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# THE FINITE-SECTION APPROXIMATION FOR ILL-POSED INTEGRAL EQUATIONS ON THE HALF-LINE

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ABSTRACT. Integral equations on the half-line are commonly approximated by the finite-section approximation, in which the infinite upper limit is replaced by a positive number called the finite-section parameter. In this paper we consider the finite-section approximation for the first kind integral equations, which are typically ill-posed and call for regularization. For some classes of such equations corresponding to inverse problems from optics and astronomy, we indicate the finite-section parameters that allow us to apply standard regularization techniques. Two discretization schemes for the finite-section equations are also proposed and their efficiency is studied.

**1. Introduction.** In this paper we consider integral equations of the form

$$Kx(t) := \int_0^\infty k(t,\tau)b(\tau)x(\tau)\,d\tau = y(t), \quad t \geq 0, \tag{1.1}$$

under the assumptions that $x(t),\ y(t) \in L_2(0,\infty)$, and $k(t,\tau),\ b(\tau)$ are continuous functions such that for $t,\tau \to \infty$ $|k(t,\tau)| \sim (t\tau)^{-\kappa},\ |b(\tau)| \sim \tau^\beta,\ \kappa,\ \beta > 0$. More precisely, we assume that there are the constants $c_k, c_b$ such that, for any $t,\tau \in [0,\infty)$,

$$|k(t,\tau)| \leq \frac{c_k}{[(1+t)(1+\tau)]^\kappa}, \quad |b(\tau)| \leq c_b \tau^\beta. \tag{1.2}$$

*Example.* Many inverse problems in optics and astronomy can be modeled, at least approximately, by the problem of solving an integral equation of the type (1.1) and (1.2). An example is an equation which determines the particle size distribution of spherical particles by scattering methods (cf. [**2, 4, 5**]) which is given by

$$\int_0^\infty x(\tau)\tau^4 \left[\frac{2J_1(t\tau)}{t\tau}\right]^2 d\tau = y(t), \tag{1.3}$$

where $J_1$ is a Bessel function of order 1 and $x$ is the unknown distribution of the particle size. Keeping in mind that for large $u$, the asymptotic representation

$$J_\nu(u) \sim \sqrt{\frac{2}{\pi u}} \cos\left(u - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) \tag{1.4}$$

holds, the above-mentioned equation belongs to our class for $b(\tau) = \tau^4$,

$$k(t,\tau) = \left[\frac{2J_1(t\tau)}{t\tau}\right]^2 \sim \frac{8}{\pi}(t\tau)^{-3} \sin^2\left(t\tau - \frac{\pi}{4}\right).$$

This means that for equation (1.3) conditions (1.2) are fulfilled with $\kappa = 3,\ \beta = 4$.

To consider integral equations (1.1) and (1.2) in a real Hilbert space $L_2(0,\infty)$, we will use a Hilbert scale of spaces $\{L_{2,s}\}_{s\in\mathbf{R}}$ generated by an unbounded strictly positive self-adjoint operator $J$ defined by $Jf(t) = (1+t)^{1/2}f(t)$. To be more precise, $L_{2,s}$ is defined as the completion of the intersection of domains of the operators $J^\nu : f(t) \to (1+t)^{\nu/2}f(t)$, $\nu \geq 0$, accomplished with norm $\|\cdot\|_s$ defined as $\|f\|_s := \langle f,f\rangle_s^{1/2}$, where

$$\langle f,g\rangle_s = \langle J^s f, J^s g\rangle = \int_0^\infty (1+t)^s f(t)g(t)\,dt.$$

Note that under the assumption (1.2)

$$\begin{aligned}\|Kx\|_s^2 &\leq c\int_0^\infty (1+t)^{s-2\kappa}\ dt\left(\int_0^\infty (1+\tau)^{\beta-\kappa-r/2}x(\tau)(1+\tau)^{r/2}\,d\tau\right)^2\\ &\leq c\int_0^\infty (1+t)^{s-2\kappa}\ dt\int_0^\infty (1+\tau)^{2\beta-2\kappa-r}\ d\tau\,\|x\|_r^2\end{aligned}$$

(here and throughout the paper $c$ denotes generic constants that can vary from appearance to appearance). Thus, in general one can guarantee that for fixed $\kappa$, $\beta > 0$ the operator $K$ acts continuously from $L_{2,r}$ into $L_{2,s}$ only if $s < 2\kappa - 1$, $r > 2\beta - 2\kappa + 1$. In particular, considering an equation (1.1) in the space $L_2(0,\infty) = L_{2,0}$ one should assume that

$$\kappa > \frac{1}{2}, \quad x \in L_{2,s}, \quad s > 2\beta - 2\kappa + 1. \tag{1.5}$$

This means that $x(t) = J^{-s}z(t) = (1+t)^{-s/2}z(t)$ where $z(t)$ is the new unknown element from $L_2(0,\infty)$. Substituting this representation in (1.1) we arrive at the following equation for $z$

$$Az(t) = y(t), \tag{1.6}$$

where

$$Az(t) = KJ^{-s}z(t) = \int_0^\infty k(t,\tau)\frac{b(\tau)}{(1+\tau)^{s/2}}\, z(\tau)\, d\tau. \tag{1.7}$$

Under the assumptions (1.2) and (1.5), the kernel $a(t,\tau) = k(t,\tau)b(\tau)(1+\tau)^{-s/2}$ is square-summable, i.e.,

$$\int_0^\infty \int_0^\infty |a(t,\tau)|^2\, dtd\tau \le c\int_0^\infty (1+t)^{-2\kappa}\, dt \int_0^\infty (1+\tau)^{2\beta-2\kappa-s}\, d\tau < \infty.$$

This implies that $A$ is a compact operator from $L_2(0,\infty)$ to $L_2(0,\infty)$. Therefore, the equation (1.6) is ill-posed in the sense of Hadamard, because of the inverse operator $A^{-1}$ (even if it exists) is not continuous in the topology $L_2(0,\infty)$ and the crux of the difficulty is that only $y_\delta \in L_2(0,\infty)$ is available such that

$$\|y - y_\delta\|_{L_2(0,\infty)} \le \delta, \tag{1.8}$$

where the parameter $\delta \in (0,1)$ characterizes the level of the noise in the data.

*Remark.* As has been proposed in [**1, 2**] and [**8**], introducing new unknown $f(\tau) = x(\tau)\tau^4$ and changing the variables $t = e^{-u}$, $\tau = e^v$, $u, v \in (-\infty,\infty)$, it is possible to reduce the equation (1.3) to the convolution equation

$$\int_{-\infty}^\infty K(u-v)F(v)\, dv = Y(u), \tag{1.9}$$

where $K(u) = 4J_1^2(e^{-u})e^{-3u}$, $F(v) = f(e^v) = x(e^v)e^{4v}$, $Y(u) = e^{-u}y(e^{-u})$. Then the powerful scheme connected with Fourier transform can be applied to (1.9). However, such a scheme a priori assumes that $F \in L_2(-\infty,\infty)$, i.e.,

$$\int_{-\infty}^\infty |F(v)|^2\, dv = \int_{-\infty}^\infty x^2(e^v)e^{7v}\, de^v = \int_0^\infty x^2(\tau)\tau^7\, d\tau < \infty.$$

This implies that one should assume $x \in L_{2,s}$ for $s \geq 7$. Keeping in mind that for (1.3) $\kappa = 3$ and $\beta = 4$, the latter assumption is more restrictive than (1.5). Moreover, if instead of $y(t)$ only $y_\delta(t)$ is available, then the corresponding noisy free term for (1.9) has the form $Y_\delta(u) = e^{-u} y_\delta(e^{-u})$ and

$$\|Y - Y_\delta\|^2_{L_2(-\infty,\infty)} = \int_0^\infty t(y(t) - y_\delta(t))^2 \, dt \sim \|y - y_\delta\|^2_{L_{2,1}} .$$

This means that in dealing with convolution equation (1.9) one should measure the data error for initial equation (1.3) in a stronger than usual norm (see (1.8)).

**2. The finite-section approximation.** Integral equations on the half-line are commonly approximated by the finite-section approximation, in which the infinite upper limit of the integral is replaced by some positive number $M$. For the case of well-posed second kind integral equations having the kernel of the Wiener-Hopf form plus some "short-ranged" kernel, the theory of the finite-section approximation has been developed in [**7, 11, 12**]. In the case of ill-posed first kind integral equations, an investigation of the finite-section approximation for them originated in [**3**]. The present paper is an extension of the latter article.

Let $P_M$ denote the orthogonal projection of $L_2(0,\infty)$ onto $L_2(0,M)$ given by $P_M u(t) = \chi_{[0,M)}(t)u(t)$, where $\chi_{[0,M)}(t)$ is the characteristic function of the interval $[0,M)$. Within the framework of the finite-section approximation we pass from the initial problem (1.6), (1.8) to the equation $P_N A P_M z(t) = P_N y_\delta(t)$ that can be represented as

(2.1)
$$P_N A P_M z(t) := \int_0^M k(t,\tau) \frac{b(\tau)}{(1+\tau)^{s/2}} z(\tau) \, d(\tau) = y_\delta(t), \quad t \in [0,N).$$

But (2.1) is still an infinite-dimensional and ill-posed problem. Therefore, the following important and natural questions arise. Namely,

$Q_1$: how to choose the finite-section parameters $M$ and $N$ against the level of the data error $\delta$;

$Q_2$: how to discretize adequately the finite-section equation;

$Q_3$: how to regularize the equation obtained after discretization.

Regularization of discretized integral equations on finite intervals has been intensively studied by a number of authors. A few selected references on this topic are [**10, 6**] and [**9**]. For this reason we will not discuss the question $Q_3$ in the present paper. Note only that under the standard assumption that the solution of (1.6) belongs to the source set $\operatorname{Range}(A^*A)^p$ one can reach the best possible order of accuracy $O\left(\delta^{p/p+1}\right)$ even if the smoothness index $p$ is unknown a priori. It is necessary only that the operator $A^{\mathrm{disc}}$ and free term $y^{\mathrm{disc}}$ of discretized equation

$$A^{\mathrm{disc}} z(t) = y^{\mathrm{disc}}(t) \tag{2.2}$$

would be such that

$$\left\|A - A^{\mathrm{disc}}\right\|_{L_2(0,\infty)\to L_2(0,\infty)} \le c_1\delta, \quad \left\|y - y^{\mathrm{disc}}\right\|_{L_2(0,\infty)} \le c_2\delta. \tag{2.3}$$

Thus, our focus will be on the two questions $Q_1$ and $Q_2$, and we will provide (2.3).

For $\kappa, \beta$ and $s$ from (1.5) we put

$$\mu = \kappa - \frac{1}{2} > 0, \quad \lambda = \kappa + \frac{s}{2} - \beta - \frac{1}{2} > 0.$$

**Lemma 2.1.** *For any $N, M > 0$*

$$\|(I-P_N)\,A\,(I-P_M)\| \le cN^{-\mu}M^{-\lambda}, \qquad \|(I-P_N)\,A\| \le cN^{-\mu},$$
$$\|A\,(I-P_M)\| \le cM^{-\lambda}, \qquad \|A - P_N A P_M\| \le c\left(N^{-\mu} + M^{-\lambda}\right),$$

*where the constant $c$ depends only on $c_\kappa, c_b$ from* (1.2).

*Proof.* We prove only the first inequality because the others can be proved in a similar manner.

It follows from (1.2) that for any $u \in L_2(0,\infty)$

$$\begin{aligned}
&\|(I-P_N)A(I-P_M)u\|^2 \\
&\qquad \le \int_N^\infty \left( \int_M^\infty \frac{|k(t,\tau)|\,|b(\tau)|}{(1+\tau)^{s/2}}\,|u(\tau)|\,d\tau \right)^2 dt \\
&\qquad \le c\int_N^\infty (1+t)^{-2\kappa}\,dt \int_M^\infty (1+\tau)^{2\beta-2\kappa-s}\,d\tau \int_M^\infty |u(\tau)|^2\,d\tau \\
&\qquad \le cN^{-2\kappa+1}M^{-2\kappa-s+2\beta+1}\,\|u\|^2,
\end{aligned}$$

as claimed. □

The next theorem answers the question $Q_1$.

**Theorem 2.1.** *Let the solutions of the equations* (1.1) *and* (1.6) *be such that for some* $d > 0$, $\|x\|_s = \|z\| \leq d$. *Then for* $M \asymp \delta^{-1/\lambda}$, $N \asymp \delta^{-1/\mu}$ *the order of the error in the operator and in the free term of the finite-section equation* (2.1) *is the same as in the initial equation* (1.6), (1.8), *i.e.*

$$\|A - P_N A P_M\| \leq c\delta, \quad \|y - P_N y_\delta\| \leq c\delta. \tag{2.4}$$

*Proof.* The first inequality in (2.4) follows directly from Lemma 2.1. For the second inequality we use this lemma in the following way:

$$\begin{aligned}\|y - P_N y_\delta\| &\leq \|y - P_N y\| + \|P_N(y - y_\delta)\| \leq \|(I - P_N)\, Az\| + \delta \\ &\leq cdN^{-\mu} + \delta \leq c\delta. \qquad \square\end{aligned}$$

**3. The discretization.** To discretize the finite-section equation (2.1) we modify the scheme proposed in [**3**]. To this end we represent the finite-section parameters as $M = 2^m$, $N = 2^n$, where $m, n$ are some integer numbers, and consider the following system of knots

$$\begin{aligned}&t^\theta_{k,q,i} = 2^{k-1} + i2^{k-1}M^{-1}_{k,\theta}, \quad i = 0, 1, \dots, M_{k,\theta},\ k = 1, 2, 3, \dots; \\ &t^\theta_{0,q,i} = i2^{-q}, \quad i = 0, 1, \dots, M_{0,\theta},\end{aligned}$$

where

$$M_{k,\theta} = \left[2^{k(1-\theta)+q}\right], \quad M_{0,\theta} = 2^q,$$

$\theta$ is some fixed number from $(0, 1]$, $q$ is a fixed integer number, and $[u]$ denotes an integer part of $u$. Note that each interval $R_k = \left[2^{k-1}, 2^k\right)$ contains $M_{k,\theta}$ knots of the system above.

For $t \in R_k$, $\tau \in R_l$ we denote by $S^q_k(y; t)$ and $S^q_{k,l}(a; t, \tau)$ the piecewise constant functions interpolating $y(t)$ and $a(t, \tau)$ at the points

$\left\{t_{k,q,i}^{\theta_1}\right\}$, $\left\{t_{l,q,j}^{\theta_2}\right\}$, i.e.,

$$S_{k,l}^{q}\left(a;t,\tau\right) = \sum_{i=1}^{M_k,\theta_1} \sum_{j=1}^{M_l,\theta_2} a\left(t_{k,q,i-1}^{\theta_1}, t_{l,q,j-1}^{\theta_2}\right) S_{k,i}(t) S_{l,j}(\tau),$$

$$S_{k}^{q}\left(y;t\right) = \sum_{i=1}^{M_k,\theta_1} y\left(t_{k,q,i-1}^{\theta_1}\right) S_{k,i}(t),$$

where $S_{k,l}$ and $S_{l,j}$ are the characteristic functions of the intervals $[t_{k,q,i-1}^{\theta_1}, t_{k,q,i}^{\theta_1})$ and $[t_{l,q,j-1}^{\theta_2}, t_{l,q,j}^{\theta_2})$, respectively.

In our further analysis we will refer to the following simple estimates

(3.1)

$$\|y - S_k^q(y,\cdot)\|_{L_2(R_k)} \le c2^{k-1} M_{k,\theta_1}^{-1} \|y^{'}\|_{L_2(R_k)} \le c2^{k\theta_1 - q} \|y^{'}\|_{L_2(R_k)},$$

$$\|a - S_{k,l}^q(a,\cdot,\cdot)\|_{L_2(R_{k,l})} \le c[2^{k\theta_1 - q}\|a^{(1,0)}\|_{L_2(R_{k,l})} + 2^{l\theta_2 - q}\|a^{(0,1)}\|_{L_2(R_{k,l})} + 2^{k\theta_1 + l\theta_2 - 2q}\|a^{(1,1)}\|_{L_2(R_{k,l})}],$$

where $R_{k,l} = R_k \times R_l = [2^{k-1}, 2^k) \times [2^{l-1}, 2^l)$ and $a^{(i,j)} = \frac{\partial^{i+j} a}{\partial t^i \partial \tau^j}$. Assume now that the kernel $a(t,\tau) = k(t,\tau)b(\tau)(1+\tau)^{-s/2}$ is such that for some $\alpha, \omega > 1/2$

$$\left|a^{(i,j)}(t,\tau)\right| \le c(1+t)^{-\alpha}(1+\tau)^{-\omega}, \quad i,j = 0,1. \tag{3.2}$$

In principle, $\alpha$ and $\omega$ depend on $\kappa, \beta, s$ from (1.2), (1.5), and on the specific form of $k(t,\tau)$ and $b(\tau)$. In particular, from (1.4) and the recurrence formula

$$\frac{dJ_\nu(u)}{du} = \frac{1}{2}[J_{\nu-1}(u) - J_{\nu-1}(u)]$$

it follows that the kernel $a(t,\tau)$ corresponding to the equation (1.3) meets the condition (3.2) with $\alpha = 2$ and $\omega = (s/2) - 2$, $s > 5$.

Using (3.2) we have

$$\left\|a^{(i,j)}\right\|_{L_2(R_{k,l})}^2 \le \int_{2^{k-1}}^{2^k} \int_{2^{l-1}}^{2^l} (1+t)^{-2\alpha}(1+\tau)^{-2\omega}\, dt\, d\tau \le c2^{-(2\alpha-1)k} 2^{-(2\omega-1)l}. \tag{3.3}$$

Let us choose $\theta_1, \theta_2$ in (3.1) as

$$\theta_1 = \min\left\{1, \frac{2\alpha - 1}{2}\right\}, \quad \theta_2 = \min\left\{1, \frac{2\omega - 1}{2}\right\}. \tag{3.4}$$

Then from (3.1)–(3.4) we obtain

$$\left\| a - S^q_{k,l}(a;\cdot,\cdot)\right\|_{L_2(R_{k,l})} \le c2^{-q}\left(2^{-(\omega-\frac{1}{2})\ell} + 2^{-(\alpha-\frac{1}{2})k} + 2^{-q}\right). \tag{3.5}$$

Consider now the discretized equation (2.2) with $y^{\text{disc}}(t) = S^q_k(y;t)$, $t \in R_k$, $k = 0, 1, \dots, n$, and

$$A^{\text{disc}} z(t) = \int_0^{2^m} a^{\text{disc}}(t,\tau) z(\tau)\, d\tau, \quad t \in [0, 2^n],$$

where

$$a^{\text{disc}}(t,\tau) = \sum_{k=1}^{n}\sum_{l=1}^{m} S^q_{k,l}(a;t,\tau) + \sum_{k=1}^{n} S^q_{k,0}(a;t,\tau) + \sum_{l=1}^{m} S^q_{0,l}(a;t,\tau) + S^q_{0,0}(a;t,\tau).$$

The next result shows how to choose the parameters in the discretization scheme [**3**] to meet the key conditions (2.3).

**Theorem 3.1.** *Assume that the conditions of Theorem* 2.1 *and* (1.2), (1.5), (3.2) *hold. If $m, n, q$ are such that*

$$2^m \asymp \delta^{-1/\lambda}, \quad 2^n \asymp \delta^{-1/\mu}, \quad 2^q \asymp \delta^{-1}\sqrt{\log\frac{1}{\delta}}, \tag{3.6}$$

*and $\theta_1$, $\theta_2$ are chosen as* (3.4), *then $A^{\text{disc}}$ and $y^{\text{disc}}$ determined above meet the conditions* (2.3).

*Proof.* We will check only the first condition in (2.3), because for the second one the argument is even simpler.

Using Theorem 2.1 with $M = 2^m,\ N = 2^n$, we obtain

$$\begin{aligned}\left\|A - A^{\rm disc}\right\| &\le \|A - P_{2^n}AP_{2^m}\| + \left\|P_{2^n}AP_{2^m} - A^{\rm disc}\right\| \\ &\le c\delta + \left\|P_{2^n}AP_{2^m} - A^{\rm disc}\right\|.\end{aligned}$$

Moreover, it is easy to see that

$$\left\|P_{2^n}AP_{2^m} - A^{\rm disc}\right\| \le \left\|a - a^{disc}\right\|_{L_2([0,2^n)\times[0,2^m))}.$$

Because of (3.5) the latter norm can be estimated as

$$\begin{aligned}\left\|a - a^{\rm disc}\right\|^2 = &\sum_{k=1}^{n}\sum_{l=1}^{m}\int_{2^{k-1}}^{2^k}\int_{2^{l-1}}^{2^l}\Big(a\left(t,\tau\right) - S_{k,l}^q\left(a;t,\tau\right)\Big)^2\,d\tau\,dt \\ &+\sum_{k=1}^{n}\int_{2^{k-1}}^{2^k}\int_0^1\Big(a\left(t,\tau\right) - S_{k,0}^q\left(a;t,\tau\right)\Big)^2\,d\tau\,dt \\ &+\sum_{l=1}^{m}\int_0^1\int_{2^{l-1}}^{2^l}\Big(a\left(t,\tau\right) - S_{0,l}^q\left(a;t,\tau\right)\Big)^2\,d\tau\,dt \\ &+\int_0^1\int_0^1\left(a\left(t,\tau\right) - S_{0,0}^q\left(a;t,\tau\right)\right)^2\,d\tau\,dt \\ \le\ & c2^{-2q}\left\{\sum_{k=1}^{n}\sum_{l=1}^{m}\Big(2^{-(2\omega-1)l}+2^{-(2\alpha-1)k}+2^{-2q}\Big)+n+m\right\} \\ \le\ & c2^{-2q}\left(n+m+2^{-2q}mn\right).\end{aligned}$$

On the other hand, under the condition (3.6) $n \asymp m \asymp \log(1/\delta)$, $2^{-2q} \asymp \delta^2\log^{-1}(1/\delta)$, and summing up we get the assertion of the theorem. □

**4. The efficiency of discretization schemes.** The straightforward approach to the discretization of the integral operator $P_NAP_M$ from the finite-section equation (2.1) consists in using the information $a\big((iN/n),(jM/m)\big)$, $i = 0,\dots,n$, $j = 0,\dots,m$. In this case one usually takes as $a^{\rm disc}(t,\tau)$ the spline or polynomial interpolating $a(t,\tau)$ at the equidistant points $\big((iN/n),(jM/m)\big)$. Under the assumption that $a^{(k,l)} \in L_2\left([0,N)\times[0,M)\right)$, $k,l = 0,1$, the standard estimation yields

$$\left\|A - A^{\rm disc}\right\| \le \left\|a - a^{\rm disc}\right\| \le c\bigg(\frac{N}{n}+\frac{M}{m}+\frac{NM}{nm}\bigg).$$

On the other hand, Theorem 2.1 gives $N \asymp \delta^{-1/\mu}$, $M \asymp^{-1/\lambda}$, and to meet the condition (2.3) with such $A^{\text{disc}}$ one should take $n \asymp N\delta^{-1} \asymp \delta^{-1-(1/\mu)}$, $m \asymp M\delta^{-1} \asymp \delta^{-1-\frac{1}{\lambda}}$.

Then the amount Card $\big\{a\big((iN/n),(jM/m)\big)\big\}$ of the used discrete information is estimated as

$$\text{Card}\left\{a\left(\frac{iN}{n},\frac{jM}{m}\right)\right\} = (n+1)\,(m+1) \asymp \delta^{-2-\frac{1}{\lambda}-\frac{1}{\mu}}. \tag{4.1}$$

Let us compare it with the amount of information used within the framework of the scheme from the previous section. If $\theta_1, \theta_2 < 1$ and $m, n, q$ are chosen as in (3.6) then

(4.2)

$$\begin{aligned}\text{Card}\left\{a\left(t^{\theta_1}_{k,q,i}, t^{\theta_2}_{l,q,j}\right)\right\} &= \sum_{k=1}^{n}\sum_{l=1}^{m} M_{k,\theta_1}M_{l,\theta_2} + M_{0,\theta_1}\sum_{l=1}^{m} M_{l,\theta_2}\\ &\quad + M_{0,\theta_2}\sum_{k=1}^{n} M_{k,\theta_1} + M_{0,\theta_1}M_{0,\theta_2}\\ &\asymp 2^{2q}\left(2^{n(1-\theta_1)}2^{m(1-\theta_2)} + 2^{n(1-\theta_1)} + 2^{m(1-\theta_2)}\right)\\ &\asymp \delta^{-2-\frac{1-\theta_1}{\mu}-\frac{1-\theta_2}{\lambda}}\log\frac{1}{\delta}.\end{aligned}$$

Moreover, it is easy to check that

$$\text{Card}\left\{a\left(t^{\theta_1}_{k,q,i}, t^{\theta_2}_{l,q,j}\right)\right\} \asymp \begin{cases}\delta^{-2}\log^3\frac{1}{\delta}, & \theta_1=\theta_2=1,\\ \delta^{-2-\frac{1-\theta_1}{\mu}}\log^2\frac{1}{\delta}, & \theta_1<1,\ \theta_2=1,\\ \delta^{-2-\frac{1-\theta_2}{\lambda}}\log^2\frac{1}{\delta}, & \theta_1=1,\ \theta_2<1.\end{cases} \tag{4.3}$$

Comparing (4.2) and (4.3) with (4.1), one can see the advantage of the modified scheme [3] over the straightforward approach to the discretization of the finite-section integral operator. We will show that for the case $\theta_1, \theta_2 < 1$ this advantage can be even reinforced.

Instead of discretizing the kernel of the finite section integral operator $P_{2^n}AP_{2^m}$ on the rectangle $[0,2^n)\times[0,2^m)$ we will do it only on the following part of this rectangle

$$\Gamma_{m,n} = [0,1)\times[0,2^m)\bigcup_{k=1}^{n}[2^{k-1},2^k)\times[0,2^{m-k}).$$

For the definiteness we assume here that $\theta_2 < \theta_1 < 1$. The kernel of the discretized integral operator $A^{\mathrm{disc}}$ now has the form

$$
\begin{aligned}
a^{\mathrm{disc}}(t,\tau) = \sum_{k=1}^{n}\sum_{l=1}^{m-k} S^q_{k,l}(a;t,\tau) + \sum_{k=1}^{n} S^q_{k,0}(a;t,\tau) \\
+ \sum_{l=1}^{m} S^q_{0,l}(a;t,\tau) + S^q_{0,0}(a;t,\tau),
\end{aligned}
\tag{4.4}
$$

and the amount of used discrete information can be estimated as

$$
\begin{aligned}
\mathrm{Card}\,(\mathrm{Inf}) &= \sum_{k=1}^{n}\sum_{l=1}^{m-k} M_{k,\theta_1} M_{l,\theta_2} + M_{0,\theta_1}\sum_{l=1}^{m} M_{l,\theta_2} + M_{0,\theta_2}\sum_{k=1}^{n} M_{k,\theta_1} \\
&\quad + M_{0,\theta_1} M_{0,\theta_2} \\
&\asymp 2^{2q}\left(2^{m(1-\theta_2)}\sum_{k=1}^{n} 2^{-k(\theta_2-\theta_1)} + 2^{n(1-\theta_1)} + 2^{m(1-\theta_2)}\right) \\
&\asymp \delta^{-2}\log\frac{1}{\delta}\left(\delta^{-\frac{1-\theta_1}{\mu}} + \delta^{-\frac{1-\theta_1}{\lambda}}\right)
\end{aligned}
$$

it is easy to see that for $\theta_1 = \theta_2 < 1$

$$
\mathrm{Card}\,(\mathrm{Inf}) \asymp \delta^{-2}\log\frac{1}{\delta}\left(\delta^{-\frac{1-\theta_1}{\mu}} + \delta^{-\frac{1-\theta_1}{\lambda}}\log\frac{1}{\delta}\right).
$$

Comparing these estimates with (4.2) we can see the advantage of the discretization scheme determined by (4.4). Note also that if at least one of $\theta_1, \theta_2$ equals 1, then the estimate for Card (Inf) is better than (4.3), by only the factor $\log(1/\delta)$.

Now the only remaining item is to check condition (2.3).

**Theorem 4.1.** *Under the conditions of Theorem* 3.1 *the discretized integral operator* $A^{\mathrm{disc}}$ *with the kernel* (4.4) *satisfies condition* (2.3).

*Proof.* For the definiteness we assume that in (3.6) $m > n$, i.e., $\mu > \lambda$. Consider the operator

$$
A_{m,n} := \sum_{k=1}^{n} \left(P_{2^k} - P_{2^{k-1}}\right) A P_{2^{m-k}} + P_1 A P_{2^m}.
$$

Using Lemma 2.1 and (3.6) we have

$$\begin{aligned}\|A - A_{m,n}\| &\le \|A - P_{2^n}A\| + \|P_{2^n}A - A_{m,n}\| \\ &\le c2^{-n\mu} + \sum_{k=1}^{n} \|(P_{2^k} - P_{2^{k-1}})\, A\,(I - P_{2^{m-k}})\| \\ &\quad + \|P_1 A\,(I - P_{2^m})\| \\ &\le c\bigg(\delta + 2^{-m\lambda}\sum_{k=1}^{n} 2^{-k(\mu-\lambda)}\bigg) \le c\delta.\end{aligned}$$

On the other hand, $A_{m,n}$ can be represented in the form of the integral operator

$$A_{m,n}z(t) = \int_0^{2^m} a_{m,n}(t,\tau)z(\tau)\,d\tau$$

with the kernel

$$a_{m,n}(t,\tau) = \begin{cases} a(t,\tau), & (t,\tau)\in\Gamma_{m,n}, \\ 0, & (t,\tau)\notin\Gamma_{m,n}. \end{cases}$$

Then, as in the proof of Theorem 3.1,

$$\begin{aligned}\left\|A_{m,n} - A^{\mathrm{disc}}\right\| &\le \left\|a_{m,n} - a^{\mathrm{disc}}\right\|_{L_2(\Gamma_{m,n})} \\ &= \left\|a - a^{\mathrm{disc}}\right\|_{L_2(\Gamma_{m,n})} \le c\delta.\end{aligned}$$

Summing up, we get the assertion of the theorem. □

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