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ON THE NON-EXPONENTIAL CONVERGENCE OF ASYMPTOTICALLY STABLE SOLUTIONS OF LINEAR SCALAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

JOHN A.D. APPLEBY AND DAVID W. REYNOLDS

ABSTRACT. We study the stability of the scalar linear Volterra equation

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,ds, \quad x(0) = x_0$$

under the assumption that all solutions satisfy $x(t) \to 0$ as $t \to \infty$. It is shown that if k is a continuously differentiable, positive, integrable function which is subexponential in the sense that $k'(t)/k(t) \to 0$ as $t \to \infty$, then $x(t)$ cannot converge to 0 as $t \to \infty$ faster than $k(t)$.

1. Introduction. In this note we consider the asymptotic stability of the scalar linear Volterra integro-differential equation

$$x'(t) = -ax(t) + \int_0^t k(t-s)x(s)\,ds, \quad t > 0, \tag{1}$$

$$x(0) = x_0. \tag{2}$$

In [**6**] Lakshmikantham and Corduneanu asked if all solutions of (1) satisfy $x(t) \to 0$ as $t \to \infty$, whether that convergence is exponentially fast. The question was natural in view of the fact that asymptotic stability of the zero solution of equations with bounded delay implies exponential asymptotic stability of the zero solution. In [**9**] Murakami showed that exponential asymptotic stability does not automatically follow from the property of (uniform) asymptotic stability of the zero solution. His result assumes that $k \in L^1(0,\infty) \cap C[0,\infty)$ and is of

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one sign, and that the zero solution of (1) is uniformly asymptotically stable. It asserts that the zero solution of (1) is exponentially asymptotically stable if and only if k is exponentially integrable. We term a function $k \in L^1(0,\infty)$ exponentially integrable if $\int_0^\infty |k(s)|e^{\gamma s}\,ds < \infty$ for some $\gamma > 0$. The natural question to ask is at what rate does $x(t) \to 0$ as $t \to \infty$ if k is not exponentially integrable.

If $x(t) \to 0$ as $t \to \infty$, we establish here using elementary analysis a positive lower bound for $\liminf_{t\to\infty} x(t)/k(t)$ for a class of kernels which cannot be exponentially integrable.

2. Technical discussion and results. A clue to the asymptotic behavior of $x(t)/k(t)$ as $t \to \infty$ is provided by a result of Burton [**4**, Theorem 1.3.7]. Our main result is analogous to it.

Theorem 1. *Let $k(t) \geq 0$ and $k \in L^1(0,\infty)$. Suppose that x is an integrable solution of* (1). *Then there is a constant $\beta > 0$ such that*

$$\int_t^\infty |x(s)|\,ds \geq \beta|x_0| \int_t^\infty k(s)\,ds, \quad t \geq 0. \tag{3}$$

Moreover if $\int_t^\infty k(s)\,ds > 0$ for large t,

$$\liminf_{t\to\infty} \frac{\int_t^\infty |x(s)|\,ds}{\int_t^\infty k(s)\,ds} \geq \frac{|x_0|}{a\big(a - \int_0^\infty k(s)\,ds\big)}. \tag{4}$$

Proof. Suppose that $x_0 > 0$. In Burton [**4**], an estimate of the form (3) is established on $t \geq 1$ from the inequality

$$a\int_t^\infty x(s)\,ds \geq \int_0^t x(s)\,ds \int_t^\infty k(s)\,ds, \quad t \geq 0. \tag{5}$$

An estimate of the form (3) is also true for $0 \leq t \leq 1$. It follows from (5) under the additional hypothesis of the theorem that

$$\liminf_{t\to\infty} \frac{\int_t^\infty x(s)\,ds}{\int_t^\infty k(s)\,ds} \geq \frac{1}{a}\int_0^\infty x(s)\,ds.$$

Since $x \in L^1(0,\infty)$ and is a solution of (1), $x(t) \to 0$ as $t \to \infty$. Thus integration of (1) shows that

$$x_0 = \left(\int_0^\infty x(s)\, ds \right) \left(a - \int_0^\infty k(s)\, ds \right),$$

from which (4) follows. The argument is similar if $x_0 < 0$. $\square$

It is shown in Appleby and Reynolds [**1**, Theorem 6.5] under additional hypothesis on k, which do not allow it be exponentially integrable, that

$$\lim_{t\to\infty} \frac{\int_t^\infty |x(s)|\, ds}{\int_t^\infty k(s)\, ds} = \frac{x_0}{\left(a - \int_0^\infty k(s)\, ds\right)^2},$$

if $a > \int_0^\infty k(s)\, ds$. Thus the exact value of the righthand side of (4) is known in that case.

Grossman and Miller [**7**] proved using Laplace transforms, a well-known result providing a necessary and sufficient condition for the solution of (1) to be integrable. It is that $x \in L^1(0,\infty)$ if and only if the equation $p + a - \bar{k}(p) = 0$ has no solutions with $\mathbf{Re}\, p \ge 0$, where $\bar{k}$ is the Laplace transform of k. Under the hypotheses on k in Theorem 1, this simplifies to $x \in L^1(0,\infty)$ if and only if $a > \int_0^\infty k(s)\, ds$.

Staffans [**10**] considers linear Volterra integro-differential equations for which the ordinary differential part of the equation dominates. There the dominant instantaneous term is used to construct Lyapunov functions with which results on the boundedness and asymptotic behavior of solutions are obtained. Also in [**10**] Laplace transform and Lyapunov function methods are compared for investigating the asymptotic behavior of convolution equations. It is remarked that for this class of equations the results for uniform stability or uniform asymptotic stability obtained using transform methods generally supersede those obtained with Lyapunov theory. In the present work, the convolution equation (1) is studied under the assumption that $a \ge \int_0^\infty k(s)\, ds$, so the ordinary differential part dominates.

The scalar Volterra integro-differential equation (1) has been extensively studied under the hypotheses that the kernel k satisfies

$$k(t) \ge 0, \quad k \in L^1(0,\infty), \quad k \in C[0,\infty). \tag{6}$$

A rough classification of the stability of the zero solution of (1) can be found using the values of a and $\int_0^\infty k(s)\,ds$. Brauer [**2**] showed that the solution could not be stable if $a < \int_0^\infty k(s)\,ds$, and therefore cannot be asymptotically stable. A slight modification of the argument in Kordonis and Philos [**8**] shows that the zero solution is stable if $a = \int_0^\infty k(s)\,ds$. This is a sharpening of Burton [**3**, Theorem 5.2.3]. In the other case $a > \int_0^\infty k(s)\,ds$, the zero solution is asymptotically stable; moreover, every solution tends to zero. This is proved in Burton and Mahfoud [**5**, Theorem 1], which contains an insightful discussion of other important papers. Therefore a necessary condition for $\lim_{t\to\infty} x(t) = 0$ for all solutions of (1) is that

$$a \ge \int_0^\infty k(s)\,ds, \tag{7}$$

a simple consequence of which is that $a > 0$.

We consider the condition

$$k \in C^1[0,\infty), \quad k(t) > 0 \quad \text{for } t \ge 0, \quad \lim_{t\to\infty} \frac{k'(t)}{k(t)} = 0. \tag{8}$$

An example is $k(t) = (1+t)^{-\alpha}$ for $\alpha > 1$. It is straightforward to infer from (8) that

$$\lim_{t\to\infty} k(t)e^{\varepsilon t} = \infty, \tag{9}$$

for every $\varepsilon > 0$. Therefore, k cannot be exponentially integrable. We now state our main result.

Theorem 2. *Suppose that k satisfies* (6) *and* (8). *Suppose that x is a solution of* (1) *satisfying $x(t) \to 0$ as $t \to \infty$. Then there is a constant $\alpha > 0$ such that*

$$|x(t)| \ge \alpha |x_0| k(t), \quad t \ge 0. \tag{10}$$

Moreover if $x_0 \ne 0$

$$\liminf_{t\to\infty} \frac{|x(t)|}{k(t)} \ge \frac{|x_0|}{a\big(a - \int_0^\infty k(s)\,ds\big)}, \tag{11}$$

where the righthand side is interpreted as ∞ *if* $a = \int_0^\infty k(s)\,ds$.

We have preferred to posit $x(t) \to 0$ as $t \to \infty$ as a hypothesis of this theorem. However, we only use it to infer that $a \geq \int_0^\infty k(s)\,ds$; the theorem and our proof remain valid if this condition replaces $x(t) \to 0$ as $t \to \infty$.

It is shown in Appleby and Reynolds [**1**, Theorem 6.2] that

$$\lim_{t\to\infty} \frac{x(t)}{k(t)} = \frac{x_0}{\left(a - \int_0^\infty k(s)\,ds\right)^2},$$

if $a > \int_0^\infty k(s)\,ds$, and in addition to (6) k satisfies

$$\text{(12)} \qquad \lim_{t\to\infty} \frac{\int_0^t k(t-s)k(s)\,ds}{k(t)} = 2\int_0^\infty k(s)\,ds, \quad \lim_{t\to\infty} \frac{k(t-s)}{k(t)} = 1$$

for each fixed $s > 0$. Thus the exact value of the lefthand side of (11) is known in that case. The relationship between these hypotheses and the present paper is that (8) implies the second condition in (12), which in turn implies (9).

It is easy to see that, under the hypotheses of Theorem 2, solutions of (1) can never decay exponentially.

Corollary 3. *Suppose that* k *satisfies* (6) *and* (8). *If* $x_0 \neq 0$, *then for every* $\varepsilon > 0$,

$$\text{(13)} \qquad \liminf_{t\to\infty} |x(t)|e^{\varepsilon t} = \infty.$$

To see that this limiting value is unbounded, note that (8) implies (9), and so

$$\liminf_{t\to\infty} |x(t)|e^{\varepsilon t} = \liminf_{t\to\infty} \frac{|x(t)|}{k(t)} \cdot k(t)e^{\varepsilon t} \geq \liminf_{t\to\infty} \frac{|x(t)|}{k(t)} \cdot \liminf_{t\to\infty} k(t)e^{\varepsilon t} = \infty,$$

for every $\varepsilon > 0$, provided $x_0 \neq 0$. Incidentally, this furnishes a slight improvement to a straightforward application of Murakami [**9**, Theorem 3], which yields $\limsup_{t\to\infty} |x(t)|e^{\varepsilon t} = \infty$ for every $\varepsilon > 0$.

3. Proof of Theorem 2. Suppose that the hypotheses of Theorem 2 hold. To investigate (1), we first represent the solution in terms of the resolvent of a linear convolution Volterra equation. In fact if h is given by

$$h(t) = \int_0^t e^{-a(t-s)} k(s)\, ds, \quad t \geq 0, \tag{14}$$

and r is the unique continuous solution of

$$r(t) = h(t) + \int_0^t h(t-s) r(s)\, ds, \quad t \geq 0, \tag{15}$$

then the unique continuous solution of the initial-value problem (1) and (2) is given by

$$x(t) = e^{-at}\left(1 + \int_0^t e^{as} r(s)\, ds\right) x_0, \quad t \geq 0. \tag{16}$$

To demonstrate this assertion it is sufficient to verify that (16) provides a solution of (1). By substituting (14) into (15) and using Fubini's theorem, it is seen that

$$\begin{aligned}
r(t)x_0 &= \left(h(t) + \int_0^t h(t-s) r(s)\, ds\right) x_0 \\
&= \left(\int_0^t k(t-s) e^{-as}\, ds + \int_0^t \int_s^t k(t-u) e^{-a(u-s)} r(s)\, du\, ds\right) x_0 \\
&= \left(\int_0^t k(t-s) e^{-as}\, ds + \int_0^t k(t-u) \int_0^u e^{-a(u-s)} r(s)\, ds\, du\right) x_0 \\
&= \int_0^t k(t-s) e^{-as} \left(1 + \int_0^s e^{au} r(u)\, du\right) x_0\, ds \\
&= \int_0^t k(t-s) x(s)\, ds.
\end{aligned}$$

It follows from this and $x'(t) = -ax(t) + r(t)x_0$ that x given by (16) is a solution of (1).

We establish now some of the properties of the function r occurring in the representation (16). As a preliminary step we note that the function h defined in (14) is a continuously differentiable, integrable function with $h(0) = 0$ and $h(t) > 0$ for all $t > 0$. It has already been observed that $x(t) \to 0$ as $t \to \infty$ implies (7). From this and Fubini's Theorem, we find that

$$\mu := \int_0^\infty h(s)\,ds = \frac{1}{a}\int_0^\infty k(s)\,ds \le 1. \tag{17}$$

Due to the dominated convergence theorem, $h(t) \to 0$ as $t \to \infty$. By L'Hôpital's rule and (8),

$$\begin{aligned}\lim_{t\to\infty}\frac{h(t)}{k(t)} &= \lim_{t\to\infty}\frac{\int_0^t e^{as}k(s)\,ds}{e^{at}k(t)} = \lim_{t\to\infty}\frac{e^{at}k(t)}{e^{at}k'(t)+ae^{at}k(t)}\\ &= \lim_{t\to\infty}\frac{1}{[k'(t)/k(t)]+a} = \frac{1}{a}.\end{aligned} \tag{18}$$

This also implies that

$$\frac{h'(t)}{h(t)} = \frac{-ah(t)+k(t)}{h(t)} = -a + \frac{k(t)}{h(t)} \to 0 \quad \text{as } t \to \infty. \tag{19}$$

It is a standard result that (15) has a unique continuous solution r with $r(t) > 0$ for all $t > 0$. It is convenient to record here an estimate detailing the asymptotic behavior of $r(t)/h(t)$ as $t \to \infty$:

$$\liminf_{t\to\infty}\frac{r(t)}{h(t)} \ge \frac{1}{1-\mu}, \tag{20}$$

where the righthand side is interpreted as ∞ if $\mu = 1$. The proof is postponed to the end.

To complete the proof note that since $r(t) > 0$ for all $t > 0$, (16) implies that

$$\frac{|x(t)|}{|x_0|} = e^{-at}\left(1 + \int_0^t e^{as}r(s)\,ds\right).$$

Using (6) we get

$$\frac{|x(t)|}{|x_0|k(t)} = \frac{1}{k(t)e^{at}} + \frac{\int_0^t e^{as}r(s)\,ds}{e^{at}k(t)} \ge \frac{\int_0^t e^{as}r(s)\,ds}{e^{at}k(t)}. \tag{21}$$

Suppose that $\mu < 1$. Let $0 < \varepsilon < 1$. Equation (20) implies that there exists $T > 0$ such that

$$r(t) > \frac{(1-\varepsilon)h(t)}{1-\mu}, \quad t > T.$$

Therefore for $t > T$,

$$\frac{\int_0^t e^{as} r(s)\, ds}{e^{at} k(t)} \geq \frac{\int_T^t e^{as} r(s)\, ds}{e^{at} k(t)} > \left(\frac{1-\varepsilon}{1-\mu}\right) \frac{\int_T^t e^{as} h(s)\, ds}{e^{at} k(t)}.$$

It follows from L'Hôpital's rule, (8) and (18)s that

$$\begin{aligned} \lim_{t\to\infty} \frac{\int_T^t e^{as} h(s)\, ds}{e^{at} k(t)} &= \lim_{t\to\infty} \frac{e^{at} h(t)}{a e^{at} k(t) + e^{at} k'(t)} \\ &= \lim_{t\to\infty} \frac{h(t)/k(t)}{a + (k'(t)/k(t))} = \frac{1}{a^2}. \end{aligned}$$

Hence

$$\liminf_{t\to\infty} \frac{\int_0^t e^{as} r(s)\, ds}{e^{at} k(t)} \geq \frac{1-\varepsilon}{a^2(1-\mu)} = \frac{1-\varepsilon}{a\big(a - \int_0^\infty k(s)\, ds\big)}.$$

Therefore by taking the lim inf of both sides of (21) as $t \to \infty$ and letting $\varepsilon \to 0$, we obtain (11) in the case $\mu < 1$. The result for $\mu = 1$ is similar. Notice that (10) follows from (21) and (11).

Equation (20) remains to be established. Since $h(t) > 0$ for $t > 0$, it can be inferred from the Neumann series representation of the resolvent r that $r(t) \uparrow \sum_{j=1}^N h^{*j}(t)$ as $N \uparrow \infty$, where h^{*j} is the j-fold convolution of h on $[0, \infty)$. Hence for $N \geq 1$

$$\liminf_{t\to\infty} \frac{r(t)}{h(t)} \geq \sum_{j=1}^N \liminf_{t\to\infty} \frac{h^{*j}(t)}{h(t)}. \tag{22}$$

If we can prove that

$$\liminf_{t\to\infty} \frac{h^{*j}(t)}{h(t)} \geq \mu^{j-1}, \quad j \geq 1, \tag{23}$$

it follows that

$$\liminf_{t\to\infty} \frac{r(t)}{h(t)} \geq \sum_{j=1}^{N} \mu^{j-1}.$$

Letting $N \to \infty$, we obtain (20).

The proof of (23) begins by observing that for any $0 < T \leq t$

$$\begin{aligned}
\frac{h^{*j}(t)}{h(t)} &= \int_0^T \frac{h(t-s)}{h(t)} h^{*(j-1)}(s)\,ds + \int_T^t \frac{h(t-s)}{h(t)} h^{*(j-1)}(s)\,ds \\
&\geq \int_0^T \frac{h(t-s)}{h(t)} h^{*(j-1)}(s)\,ds \\
&= \int_0^T \left(\frac{h(t-s)}{h(t)} - 1\right) h^{*(j-1)}(s)\,ds + \int_0^T h^{*(j-1)}(s)\,ds.
\end{aligned}$$

Thus, since $\int_0^\infty h^{*(j-1)}(s)\,ds = \mu^{j-1}$,

(24)
$$\frac{h^{*j}(t)}{h(t)} - \mu^{j-1} \geq \int_0^T \left(\frac{h(t-s)}{h(t)} - 1\right) h^{*(j-1)}(s)\,ds - \int_T^\infty h^{*(j-1)}(s)\,ds.$$

T can be chosen large enough for the last integral to be arbitrarily small. Hence to establish (23) it is enough to prove that the first term tends to 0 as $t \to \infty$. Due to (19), $h'(t) = p(t)h(t)$ for a continuous function p with $\lim_{t\to\infty} p(t) = 0$. Let $s > 0$ be fixed. Then $\int_{t-s}^t p(u)\,du \to 0$ as $t \to \infty$. Since $h(t) = h(1)\exp(\int_1^t p(u)\,du)$,

$$\frac{h(t-s)}{h(t)} = \exp\left(-\int_{t-s}^t p(u)\,du\right) \longrightarrow 1 \quad \text{as } t \to \infty.$$

This implies that

$$\begin{aligned}
\left|\int_0^T \left(\frac{h(t-s)}{h(t)} - 1\right) h^{*(j-1)}(s)ds\right| &\leq \sup_{0\leq s\leq T} \left|\frac{h(t-s)}{h(t)} - 1\right| \int_0^\infty h^{*(j-1)}(s)ds \\
&\to 0 \text{ as } t \to \infty.
\end{aligned}$$

This finishes the proof of Theorem 2.

REFERENCES

1. J.A.D. Appleby and D.W. Reynolds, *Subexponential solutions of linear Volterra integro-differential equations and transient renewal equations*, Proc. Roy. Soc. Edinburgh **132A** (2002), 521–543.

2. F. Brauer, *Asymptotic stability of a class of integro-differential equations*, J. Differential Equations **28** (1978), 180–188.

3. T.A. Burton, *Volterra integral and differential equations*, Mathematics in Science and Engineering, Academic Press, 1983.

4. ———, *Stability and periodic solutions of functional differential equations*, Mathematics in Science and Engineering, Academic Press, Orlando, Florida, 1985.

5. T.A. Burton and W.E. Mahfoud, *Stability criterion for Volterra equations*, Trans. Amer. Math. Soc. **279** (1983), 143–174.

6. C. Corduneanu and V. Lakshmikantham, *Equations with unbounded delay: A survey*, Nonlinear Anal. **4** (1980), 831–877.

7. S.I. Grossman and R.K. Miller, *Nonlinear Volterra integrodifferential equations with L^1-kernels*, J. Differential Equations **13** (1973), 551–566.

8. I.-G.E. Kordonis and Ch.G. Philos, *The behavior of solutions of linear integro-differential equations with unbounded delay*, Comput. Math. Appl. **38** (1999), 45–50.

9. S. Murakami, *Exponential asymptotic stability of scalar linear Volterra equations*, Differential Integral Equations **4** (1991), 519–525.

10. Olof J. Staffans, *A direct Lyapunov approach to Volterra integro-differential equations*, SIAM J. Math. Anal. **19** (1988), 879–901.

School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland
Email address: john.appleby@dcu.ie

School of Mathematical Sciences, Dublin City University, Dublin 9, Ireland
Email address: david.reynolds@dcu.ie