\binom{u_j}{v_j}$, $j = 1, 2$, defined by

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} := 1/2(I + A_1) \begin{pmatrix} u \\ v \end{pmatrix}$$

and

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} := 1/2(I - A_2) \left[M \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right]$$

are the Cauchy data of the solution of (2.1). They give, by the usual representation formulas, the solution u of (2.1) in $\mathbf{R}^2$, i.e., we have

$$u(x) = (-1)^j 1/2(K_{\Omega_j} u_j(x) - V_{\Omega_j} v_j(x)), \quad x \in \Omega_j,$$

where $V_{\Omega_j} v_j(x) := -2 \int_\Gamma g_j(x, y) v_j(y)\,ds_y$.

The transmission problem also includes scattering problems. Let an incident field u_I on $\mathbf{R}^2$ be given with $(\Delta + k_2^2)u_I = 0$ in Ω_1, and let u_0, v_0 represent the body trace of the field u_I, i.e.,

$$u_0 = u_I\Big|_\Gamma \quad \text{and} \quad v_0 = \frac{\partial u_I}{\partial n}\Big|_\Gamma.$$

Then the solution (u_1, u_2) of the corresponding transmission problem (2.4) yields the total field u. In Ω_1 one has $u|_{\Omega_1} = u_1$ and in Ω_2 one has $u|_{\Omega_2} = u_2 + u_I$. The transmission conditions on Γ are

$$u_1 = u \quad \text{and} \quad \mu\frac{\partial u_1}{\partial n} = \frac{\partial u}{\partial n}.$$

Clearly the scattered field in the exterior domain Ω_2 is nothing else than u_2. Further, the right side of equation (2.4) can be simplified since

$$(I - A_2)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0,$$

which leads to

$$H\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v_0 \\ \mu^{-1}u_0 \end{pmatrix}.$$

Therefore convergence properties derived for the transmission problem also apply in an effective way on the scattering problem and its approximate solutions.

2.2. *Existence and uniqueness results.* Existence and uniqueness results are based on the observation that the operator H satisfies a Gårding inequality. For its formulation we introduce the notion

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix} \right\rangle_\Gamma := \langle u, w\rangle + \langle v, z\rangle$$

for $\binom{u}{v} \in H^s(\Gamma) \times H^{-s}(\Gamma)$, $\binom{w}{z} \in H^{-s}(\Gamma) \times H^s(\Gamma)$, $s \in \mathbf{R}$.

Theorem 2.1. *If* $\operatorname{Re}(1 + 1/\mu) > 0$ *and* $\operatorname{Re}(1 + \mu) > 0$ *there exists a compact operator* $\mathcal{C} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *and a constant* $\gamma > 0$ *such that*

$$\operatorname{Re}\left\langle (H + \mathcal{C})\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\rangle_\Gamma \geq \gamma(\|u\|_{1/2}^2 + \|v\|_{-1/2}^2) \tag{2.5}$$

for $\binom{u}{v} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$.

For a proof, cf. [**12**]. If $k_1 = k_2 = 0$, i.e., for Laplacians in Ω_1 and Ω_2 one can give a simpler proof by using results from [**11**] which also apply to general Lipschitz domains in arbitrary dimensions: Let us denote the integral operators with respect to $k_1 = k_2 = 0$ by V_0, K_0, K_0' and D_0. Taking into account that the kernel functions of the operators K_0 and K_0' are real, we get

$$\left\langle H\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\rangle_\Gamma = 1/2(1+1/\mu)\langle D_0 u, \bar{u}\rangle + 1/2(1+\mu)\langle V_0 v, \bar{v}\rangle + 2i\mathrm{Im}\,\langle v, K_0\bar{u}\rangle. \tag{2.6}$$

With respect to D_0 and V_0 there are compact operators $\mathcal{C}_{D_0} : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, $\mathcal{C}_{V_0} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and constants $c_1, c_2 > 0$ such that

$$\langle (D_0 + \mathcal{C}_{D_0})u, \bar{u}\rangle \geq c_1 \|u\|_{1/2}^2, \quad u \in H^{1/2}(\Gamma), \tag{2.7}$$

$$\langle (V_0 + \mathcal{C}_{V_0})v, \bar{v}\rangle \geq c_2 \|v\|_{-1/2}^2, \quad v \in H^{-1/2}(\Gamma), \tag{2.8}$$

cf. [**11**]. Because of the symmetry relations (2.2), the quantities $\langle D_0 u, \bar{u}\rangle$ and $\langle V_0 v, \bar{v}\rangle$ are real. Thus we get, by (2.6)–(2.8),

$$\mathrm{Re}\left\langle \left(H + \begin{pmatrix} 1/2(1+1/\mu)\mathcal{C}_{D_0} & 0 \\ 0 & 1/2(1+\mu)\mathcal{C}_{V_0} \end{pmatrix} \right) \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} \right\rangle \geq c_1 \mathrm{Re}\,((1+1/\mu)/2)\|u\|_{1/2}^2 + c_2 \mathrm{Re}\,((1+\mu)/2)\|v\|_{-1/2}^2).$$

The Gårding inequality (2.5) implies Fredholm's alternative for the transmission problem (2.4), i.e., there are unique solutions $\binom{u}{v} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ for arbitrary $\binom{u_0}{v_0} \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ if the operator H is injective. In [**12**], several sufficient conditions for uniqueness are given. Generally, uniqueness does not hold, cf. [**10**] for a counterexample.

Corollary 2.2. *If the matrix operator H is injective, then problem* (2.4) *is uniquely solvable and H is continuously invertible, i.e., there are constants $c_1, c_2 > 0$ such that*

$$c_1 \left\| H\begin{pmatrix} u \\ v \end{pmatrix} \right\|_{-1/2,1/2} \leq \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|_{1/2,-1/2} \leq c_2 \left\| H\begin{pmatrix} u \\ v \end{pmatrix} \right\|_{-1/2,1/2}. \tag{2.9}$$

Investigating Galerkin approximations with respect to weaker norms than $\|\cdot\|_{1/2,-1/2}$ by the Aubin-Nitsche trick uses the adjoint problem, which reads as follows: Find for given $f \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ functions $w \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ such that

$$\langle v, H'w \rangle_\Gamma = \langle v, f \rangle_\Gamma, \quad v \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma),$$

where

$$H' := \begin{pmatrix} D & -K' \\ K & V \end{pmatrix}.$$

The adjoint operator H' has the same continuity properties as H and also holds a Gårding inequality. In particular, if uniqueness holds, we have that

$$H, H' : H^s(\Gamma) \times H^{s-1}(\Gamma) \longrightarrow H^{s-1}(\Gamma) \times H^s(\Gamma) \tag{2.10}$$

are continuous invertible operators for $s \in [1/2, \tilde{s})$ with $\tilde{s} \in (1, 3/2]$ determined by

$$\tilde{s} := \min\Bigg\{3/2, 1/2 + \min\Bigg\{\operatorname{Re}\alpha \mid \operatorname{Re}\alpha > 0, \left(\frac{\sin(\pi - w_j)\alpha}{\sin \pi\alpha}\right)^2 = \left(\frac{\mu+1}{\mu-1}\right)^2\Bigg\}\Bigg\}, \tag{2.11}$$

where ω_j denote the angles in the corners of the polygon Γ, cf. [**12**]. For $s \geq \tilde{s}$ one has to consider singularity functions which can be obtained by Mellin-transform techniques. Then one gets continuity properties for so-called augmented function spaces. Those singularity functions can be used for establishing Galerkin methods with higher convergence orders and lead to the so-called Fix-Strang method. An aim in applying adaptive methods which are controlled by a posteriori estimates is to overcome the explicit use of those singularity functions. One is interested in discretization spaces reflecting the lack in the regularity of the solution automatically.

3. Galerkin methods based on multiscale bases.

3.1. *Multiscale bases.* The goal of this note is to investigate Galerkin schemes with respect to multiscale bases discretizations of (2.4). In this subsection we formulate facts and assumptions about multiscale bases which seem appropriate for the underlying boundary integral equation problem (2.4). Instead of reproducing ideas and constructions made, e.g., in [**18**], we only pose those properties of multiscale bases upon which our results are based.

To this end, let the polygonal boundary $\Gamma \subset \mathbf{R}^2$ be given by

$$\Gamma = \bigcup_{i=1}^{N} \overline{\Gamma}_i, \qquad \Gamma_i = \gamma_i(0,1), \quad i = 1, \dots, N,$$

where $\gamma_i : \mathbf{R} \to \mathbf{R}^2$ are affine functions with $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, \dots, N$. We set $\gamma_{N+1} := \gamma_1$. The corners of the polygonal Γ are denoted by $c_i := \overline{\Gamma}_{i+1} \cap \overline{\Gamma}_i$. Obviously, the mappings $\gamma : (-1,1) \to \mathbf{R}^2$ defined by

$$\gamma(t) := \begin{cases} \gamma_i(t+1) & t \in (-1,0), \\ \gamma_{i+1}(t) & t \in [0,1), \end{cases}$$

$i = 1, \dots, N$, are Lipschitz continuous in $(-1,1)$.

Generally the starting point for the construction of multiscale bases are sequences of closed nested subspaces $\mathcal{S} = \{S_j\}_{j=j_0}^{\infty}$ of $L_2(\Gamma)$ whose union is dense in $L_2(\Gamma)$, i.e.,

$$S_{j_0} \subset S_{j_0+1} \subset \cdots \subset L_2(\Gamma), \quad \mathrm{clos}_{L_2}\left(\bigcup_{j=j_0}^{\infty} S_j \right) = L_2(\Gamma). \tag{3.12}$$

We assume that S_j are spanned by $\Phi_j = \{\phi_{j,k} : k \in I_j\}$ where these bases are *uniformly stable*, i.e.,

$$\|c\|_{l_2(I_j)} \sim \left\| \sum_{k \in I_j} c_k \phi_{j,k} \right\|_0 \tag{3.13}$$

uniformly in $j \geq j_0$ with $\|c\|_{l_2(I_j)} = (\sum_{k \in I_j} |c_k|^2)^{1/2}$.

Successively updating a current approximation in S_{j-1} to a better one in S_j can be facilitated if stable bases

$$\Psi_j = \{\psi_{j,k} \mid k \in J_j\}$$

for complements W_j of S_{j-1} in S_j are available. Defining for convenience $\Psi_{j_0} := \Phi_{j_0}$, $W_{j_0} := S_{j_0}$, any $v_n = \sum_{k\in I_n} c_k\phi_{n,k} \in S_n$ has an alternative multiscale representation

$$v_n = \sum_{j=j_0}^{n} \sum_{k\in J_j} d_{j,k}\psi_{j,k},$$

which corresponds to the direct sum decomposition

$$S_n = \bigoplus_{j=j_0}^{n} W_j.$$

The transformation that takes the coefficients $d_{j,k}$ in the multiscale representation of v_n into the coefficients c_k of the single scale representation is well-conditioned if and only if $\Psi = \cup_{j\geq j_0}\Psi_j$ forms a Riesz-basis of $L_2(\Gamma)$. This means that every $v \in L_2(\Gamma)$ has a unique expansion

$$v = \sum_{j=j_0}^{\infty} \sum_{k\in J_j} \langle v, \tilde{\psi}_{j,k}\rangle \psi_{j,k}$$

such that

$$\|v\|_0 \sim \left(\sum_{j=j_0}^{\infty} \sum_{k\in J_j} |\langle v, \tilde{\psi}_{j,k}\rangle|^2 \right)^{1/2}, \quad v \in L_2(\Gamma),$$

where $\tilde{\Psi} = \{\tilde{\psi}_{j,k} \mid k \in J_j, j \geq j_0\}$ forms a biorthogonal system, $\tilde{S}_n = \operatorname{span}\{\tilde{\psi}_{j,k} \mid k \in J_j, j_0 \leq j \leq n\}$,

$$\langle \psi_{j,k}, \tilde{\psi}_{j',k'}\rangle = \delta_{j,j'}\delta_{k,k'}, \qquad j, j' \geq j_0,\ k \in J_j,\ k' \in J_{j'}$$

and is in fact also a Riesz-basis for $L_2(\Gamma)$, cf. [**15**].

Moreover, we introduce projectors $Q_n : L_2(\Gamma) \to S_n$ and $Q_n' : L_2(\Gamma) \to \tilde{S}_n$ by

$$Q_n v := \sum_{j=j_0}^{n} \sum_{k\in J_j} \langle v, \tilde{\psi}_{j,k}\rangle \psi_{j,k}$$

and

$$Q'_n v := \sum_{j=j_0}^{n} \sum_{k \in J_j} \langle v, \psi_{j,k} \rangle \tilde{\psi}_{j,k}.$$

For our applications it will be important to work with local bases, i.e., we will always assume that

$$\operatorname{diam}(\operatorname{supp} \psi_{j,k}) \sim 2^{-j}, \qquad j \geq j_0.$$

Furthermore, it is desirable that the biorthogonal system has the same property.

We shall call Ψ a multiscale basis of type M_d^γ, $\gamma = (\gamma_1, \gamma_2) \in \mathbf{R}^2$, $\gamma_1 < 0 < \gamma_2$, $d \in \mathbf{N}$, if the following additional requirements are fulfilled:

1. There hold *norm-equivalences*, i.e., for $s \in (\gamma_1, \gamma_2)$, one has

$$\|v\|_s \sim \left(\sum_{j=j_0}^{\infty} \sum_{k \in J_j} 2^{2sj} |\langle v, \tilde{\psi}_{j,k} \rangle|^2 \right)^{1/2}, \quad v \in H^s(\Gamma), \tag{3.14}$$

and for $s \in (-\gamma_2, -\gamma_1)$ one has

$$\|v\|_s \sim \left(\sum_{j=j_0}^{\infty} \sum_{k \in J_j} 2^{2sj} |\langle v, \psi_{j,k} \rangle|^2 \right)^{1/2}, \quad v \in H^s(\Gamma). \tag{3.15}$$

2. The basis functions $\psi_{j,k}$, $j > j_0$, have *vanishing moments* of order d, i.e., for $k \in J_j$ and $i \in \{1, \ldots, N\}$ one has

$$\int_0^1 p(s) \psi_{j,k}\Big|_{\Gamma_i} \circ \gamma_i(s)\, ds = 0, \quad p \in \Pi_{d-1}(0,1), \tag{3.16}$$

where $\Pi_{d-1}(0,1)$ denotes the space of polynomials with order smaller than or equal to $d-1$ on $(0,1)$.

3. The basis functions $\psi_{j,k}$ are essentially piecewise polynomials, i.e., there is a $d' \in \mathbf{N}$, $d' \leq d$, such that for $j \geq j_0$, $k \in J_j$, $i \in \{1, \ldots, N\}$

$$\psi_{j,k}\Big|_{\Gamma_i} \circ \gamma_i \in \Pi_{d'-1,\mathbf{pw}}(0,1). \tag{3.17}$$

Furthermore, we suppose that if $c_i \in \operatorname{supp} \psi_{j,k}$, $k \in J_j$, $j \geq j_0$, then $c_i \in \operatorname{sing\,supp} \psi_{j,k}$. Because of our assumptions d' does not appear explicitly in any of the following considerations of our estimates. But remember that the order d' is crucial, e.g., for establishing Jackson type estimates which are, besides Bernstein inequalities, the essential properties of a multi-resolution analysis implying norm-equivalences like (3.14). For the existence of bases for appropriate parameters γ and d, we refer to [**18**]. We remark that for multiscale bases functions with different properties most of the proofs in this note can be adjusted to provide results which are similar to the presented ones. Additionally, we refer to [**15**] for more information about the functionalanalytic background.

As mentioned before, Jackson and Bernstein estimates seem to be the key points for establishing norm equivalences. Otherwise, it is possible to derive Jackson and Bernstein estimates from norm equivalences, cf. [**15**].

Proposition 3.1. *Let Ψ be an M_d^γ-basis. Then one has, for $\gamma_1 < s \leq t < \gamma_2$ and $n \geq j_0$,*

$$\|v - Q_n v\|_s \lesssim 2^{-n(t-s)} \|v\|_t, \quad v \in H^t(\Gamma), \tag{3.18}$$

and

$$\|v_n\|_t \lesssim 2^{n(t-s)} \|v_n\|_s, \quad v_n \in S_n. \tag{3.19}$$

Clearly, (3.18) holds typically not only for the described parameters s and t, but also for $-d \leq s < \gamma_2$, $s \leq t$, $\gamma_1 < t \leq d'$, cf. [**16, 26**].

For the discretization of the transmission problem (2.4), we have to take two multi-scale bases Ψ^1 and Ψ^2. Here the $M_{d^1}^{\gamma^1}$-basis Ψ^1 is related to $H^{1/2}(\Gamma)$, hence $1/2 < \gamma_2^1$ and the $M_{d^2}^{\gamma^2}$-basis Ψ^2 to $H^{-1/2}(\Gamma)$, hence $\gamma_1^1 < -1/2$. We set $\Psi := \Psi^1 \times \Psi^2$, $J := J^1 \times J^2$, $W_j := \operatorname{span}\{\psi_{j_1,k}^1 \mid k \in J_{j_1}^1\} \times \operatorname{span}\{\psi_{j_2,k}^2 \mid k \in J_{j_2}^2\}$ for $j = (j_1, j_2) \in \mathbf{N}_0^2$, $j_i \geq j^i$ and $S_n := S_{n_1} \times S_{n_2} = \cup_{j \leq n} W_j$ for $n = (n_1, n_2) \in \mathbf{N}_0^2$, where we write $j \leq n$ if and only if $j_i \leq n_i$ for $i = 1, 2$. Associated projectors $Q_n : L_2(\Gamma) \times L_2(\Gamma) \to S_n$ and $Q_n' : L_2(\Gamma) \times L_2(\Gamma) \to \tilde{S}_n := \tilde{S}_{n_1} \times \tilde{S}_{n_2}$

are defined by

$$Q_n v := \begin{pmatrix} Q_{n_1} v^1 \\ Q_{n_2} v^2 \end{pmatrix} \quad \text{and} \quad Q'_n v := \begin{pmatrix} Q'_{n_1} v^1 \\ Q'_{n_2} v^2 \end{pmatrix}, \tag{3.20}$$
$$v = (v^1, v^2).$$

Obviously, Q'_n are the adjoints of the projectors Q_n since Q'_{n_i} are the adjoints of Q_{n_i}.

As an immediate consequence from Proposition 3.1, we obtain Jackson and Bernstein estimates for the product spaces.

Proposition 3.2. *Let Ψ^1 be an $M_{d^1}^{\gamma^1}$- and Ψ^2 an $M_{d^2}^{\gamma^2}$-basis. Then one has, for $\gamma_1^i < s_i \le t_i < \gamma_2^i$,*

$$\|v - Q_n v\|_{s_1,s_2} \lesssim 2^{-n_1(t_1-s_1)}\|v^1\|_{t_1} + 2^{-n_2(t_2-s_2)}\|v^2\|_{t_2},$$

$v = \binom{v^1}{v^2} \in H^{t_1}(\Gamma) \times H^{t_2}(\Gamma)$, and

$$\|v_n\|_{t_1,t_2} \lesssim 2^{n_1(t_1-s_1)}\|v^1_{n_1}\|_{s_1} + 2^{n_2(t_2-s_2)}\|v^2_{n_2}\|_{s_2},$$

$v_n = \binom{v^1_{n_1}}{v^2_{n_2}} \in S_{n_1} \times S_{n_2}$.

Analogously, one gets norm equivalences in the following manner. For an $M_{d^1}^{\gamma^1}$-basis Ψ^1 and an $M_{d^2}^{\gamma^2}$-basis Ψ^2, follow with $\gamma_1^1 < s_1 < \gamma_2^1$ and $\gamma_1^2 < s_2 < \gamma_2^2$,

$$\|v\|_{s_1,s_2} \sim \Big(\sum_{j=j^1}^{\infty} \sum_{k\in J_j^1} 2^{2s_1 j} |\langle v^1, \tilde{\psi}^1_{j,k}\rangle|^2\Big)^{1/2} + \Big(\sum_{j=j^2}^{\infty} \sum_{k\in J_j^2} 2^{2s_2 j} |\langle v^2, \tilde{\psi}^2_{j,k}\rangle|^2\Big)^{1/2},$$

and with $-\gamma_2^1 < s_1 < -\gamma_1^1$, $-\gamma_2^2 < s_2 < -\gamma_1^2$,

$$\|v\|_{s_1,s_2} \sim \Big(\sum_{j=j^1}^{\infty} \sum_{k\in J_j^1} 2^{2s_1 j} |\langle v^1, \psi^1_{j,k}\rangle|^2\Big)^{1/2} + \Big(\sum_{j=j^2}^{\infty} \sum_{k\in J_j^2} 2^{2s_2 j} |\langle v^2, \psi^2_{j,k}\rangle|^2\Big)^{1/2}$$

for $v = \binom{v^1}{v^2} \in H^{s_1}(\Gamma) \times H^{s_2}(\Gamma)$.

3.2. *The Galerkin scheme.* We return to the transmission problem and assume in the following that the wave numbers k_1 and k_2 and the transmission coefficient μ are chosen in such a way that H is injective, i.e., Corollary 2.2 can be applied and (2.9) holds.

The standard Galerkin procedure requires that we find $u_n \in S_n$ with

$$\langle Hu_n, v_n\rangle_\Gamma = \langle g, v_n\rangle_\Gamma, \quad v_n \in S_n, \tag{3.21}$$

which is equivalent to

$$Q'_n H u_n = Q'_n g \tag{3.22}$$

or, respectively,

$$\begin{pmatrix} Q'_{n_1} D u^1_{n_1} + Q'_{n_1} K' u^2_{n_2} \\ -Q'_{n_2} K u^1_{n_1} + Q'_{n_2} V u^2_{n_2} \end{pmatrix} = \begin{pmatrix} Q'_{n_1} g_1 \\ Q'_{n_2} g_2 \end{pmatrix}$$

for $u_n = \binom{u^1_{n_1}}{u^2_{n_2}}$.

Theorem 2.1 and Corollary 2.2 give by standard results the quasi-optimal approximation property for the Galerkin scheme, cf. [**21**]. Since $\gamma^1_2 > 1/2$ and $\gamma^2_2 > 0$ Proposition 3.2 gives convergence rates related to γ^1_2 and γ^2_2. Moreover, the Galerkin scheme is stable, cf. [**21**], i.e., there exists $m = (m_1, m_2)$ such that, for $n \geq m$,

$$\|Q'_n H v_n\|_{-1/2,1/2} \gtrsim \|v_n\|_{1/2,-1/2}, \quad v_n \in S_n, \tag{3.23}$$

i.e., with $v_n = (v^1_{n_1}, v^2_{n_2}) \in S_{n_1} \times S_{n_2}$

$$\begin{aligned} \|Q'_{n_1} D v^1_{n_1} + Q'_{n_1} K' v^2_{n_2}\|^2_{-1/2} + \| - Q'_{n_2} K v^1_{n_1} + Q'_{n_2} V v^2_{n_2}\|^2_{1/2} \\ \gtrsim \|v^1_{n_1}\|^2_{1/2} + \|v^2_{n_2}\|^2_{-1/2}. \end{aligned}$$

In most cases, Jackson and Bernstein inequalities and the Aubin-Nitsche trick lead to stability estimates and stronger convergence results with respect to weaker norms. Following standard lines, cf. [**16**], one can show that these ideas also apply to the transmission problem.

Theorem 3.3. *For* $1-\min(\tilde{s},\gamma_2^1) < s \le \tau < \min(\tilde{s},\gamma_2^1)$, $s > -\gamma_2^2$, $\tau < 1+\gamma_2^2$ *and sufficiently great* $n=(m,m)$, *the convergence estimates*

$$\|u-u_n\|_{s,s-1} \lesssim 2^{-m(\tau-s)}\|u\|_{\tau,\tau-1} \tag{3.24}$$

hold, and for $1-\min(\tilde{s},\gamma_2^1) < s \le 1/2$, $s > -\gamma_2^2$, *the stability estimates*

$$\|Q_n' H v_n\|_{s-1,s} \gtrsim \|v_n\|_{s,s-1}, \quad v_n \in S_n \tag{3.25}$$

hold.

Clearly, considering the remark after Proposition 3.1, the estimates (3.24) and (3.25) can be generalized to $\tau < \min(\tilde{s}, d'+1)$, etc. But this fact does not matter to our investigations of a posteriori estimates and adaptive schemes.

For further considerations it will be convenient to introduce the following notions. For $\lambda = (\lambda_1,\lambda_2) = ((j_1,k_1),(j_2,k_2)) \in J^1 \times J^2$ define $|\lambda_1| := j_1$, $|\lambda_2| := j_2$, $\psi_\lambda := (\psi^1_{\lambda_1}, \psi^2_{\lambda_2})$, and for $n=(m,m)$, let

$$H_n := \begin{pmatrix} (\langle D\psi^1_{\lambda_1'}, \psi^1_{\lambda_1}\rangle)_{|\lambda_1|,|\lambda_1'|\le m} & (\langle K'\psi^2_{\lambda_2'}, \psi^1_{\lambda_1}\rangle)_{|\lambda_1|,|\lambda_2'|\le m} \\ (-\langle K\psi^1_{\lambda_1'}, \psi^2_{\lambda_2}\rangle)_{|\lambda_2|,|\lambda_1'|\le m} & (\langle V\psi^2_{\lambda_2'}, \psi^2_{\lambda_2}\rangle)_{|\lambda_2|,|\lambda_2'|\le m} \end{pmatrix},$$

$$G_n := \begin{pmatrix} (\langle g_1, \psi^1_{\lambda_1}\rangle)_{|\lambda_1|\le m} \\ (\langle g_2, \psi^2_{\lambda_2}\rangle)_{|\lambda_2|\le m} \end{pmatrix},$$

so that (3.21) and (3.22) are equivalent to the linear system of equations

$$H_n U = G_n.$$

Moreover, introducing the diagonal matrix D_n defined by

$$D_n := \begin{pmatrix} (2^{-|\lambda_1|/2}\delta_{\lambda_1',\lambda_1})_{|\lambda_1|,|\lambda_1'|\le m} & 0 \\ 0 & (2^{|\lambda_2|/2}\delta_{\lambda_2',\lambda_2})_{|\lambda_2|,|\lambda_2'|\le m} \end{pmatrix}, \tag{3.26}$$

one can show that

$$\mathrm{cond}_2(D_n H_n D_n) \sim 1.$$

Notice that the usual pCG is not applicable because H is not symmetric. But at least for smooth boundaries Γ one can strictly justify the application of, e.g., GMRES. To be more precise, the splitting of H described later in the convergence proof of the adaptive scheme in Section 5 is just appropriate to apply the considerations made in [**32**]. It follows that the diagonal preconditioner (3.26) can also be used efficiently in the transmission problem if a certain correction on the coarsest level is taken into account.

H_n is by definition fully populated but can be approximated very well by sparse matrices using the estimates in the next section. In this way one gets compressed systems where the accuracy of the corresponding solutions is still asymptotically optimal, cf. [**16, 17, 26**].

Until now, we applied spaces S_n which correspond to uniform refinements. However, in the present note our main concern is not to find possibly sparse representations of the operator H relative to a priori fixed trial spaces but to find possibly economical trial spaces leading to as small linear systems as possible. More precisely, we wish to determine step-by-step possibly small subspaces of the full spaces S_n which recover the solution as well as possible. To this end, we set for any nonempty index set $\Lambda = \Lambda^1 \times \Lambda^2 \subset J^1 \times J^2 = J$,

$$S_\Lambda := \operatorname{span}\{\psi^1_{\lambda_1} \mid \lambda_1 \in \Lambda^1\} \times \operatorname{span}\{\psi^2_{\lambda_2} \mid \lambda_2 \in \Lambda^2\}, \tag{3.27}$$

$$Q_\Lambda v := Q_\Lambda \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} := \begin{pmatrix} \sum_{\lambda_1 \in \Lambda^1} \langle v^1, \tilde{\psi}^1_{\lambda_1} \rangle \psi^1_{\lambda_1} \\ \sum_{\lambda_2 \in \Lambda^2} \langle v^2, \tilde{\psi}^2_{\lambda_2} \rangle \psi^2_{\lambda_2} \end{pmatrix} \tag{3.28}$$

and

$$H_\Lambda := \begin{pmatrix} (\langle D\psi^1_{\lambda_1'}, \psi^1_{\lambda_1} \rangle)_{\lambda_1, \lambda_1' \in \Lambda^1} & (\langle K'\psi^2_{\lambda_2'}, \psi^1_{\lambda_1} \rangle)_{\lambda_1 \in \Lambda^1, \lambda_2' \in \Lambda^2} \\ (-\langle K\psi^1_{\lambda_1'}, \psi^2_{\lambda_2} \rangle)_{\lambda_2 \in \Lambda^2, \lambda_1' \in \Lambda^1} & (\langle V\psi^2_{\lambda_2'}, \psi^2_{\lambda_2} \rangle)_{\lambda_2, \lambda_2' \in \Lambda^2} \end{pmatrix}. \tag{3.29}$$

We assume that

$$\|Q_\Lambda' H Q_\Lambda v_\Lambda\|_{-1/2,1/2} \sim \|v_\Lambda\|_{1/2,-1/2}, \quad v_\Lambda \in S_\Lambda, \tag{3.30}$$

especially that

$$Q_\Lambda' H u_\Lambda = Q_\Lambda' g \tag{3.31}$$

possess unique solutions $u_\Lambda \in S_\Lambda$. By arguments which are analogous to [**14**], it follows from (3.30) that

$$\operatorname{cond}_2(D_\Lambda H_\Lambda D_\Lambda) \sim 1, \tag{3.32}$$

where

$$D_\Lambda := \begin{pmatrix} (2^{-|\lambda_1|/2}\delta_{\lambda_1',\lambda_1})_{\lambda_1,\lambda_1'\in\Lambda^1} & 0 \\ 0 & (2^{|\lambda_2|/2}\delta_{\lambda_2',\lambda_2})_{\lambda_2,\lambda_2'\in\Lambda^2} \end{pmatrix}.$$

In fact, defining for $s = (s_1, s_2) \in \mathbf{R}^2$,

$$\Sigma_s v := \begin{pmatrix} \sum_{n_1=j^1}^\infty 2^{n_1 s_1}(Q^1_{n_1} - Q^1_{n_1-1})v_1 \\ \sum_{n_2=j^2}^\infty 2^{n_2 s_2}(Q^2_{n_2} - Q^2_{n_2-1})v_2 \end{pmatrix},$$

the norm equivalences imply for $w_\Lambda := \Sigma_{(1/2,-1/2)} v_\Lambda$, $v_\Lambda \in S_\Lambda$,

$$\begin{aligned} \|w_\Lambda\|_0 \sim \|v_\Lambda\|_{1/2,-1/2} &\sim \|Q'_\Lambda H v_\Lambda\|_{-1/2,1/2} \\ &\sim \|\Sigma'_{(-1/2,1/2)} Q'_\Lambda H \Sigma_{(-1/2,1/2)} w_\Lambda\|_0, \end{aligned}$$

which means that the operators $\Sigma'_{(-1/2,1/2)} Q'_\Lambda H \Sigma_{(-1/2,1/2)}$ and their inverses are uniformly bounded on $L_2(\Gamma)$. Moreover, it is easy to see that $D_\Lambda H_\Lambda D_\Lambda$ is the matrix representation of $\Sigma'_{(-1/2,1/2)} Q'_\Lambda H \Sigma_{(-1/2,1/2)}$ which confirms (3.32). The above remark about GMRES applies analogously.

4. A posteriori estimates.

4.1. *A posteriori estimates by infinite series.* Once a Galerkin approximation $u_\Lambda \in S_\Lambda$ for the solution $u \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ of (2.4) has been calculated one can evaluate the residual

$$r_\Lambda := (r^1_\Lambda, r^2_\Lambda) := H u_\Lambda - g = H(u_\Lambda - u). \tag{4.1}$$

On account of Corollary 2.2, there are constants $c_1, c_2 \in \mathbf{R}^+$ such that

$$c_1 \|r_\Lambda\|_{-1/2,1/2} \le \|u - u_\Lambda\|_{1/2,-1/2} \le c_2 \|r_\Lambda\|_{-1/2,1/2}. \tag{4.2}$$

Making essential use of the norm equivalences (3.15), we get an estimate for $\|r_\Lambda\|_{-1/2,1/2}$ by a weighted sequence norm of multiscale basis coefficients, i.e., we get

$$\|u - u_\Lambda\|_{1/2,-1/2} \sim \Bigg(\sum_{\lambda_1 \in J^1 \setminus \Lambda^1} 2^{-|\lambda_1|} |\langle r^1_\Lambda, \psi^1_{\lambda_1}\rangle|^2 + \sum_{\lambda_2 \in J^2 \setminus \Lambda^2} 2^{|\lambda_2|} |\langle r^2_\Lambda, \psi^2_{\lambda_2}\rangle|^2 \Bigg)^{1/2}. \tag{4.3}$$

This is a consequence of

$$r_\Lambda = \bigg(\sum_{\lambda_1 \in J^1} \langle r^1_\Lambda, \psi^1_{\lambda_1} \rangle \tilde{\psi}^1_{\lambda_1}, \sum_{\lambda_2 \in J^2} \langle r^2_\Lambda, \psi^2_{\lambda_2} \rangle \tilde{\psi}^2_{\lambda_2} \bigg)$$
$$= \bigg(\sum_{\lambda_1 \in J^1 \setminus \Lambda^1} \langle r^1_\Lambda, \psi^1_{\lambda_1} \rangle \tilde{\psi}^1_{\lambda_1}, \sum_{\lambda_2 \in J^2 \setminus \Lambda^2} \langle r^2_\Lambda, \psi^2_{\lambda_2} \rangle \tilde{\psi}^2_{\lambda_2} \bigg),$$

where we used that u_Λ is the Galerkin solution with respect to $S_\Lambda = S_{\Lambda^1} \times S_{\Lambda^2}$.

Let us abbreviate

$$\begin{aligned} \delta^1_{\lambda_1} &:= 2^{-|\lambda_1|/2} |\langle r^1_\Lambda, \psi^1_{\lambda_1} \rangle|, \\ \delta^2_{\lambda_2} &:= 2^{|\lambda_2|/2} |\langle r^2_\Lambda, \psi^2_{\lambda_2} \rangle| \end{aligned} \tag{4.4}$$

and note that inserting the representation

$$u_\Lambda = (u^1_\Lambda, u^2_\Lambda) = \bigg(\sum_{\lambda'_1 \in \Lambda^1} u^1_{\lambda'_1} \psi^1_{\lambda'_1}, \sum_{\lambda'_2 \in \Lambda^2} u^2_{\lambda'_2} \psi^2_{\lambda'_2} \bigg) \tag{4.5}$$

yields the expressions

$$\delta^1_{\lambda_1} = 2^{-|\lambda_1|/2} \bigg| g^1_{\lambda_1} - \sum_{\lambda'_1 \in \Lambda^1} \langle D\psi^1_{\lambda'_1}, \psi^1_{\lambda_1} \rangle u^1_{\lambda'_1} - \sum_{\lambda'_2 \in \Lambda^2} \langle K'\psi^2_{\lambda'_2}, \psi^1_{\lambda_1} \rangle u^2_{\lambda'_2} \bigg|,$$
$$\delta^2_{\lambda_2} = 2^{|\lambda_2|/2} \bigg| g^2_{\lambda_2} + \sum_{\lambda'_1 \in \Lambda^1} \langle K\psi^1_{\lambda'_1}, \psi^2_{\lambda_2} \rangle u^1_{\lambda'_1} - \sum_{\lambda'_2 \in \Lambda^2} \langle V\psi^2_{\lambda'_2}, \psi^2_{\lambda_2} \rangle u^2_{\lambda'_2} \bigg|,$$

with $g^1_{\lambda_1} := \langle g^1, \psi^1_{\lambda_1} \rangle$ and $g^2_{\lambda_2} := \langle g^2, \psi^2_{\lambda_2} \rangle$. Moreover, (4.3) gives constants $c_3, c_4 \in \mathbf{R}^+$, which reflect the norm equivalences, such that

$$\begin{aligned} c_1 c_3 \bigg(\sum_{\lambda_1 \in J^1 \setminus \Lambda^1} \delta^2_{\lambda_1} + \sum_{\lambda_2 \in J^2 \setminus \Lambda^2} \delta^2_{\lambda_2} \bigg)^{1/2} & \\ \le \|u - u_\Lambda\|_{1/2,-1/2} & \\ \le c_2 c_4 \bigg(\sum_{\lambda_1 \in J^1 \setminus \Lambda^1} \delta^2_{\lambda_1} + \sum_{\lambda_2 \in J^2 \setminus \Lambda^2} \delta^2_{\lambda_2} \bigg)^{1/2}. & \end{aligned} \tag{4.6}$$

Obviously, the estimates (4.6) provide an efficient and reliable error bound. Further estimates for the error with respect to other norms can be received in the same way if H is, with respect to those norms, bounded and boundedly invertible, and the multi-scale bases yield norm equivalences.

However, the estimates (4.6) seem practically useless since they involve infinitely many terms, and the question arises as to how to get finite sums. Replacing the magnitudes $\delta_\lambda := (\delta_{\lambda_1}, \delta_{\lambda_2})$ by finitely many ones requires some information about the given data and about the behavior of the entities $\langle D\psi^1_{\lambda'_1}, \psi^1_{\lambda_1}\rangle$, $\langle K'\psi^1_{\lambda'_2}, \psi^1_{\lambda_1}\rangle$, $\langle K\psi^1_{\lambda'_1}, \psi^2_{\lambda_2}\rangle$ and $\langle V\psi^2_{\lambda'_2}, \psi^2_{\lambda_2}\rangle$. We shall show that, for almost all $\lambda_1 \in J^1\backslash\Lambda^1$ the terms $\langle D\psi^1_{\lambda'_1}, \psi^1_{\lambda_1}\rangle$, $\langle K'\psi^2_{\lambda'_2}, \lambda^1_{\lambda_1}\rangle$ and for almost all $\lambda_2 \in J^2\backslash\Lambda^2$ the terms $(K\psi^1_{\lambda'_1}, \psi^2_{\lambda_2}\rangle$ and $\langle V\psi^2_{\lambda'_2}, \psi^2_{\lambda_2}\rangle$ can actually be neglected.

To this end, we prove in Section 4.2 decay estimates for those terms using ideas from [**16, 17, 24, 26**]. Also some estimates which seem to be new will be derived.

4.2. *Decay estimates.* We split the investigation of the decay estimates into two parts. Lemma 4.1 presents estimates which essentially require properties of the involved kernel functions whereas the results in Lemmas 4.3–4.6 also make use of mapping properties of the integral operators involved in H. To formulate the decay estimates we introduce, for $\psi^i_{\lambda_i}$, $\lambda_i \in J^i$, $i = 1, 2$, the notions $\Omega^i_{\lambda_i} := \operatorname{supp}\psi^i_{\lambda_i}$ and $\Omega^{i,s}_{\lambda_i} := \operatorname{sing\,supp}\psi^i_{\lambda_i}$.

Lemma 4.1. *Let $\Psi^1 = (\psi^1_{\lambda_1})_{\lambda_1\in J^1}$ be an $M^{\lambda^1}_{d^1}$-basis and $\Psi^2 = (\psi^2_{\lambda_2})_{\lambda_2\in J^2}$ an $M^{\gamma^2}_{d^2}$-basis. Then, for an integral operator*

$$Au(x) = \int_\Gamma a(x,y)u(y)\,ds_y, \quad x \in \Gamma,$$

with a kernel function $a \in C^\infty(\mathbf{R}^2 \times \mathbf{R}^2\backslash\{(x,y) \mid x = y\})$, that satisfies for a fixed $\rho \in [-1,1]$ with $\rho/2 \in (\gamma^1_1, \gamma^1_2)$, $-\rho/2 \in (\gamma^2_1, \gamma^2_2)$, and arbitrary $\alpha, \beta \in \mathbf{N}_0$ with $1 + \rho + \alpha + \beta > 0$

$$|\partial^\beta_y \partial^\alpha_x a(x,y)| \lesssim |x-y|^{-1-\rho-\alpha-\beta}, \quad (x,y) \in \mathbf{R}^2 \times \mathbf{R}^2\backslash\{(x,y) \mid x = y\}, \tag{4.7}$$

one has for $\operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) > 0$ *and* $|\lambda_2| > j^2$

$$
(4.8)\quad |\langle A\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim \begin{cases} \operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2})^{-1-\rho-d^2} 2^{-j^1/2} 2^{-(d^2+1/2)|\lambda_2|}, & |\lambda_1| = j^1, \\ \operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2})^{-1-\rho-d^1-d^2} 2^{-(d^1+1/2)|\lambda_1|-(d^2+1/2)|\lambda_2|}, & |\lambda_1| > j^1. \end{cases}
$$

If $\operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) = 0$*, one has for* $|\lambda_2| > j^2$*,*

$$
(4.9)\quad |\langle A\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim \begin{cases} \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2})^{-\rho-d^2} 2^{-d^2|\lambda_2|}, & |\lambda_1| = j^1, \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2}) > 0, \\ \operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^{2,s}_{\lambda_2})^{-\rho-d^1} 2^{-(d^1+1/2)|\lambda_1|} 2^{|\lambda_2|/2}, & |\lambda_1| > j^1, \operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^{2,s}_{\lambda_2}) > 0. \end{cases}
$$

Proof. For $|\lambda_1| \geq j^1$ and $\operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) > 0$, the estimates follow essentially by Taylor formula arguments, cf. [**16, 18, 26**].

Let $|\lambda_1| = j^1$ but $\operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) = 0$ and $\operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2}) > 0$. These assumptions imply $\Omega^2_{\lambda_2} \subset \Omega^1_{\lambda_1}$ and the existence of a Γ_i, $i \in \{1, \dots, n\}$ with $\overline{\Omega^2_{\lambda_2}} \subset \Gamma_i$. We introduce a smooth cut-off function χ with $0 \leq \chi \leq 1$, $\chi \circ \gamma_i \in C^\infty_0(0,1)$,

$$
\|\chi \circ \gamma_i\|_{d^2+\rho;(0,1)} \lesssim \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2})^{-d^2-\rho}
$$

and

$$
\chi(y) = \begin{cases} 1 & y \in U_\varepsilon(\Omega^2_{\lambda_2}), \\ 0 & y \in \Gamma_i \backslash U_{2\varepsilon}(\Omega^2_{\lambda_2}), \end{cases}
$$

where $U_\varepsilon(\Omega^2_{\lambda_2}) \subset \Gamma_i$ denotes an ε-neighborhood and, respectively, $U_{2\varepsilon}(\Omega^2_{\lambda_2}) \subset \Gamma_i$ a 2ε-neighborhood of $\Omega^2_{\lambda_2}$ with respect to $\varepsilon := \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2})/3$. With

$$
I_1(x) := \int_{\Gamma_i} a(x,y)\chi(y)\psi^1_{\lambda_1}(y)\, ds_y
$$

and

$$I_2(x) := \int_\Gamma a(x,y)(1-\chi(y))\psi^1_{\lambda_1}(y)\,ds_y$$

we get

$$|\langle A\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \le |\langle I_1, \psi^2_{\lambda_2}\rangle| + |\langle I_2, \psi^2_{\lambda_2}\rangle|. \tag{4.10}$$

The first term in (4.10) gives

$$\begin{aligned} |\langle I_1, \psi^2_{\lambda_2}\rangle| &= \inf{}_{p\in\Pi_{d^2-1}(0,1)} |\langle I_1 - p\circ\gamma_i^{-1}, \psi^2_{\lambda_2}\rangle| \\ &\lesssim \inf{}_{p\in\Pi_{d^2-1}(0,1)} \|I_1 - p\circ\gamma_i^{-1}\|_{0;\Omega^2_{\lambda_2}} \\ &\lesssim 2^{-d^2|\lambda_2|}\|I_1\|_{d^2;\Omega^2_{\lambda_2}} \\ &\lesssim 2^{-d^2|\lambda_2|}\|\chi\psi^1_{\lambda_1}\|_{d^2+\rho;\Gamma_i} \\ &\lesssim \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2})^{-d^2-\rho} 2^{-d^2|\lambda_2|}, \end{aligned} \tag{4.11}$$

where we used a Whitney-type estimate and the fact that the kernel function generates an operator which acts with respect to the involved functions as a pseudodifferential operator, i.e., in particular, one has

$$\|I_1\|_{d^2;\Omega^2_{\lambda_2}} \lesssim \|\chi\psi^1_{\lambda_1}\|_{d^2+\rho;\Gamma_i}.$$

On the other hand, we have

$$\frac{d^{d^2} I_2\circ\gamma_i(s)}{ds^{d^2}} = \int_\Gamma \frac{d^{d^2}}{ds^{d^2}} a(\gamma_i(s), y)(1-\chi(y))\psi^1_{\lambda_1}(y)\,ds_y \tag{4.12}$$

for $s \in \gamma_i^{-1}(\Omega^2_{\lambda_2})$, which gives by (4.7),

$$\left|\frac{d^{d^2} I_2\circ\gamma_i(s)}{ds^{d^2}}\right| \le \int_\Gamma |\gamma_i(s)-y|^{-1-d^2-\rho}(1-\chi(y))\psi^1_{\lambda_1}(y)\,ds_y,$$

hence

$$\left|\frac{d^{d^2} I_2\circ\gamma_i(s)}{ds^{d^2}}\right|^2 \lesssim \operatorname{dist}(\gamma_i(s), \operatorname{supp}((1-\chi)\psi^1_{\lambda_1}))^{-1-2(d^2-\rho)}.$$

Integration yields

$$\|I_2\|_{d^2;\Omega^2_{\lambda_2}} \lesssim \operatorname{dist}(\Omega^2_{\lambda_2}, \operatorname{supp}(1-\chi)\psi^1_{\lambda_1})^{-d^2-\rho},$$

hence

$$\|I_2\|_{d^2;\Omega^2_{\lambda_2}} \lesssim \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2})^{-d^2-\rho}. \tag{4.13}$$

By (4.13), we obtain

$$\begin{aligned} |\langle I_2, \psi^2_{\lambda_2}\rangle| &= \inf{}_{p\in\Pi_{d^2-1}(0,1)} |\langle I_2 - p\circ\gamma_i^{-1}, \psi^2_{\lambda_2}\rangle| \\ &\lesssim \inf{}_{p\in\Pi_{d^2-1}(0,1)} \|I_2 - p\circ\gamma_i^{-1}\|_{0;\Omega^2_{\lambda_2}} \\ &\lesssim 2^{-d^2|\lambda_2|}\|I_2\|_{d^2;\Omega^2_{\lambda_2}} \\ &\lesssim \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2})^{-d^2-\rho} 2^{-d^2|\lambda_2|}. \end{aligned} \tag{4.14}$$

Combining (4.11) and (4.14) gives

$$|\langle A\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2})^{-d^2-\rho} 2^{-d^2|\lambda_2|}.$$

Now let $j^1 < |\lambda_1|$, $\operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) = 0$ and $\operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^{2,s}_{\lambda_2}) > 0$. Moreover, let $\Omega^2_{\lambda_2} \subset \overline{\Gamma}_i$ for some $i \in \{1,\dots,N\}$, i.e., especially $\Omega^1_{\lambda_1} \subset \Omega^2_{\lambda_2}$. The following arguments are essentially due to [**26**]. Because the functions $\psi^2_{\lambda_2}\circ\gamma_i$ are piecewise polynomials on $(0,1)$, there are subintervals $\Sigma_1,\dots,\Sigma_m$ of $\Sigma := \gamma_i^{-1}(\Omega^2_{\lambda_2}) \subset [0,1]$, $\Sigma = \cup_{l=1}^m \Sigma_l$, such that

$$\psi^2_{\lambda_2}\circ\gamma_i|_{\Sigma_l}(t) = 2^{|\lambda_2|/2} \sum_{k\le d^2-1} c_{l,k}[2^{|\lambda_2|}(t-t_l)]^k, \qquad t\in\Sigma_l, \quad l = 1,\dots,m,$$

with uniformly bounded coefficients $c_{l,k} \in \mathbf{C}$ and $t_l \in \Sigma_l$. Since $\operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^{2,s}_{\lambda_2}) > 0$ there is a unique subinterval Σ_s, $s\in\{1,\dots,m\}$, with $\gamma_i^{-1}(\Omega^1_{\lambda_1}) \subset \Sigma_s$. Denoting by P_{d^2-1} the $L_2(\Sigma)$-orthogonal projection on polynomials of degree smaller than d^2, one obtains

$$\int_0^1 \psi^2_{\lambda_2}(\gamma_i(t)) P_{d^2-1} A\psi^1_{\lambda_1}(\gamma_i(t)) |\gamma_i'(t)|\, dt = 0,$$

because $\psi^2_{\lambda_2}(\gamma_i(t))$ has vanishing moments of order d^2 and γ_i is an affine function; hence $|\gamma_i'(t)|$ is constant on $(0,1)$. Therefore, it follows that

$$\begin{aligned}
|\langle A\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| &= \left| \int_0^1 \psi^2_{\lambda_2}(\gamma_i(t)) A\psi^1_{\lambda_1}(\gamma_i(t))|\gamma_i'(t)|dt \right| \\
&= \left| \int_0^1 \psi^2_{\lambda_2}(\gamma_i(t))(Id - P_{d^2-1})A\psi^1_{\lambda_1}(\gamma_i(t))|\gamma_i'(t)|dt \right| \\
&= \left| \sum_{l=1}^m \int_{\Sigma_l} 2^{|\lambda_2|/2} \sum_{k\le d^2-1} c_{l,k}[2^{|\lambda_2|}(t-t_l)]^k \right. \\
&\qquad \left. \cdot (Id - P_{d^2-1})A\psi^1_{\lambda_1}(\gamma_i(t))|\gamma_i'(t)|dt \right| \\
&\lesssim \left| \int_{\Sigma\setminus\Sigma_s} 2^{|\lambda_2|/2} \sum_{k\le d^2-1} c_{s,k}[2^{|\lambda_2|}(t-t_s)]^k \right. \\
&\qquad \left. \cdot (Id - P_{d^2-1})A\psi^1_{\lambda_1}(\gamma_i(t))|\gamma_i'(t)|dt \right| \\
&\quad + \left| \sum_{\substack{l=1\\ l\ne s}}^m 2^{|\lambda_2|/2} \int_{\Sigma_l} \sum_{k\le d^2-1} c_{l,k}[2^{|\lambda_2|}(t-t_l)]^k \right. \\
&\qquad \left. \cdot (Id - P_{d^2-1})A\psi^1_{\lambda_1}(\gamma_i(t))|\gamma_i'(t)|dt \right|,
\end{aligned}$$

since

$$\int_\Sigma 2^{|\lambda_2|/2} \sum_{k\le d^2-1} c_{s,k}[2^{|\lambda_2|}(t-t_s)]^k (Id - P_{d^2-1})A\psi^1_{\lambda_1}(\gamma_i(t))|\gamma_i'(t)|\,dt = 0.$$

Next we estimate for $l \ne s$ the term $(Id - P_{d^2-1})A\psi^1_{\lambda_1}(\gamma_i(t))$ on Σ_l. Because $\operatorname{dist}(\Sigma_l, \gamma_i^{-1}(\Omega^1_{\lambda_1})) > 0$ the function $(Id - P_{d^2-1})A\psi^1_{\lambda_1}(\gamma_i(\cdot))$ can be expressed with respect to a smooth Schwarz kernel $k(\cdot,\cdot)$ related to the operator $(Id - P_{d^2-1})A$, cf. [**26**]. Then a Taylor expansion argument gives, for $t \in \Sigma_l$,

(4.15)
$$\begin{aligned}
|(Id - P_{d^2-1})A\psi^1_{\lambda_1}(\gamma_i(t))| &= \left| \int_{\mathbf{R}} k(s,t)\psi^1_{\lambda_1}(\gamma_i(s))\,ds \right| \\
&\lesssim \operatorname{dist}(\Omega^1_{\lambda_1}, \gamma_i(t))^{-1-\rho-d^1} 2^{-(d^1+1/2)|\lambda_1|}.
\end{aligned}$$

Inserting (4.15), we obtain for $l \neq s$,

$$\begin{aligned} 2^{|\lambda_2|/2} \int_{\Sigma_l} c_{l,k} [2^{|\lambda_2|} |t - t_l|]^k |(Id - P_{d^2-1}) A \psi^1_{\lambda_1}(\gamma_i(t))| |\gamma_i'(t)| \, dt \\ \leq 2^{|\lambda_2|/2} 2^{-(d^1+1/2)|\lambda_1|} \mathrm{dist}\,(\Omega^1_{\lambda_1}, \Omega^{2,s}_{\lambda_2})^{-\rho-d^1} \end{aligned} \tag{4.16}$$

and analogously,

$$\begin{aligned} 2^{|\lambda_2|/2} \int_{\Sigma_l} c_{s,k} [2^{|\lambda_2|} |t - t_s|]^k (Id - P_{d^2-1}) A \psi^1_{\lambda_1}(\gamma_i(t))| |\gamma_i'(t)| \, dt \\ \leq 2^{|\lambda_2|/2} 2^{-(d^1+1/2)|\lambda_1|} \mathrm{dist}\,(\Omega^1_{\lambda_1}, \Omega^{2,s}_{\lambda_2})^{-\rho-d^1}, \end{aligned}$$

which give

$$|\langle A\psi^1_{\lambda_1}, \psi^2_{\lambda_2} \rangle| \lesssim 2^{|\lambda_2|/2} 2^{-(d^1+1/2)|\lambda_1|} \mathrm{dist}\,(\Omega^1_{\lambda_1}, \Omega^{2,s}_{\lambda_2})^{-\rho-d^1}. \tag{4.17}$$

Next, let $\Omega^2_{\lambda_2} \cap \Gamma_i$, $\Omega^2_{\lambda_2} \cap \Gamma_{i+1} \neq \varnothing$ and $\mathrm{dist}\,(\Omega^1_{\lambda_1}, \Omega^{2,s}_{\lambda_2}) > 0$. We define

$$\gamma(t) := \begin{cases} \gamma_i(t+1) & t \in (-1, 0], \\ \gamma_{i+1}(t) & t \in [0, 1), \end{cases} \qquad \Sigma := \gamma^{-1}(\Omega^2_{\lambda_2}) \subset (-1, 1),$$

subintervals $\Sigma_1, \dots, \Sigma_m$ of Σ with $\Sigma = \cup_{l=1}^m \Sigma_l$ and such that $\psi^2_{\lambda_2} \circ \gamma^{-1}|_{\Sigma_l}$ are polynomials of degree less than d^2; finally Σ_s and P_{d^2-1} are analogous to the above. Then we have

$$\int_{-1}^1 \psi^2_{\lambda_2}(\gamma(t)) P_{d^2-1} A \psi^1_{\lambda_1}(\gamma(t)) |\gamma'(t)| \, dt = 0.$$

Because the distance between $\Omega^1_{\lambda_1}$ and the corners of Γ is positive, there is again an appropriate Schwartz kernel related to the pseudo differential operator $(Id - P_{d^2-1})A$. Therefore, applying similar arguments as above gives again (4.17). □

Remark 4.2. Decay estimates for the more interesting case $\mathrm{dist}\,(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) = 0$ but $\mathrm{dist}\,(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2}) > 0$ can be immediately derived from the proved result and the identity $|\langle A\psi^1_{\lambda_1}, \psi^2_{\lambda_2} \rangle| = |\langle \psi^1_{\lambda_1}, A'\psi^2_{\lambda_2} \rangle|$. In the underlying problem the adjoint operators are explicitly given by the symmetry relations (2.2). Thus we only have to permute indices for getting the desired estimates.

After Lemma 4.1 it remains to estimate terms $|\langle A\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle|$ with $\mathrm{dist}\,(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) \ll 1$. We handle these terms by proving estimates for each operator appearing in H separately. We shall use mapping properties, the fact that $|\lambda_2| > j^2$, i.e., that $\psi^2_{\lambda_2}$ have vanishing moments of some order and the validity of inverse estimates. The derived decay estimates hold generally. Therefore, if distances in the former estimates become small they give possibly better estimates especially for low levels $|\lambda_2|$. This fact will be taken into account in the subsequent Schur lemma argument.

Generally, let there be given an $M^{\gamma_1}_{d^1}$-basis Ψ^1 and an $M^{\gamma_2}_{d^2}$-basis Ψ^2. Besides the vanishing moment property of the basis functions the validity of inverse estimates plays an important role. We say that Ψ^i have inverse estimates up to some $s^i > 0$ if, for $s \in (0, s^i]$, the estimates

$$\|\psi^i_{\lambda_i}\|_s \lesssim 2^{|\lambda_i|s}, \quad \lambda_i \in J^i, \tag{4.18}$$

hold.

Lemma 4.3. *If $s^1 \in (0, 1/2)$, one has for $d^2 \geq 2$ and $|\lambda_2| > j^2$,*

$$2^{(|\lambda_1|+|\lambda_2|)/2}|\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{-(|\lambda_2|-|\lambda_1|)(s^1+1/2)}. \tag{4.19}$$

For $s^1 \geq 1/2$, $d^2 \geq s^1+1$, the estimate (4.19) holds also if $\Omega^1_{\lambda_1}$ contains no corner c_i of Γ with $\mathrm{dist}\,(c_i, \Omega^2_{\lambda_2}) \lesssim 2^{-|\lambda_1|}$.

Proof. If V_0 denotes the single layer operator with respect to the Laplacian, i.e., the wave number is set to zero, we get

$$V = 1/2(1+\mu)V_0 + 1/2(V_1 - V_0 + \mu(V_2 - V_0)).$$

Results from [**12**] and [**13**] show the continuity of the mappings

$$V_0 : H^s(\Gamma) \to H^{s+1}(\Gamma), \quad s \in (-3/2, 1/2],$$

as well as

$$V_1 - V_0, V_2 - V_0 : H^s(\Gamma) \to H^{s+1}(\Gamma), \quad s \in (-3/2, 1/2).$$

Hence the operator $V : H^s(\Gamma) \to H^{s+1}(\Gamma)$ is continuous for $s \in (-3/2, 1/2)$.

Let $j^2 < |\lambda_2|$ and suppose that $\Omega^2_{\lambda_2} \subset \overline{\Gamma}_i$. Then we get, with $s^1 \in (0, 1/2)$ and $d^2 \geq 2$,

$$\begin{aligned}|\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| &= \inf{}_{p\in\Pi_{d^2-1}(0,1)} |\langle V\psi^1_{\lambda_1} - p\circ\gamma_i^{-1}, \psi^2_{\lambda_2}\rangle| \\ &\lesssim 2^{-|\lambda_2|(s^1+1)} \|V\psi^1_{\lambda_1}\|_{s^1+1} \\ &\lesssim 2^{|\lambda_1|s^1} 2^{-|\lambda_2|(s^1+1)},\end{aligned}$$

where we used the vanishing moment property of $\psi^2_{\lambda_2}$, a Whitney type estimate and inverse estimates with respect to $\psi^1_{\lambda_1}$. Clearly the estimate also holds if $\Omega^2_{\lambda_2}$ lives on two adjacent straight lines.

Next we assume that either $\Omega^1_{\lambda_1}$ contains no corner of the boundary Γ or, if $\Omega^1_{\lambda_1}$ contains the corner c_i, $i \in \{1, \ldots, n\}$, then one has $\operatorname{dist}(\Omega^2_{\lambda_2}, c_i) \gtrsim 2^{-|\lambda_1|}$. At first we consider the case where $\Omega^1_{\lambda_1} \subset \Gamma_i$ for some $i \in \{1, \ldots, n\}$. With respect to Γ_i the operator V can be interpreted as a usual pseudodifferential operator of order -1 and we get

$$|\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \leq |\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}|_{\Gamma_i}\rangle| + |\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}|_{\Gamma\backslash\Gamma_i}\rangle|.$$

For $\Omega^2_{\lambda_2} \cap \Gamma_i \neq \varnothing$ and $s^1 \geq 1/2$, $d^2 \geq s^1 + 1$, we estimate the first part by

$$\begin{aligned}|\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| &= \inf{}_{p\in\Pi_{d^2-1}(0,1)} |\langle V\psi^1_{\lambda_1} - p\circ\gamma_i^{-1}, \psi^2_{\lambda_2}\rangle| \\ &\lesssim 2^{-|\lambda_2|(s^1+1)} \|V\psi^1_{\lambda_1}\|_{s^1+1;\Omega^2_{\lambda_2}} \\ &\lesssim 2^{-|\lambda_2|(s^1+1)} 2^{|\lambda_1|s^1}.\end{aligned}$$

For an estimate of the second part when $\Omega^2_{\lambda_2} \cap (\Gamma\backslash\Gamma_i) \neq \varnothing$, we use the fact that $\operatorname{dist}(\Gamma\backslash\Gamma_i, \Omega^1_{\lambda_1}) \gtrsim 2^{-|\lambda_1|}$. We argue as in the proof of (4.14) and conclude

$$\|V\psi^1_{\lambda_1}\|_{s^1+1;\Gamma\backslash\Gamma_i} \lesssim 2^{|\lambda_1|s^1},$$

which leads as above to

$$|\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}|_{\Gamma\backslash\Gamma_i}\rangle| \lesssim 2^{|\lambda_1|s^1} 2^{-|\lambda_2|(s^1+1)}.$$

Summarizing we obtain

$$|\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{|\lambda_1|s^1} 2^{-|\lambda_2|(s^1+1)}.$$

Now let $\Omega^1_{\lambda_1}$ contain the corner c_i, but suppose that $\operatorname{dist}(c_i, \Omega^2_{\lambda_2}) \gtrsim 2^{-|\lambda_1|}$. Without restriction we assume that $\Omega^2_{\lambda_2} \subset \Gamma_i$. Then we introduce a suitably smooth cut-off function χ, which is 1 in an ε-neighborhood U_ε of c_i with $\varepsilon \sim 2^{-|\lambda_1|}$ and $\operatorname{dist}(U_\varepsilon, \Omega^2_{\lambda_2}) \gtrsim 2^{-|\lambda_1|}$ and vanishes elsewhere. With that at hand, we obtain

$$|\langle V\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \leq |\langle V(\chi\psi^1_{\lambda_1}), \psi^2_{\lambda_2}\rangle| + |\langle V((1-\chi)\psi^1_{\lambda_1}), \psi^2_{\lambda_2}\rangle|.$$

For the first part we get, since $\operatorname{dist}(U_\varepsilon, \Omega^2_{\lambda_2}) \gtrsim 2^{-|\lambda_1|}$,

$$|\langle V(\chi\psi^1_{\lambda_1}), \psi^2_{\lambda_2}\rangle| \lesssim 2^{-|\lambda_2|(s^1+1)} 2^{|\lambda_1|s^1}.$$

From $\operatorname{dist}(\operatorname{supp}((1-\chi)\psi^1_{\lambda_1}) \cap \Gamma_{i+1}, \Omega^2_{\lambda_2}) \gtrsim 2^{-|\lambda_1|}$ follows

$$|\langle V((1-\chi)\psi^1_{\lambda_1})|_{\Gamma_{i+1}}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{-|\lambda_2|(s^1+1)} 2^{|\lambda_1|s^1}.$$

Finally again the interpretation of V as a pseudo differential operator on Γ_i yields

$$|\langle V((1-\chi)\psi^1_{\lambda_1}|_{\Gamma_i}), \psi^2_{\lambda_2}\rangle| \lesssim 2^{-|\lambda_2|(s^1+1)} 2^{|\lambda_1|s^1}. \qquad \square$$

Lemma 4.4. *If $s^1 \in (1, 3/2)$, one has for $d^2 \geq 2$ and $|\lambda_2| > j^2$,*

$$2^{(-|\lambda_1|+|\lambda_2|)/2} |\langle K\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{-(|\lambda_2|-|\lambda_1|)(s^1-1/2)}. \tag{4.20}$$

For $s^1 \geq 3/2$, $d^2 \geq s^1$, the estimate (4.20) *holds also if $\Omega^1_{\lambda_1}$ contains no corner c_i of Γ with* $\operatorname{dist}(c_i, \Omega^2_{\lambda_2}) \lesssim 2^{-|\lambda_1|}$.

Proof. Denoting K_0 the double layer operator with respect to the Laplacian, one has $K = K_0 + \tilde{K}_0$ with $\tilde{K}_0 = 1/2(K_1 - K_0) + 1/2(K_2 - K_0)$. The operator $\tilde{K}_0 : H^s(\Gamma) \to H^{s+2}(\Gamma)$ is continuous for $s \in (-1/2, 1/2)$, cf. [**12**]. Thus the continuity properties of K_0, cf. [**13**], show that $K : H^{s^1}(\Gamma) \to H^{s^1}(\Gamma)$ is continuous for $s^1 \in (1, 3/2)$. Hence we obtain for $s^1 \in (1, 3/2)$ and $d^2 \geq 2$,

$$|\langle K\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{-|\lambda_2|s^1} 2^{|\lambda_1|s^1}.$$

If either $\Omega^1_{\lambda_1}$ contains no corner or $\operatorname{dist}(\Omega^2_{\lambda_2}, c_i) \gtrsim 2^{-|\lambda_1|}$, one argues analogously to Lemma 4.3. $\square$

For example, the estimate (4.20) can be sharpened. If either $\Omega^1_{\lambda_1}$ contains no corner or $\operatorname{dist}(\Omega^2_{\lambda_2}, c_i) \gtrsim 2^{-|\lambda_1|}$ one argues as follows. Suppose that $\Omega^2_{\lambda_2} \subset \Gamma_i$ for some $i \in \{1, \dots, n\}$ and $\operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2}) = 0$. Then the introduction of a cut-off function χ with respect to $\Omega^2_{\lambda_2}$ leads for $s^1 \in (1, 3/2)$ to

$$\begin{aligned}
|\langle K(\chi\psi^1_{\lambda_1}), \psi^2_{\lambda_2}\rangle| &= |\langle \tilde{K}_0(\chi\psi^1_{\lambda_1}), \psi^2_{\lambda_2}\rangle| \\
&= \inf{}_{p\in\Pi_{d^2-1}(0,1)} |\langle \tilde{K}_0(\chi\psi^1_{\lambda_1}) - p\circ\gamma_i^{-1}, \psi^2_{\lambda_2}\rangle| \\
&\lesssim 2^{-|\lambda_2|(s^1+1)} \|\tilde{K}_0(\chi\psi^1_{\lambda_1})\|_{s^1+1;\Omega^2_{\lambda_2}} \\
&\lesssim 2^{-|\lambda_2|(s^1+1)} \|\chi\psi^1_{\lambda_1}\|_{s^1-1;\Gamma_i} \\
&\lesssim 2^{-|\lambda_2|(s^1+1)} 2^{|\lambda_1|(s^1-1)},
\end{aligned}$$

where we used the fact that, since $\operatorname{supp}(\chi\psi^1_{\lambda_1})$, $\Omega^2_{\lambda_2} \subset \Gamma_i$ and Γ_i is a straight line, $\langle K_0(\chi\psi^1_{\lambda_1}), \psi^2_{\lambda_2}\rangle = 0$.

Lemma 4.5. *If $s^1 \in (0, 1/2)$, one has, for $d^2 \geq 1$ and $|\lambda_2| > j^2$,*

$$2^{(|\lambda_1|-\lambda_2|)/2} |\langle K'\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{-(|\lambda_2|-|\lambda_1|)(s^1+1/2)}. \tag{4.21}$$

For $s^1 \geq 1/2$, $d^2 \geq s^1$, the estimate (4.21) also holds if $\Omega^1_{\lambda_1}$ contains no corner c_i of Γ with $\operatorname{dist}(c_i, \Omega^2_{\lambda_2}) \lesssim 2^{-|\lambda_1|}$.

Proof. Again, nearly the same arguments can be applied to K'. In particular, we introduce K'_0 and apply the continuity of

$$K' - K'_0 : H^{s^1}(\Gamma) \longrightarrow \mathcal{H}^{s^1+2}(\Gamma), \quad s^1 \in (0, 1/2),$$

where $\mathcal{H}^{s^1}(\Gamma) := \{u \in L^2(\Gamma) \mid u|_{\Gamma_i} \in H^{s^1}(\Gamma_i), i = 1, 2, \dots, n\}$, cf. [**12**]. Then we get the estimates

$$|\langle K'\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{-|\lambda_2|s^1} 2^{|\lambda_1|s^1}. \qquad \square$$

Lemma 4.6. *If $s^1 \in (1,3/2)$, one has for $d^2 \geq 1$ and $|\lambda_2| > j^2$,*

$$2^{-(|\lambda_1|+|\lambda_2|)/2}|\langle D\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{-(|\lambda_2|-|\lambda_1|)(s^1-1/2)}. \tag{4.22}$$

For $s^1 \geq 3/2$, $d^2 \geq s^1-1$, the estimate (4.22) also holds if $\Omega^1_{\lambda_1}$ contains no corner c_i of Γ with $\operatorname{dist}(c_i, \Omega^2_{\lambda_2}) \lesssim 2^{-|\lambda_1|}$.

Proof. For example, if $\Omega^2_{\lambda_2} \subset \overline{\Gamma}_i$, we get

$$\begin{aligned} |\langle D\psi^1_{\lambda_1}, \psi^2_{\lambda_2}\rangle| &\lesssim \inf{}_{p\in\Pi_{d^2-1}(0,1)} \|D\psi^1_{\lambda_1}\circ\gamma_i - p\|_{0;\Omega^2_{\lambda_2}} \\ &\lesssim 2^{-|\lambda_2|(s^1-1)}\|D\psi^1_{\lambda_1}\|_{s^1-1;\Gamma} \\ &\lesssim 2^{|\lambda_1|s^1}2^{-|\lambda_2|(s^1-1)}. \qquad \square \end{aligned}$$

Combining Lemma 4.1 and 4.3–4.6 and the symmetry relations (2.2), we obtain the following corollary.

Corollary 4.7. *Suppose that Ψ^1 is an $M^{\gamma^1}_{d^1}$-basis with $s^1 \in (1,3/2)$, $d^1 \geq 1$, and Ψ^2 an $M^{\gamma^2}_{d^2}$-basis with $s^2 \in (0,1/2)$, $d^2 \geq 2$. Then we have, for the operator D and $|\lambda_1| > j^1$,*

$$\begin{aligned} &2^{-(|\lambda_1'|+|\lambda_1|)/2}|\langle D\psi^1_{\lambda_1'}, \psi^1_{\lambda_1}\rangle| \\ &\qquad\lesssim \begin{cases} \operatorname{dist}(\Omega^1_{\lambda_1'}, \Omega^1_{\lambda_1})^{-2-d^1}2^{-j^1}2^{-(d^1+1)|\lambda_1|}, & |\lambda_1'| = j^1, \\ \operatorname{dist}(\Omega^1_{\lambda_1'}, \Omega^1_{\lambda_1})^{-2-2d^1}2^{-(d^1+1)(|\lambda_1'|+|\lambda_1|)}, & |\lambda_1'| > j^1, \end{cases} \end{aligned}$$

if $\operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^1_{\lambda_1}) > 0$,

$$\begin{aligned} &2^{-(|\lambda_1'|+|\lambda_1|)/2}|\langle D\psi^1_{\lambda_1'}, \psi^1_{\lambda_1}\rangle| \\ &\qquad\lesssim \begin{cases} \operatorname{dist}(\Omega^{1,s}_{\lambda_1'}, \Omega^1_{\lambda_1})^{-1-d^1}2^{-(d^1+1/2)|\lambda_1|}2^{-1/2j^1}, & |\lambda_1'| = j^1, \\ \operatorname{dist}(\Omega^{1,s}_{\lambda_1'}, \Omega^1_{\lambda_1})^{-1-d^1}2^{-(d^1+1)|\lambda_1|}, & |\lambda_1'| > j^1, \end{cases} \end{aligned}$$

if $\operatorname{dist}(\Omega^1_{\lambda_1'}, \Omega^1_{\lambda_1}) = 0$ *but* $\operatorname{dist}(\Omega^{1,s}_{\lambda_1'}, \Omega^1_{\lambda_1}) > 0$, *and generally*

$$2^{-(|\lambda_1'|+|\lambda_1|)/2}|\langle D\psi^1_{\lambda_1'}, \psi^1_{\lambda_1}\rangle| \lesssim 2^{-||\lambda_1'|-|\lambda_1||(s^1-1/2)}. \tag{4.23}$$

For the operator K' we have, for $|\lambda_1| > j^1$,

$$2^{(|\lambda_2'|-|\lambda_1|)/2}|\langle K'\psi^2_{\lambda_2'}, \psi^1_{\lambda_1}\rangle| \lesssim \begin{cases} \operatorname{dist}(\Omega^2_{\lambda_2'}, \Omega^1_{\lambda_1})^{-1-d^1} 2^{-(d^1+1)|\lambda_1|}, & |\lambda_2'| = j^2, \\ \operatorname{dist}(\Omega^2_{\lambda_2'}, \Omega^1_{\lambda_1})^{-1-d^1-d^2} 2^{-(d^1+1)|\lambda_1|-d^2|\lambda_2'|}, & |\lambda_2'| > j^2, \end{cases}$$

if $\operatorname{dist}(\Omega^2_{\lambda_2'}, \Omega^1_{\lambda_1}) > 0$*,*

$$2^{(|\lambda_2'|-|\lambda_1|)/2}|\langle K'\psi^2_{\lambda_2'}, \psi^1_{\lambda_1}\rangle| \lesssim \begin{cases} \operatorname{dist}(\Omega^{2,s}_{\lambda_2'}, \Omega^1_{\lambda_1})^{-d^1} 2^{-(d^1+1/2)|\lambda_1|} 2^{1/2 j^2}, & |\lambda_2'| = j^2, \\ \operatorname{dist}(\Omega^{2,s}_{\lambda_2'}, \Omega^1_{\lambda_1})^{-d^1} 2^{-(d^1+1)|\lambda_1|} 2^{|\lambda_2'|}, & |\lambda_2'| > j^2, \end{cases}$$

if $\operatorname{dist}(\Omega^2_{\lambda_2'}, \Omega^1_{\lambda_1}) = 0$ *but* $\operatorname{dist}(\Omega^{2,s}_{\lambda_2'}, \Omega^1_{\lambda_1}) > 0$*, and generally*

(4.24)
$$2^{(|\lambda_2'|-|\lambda_1|)/2}|\langle K'\psi^2_{\lambda_2'}, \psi^1_{\lambda_1}\rangle| \lesssim 2^{-||\lambda_2'|-|\lambda_1|| \begin{cases} s^2+1/2, & |\lambda_2'| \le |\lambda_1|, \\ s^1-1/2, & |\lambda_2'| \ge |\lambda_1|. \end{cases}}$$

For the operator K we have, for $|\lambda_2| > j^2$,

$$2^{(-|\lambda_1'|+|\lambda_2|)/2}|\langle K\psi^1_{\lambda_1'}, \psi^2_{\lambda_2}\rangle| \lesssim \begin{cases} \operatorname{dist}(\Omega^1_{\lambda_1'}, \Omega^2_{\lambda_2})^{-1-d^2} 2^{-j^1} 2^{-d^2|\lambda_2|}, & |\lambda_1'| = j^1, \\ \operatorname{dist}(\Omega^1_{\lambda_1'}, \Omega^2_{\lambda_2})^{-1-d^1-d^2} 2^{-(d^1+1)|\lambda_1'|-d^2|\lambda_2|}, & |\lambda_1'| > j^1, \end{cases}$$

if $\operatorname{dist}(\Omega^1_{\lambda_1'}, \Omega^2_{\lambda_2}) > 0$*,*

$$2^{(-|\lambda_1'|+|\lambda_2|)/2}|\langle K\psi^1_{\lambda_1'}, \psi^2_{\lambda_2}\rangle| \lesssim \begin{cases} \operatorname{dist}(\Omega^{1,s}_{\lambda_1'}, \Omega^2_{\lambda_2})^{-d^2} 2^{(-d^2+1/2)|\lambda_2|} 2^{-1/2 j^1}, & |\lambda_1'| = j^1, \\ \operatorname{dist}(\Omega^{1,s}_{\lambda_1'}, \Omega^2_{\lambda_2})^{-d^2} 2^{-d^2|\lambda_2|}, & |\lambda_1'| > j^1, \end{cases}$$

if $\operatorname{dist}(\Omega^1_{\lambda_1'}, \Omega^2_{\lambda_2}) = 0$ *but* $\operatorname{dist}(\Omega^{1,s}_{\lambda_1'}, \Omega^2_{\lambda_2}) > 0$*, and generally,*

(4.25)
$$2^{(-|\lambda_1'|+|\lambda_2|)/2}|\langle K\psi^1_{\lambda_1'}, \psi^2_{\lambda_2}\rangle| \lesssim 2^{-||\lambda_2|-|\lambda_1'|| \begin{cases} s^2+1/2, & |\lambda_2| \le |\lambda_1'|, \\ s^1-1/2, & |\lambda_2| \ge |\lambda_1'|. \end{cases}}$$

For the operator V we have

$$2^{(|\lambda_2'|+|\lambda_2|)/2}|\langle V\psi^2_{\lambda_2'},\psi^2_{\lambda_2}\rangle| \lesssim \begin{cases} \operatorname{dist}(\Omega^2_{\lambda_2'},\Omega^2_{\lambda_2})^{-d^2}2^{-d^2|\lambda_2|}, & |\lambda_2'|=j^2,\\ \operatorname{dist}(\Omega^2_{\lambda_2'},\Omega^2_{\lambda_2})^{-2d^2}2^{-d^2(|\lambda_2|+|\lambda_2'|)}, & |\lambda_2'|>j^2,\end{cases}$$

if $\operatorname{dist}(\Omega^2_{\lambda_2'},\Omega^2_{\lambda_2})>0$,

$$2^{(|\lambda_2'|+|\lambda_2|)/2}|\langle V\psi^2_{\lambda_2'},\psi^2_{\lambda_2}\rangle| \lesssim \begin{cases} \operatorname{dist}(\Omega^{2,s}_{\lambda_2'},\Omega^2_{\lambda_2})^{1-d^2}2(-d^2+1/2)|\lambda_2|2^{1/2j^2}, & |\lambda_2'|=j^2,\\ \operatorname{dist}(\Omega^{2,s}_{\lambda_2'},\Omega^2_{\lambda_2})^{1-d^2}2^{-d^2|\lambda_2|}2^{|\lambda_2'|}, & |\lambda_2'|>j^2,\end{cases}$$

if $\operatorname{dist}(\Omega^2_{\lambda_2'},\Omega^2_{\lambda_2})=0$ *but* $\operatorname{dist}(\Omega^{2,s}_{\lambda_2'},\Omega^2_{\lambda_2})>0$, *and generally*

$$2^{(|\lambda_2'|+|\lambda_2|)/2}|\langle V\psi^2_{\lambda_2'},\psi^2_{\lambda_2}\rangle| \lesssim 2^{-||\lambda_2|-|\lambda_2'||(s^2+1/2)}. \tag{4.26}$$

In associated "no-corner cases" the estimates (4.23)–(4.26) also hold for $s^1\geq 3/2$, $d^1\geq\max(s^1-1,s^2)$ and $s^2\geq 1/2$, $d^2\geq\max(s^2+1,s^1)$.

4.3. *The Schur lemma argument.* Because of the diversity of derived decay estimates, it does not seem to be useful to combine all possibilities for suitable index sets in a single theorem. Instead of that, we shall use Schur's lemma to estimate the l_2-norm of a typical rectangular matrix. Given then an infinite rectangular matrix A with a splitting $A=\sum_{k=1}^m A_k$, one can estimate the norm of A by the sum of norms of A_k. For estimates of A_k, we apply Lemma 4.8. In this manner we obtain index sets such that the magnitudes which are neglected in the final a posteriori estimate can be controlled in an appropriate way.

To be more precise, we consider matrices with entries weighted by factors $2^{\delta_1|\lambda_1|+\delta_2|\lambda_2|}$ with $\delta_1,\delta_2\in\{-1,2,1/2\}$ and $\delta_1+\delta_2+\rho=0$, where ρ denotes the order of the involved operator. Apart from that the entries are given with respect to bases Ψ^1 and Ψ^2. To simplify the presentation, we assume that there are a finite index set Λ^1 with $J^1_{j^1} \subset \Lambda^1 \subset J^1$ and an infinite index set $T^2 \subset J^2$ with $|\lambda_1| < |\lambda_2|$ for $\lambda_1 \in \Lambda^1$, $\lambda_2 \in T^2$. Further, we suppose that the matrix entries

$$(4.27)\quad (A_{\lambda_1,\lambda_2})_{\lambda_1\in\Lambda^1,\lambda_2\in T^2} := (2^{\delta_1|\lambda_1|+\delta_2|\lambda_2|}|\langle A\psi^1_{\lambda_1},\psi^2_{\lambda_2}\rangle|)_{\lambda_1\in\Lambda^1,\lambda_2\in T^2}$$

satisfy the estimates

$$(4.28)\quad A_{\lambda_1,\lambda_2} \lesssim \begin{cases} \operatorname{dist}(\Omega^1_{\lambda_1},\Omega^2_{\lambda_2})^{-1-\rho-d^2} 2^{j^1(\delta_1-1/2)} 2^{-(d^2+1/2-\delta_2)|\lambda_2|}, & |\lambda_1| = j^1, \\ \operatorname{dist}(\Omega^1_{\lambda_1},\Omega^2_{\lambda_2})^{-1-\rho-d^1-d^2} 2^{-(d^1+1/2-\delta_1)|\lambda_1|-(d^2+1/2-\delta_2)|\lambda_2|}, & |\lambda_1| > j^1, \end{cases}$$

if $\operatorname{dist}(\Omega^1_{\lambda_1},\Omega^2_{\lambda_2}) > 0$,

$$(4.29)\quad A_{\lambda_1,\lambda_2} \lesssim \begin{cases} \operatorname{dist}(\Omega^{1,s}_{\lambda_1},\Omega^2_{\lambda_2})^{-\rho-d^2} 2^{-(d^2-\delta_2)|\lambda_2|} 2^{j^1\delta_1}, & |\lambda_1| = j^1, \\ \operatorname{dist}(\Omega^{1,s}_{\lambda_1},\Omega^2_{\lambda_2})^{-\rho-d^2} 2^{-(d^2+1/2-\delta_2)|\lambda_2|} 2^{|\lambda_1|(1/2+\delta_1)}, & |\lambda_1| > j^1, \end{cases}$$

if $\operatorname{dist}(\Omega^1_{\lambda_1},\Omega^2_{\lambda_2}) = 0$ but $\operatorname{dist}(\Omega^{1,s}_{\lambda_1},\Omega^2_{\lambda_2}) > 0$, and generally

$$(4.30)\qquad A_{\lambda_1,\lambda_2} \lesssim 2^{-||\lambda_2|-|\lambda_1||d}$$

with some $d > 1/2$. In the proof of Theorem 4.9, we shall use several times the following simplified version of the well-known Schur lemma, cf., e.g., [**26**].

Lemma 4.8. *Let $A = (a_{i,j})_{i\in I,j\in J}$ be a possibly bi-infinite rectangular matrix with countable index sets $I, J \subset \mathbf{Z}$. Then there exists a constant $c > 0$ such that, for arbitrary $u \in l_2(J)$,*

$$\|Au\|_{l_2(I)} \leq c\Big[\sup_{i\in I}\sum_{j\in J} 2^{1/2(i-j)}|a_{i,j}|\Big]^{1/2}\Big[\sup_{j\in J}\sum_{i\in I} 2^{1/2(j-i)}|a_{i,j}|\Big]^{1/2}\|u\|_{l_2(J)}.$$

Theorem 4.9. *Set, for $\lambda_2 \in T^2$,*

$$M_{\lambda_2;1} := J^1_{j^1}, \qquad M_{\lambda_2;2} := \Lambda^1 \backslash J^1_{j^1}$$

and let $\varepsilon > 0$. Choose constants $a_{i,j} \in [0,1]$, $b_{i,j} \in [0,1)$, small $\delta_{i,j} > 0$, $i, j = 1, 2$, and $k_{i,j} \in \mathbf{N}$, $i = 1, 2$, $j = 1, 2, 3, 4$, with $d^2 - \delta_2 - b_{1,1}(\rho + d^2 + 1) > 0$, $d^2 - \delta_2 - 1/2 - b_{1,2}(\rho + d^2) > 0$, $d^2 - \delta_2 - b_{2,1}(1 + \rho + d^1 + d^2) > 0$, $d^2 - \delta_2 - b_{2,2}(p + d^2) > 0$, $(a_{2,1} + b_{2,1} - 1)(\rho + d^1 + d^2) + \max(a_{2,1}, b_{2,1}) \leq 0$, $(a_{2,2} + b_{2,2} - 1)(d^2 + \rho) - b_{2,2} + 1 \leq 0$, $a_{2,1} + b_{2,1} - 1 \leq 0$ and $a_{2,2} + b_{2,2} - 1 \leq 0$, such that with $j_{\Lambda^1} := \max\{|\lambda_1| | \lambda_1 \in \Lambda^1\}$
(4.31)

$$\begin{aligned}
&\delta_{1,1}^{2(\rho+d^2)} 2^{[2j^1(a_{1,1}+b_{1,1}-1)(\rho+d^2)+a_{1,1}+b_{1,1}]} 2^{-2k_{1,1}(d^2-\delta_2-b_{1,1}(\rho+d^2+1/2))} \\
&\quad + \delta_{1,2}^{2(\rho+d^2)-1} 2^{j^1[2(a_{1,2}+b_{1,2}-1)(\rho+d^2)+1-b_{1,2}]} \\
&\qquad\qquad \cdot 2^{-2k_{1,2}(d^2-\delta_2-1/2-b_{1,2}(\rho+d^2-1/2))} \\
&\quad + \delta_{1,2}^{-1} 2^{j^1(1-a_{1,2}-b_{1,2})} 2^{-k_{1,3}(2d-1+b_{1,2})} \\
&\quad + \delta_{1,1}^{-1} 2^{j^1(1-a_{1,1}-b_{1,1})} 2^{-k_{1,4}(2d-1+b_{1,1})} \\
&< \varepsilon^2
\end{aligned}$$

and
(4.32)

$$\begin{aligned}
&\delta_{2,1}^{2(\rho+d^1+d^2)} 2^{j^1[2(a_{2,1}+b_{2,1}-1)(\rho+d^1+d^2)+a_{2,1}+b_{2,1}]} \\
&\qquad\qquad \cdot 2^{-2k_{2,1}(d^2-\delta_2-b_{2,1}(\rho+d^1+d^2+1/2))} \\
&\quad + \delta_{2,2}^{2(\rho+d^2)-1} 2^{j^1[2(a_{2,2}+b_{2,2}-1)(d^2+\rho)-b_{2,2}+1]} 2^{-2k_{2,2}(d^2-\delta_2-b_{2,2}(\rho+d^2-1/2))} \\
&\quad + \delta_{2,2}^{-1} 2^{j_{\Lambda^1}(-a_{2,2}-b_{2,2}+1)} 2^{-k_{2,3}(2d-1+b_{2,2})} \\
&\qquad\qquad \cdot \delta_{2,1}^{-1} 2^{j_{\Lambda^1}(-a_{2,1}-b_{2,1}+1)} 2^{-k_{2,4}(2d-1+b_{2,1})} \\
&< \varepsilon^2.
\end{aligned}$$

Then, for the following index sets,

$$\begin{aligned} N_{\lambda_2;i,1} &:= \{\lambda_1 \in M_{\lambda_2;i} \mid \operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) \\ &\qquad \geq \delta_{i,1}^{-1} 2^{-|\lambda_1| a_{i,1}} 2^{-|\lambda_2| b_{i,1}}, |\lambda_2| - |\lambda_1| \geq k_{i,1}\}, \\ N_{\lambda_2;i,2} &:= \{\lambda_1 \in M_{\lambda_2;i} \mid \Omega^2_{\lambda_2} \subset \Omega^1_{\lambda_1}, \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2}) \\ &\qquad \geq \delta_{i,2}^{-1} 2^{-|\lambda_1| a_{i,2}} 2^{-|\lambda_2| b_{i,2}}, |\lambda_2| - |\lambda_1| \geq k_{i,2}\}, \\ N_{\lambda_2;i,3} &:= \{\lambda_1 \in M_{\lambda_2,i} \mid \Omega^2_{\lambda_2} \subset \Omega^1_{\lambda_1}, \operatorname{dist}(\Omega^{1,s}_{\lambda_1}, \Omega^2_{\lambda_2}) \\ &\qquad < \delta_{i,2}^{-1} 2^{-|\lambda_1| a_{i,2}} 2^{-|\lambda_2| b_{i,2}}, |\lambda_2| - |\lambda_1| \geq k_{i,3}\}, \\ N_{\lambda_2;i,4} &:= \{\lambda_1 \in M_{\lambda_2,i} \mid \Omega^2_{\lambda_2} \not\subset \Omega^1_{\lambda_1}, \operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2}) \\ &\qquad < \delta_{i,1}^{-1} 2^{-|\lambda_1| a_{i,1}} 2^{-|\lambda_2| b_{i,1}}, |\lambda_2| - |\lambda_1| \geq k_{i,4}\}, \end{aligned}$$

$i = 1, 2$, *and*

$$N^{\varepsilon}_{\lambda_2} := \Lambda^1 \Big\backslash \bigcup_{i=1}^{2} \bigcup_{j=1}^{4} N_{\lambda_2;i,j},$$

$\lambda_2 \in T^2$, *there exists a constant* $\beta > 0$ *such that, for*

$$e_{\lambda_2} := \sum_{\lambda_1 \in \Lambda^1 \setminus N^{\varepsilon}_{\lambda_2}} \langle A\psi_{\lambda_1}, \psi_{\lambda_2} \rangle u_{\lambda_1}, \quad \lambda_2 \in T^2,$$

one has

$$\left(\sum_{\lambda_2 \in T^2} 2^{2\delta_2 |\lambda_2|} |e_{\lambda_2}|^2 \right)^{1/2} \leq \beta \varepsilon \|u_{\Lambda^1}\|_{-\delta_1}. \tag{4.33}$$

Proof. At first we consider $N_{\lambda_2;1,j}$ for $j \in \{1, 2, 3, 4\}$. For an arbitrary but fixed $\lambda_1 \in J_{j^1}$ we sum up λ_2 with $\lambda_1 \in N_{\lambda_2;1,1}$ and get, by (4.28),

$$\begin{aligned} &\sum_{\lambda_2} 2^{(j^1 - |\lambda_2|)/2} A_{\lambda_1,\lambda_2} \\ &\qquad \lesssim 2^{j^1 \delta_1} \sum_{\lambda_2} 2^{-|\lambda_2|(d^2+1-\delta_2)} \operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2})^{-1-\rho-d^2} \\ &\qquad \lesssim \delta_{1,1}^{\rho+d^2} 2^{j^1(\delta_1 + a_{1,1}(1+\rho+d^2))} \sum_{l} 2^{-l(d^2-\delta_2-b_{1,1}(\rho+d^2))} \\ &\qquad \lesssim \delta_{1,1}^{\rho+d^2} 2^{j^1[(a_{1,1}+b_{1,1}-1)(\rho+d^2)+a_{1,1}]} 2^{-k_{1,1}(d^2-\delta_2-b_{1,1}(\rho+d^2))}, \end{aligned}$$

where we used that, with $b_{1,1} \in [0,1)$,

$$\sum_{|\lambda_2|=l} \operatorname{dist}(\Omega^1_{\lambda_1}, \Omega^2_{\lambda_2})^{-1-\rho-d^2} \lesssim \delta_{1,1}^{1+\rho+d^2} 2^{j^1 a_{1,1}(1+\rho+d^2)} 2^{l b_{1,1}(1+\rho+d^2)} + 2^l \int_{x \geq \delta_{1,1}^{-1} 2^{-j^1 a_{1,1}} 2^{-|\lambda_2| b_{1,1}}} x^{-1-\rho-d^2}\, dx \lesssim \delta_{1,1}^{\rho+d^2} 2^{j^1 a_{1,1}(1+\rho+d^2)} 2^{l(1+b_{1,1}(\rho+d^2))}.$$

Analogously, we sum up λ_2 with $\lambda_1 \in N_{\lambda_2;1,2}$ and obtain by (4.29) and $b_{1,2} \in [0,1)$,

$$\sum_{\lambda_2} 2^{(j^1-|\lambda_2|)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{1,2}^{p+d^2-1} 2^{j^1[(a_{1,2}+b_{1,2}-1)(\rho+d^2-1)+a_{1,2}]} \cdot 2^{-k_{1,2}(d^2-\delta_2-1/2-b_{1,2}(\rho+d^2-1))}.$$

Then we sum up λ_2 with $\lambda_1 \in N_{\lambda_2;1,3}$ and obtain by (4.30), since $\sharp\{|\lambda_2| = l \mid \lambda_1 \in N_{\lambda_2;1,3}\} \lesssim \delta_{1,2}^{-1} 2^{-j^1 a_{1,2}} 2^{l(1-b_{1,2})}$,

$$\sum_{\lambda_2} 2^{(j^1-|\lambda_2|)/2} A_{\lambda_1,\lambda_2} \lesssim 2^{j^1(d+1/2)} \sum_{\lambda_2} 2^{-|\lambda_2|(d+1/2)} \lesssim \delta_{1,2}^{-1} 2^{j^1(d+1/2-a_{1,2})} \sum_l 2^{-l(d-1/2+b_{1,2})} \lesssim \delta_{1,2}^{-1} 2^{j^1(1-a_{1,2}-b_{1,2})} 2^{-k_{1,3}(d-1/2+b_{1,2})}.$$

Finally we sum up λ_2 with $\lambda_1 \in N_{\lambda_2;1,4}$ and obtain by (4.30), since $\sharp\{|\lambda_2| = l \mid \lambda_1 \in N_{\lambda_2;1,4}\} \lesssim \delta_{1,1}^{-1} 2^{-j^1 a_{1,1}} 2^{l(1-b_{1,1})}$,

$$\sum_{\lambda_2} 2^{(j^1-|\lambda_2|)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{1,1}^{-1} 2^{j^1(1-a_{1,1}-b_{1,1})} 2^{-k_{1,4}(d-1/2+b_{1,1})}.$$

Then we take an arbitrary λ_2 with $\cup_{j=1}^4 N_{\lambda_2;1,j} \neq \varnothing$. At first we sum up $\lambda_1 \in N_{\lambda_2;1,1}$ and get with $a_{1,1} \in [0,1]$ and $b_{1,1} \in [0,1)$

$$\sum_{\lambda_1} 2^{(|\lambda_2|-j^1)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{1,1}^{\rho+d^2} 2^{j^1[(a_{1,1}+b_{1,1}-1)(\rho+d^2)+b_{1,1}]} \cdot 2^{-k_{1,1}(d^2-\delta_2-b_{1,1}(\rho+d^2+1))}.$$

Analogously, we sum up $\lambda_1 \in N_{\lambda_2;1,2}$ and get with $a_{1,2} \in [0,1]$,

$$\sum_{\lambda_1} 2^{(|\lambda_2|-j^1)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{1,2}^{\rho+d^2} 2^{j^1(a_{1,2}+b_{1,2}-1)(\rho+d^2)} \cdot 2^{-k_{1,2}(d^2-\delta_2-1/2-b_{1,2}(\rho+d^2))}.$$

Then we sum up $\lambda_1 \in N_{\lambda_2;1,3}$ and get, because $\sharp\{|\lambda_1| = l \mid \lambda_1 \in N_{\lambda_2;1,3}\} = \mathcal{O}(1)$,

$$\sum_{\lambda_1} 2^{(|\lambda_2|-j^1)/2} A_{\lambda_1,\lambda_2} \lesssim 2^{-|\lambda_2|(d-1/2)} 2^{j_1(d-1/2)} \lesssim 2^{-k_{1,3}(d-1/2)}.$$

Finally, we sum up $\lambda_1 \in N_{\lambda_2;1,4}$ and get, because $\sharp\{|\lambda_1| = l \mid \lambda_1 \in N_{\lambda_2;1,4}\} = \mathcal{O}(1)$,

$$\sum_{\lambda_1} 2^{(|\lambda_2|-j^1)/2} A_{\lambda_1,\lambda_2} \lesssim 2^{-k_{1,4}(d-1/2)}.$$

Multiplying related terms, we get, by the assumption (4.31) and Lemma 4.8, an estimate like (4.33), cf. [**14**].

Next we consider $N_{\lambda_2;2,j}$, $j \in \{1,2,3,4\}$. For an arbitrary but fixed $\lambda_1 \in \Lambda_1 \backslash J_{j^1}$ we sum up λ_2 with $\lambda_1 \in N_{\lambda_2;2,1}$ and get by (4.28), $(a_{2,1}+b_{2,1}-1)(\rho+d^1+d^2)+a_{2,1} \leq 0$ and $b_{2,1} \in [0,1)$

$$\sum_{\lambda_2} 2^{(|\lambda_1|-|\lambda_2|)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{2,1}^{\rho+d^1+d^2} 2^{j^1[(a_{2,1}+b_{2,1}-1)(\rho+d^1+d^2)+a_{2,1}]} \cdot 2^{-k_{2,1}(d^2-\delta_2-b_{2,1}(\rho+d^1+d^2))}.$$

Analogously we sum up λ_2 with $\lambda_1 \in N_{\lambda_2;2,2}$ and obtain, by (4.29), $b_{2,2} \in [0,1)$ and $(a_{2,2}+b_{2,2}-1)(d^2+\rho-1)+a_{2,2} = (a_{2,2}+b_{2,2}-1)(d^2+\rho)-b_{2,2}+1 \leq 0$,

$$\sum_{\lambda_2} 2^{(|\lambda_1|-|\lambda_2|)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{2,2}^{\rho+d^2-1} 2^{j^1[(a_{2,2}+b_{2,2}-1)(d^2+\rho-1)+a_{2,2}]} \cdot 2^{-k_{2,2}(d^2-\delta_2-b_{2,2}(\rho+d^2-1))}.$$

Then we sum up λ_2 with $\lambda_1 \in N_{\lambda_2;2,3}$ and obtain by (4.30) and $-a_{2,2}-b_{2,2}+1 \geq 0$, since $\sharp\{|\lambda_2| = l|\lambda_1 \in N_{\lambda_2;2,3}\} \lesssim \delta_{2,2}^{-1} 2^{l(1-b_{2,2})} 2^{-|\lambda_1|a_{2,2}}$,

$$\sum_{\lambda_2} 2^{(|\lambda_1|-|\lambda_2|)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{2,2}^{-1} 2^{j_{\Lambda^1}(-a_{2,2}-b_{2,2}+1)} 2^{-k_{2,3}(d-1/2+b_{2,2})}.$$

Then we sum up λ_2 with $\lambda_1 \in N_{\lambda_2;2,4}$ and obtain by (4.30) and $-a_{2,1} - b_{2,1} + 1 \geq 0$ since $\sharp\{|\lambda_2| = l \mid \lambda_1 \in N_{\lambda_2;2,4}\} \lesssim \delta_{2,1}^{-1} 2^{l(1-b_{2,1})} 2^{-|\lambda_1| a_{2,1}}$

$$\sum_{\lambda_2} 2^{(|\lambda_1|-|\lambda_2|)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{2,1}^{-1} 2^{j_{\Lambda^1}(-a_{2,1}-b_{2,1}+1)} 2^{-k_{2,4}(d-1/2+b_{2,1})}.$$

Finally we take an arbitrary λ_2 with $\cup_{j=1}^4 N_{\lambda_2;2,j} \neq \varnothing$. At first, we sum up $\lambda_1 \in N_{\lambda_2;2,1}$ and get, by (4.28), $(a_{2,1} + b_{2,1} - 1)(\rho + d^1 + d^2) + b_{2,1} \leq 0$ and $a_{2,1} \in [0,1]$,

$$\sum_{\lambda_1} 2^{(|\lambda_2|-|\lambda_1|)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{2,1}^{\rho+d^1+d^2} 2^{j^1[(a_{2,1}+b_{2,1}-1)(\rho+d^1+d^2)+b_{2,1}]} \cdot 2^{-k_{2,1}(d^2-\delta_2-b_{2,1}(1+\rho+d^1+d^2))}.$$

Then we sum up $\lambda_1 \in N_{\lambda_2;2,2}$ and obtain by (4.29) and $(a_{2,2} + b_{2,2} - 1)(d^2 + \rho) \leq 0$,

$$\sum_{\lambda_1} 2^{(|\lambda_2|-|\lambda_1|)/2} A_{\lambda_1,\lambda_2} \lesssim \delta_{2,2}^{\rho+d^2} 2^{j^1(a_{2,2}+b_{2,2}-1)(d^2+\rho)} \cdot 2^{-k_{2,2}(d^2-\delta_2-b_{2,2}(\rho+d^2))}.$$

Then we sum up $\lambda_1 \in N_{\lambda_2;2,3}$ and obtain by (4.30), since $\sharp\{|\lambda_1| = l \mid \lambda_1 \in N_{\lambda_2;2,3}\} = \mathcal{O}(1)$,

$$\sum_{\lambda_1} 2^{(|\lambda_2|-|\lambda_1|)/2} A_{\lambda_1,\lambda_2} \lesssim 2^{-k_{2,3}(d-1/2)}.$$

Then we sum up $\lambda_1 \in N_{\lambda_2;2,4}$ and obtain by (4.30), since $\sharp\{|\lambda_1| = l \mid \lambda_1 \in N_{\lambda_2;2,4}\} = \mathcal{O}(1)$,

$$\sum_{\lambda_1} 2^{(|\lambda_2|-|\lambda_1|)/2} A_{\lambda_1,\lambda_2} \lesssim 2^{-k_{2,4}(d-1/2)}.$$

Thus, we get by (4.32) an estimate like (4.33). □

Concealing the δ, 2^{j^1} and $2^{j_{\Lambda^1}}$-terms in generic constants, one obtains the simplified conditions

$$2^{-2k_{1,1}(d^2-\delta_2-b_{1,1}(\rho+d^2+1/2))} + 2^{-2k_{1,2}(d^2-\delta_2-1/2-b_{1,2}(\rho+d^2-1/2))} + 2^{-k_{1,3}(2d-1+b_{1,2})} + 2^{-k_{1,4}(2d-1+b_{1,1})} < \varepsilon^2$$

and

$$2^{-2k_{2,1}(d^2-\delta_2-b_{2,1}(\rho+d^1+d^2+1/2))} + 2^{-2k_{2,2}(d^2-\delta_2-b_{2,2}(\rho+d^2-1/2))}$$
$$2^{-k_{2,3}(2d-1+b_{2,2})} + 2^{-k_{2,4}(2d-1+b_{2,1})} < \varepsilon^2.$$

Moreover, neglecting the decay estimates with respect to the singular supports, we get with $b_{1,1} = b_{1,2} = 0$, $k_{1,1} = k_{1,2}$ and $k_{1,3} = k_{1,4}$

$$2^{-2k_{1,1}(d^2-\delta_2)} + 2^{-2k_{1,4}(d-1/2)} < \varepsilon^2$$

and, respectively, with $b_{2,1} = b_{2,2} = 0$, $k_{2,1} = k_{2,2}$ and $k_{2,3} = k_{2,4}$

$$2^{-2k_{2,1}(d^2-\delta_2)} + 2^{-2k_{2,4}(d-1/2)} < \varepsilon^2.$$

If one chooses, e.g., $a_{2,2}, b_{2,2}$ such that $(a_{2,2} + b_{2,2} - 1)(d^2 + \rho) > 0$, then the term $2^{j^1[2(a_{2,2}+b_{2,2}-1)(d^2+\rho)-b_{2,2}+1]}$ has to be replaced by $2^{j_{\Lambda^1}[2(a_{2,2}+b_{2,2}-1)(d^2+\rho)-b_{2,2}+1]}$. If $a_{2,2} + b_{2,2} - 1 > 0$, then one gets instead of $2^{j_{\Lambda^1}(1-a_{2,2}-b_{2,2})}$ the magnitude $2^{j^1(1-a_{2,2}-b_{2,2})}$, etc.

Clearly further variants are possible. Especially matrix entries with respect to associated "no-corner" basis functions can be collected in sub-matrices with a greater d, which lead to $N^{\varepsilon}_{\lambda_2}$ with less elements.

4.4. *A posteriori estimates by finite sums.* We combine Corollary 4.7 and Theorem 4.9 to obtain the following result.

Corollary 4.10. *Under the assumptions of Corollary* 4.7 *there exists a constant* $c_5 > 0$ *such that, for arbitrary* $\varepsilon > 0$ *and* $\varepsilon_D, \varepsilon_K, \varepsilon_{K'}, \varepsilon_V > 0$ *with* $\varepsilon_D + \varepsilon_K \le \varepsilon$, $\varepsilon_{K'} + \varepsilon_V \le \varepsilon$, *there are index sets* $N^{D,\varepsilon_D}_{\lambda_1}$, $N^{K,\varepsilon_K}_{\lambda_2}$, $N^{K',\varepsilon_{K'}}_{\lambda_1}$, $N^{V,\varepsilon_V}_{\lambda_2}$ *with*

$$(N^{D,\varepsilon_D}_{\lambda_1} \cup N^{K,\varepsilon_K}_{\lambda_2}) \times (N^{K',\varepsilon_{K'}}_{\lambda_1} \cup N^{V,\varepsilon_V}_{\lambda_2}) \subset \Lambda^1 \times \Lambda^2, \quad (\lambda_1, \lambda_2) \in J\backslash\Lambda,$$

such that $N^D_{\Lambda^1,\varepsilon_D} := \{\lambda_1 \in J^1\backslash\Lambda^1 \mid N^{D,\varepsilon_D}_{\lambda_1} \neq \varnothing\}$, $N^K_{\Lambda^2,\varepsilon_K} := \{\lambda_2 \in J^2\backslash\Lambda^2 \mid N^{K,\varepsilon_K}_{\lambda_2} \neq \varnothing\}$, $N^{K'}_{\Lambda^1,\varepsilon_{K'}} := \{\lambda_1 \in J^1\backslash\Lambda^1 \mid N^{K',\varepsilon_{K'}}_{\lambda_1} \neq \varnothing\}$,

$N^V_{\Lambda^2,\varepsilon_V} := \{\lambda_2 \in J^2\backslash\Lambda^2 \mid N^{V,\varepsilon_V}_{\lambda_2} \neq \varnothing\}$ *are finite and the magnitudes*

$$d_{\lambda_1} := \sum_{\lambda_1' \in \Lambda^1 \backslash N^{D,\varepsilon_D}_{\lambda_1}} \langle D\psi^1_{\lambda_1'}, \psi^1_{\lambda_1}\rangle u^1_{\lambda_1'},$$

$$k'_{\lambda_1} := \sum_{\lambda_2' \in \Lambda^2 \backslash N^{K',\varepsilon_{K'}}_{\lambda_1}} \langle K'\psi^2_{\lambda_2'}, \psi^1_{\lambda_1}\rangle u^2_{\lambda_2'},$$

$$k_{\lambda_2} := \sum_{\lambda_1' \in \Lambda^1 \backslash N^{K,\varepsilon_K}_{\lambda_2}} \langle K\psi^1_{\lambda_1'}, \psi^2_{\lambda_2}\rangle u^1_{\lambda_1'},$$

$$v_{\lambda_2} := \sum_{\lambda_2' \in \Lambda^2 \backslash N^{V,\varepsilon_V}_{\lambda_2}} \langle V\psi^2_{\lambda_2'}, \psi^2_{\lambda_2}\rangle u^2_{\lambda_2'},$$

give

$$\Big(\sum_{\lambda_1 \in J^1\backslash\Lambda^1} 2^{-|\lambda_1|} |d_{\lambda_1} + k'_{\lambda_1}|^2 + \sum_{\lambda_2 \in J^2\backslash\Lambda^2} 2^{|\lambda_2|} |k_{\lambda_2} + v_{\lambda_2}|^2 \Big)^{1/2} \leq c_5 \varepsilon \|u_\Lambda\|_{1/2,-1/2}.$$

The local a posteriori estimator will be defined by the remaining terms, i.e.,. we set for $\lambda = (\lambda_1, \lambda_2) \in J\backslash\Lambda$,

$$a_\lambda(\Lambda,\varepsilon) := 2^{-|\lambda_1|/2} \Big| \sum_{\lambda_1' \in N^{D,\varepsilon_D}_{\lambda_1}} \langle D\psi^1_{\lambda_1'}, \psi^1_{\lambda_1}\rangle u^1_{\lambda_1'} + \sum_{\lambda_2' \in N^{K',\varepsilon_{K'}}_{\lambda_2}} \langle K'\psi^2_{\lambda_2'}, \psi^1_{\lambda_1}\rangle u^2_{\lambda_2'} \Big| + 2^{|\lambda_2|/2} \Big| \sum_{\lambda_1' \in N^{K,\varepsilon_K}_{\lambda_2}} \langle K\psi^1_{\lambda_1'}, \psi^2_{\lambda_2}\rangle u^1_{\lambda_1'} + \sum_{\lambda_2' \in N^{V,\varepsilon_V}_{\lambda_2}} \langle V\psi^2_{\lambda_2'}, \psi^2_{\lambda_2}\rangle u^2_{\lambda_2'} \Big|,$$

which implies the following estimates, cf. [**14**].

Corollary 4.11. *Under the assumptions of Corollary* 4.7, *we have*

(4.34)

$$\|u - u_\Lambda\|_{1/2,-1/2} \leq c_2 c_4 \bigg(\Big(\sum_{\lambda \in J\backslash\Lambda} a^2_\lambda(\Lambda,\varepsilon) \Big)^{1/2} + c_5 \varepsilon \|u_\Lambda\|_{1/2,-1/2} + \Big(\sum_{\lambda \in J\backslash\Lambda} 2^{-|\lambda_1|} |g^1_{\lambda_1}|^2 + 2^{|\lambda_2|} |g^2_{\lambda_2}|^2 \Big)^{1/2} \bigg),$$

as well as

(4.35)
$$\Big(\sum_{\lambda\in J\setminus\Lambda} a_\lambda^2(\Lambda,\varepsilon)\Big)^{1/2} \le \frac{1}{c_1c_3}\|u-u_\Lambda\|_{1/2,-1/2} + c_5\varepsilon\|u_\Lambda\|_{1/2,-1/2} + \Big(\sum_{\lambda\in J\setminus\Lambda} 2^{-|\lambda_1|}|g^1_{\lambda_1}|^2 + 2^{|\lambda_2|}|g^2_{\lambda_2}|^2\Big)^{1/2}.$$

Moreover, for $\Lambda\subset\tilde\Lambda\subset J$*, we have for the Galerkin solutions* u_Λ *and* $u_{\tilde\Lambda}$*,*

(4.36)
$$\Big(\sum_{\lambda\in \tilde\Lambda\setminus\Lambda} a_\lambda^2(\Lambda,\varepsilon)\Big)^{1/2} \le \frac{1}{c_1c_3}\|u_{\tilde\Lambda}-u_\Lambda\|_{1/2,-1/2} + c_5\varepsilon\|u_\Lambda\|_{1/2,-1/2} + \Big(\sum_{\lambda\in J\setminus\Lambda} 2^{-|\lambda_1|}|g^1_{\lambda_1}|^2 + 2^{|\lambda_2|}|g^2_{\lambda_2}|^2\Big)^{1/2}.$$

The estimates (4.34) and (4.35) show that

$$\eta_{\Lambda,\varepsilon} := \Big(\sum_{\lambda\in J\setminus\Lambda} a_\lambda^2(\Lambda,\varepsilon)\Big)^{1/2} \tag{4.37}$$

defines up to the tolerance $\varepsilon>0$ an efficient and reliable a posteriori estimator. The inequality (4.36) relates two successive Galerkin solutions and is crucial for the convergence proof of the adaptive scheme presented in the next section.

5. Adaptive schemes. The a posteriori estimates in Section 4 suggest to apply the following adaptive scheme which is based on the idea of equilibration of the error. For any $\Lambda\subset J$, a tolerance $\varepsilon>0$ and an index-set $\tilde\Lambda$ such that $(\sum_{\lambda\in\tilde\Lambda\setminus\Lambda} a_\lambda^2(\Lambda,\varepsilon))^{1/2}$ carries the relevant part of the error $\|u-u_\Lambda\|_{1/2,-1/2}$ in the sense of (4.34) and (4.35) one takes sufficiently many terms $a_\lambda(\Lambda,\varepsilon)$, $\lambda\in Z\subset\tilde\Lambda\setminus\Lambda$, if possible the biggest ones, i.e., $|Z|$ possibly small, such that

$$\Big(\sum_{\lambda\in Z\setminus\Lambda} a_\lambda^2(\Lambda,\varepsilon)\Big)^{1/2} \ge (1-\theta)\Big(\sum_{\lambda\in\tilde\Lambda\setminus\Lambda} a_\lambda^2(\Lambda,\varepsilon)\Big)^{1/2} \tag{5.1}$$

for some fixed $\theta \in (0,1)$. Then the next Galerkin step uses $\Lambda_Z := \Lambda \cup Z$ and gives possibly a more exact approximation u_{Λ_Z}. This approach is well known and described in several papers about a posteriori estimates and adaptive schemes. Therefore, we omit further details.

What we want to do next is to pose conditions which imply the convergence of such an adaptive scheme. That is, the following algorithm ensures that either the new Galerkin solution is really better or the a posteriori estimator shows that one has already reached an approximate solution within the desired accuracy. We shall restrict our attention to transmission problems with respect to smooth boundaries Γ. This restriction seems necessary for having a sufficiently strong relation between the Galerkin orthogonality induced by the Galerkin method and an appropriate Hilbert space norm $\|\cdot\|$ equivalent to the $\|\cdot\|_{1,2,-1/2}$-norm. We shall show that there is a self-adjoint operator S, such that the difference $H - S$ allows asymptotically better estimates than H itself. We notice that a similar idea is applied in [**14**] on non-symmetric elliptic boundary value problem which includes a convection term.

Lemma 5.1. *Let* $\mathrm{Re}\,(1+1/\mu) > 0$ *and* $\mathrm{Re}\,(1+\mu) > 0$. *Then there is an operator* $\mathcal{C} : L^2(\Gamma) \times H^{-1}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *such that* $S := H - \mathcal{C} : H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ *is self-adjoint and positive definite, i.e., one has*

$$|\langle Su, \bar{u}\rangle_\Gamma| \gtrsim \|u\|^2_{1/2,-1/2}, \quad u \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma). \tag{5.2}$$

Proof. At first we consider the operator

$$S_0 := \begin{pmatrix} 1/2(1+1/\mu)D_0 & 0 \\ 0 & 1/2(1+\mu)V_0 \end{pmatrix},$$

where D_0 and V_0 denote the integral operators with respect to the wave number zero. It is well known, cf. e.g. [**11**], that there is a self-adjoint compact operator $\mathcal{C}_0 : L^2(\Gamma) \times H^{-1}(\Gamma) \to H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ with

$$|\langle (S_0 + \mathcal{C}_0 u, \bar{u}\rangle_\Gamma| \gtrsim \|u\|^2_{1/2,-1/2}, \quad u \in H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma),$$

hence, $S := S_0 + \mathcal{C}_0$ satisfies (5.2). Therefore the assertion of the lemma is proved when we show that $\mathcal{C} := H - S : L^2(\Gamma) \times H^{-1}(\Gamma) \to$

$H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ is bounded, i.e., we have to investigate

$$\mathcal{C} = \begin{pmatrix} 1/2(D_1 - D_0 + 1/\mu(D_2 - D_0)) & K' \\ -K & 1/2(V_1 - V_0 + \mu(V_2 - V_0)) \end{pmatrix}.$$

Because of the mapping properties of $D_i - D_0$ and $V_i - V_0$, $i = 1, 2$, studied in [**12**], we only have to consider the operators K' and K. In [**29**] it is proved that $K' : H^s(\Gamma) \to H^{s+1/2}(\Gamma)$ is bounded for $s \in \mathbf{R}$ if Γ is smooth enough, e.g., if Γ is a C^l-curve with $l \geq |s| + 1/2$. Boundedness results for K then follow by duality arguments. Thus, since Γ is smooth, the operator $\mathcal{C}$ fulfills in fact the desired boundedness property. □

Clearly, the operator $\mathcal{C} = H - S$ is also bounded as an operator from $H^s(\Gamma) \times H^{-1+s}(\Gamma)$ to $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ for $s \in [0, 1/2]$.

Lemma 5.2. *Let $t \in [\max(0, 1 - \gamma_2^1), 1/2)$. Then, for $\Lambda \subset J$ with*

$$\|(I - Q_\Lambda)v\|_{1/2,-1/2} \leq \delta \|v\|_{1-t,-t}, \quad v \in H^{1-t}(\Gamma) \times H^{-t}(\Gamma), \tag{5.3}$$

and $\tilde{\Lambda} \subset J$, $\Lambda \subset \tilde{\Lambda}$, one has

$$|\langle \mathcal{C}(u - u_{\tilde{\Lambda}}), \overline{u_{\tilde{\Lambda}} - u_\Lambda} \rangle_\Gamma| \lesssim \delta(\|u - u_{\tilde{\Lambda}}\|^2_{1/2,-1/2} + \|u - u_\Lambda\|^2_{1/2,-1/2}) \tag{5.4}$$

with a constant independent of δ, Λ and $\tilde{\Lambda}$.

Proof. The mapping properties of $\mathcal{C}$ give

$$\begin{aligned} |\langle \mathcal{C}(u - u_{\tilde{\Lambda}}), \overline{u_{\tilde{\Lambda}} - u_\Lambda} \rangle_\Gamma| &\leq \|\mathcal{C}(u - u_{\tilde{\Lambda}})\|_{-1/2,1/2} \|u_{\tilde{\Lambda}} - u_\Lambda\|_{1/2,-1/2} \\ &\leq \|u - u_{\tilde{\Lambda}}\|_{t,t-1} \|u_{\tilde{\Lambda}} - u_\Lambda\|_{1/2,-1/2}. \end{aligned} \tag{5.5}$$

Furthermore, the Ansatz of the Aubin Nitsche trick provides for (5.3) the estimates

(5.6)

$$\begin{aligned}\|u-u_{\tilde\Lambda}\|_{t,t-1} &= \sup_{\|\xi\|_{-t,1-t}=1} |\langle u-u_{\tilde\Lambda},\xi\rangle_\Gamma| \\ &= \sup_{\|\xi\|_{-t,1-t}=1} |\langle H(u-u_{\tilde\Lambda}),(H')^{-1}\xi\rangle_\Gamma| \\ &\lesssim \sup_{\|\xi\|_{-t,1-t}=1} \inf_{\chi\in S_{\tilde\Lambda}} \|(H')^{-1}\xi-\chi\|_{1/2,-1/2}\|u-u_{\tilde\Lambda}\|_{1/2,-1/2} \\ &\lesssim \delta \sup_{\|\xi\|_{-t,1-t}=1} \|(H')^{-1}\xi\|_{1-t,-t}\|u-u_{\tilde\Lambda}\|_{1/2,-1/2} \\ &\lesssim \delta\|u-u_{\tilde\Lambda}\|_{1/2,-1/2}.\end{aligned}$$

We obtain (5.4) by combining (5.5) and (5.6). □

Next we define an appropriate Hilbert space norm $\|\cdot\|$ by

$$\|u\| := \sqrt{\langle Su,\bar u\rangle_\Gamma}, \quad u\in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma).$$

Lemma 5.1 gives constants $c_7, c_8 > 0$ such that

(5.7)

$$c_7\|u\|_{1/2,-1/2} \le \|u\| \le c_8\|u\|_{1/2,-1/2}, \quad u\in H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma).$$

A simple calculation yields

$$\|u-u_\Lambda\|^2 - \|u-u_{\tilde\Lambda}\|^2 - \|u_\Lambda-u_{\tilde\Lambda}\|^2 = 2\langle S(u-u_{\tilde\Lambda}),\overline{u_{\tilde\Lambda}-u_\Lambda}\rangle_\Gamma;$$

hence the Galerkin orthogonality $\langle H(u-u_{\tilde\Lambda}),\overline{u_{\tilde\Lambda}-u_\Lambda}\rangle_\Gamma = 0$ implies

$$\|u-u_{\tilde\Lambda}\|^2 = \|u-u_\Lambda\|^2 - \|u_\Lambda-u_{\tilde\Lambda}\|^2 + 2\langle \mathcal{C}(u-u_{\tilde\Lambda}),\overline{u_{\tilde\Lambda}-u_\Lambda}\rangle_\Gamma.$$

Therefore, by (5.3) and its consequence (5.4) for a fixed $t\in[\max(0, 1-\gamma_2^1), 1/2)$, we get a constant $c_6 > 0$ such that

$$(1-c_6\delta)\|u-u_{\tilde\Lambda}\|^2 \le (1+c_6\delta)\|u-u_\Lambda\|^2 - \|u_\Lambda-u_{\tilde\Lambda}\|^2. \tag{5.8}$$

Theorem 5.3. *Let there be given a tolerance* eps > 0. *Fix any* $\theta^*\in(0,1)$ *and define*

$$C_e := \left(\frac{1}{c_1c_3} + \frac{1-\theta^*}{2(2-\theta^*)c_2c_4}\right). \tag{5.9}$$

Choose any $\mu^* > 0$ such that

$$\mu^* C_e \leq \frac{1-\theta^*}{2(2-\theta^*)c_2 c_4}, \tag{5.10}$$

set

$$\epsilon := \frac{\mu^* \mathrm{eps}}{2c_5 \|u_\Lambda\|_{1/2,-1/2}} \tag{5.11}$$

and choose $\delta^* > 0$ such that

$$\delta^* < \frac{c_1^2 c_3^2 c_7^2 (1-\theta^*)^2}{4c_2^2 c_4^2 c_6 c_8^2}. \tag{5.12}$$

Suppose that $\Lambda \subset J$ satisfies

$$\left(\sum_{\lambda \in J \setminus \Lambda} 2^{-|\lambda_1|} |g^1_{\lambda_1}|^2 + 2^{|\lambda_2|} |g^2_{\lambda_2}|^2 \right)^{1/2} < \frac{1}{2} \mu^* \mathrm{eps} \tag{5.13}$$

and

$$\|(I - Q_\Lambda) v\|_{1/2,-1/2} \leq \delta^* \|v\|_{1-t,-t}, \quad v \in H^{1-t}(\Gamma) \times H^{-t}(\Gamma). \tag{5.14}$$

Then whenever $\tilde{\Lambda} \subset J$, $\Lambda \subset \tilde{\Lambda}$ fulfills

$$\left(\sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} a_\lambda(\Lambda, \varepsilon)^2 \right)^{1/2} \geq (1-\theta^*) \left(\sum_{\lambda \in J \setminus \Lambda} a_\lambda(\Lambda, \varepsilon)^2 \right)^{1/2}, \tag{5.15}$$

there exists a constant $\kappa \in (0,1)$, depending on the constants θ^*, δ^* and the constants c_i, $i = 1, \dots, 8$, such that either

$$\|u - u_{\tilde{\Lambda}}\| \leq \kappa \|u - u_\Lambda\| \tag{5.16}$$

or $(\sum_{\lambda \in J \setminus \Lambda} a_\lambda(\Lambda, \varepsilon)^2)^{1/2} \leq \mathrm{eps}$.

Proof. We first assume that $\|u - u_\Lambda\|_{1/2,-1/2} \geq \mathrm{eps}/C_e$ where the constant $C_e > 0$ is defined by (5.9). When $\tilde{\Lambda}$ satisfies (5.15) we infer from (4.34), (4.36), (5.10), (5.11) and (5.13), cf. [**14**],

$$\begin{aligned}\|u_{\tilde{\Lambda}} - u_\Lambda\|_{1/2,-1/2} \geq c_1 c_3 \Bigg(\Bigg(& \sum_{\lambda \in \tilde{\Lambda} \setminus \Lambda} a_\lambda(\Lambda, \epsilon)^2 \Bigg)^{1/2} \\ & - c_5 \epsilon \|u_\Lambda\|_{1/2,-1/2} \\ & - \Bigg(\sum_{\lambda \in J \setminus \lambda} 2^{-|\lambda_1|} |g^1_{\lambda_1}|^2 + 2^{|\lambda_2|} |g^2_{\lambda_2}|^2 \Bigg)^{1/2} \Bigg) \\ \geq & \frac{c_1 c_3 (1-\theta^*)}{2 c_2 c_4} \|u - u_\Lambda\|_{1/2,-1/2}. \end{aligned} \tag{5.17}$$

By (5.7) and (5.17) we obtain with $c_9 := (c_1 c_3 c_7 (1-\theta^*))/(2 c_2 c_4 c_8)$

$$\|u_{\tilde{\Lambda}} - u_\Lambda\| \geq c_9 \|u - u_\Lambda\|. \tag{5.18}$$

On account of (5.8) we obtain, by (5.18),

$$\|u - u_{\tilde{\Lambda}}\| \leq \sqrt{\frac{1 - c_9^2 + c_6 \delta^*}{1 - c_6 \delta^*}} \|u - u_\Lambda\|;$$

hence,

$$\|u - u_{\tilde{\Lambda}}\| \leq \kappa \|u - u_\Lambda\| \tag{5.19}$$

with $\kappa \in (0,1)$ because of (5.12).

On the other hand, $\|u - u_\Lambda\|_{1/2,-1/2} < \mathrm{eps}/C_e$ yields, in view of (4.35), (5.11) and (5.13), cf. [**14**],

$$\Bigg(\sum_{\lambda \in J \setminus \Lambda} a_\lambda(\Lambda, \epsilon)^2 \Bigg)^{1/2} \leq \frac{(1/(c_1 c_3)) + \mu^* C_e) \mathrm{eps}}{C_e}.$$

Taking (5.9) and (5.10) into account, we see that $(1/(c_1 c_3) + \mu^* C_e)/C_e \leq \Lambda$ so that

$$\Bigg(\sum_{\lambda \in J \setminus \Lambda} a_\lambda(\Lambda, \epsilon)^2 \Bigg)^{1/2} \leq \mathrm{eps}\,,$$

which completes the proof. □

Note that $(\sum_{\lambda\in J\setminus\Lambda} a_\lambda(\Lambda,\epsilon)^2)^{1/2} \le \text{eps}$ yields, by (4.34), (5.11) and (5.13),

$$\|u - u_\Lambda\|_{1/2,-1/2} \le c_2 c_4 (1+\mu^*)\varepsilon. \tag{5.20}$$

Remark 5.4. The term $\|u_\Lambda\|_{1/2,-1/2}$ in (5.11) can be replaced by

$$\Big(\sum_{\lambda_1\in\Lambda_1} 2^{|\lambda_1|} |\langle u^1_\Lambda, \tilde\psi^1_{\lambda_1}\rangle|^2 + \sum_{\lambda_2\in\Lambda_2} 2^{-|\lambda_2|} |\langle u^2_\Lambda, \tilde\psi^2_{\lambda_2}\rangle|^2 \Big)^{1/2}.$$

The constants then change in an obvious way.

An asymptotic sufficient condition for (5.14) is, e.g., that $S_\Lambda \subset S_n$, $n = (m,m)$, with

$$m \gtrsim \frac{\log_2 \delta^*}{t-1/2} - 1,$$

which follows by $(\sum_{j=m+1}^\infty 2^{(2t-1)j})^{1/2} \lesssim \delta^*$.

Theorem 5.3 suggests the following convergent adaptive algorithm:

1) Choose θ^*, μ^* and δ^*, an initial accuracy eps_1 and a final accuracy eps.

2) Set $\Lambda_{1,0} \subset J$ such that $\|(I - Q_{\Lambda_{1,0}})v\|_{1/2,-1/2} \le \delta^* \|v\|_{1-t,-t}$, $v \in H^{1-t}(\Gamma)\times H^{-t}(\Gamma)$.

3) Choose for $i = 1,\infty$ index sets $\Lambda_{i,1} \subset J$, $\Lambda_{i,1} \supset \Lambda_{i,0}$ such that

$$\Big(\sum_{\lambda\in J\setminus\Lambda_{i,1}} 2^{-|\lambda_1|} |g^1_{\lambda_1}|^2 + 2^{|\lambda_2|} |g^2_{\lambda_2}|^2 \Big)^{1/2} < \frac{1}{2}\mu^* \text{eps}_i.$$

a) Compute for $j = 1,\infty$ the Galerkin solution $u_{\Lambda_{i,j}}$ with respect to $S_{\Lambda_{i,j}}$.

Set

$$\varepsilon := \frac{\mu^* \cdot \text{eps}_i}{2c_5 \|u_{\Lambda_{i,j}}\|_{1/2,-1/2}}.$$

Determine an appropriate index set $N_{\Lambda_{i,j},\varepsilon}$ and

$$\eta_{\Lambda_{i,j,\varepsilon}} := \Big(\sum_{\lambda \in N_{\Lambda_{i,j},\varepsilon}} a_\lambda^2(\Lambda, \varepsilon) \Big)^{1/2}.$$

b) If $\eta_{\Lambda_{i,j,\varepsilon}} < \mathrm{eps}_i$, set $j := \infty$,
else choose $\Lambda_{i,j+1} \supset \Lambda_{i,j}$, $\Lambda_{i,j+1} \backslash \Lambda_{i,j} \subset N_{\Lambda_{i,j},\varepsilon}$ such that

$$\Big(\sum_{\lambda \in \Lambda_{i,j+1} \backslash \Lambda_{i,j}} a_\lambda^2(\Lambda, \varepsilon) \Big)^{1/2} \geq (1 - \theta^*) \eta_{\Lambda_{i,j},\varepsilon}.$$

If $\mathrm{eps}_i < \mathrm{eps}$, go to Stop,
else set $\mathrm{eps}_{i+1} := \mathrm{eps}_i/2$ and $\Lambda_{i+1,0} := \Lambda_{i,j}$.

4) Stop.

Finally we notice that, by mapping properties of the operators A_j, K_{Ω_j} and V_{Ω_j} one can derive error estimates for representations of Galerkin approximations as well as their convergence with respect to an adaptive scheme if therein the parameters are chosen such that the resulting $\kappa \in (0, 1)$ is appropriately small. Small $\kappa \in (0, 1)$ can be attained if the parameters θ^* and δ^* are suitably adapted.

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