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UPPER AND LOWER BOUNDS FOR SOLUTIONS OF NONLINEAR VOLTERRA CONVOLUTION INTEGRAL EQUATIONS WITH POWER NONLINEARITY

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ABSTRACT. The Volterra nonlinear integral equation

$$\varphi^m(x) = a(x)\int_0^x k(x-t)b(t)\varphi(t)\,dt + f(x),$$
$$0 < x < d \leq \infty$$

with $m > 1$ and real nonnegative functions $a(x)$, $k(u)$, $b(t)$ and $f(x)$ is studied. In the general case some upper bounds of the average

$$\frac{1}{x}\int_0^x \varphi(t)\,dt$$

of the solution are given. In the case when $a(x)$, $k(u)$, $b(t)$ and $f(x)$ have power lower estimates near the origin, lower power type bounds for solutions $\varphi(x)$ are investigated. Conditions for the uniqueness of the solution in a weighted space of continuous functions are also proved. Particular cases of the equation are specially considered.

1. Introduction. We consider the Volterra nonlinear integral equation of the form

$$\varphi^m(x) = a(x)\int_0^x k(x-t)b(t)\varphi(t)\,dt + f(x), \qquad 0 < x < d \leq \infty \tag{1.1}$$

with $m > 0$ and real-valued functions $a(x)$, $k(u)$, $b(t)$ and $f(x)$. This equation generalizes equations investigated by many authors. The equation

$$\varphi^m(x) = \int_0^x k(x-t)\varphi(t)\,dt + f(x), \quad 0 < x < d \leq \infty \tag{1.2}$$

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arising in various applications, e.g., in water percolation [**9**], [**27**], [**28**] and in the nonlinear theory of wave propagation [**15**], was studied in [**1**], [**4**], [**6**], [**11**], [**26**], [**27**], [**28**], while the more general equation

$$\varphi^m(x) = a(x) \int_0^x k(x-t)\varphi(t)\,dt + f(x), \quad 0 < x < d \le \infty \tag{1.3}$$

was studied in [**2**], [**3**], [**5**], [**7**]. When $m > 1$ the equations (1.2) and (1.3) with $f(x) = 0$ may have a nontrivial solution $\varphi(x)$ (see, for example, [**26**], [**30**]). All the papers above were devoted to investigation of problems concerning in main the existence and uniqueness of a solution $\varphi(x)$ for equations (1.2) and (1.3) with $m > 1$, in some spaces of continuous or integrable functions. The equation (1.2) with $0 < m < 1$ and a continuous kernel $k(u)$ was considered in [**1**], [**4**], where some results were given on the uniqueness of its solution $\varphi(x)$ in some spaces of continuous or integrable functions. Such a problem for equation (1.2) with $m < 0$ and nonincreasing kernel $k(u)$ in the class of almost decreasing functions was studied in [**14**]. Lower estimates and asymptotic properties near zero for the solution $\varphi(x)$ of the equation (1.3) with $m > 1$ were obtained in [**12**] provided that $a(x)$, $k(u)$ and $f(x)$ have power asymptotic behavior near zero.

The existence of the solution for the equation

$$\varphi^m(x) = a(x) \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt + f(x), \quad 0 < x < d \le \infty \tag{1.4}$$

with $m > 0$ and with the weakly singular kernel $k(u) = u^{\alpha-1}$, $0 < \alpha < 1$, was investigated in spaces of locally integrable and continuous functions in [**16**], [**17**], [**21**]. Asymptotic properties at zero of the solution $\varphi(x)$ for the equation (1.4) with $m \in \mathbf{R} = (-\infty, \infty)$, $m \ne 0, -1, -2, \dots$, in the case when $a(x)$ and $f(x)$ have special asymptotics at zero were studied in [**18**], [**19**], [**20**], [**22**], [**29**]. Special cases of (1.4), when its solution $\varphi(x)$ can be found in closed form, were investigated in [**12**], [**20**], [**21**], [**22**] and [**30**].

The equation of the form

$$\varphi^m(x) = a(x) \int_0^x b(t)\varphi(t)\,dt + f(x), \quad 0 < x < d \le \infty, \tag{1.5}$$

with real m was considered in [**24**] where existence and uniqueness results were discussed and special cases of solution in closed form were treated.

The main results in this paper are estimation of lower bounds of solutions given in Theorem 2.1, of upper bounds for averages of solutions given in Theorem 4.1 and uniqueness theorems given in Theorems 5.1 and 5.2. These results are obtained in the case of $m > 1$ and real nonnegative functions $a(x)$, $k(u)$, $b(t)$ and $f(x)$. In Section 6 the case when $a(x)k(x-t)b(t) \equiv ax^{-\alpha}(x-t)^{\alpha-1}$ is specially treated (see Theorem 6.1).

2. Lower estimates for a solution of integral equation (1.1). Let $C(0,d)$, $0 < d \leq \infty$, be the space of real valued continuous functions on $(0,d)$, and let $L_1^{\mathrm{loc}}(0,d)$ be the space of all Lebesgue measurable functions which are in $L_1(0,d_0)$ for all d_0 such that $0 < d_0 < d$. We denote by $CL_{\mathrm{loc}}(0,d)$ the intersection of $C(0,d)$ and $L_1^{\mathrm{loc}}(0,d)$:

$$CL_{\mathrm{loc}}(0,d) = C(0,d) \cap L_1^{\mathrm{loc}}(0,d), \tag{2.1}$$

so that a function in $CL_{\mathrm{loc}}(0,d)$ may have singularities only at the end points of $(0,d)$. By $CL_{\mathrm{loc}}^{+}(0,d)$, we denote the nonnegative functions in $CL_{\mathrm{loc}}(0,d)$.

Let $C[0,d)$ be the space of continuous functions on $[0,d)$. Let $b(x)$ be a nonnegative function on $[0,d)$. We denote by $C([0,d),b)$ the space of functions $g(x)$ such that $b(x)g(x) \in C[0,d)$, and by $C^{+}[0,d)$ and $C^{+}([0,d),b)$ subclasses of $C[0,d)$ and $C([0,d),b)$ composed of nonnegative functions, respectively. Similarly, $CL_{\mathrm{loc}}((0,d),b)$ and $CL_{\mathrm{loc}}^{+}((0,d),b)$ are subclasses of functions $g(x)$ such that $b(x)g(x) \in CL_{\mathrm{loc}}(0,d)$ and $b(x)g(x) \in CL_{\mathrm{loc}}^{+}(0,d)$, respectively.

Obviously,

$$C([0,d),b) \subset CL_{\mathrm{loc}}((0,d),b), \quad C^{+}([0,d),b) \subset CL_{\mathrm{loc}}^{+}((0,d),b). \tag{2.2}$$

Remark 2.1. It is clear that equation (1.1), in case $b(x) \not\equiv 0$ in a neighborhood of the origin, may be reduced to the equation with only one coefficient:

$$\varphi_1^m(x) = a_1(x)\int_0^x k(x-t)\varphi_1(t)\,dt + f_1(x), \quad 0 < x < d \leq \infty, \tag{2.3}$$

where $\varphi_1(x) = b(x)\varphi(x)$, $a_1(x) = [b(x)]^m a(x)$ and $f_1(x) = [b(x)]^m f(x)$, for which a solution $\varphi_1(x)$ is sought in the space $CL^+_{\mathrm{loc}}(0, d)$. Though equation (2.3) was investigated in [**12**], we find it is more convenient to consider the equations just in the form (1.1).

In this section we give an a priori lower estimate for a nonnegative solution $\varphi(x)$ of the equation (1.1) with $m > 1$ under the assumption that the functions $a(x), k(x), b(x), f(x) \in CL^+_{\mathrm{loc}}(0, d)$ have lower power bounds (see Theorem 2.1 below). We assume that solutions of equation (1.1) are in the space $CL^+_{\mathrm{loc}}([0, d), b)$. Since $CL^+_{\mathrm{loc}}([0, \infty))$ is a ring with respect to the Volterra convolution (see [**12**, Theorem 1]), the integral term in (1.1) is also a locally integrable function. Therefore, equation (1.1) is well posed in this class.

We note that the condition

$$f^{1/m}(x) \in CL^+_{\mathrm{loc}}([0, d), b) \tag{2.4}$$

is necessary for solvability of equation (1.1) in $CL^+_{\mathrm{loc}}([0, d), b)$. This fact is a consequence of the evident inequality $f(x) \le \varphi^m(x)$ and the assumptions on the solution $\varphi(x)$. We also suppose that $f(x)$ satisfies the condition

$$f(x) \not\equiv 0, \quad x \in (0, \varepsilon) \quad \text{for any} \quad \varepsilon > 0. \tag{2.5}$$

Otherwise, if $f(x) \equiv 0$, $x \in (0, \varepsilon_0)$, for some $\varepsilon_0 < d$ and $f(x) \not\equiv 0$, $x \in (\varepsilon_0, \varepsilon_0 + \varepsilon)$, for any $\varepsilon > 0$, we may pass to the function

$$\psi(x) = \begin{cases} 0 & 0 < x < \varepsilon_0, \\ \varphi(x + \varepsilon_0) & \varepsilon_0 < x < d - \varepsilon_0, \end{cases} \tag{2.6}$$

and obtain the same equation (1.1) with respect to $\psi(x)$ for which the condition (2.5) is satisfied. In the case when $f(x) \equiv 0$, $x \in (0, d)$, equation (1.1) may also be investigated, but in this case we need some special additional assumptions on the functions $a(x), k(x), b(x)$. We consider this case specially in Section 3 while in this section we only suppose that $f(x)$ satisfies condition (2.5).

a) *The case when $b(x)$ has a power lower estimate.*

Theorem 2.1. *Let $m, \alpha, \mu, \nu \in \mathbf{R}$ be such that*

$$m > 1, \quad 0 < \alpha \le 1, \quad \mu + \alpha + m\nu > 1, \tag{2.7}$$

and let nonnegative coefficients $a(x)$*,* $b(x)$ *and the nonnegative kernel* $k(x) \in CL^+_{\mathrm{loc}}(0,d)$ *satisfy the conditions*

$$a(x) \,\ge\, ax^{\mu}, \quad k(x) \,\ge\, kx^{\alpha-1}, \quad b(x) \,\ge\, bx^{\nu-1} \tag{2.8}$$

with positive constants a, k, b*. Also let* $f(x)$ *satisfy conditions* (2.4)–(2.5)*. If equation* (1.1) *is solvable in* $CL^+_{\mathrm{loc}}((0,d),b)$*, then its solution* $\varphi(x)$ *satisfies the estimate*

$$\varphi(x) \,\ge\, Ax^{(\mu+\alpha+\nu-1)/(m-1)}, \tag{2.9}$$

where

$$A \,=\, \left[akbB\left(\alpha, \frac{\mu+\alpha+m\nu-1}{m-1}\right)\right]^{1/(m-1)} \tag{2.10}$$

and $B(z,w)$ *is the Euler beta function. The constant* A *is precise in the sense that we have exact equality in* (2.9) *when* $f(x) \equiv 0$ *and inequalities in* (2.8) *are replaced by equalities.*

Proof. According to (2.8) and (1.1), we have

$$\varphi^m(x) \,\ge\, akbx^{\mu} \int_0^x \frac{t^{\nu-1}}{(x-t)^{1-\alpha}}\, \varphi(t)\, dt \tag{2.11}$$

and, since $0 < \alpha \le 1$, we obtain

$$\varphi^m(x) \,\ge\, akbx^{\mu+\alpha-1} \int_0^x t^{\nu-1}\, \varphi(t)\, dt. \tag{2.12}$$

We denote

$$y(x) \,=\, \int_0^x t^{\nu-1}\, \varphi(t)\, dt. \tag{2.13}$$

Since $\varphi(x) \in CL^+_{\mathrm{loc}}((0,d),b)$, then $\varphi(x) \in CL^+_{\mathrm{loc}}((0,d),t^{\nu-1})$, so the integral on the righthand side of (2.13) exists and $y'(x) \in C(0,d)$ with $y(0) = 0$. Thus $\varphi(x) = x^{1-\nu}y'(x)$, and we rewrite (2.12) in the form

$$[x^{1-\nu}y'(x)]^m \,\ge\, akbx^{\mu+\alpha-1}y(x).$$

Since $\varphi^m(x) \geq f(x)$, we have

$$y(x) = \int_0^x t^{\nu-1} \varphi(t)\, dt \geq \int_0^x t^{\nu-1} f^{1/m}(t)\, dt.$$

Therefore, $y(x) > 0$ in some neighborhood of the origin, by (2.5). Then (2.12) implies that $\varphi(x) > 0$ for all $x \in (0, d)$ and also $y(x) > 0$ for $x \in (0, d)$ by (2.13). Taking this into account, we obtain

$$y'(x) y^{-1/m}(x) \geq (akb)^{1/m} x^{[(\mu+\alpha-1)/m]+\nu-1}. \tag{2.14}$$

Integrating this inequality over $(0, x)$ and using that $y(0) = 0$ and the conditions in (2.7), we arrive at the estimate

$$y(x) \geq (akb)^{1/(m-1)} \left[\frac{m-1}{\mu+\alpha+m\nu-1}\right]^{m/(m-1)} x^{(\mu+\alpha+m\nu-1)/(m-1)}. \tag{2.15}$$

Thus the estimate

$$\varphi(x) \geq D x^{(\mu+\alpha+\nu-1)/(m-1)} \tag{2.16}$$

follows easily from (2.12) and (2.13) with the constant

$$D = \left[\frac{akb(m-1)}{\mu+\alpha+m\nu-1}\right]^{1/(m-1)}.$$

It remains to improve the constant. Applying the estimate (2.16) to the righthand side of (2.11), we obtain

$$\varphi^m(x) \geq DakbB\left(\alpha, \nu + \frac{\mu+\alpha+\nu-1}{m-1}\right) x^{(\mu+\alpha+\nu-1)m/(m-1)},$$

or

$$\varphi(x) \geq D^{1/m} A^{(m-1)/m}\, x^{(\mu+\alpha+\nu-1)/(m-1)}.$$

Substituting again this estimate into (2.11), we obtain in a similar way

$$\varphi(x) \geq D^{1/m^2} A^{(m-1)[1+1/m]/m}\, x^{(\mu+\alpha+\nu-1)/(m-1)}.$$

Repeating this operation n times, we find

$$\varphi(x) \geq D^{1/m^n} A^{(m-1)[1+1/m+\cdots+1/m^{n-1}]/m}\, x^{(\mu+\alpha+\nu-1)/(m-1)}.$$

Taking the limit as $n \to \infty$, we arrive at the inequality (2.9) with the required constant A.

Evidently $A \geq D$ since $B(z,w) > 1/z$ for $z > 0$ and $0 < w \leq 1$. The preciseness of the constant A may be checked by direct verification of the fact that the function $Ax^{(\mu+\alpha+\nu-1)/(m-1)}$ satisfies equation (1.1) when $f(x) \equiv 0$, $a(x) = ax^{\mu}$, $k(x) = kx^{\alpha-1}$ and $b(x) = bx^{\nu-1}$. $\square$

Corollary 2.1. *Let $m > 1$, $a > 0$, $0 < \alpha \leq 1$ and $\mu, \nu \in \mathbf{R}$ be such that $\mu+\alpha+m\nu > 1$, and let the function $f^{1/m}(x) \in CL^{+}_{\text{loc}}((0,d), t^{\nu-1})$ satisfy the condition* (2.5)*. If the equation*

$$\varphi^m(x) \;=\; ax^{\mu} \int_0^x \frac{t^{\nu-1}}{(x-t)^{1-\alpha}}\, \varphi(t)\, dt + f(x), \quad 0 < x < d \leq \infty \tag{2.17}$$

is solvable in $CL^{+}_{\text{loc}}((0,d), t^{\nu-1})$, then its solution $\varphi(x)$ satisfies the estimate

$$\varphi(x) \;\geq\; A_1\, x^{(\mu+\alpha+\nu-1)/(m-1)} \tag{2.18}$$

with

$$A_1 \;=\; \left[aB\left(\alpha, \frac{\mu+\alpha+m\nu-1}{m-1}\right)\right]^{1/(m-1)}.$$

We note that the condition $f^{1/m}(x) \in CL^{+}_{\text{loc}}((0,d), t^{\nu-1})$ of Corollary 2.1 is fulfilled if, for example, $f(x) \in CL^{+}_{\text{loc}}(0,d)$ and $\nu m > 1$.

Corollary 2.2. *Under the assumptions of Theorem* 2.1*, if equation* (1.1) *has a solution in $CL^{+}_{\text{loc}}((0,d), b)$ with the asymptotic behavior*

$$\varphi(x) \;=\; cx^{\gamma} + o(x^{\gamma}), \quad x \to 0;\; c \neq 0, \tag{2.19}$$

then necessarily

$$\gamma \;\leq\; \frac{\mu+\alpha+\nu-1}{m-1}. \tag{2.20}$$

Corollary 2.3. *Let $m > 1$, $\mu, \nu \in \mathbf{R}$ be such that $\mu + m\nu > 0$, and let nonnegative coefficients $a(x)$ and $b(x)$ satisfy the conditions*

$a(x) \geq ax^{\mu}$, $b(x) \geq bx^{\nu-1}$ *with* $a > 0$ *and* $b > 0$. *Also let* $f^{1/m}(x) \in CL^{+}_{\text{loc}}((0,d),b)$ *satisfy condition* (2.5). *If equation* (1.5) *is solvable in* $CL^{+}_{\text{loc}}((0,d),b)$, *then its solution* $\varphi(x)$ *satisfies the estimate*

$$\varphi(x) \;\geq\; A_2\, x^{(\mu+\nu)/(m-1)} \quad \text{for} \quad A_2 \;=\; \left[\frac{ab(m-1)}{\mu+m\nu}\right]^{1/(m-1)}. \tag{2.21}$$

Remark 2.2. By (2.2), the statements of Theorem 2.1 and Corollary 2.1 are also valid for continuous solutions in the weighted space $C([0,d),b)$.

Remark 2.3. Using the exact constant A in (2.9), we can obtain a more exact lower estimate for the solution $\varphi(x)$ of the equation (1.1). Namely if the assumptions of Theorem 2.1 are satisfied and integral equation (1.1) is solvable in $CL^{+}_{\text{loc}}((0,d),b)$, then its solution $\varphi(x)$ satisfies the estimate

$$\varphi(x) \;\geq\; (A^m\, x^{m(\mu+\alpha+\nu-1)/(m-1)} + f(x))^{1/m}. \tag{2.22}$$

Indeed, making use of (2.9), (2.11) and the exact estimate (2.9), from equation (1.1) we have

$$\begin{aligned}\varphi^m(x) \;&\geq\; f(x) + x^{\mu} abk \int_0^x \frac{t^{\nu-1}}{(x-t)^{1-\alpha}}\,\varphi(t)\,dt \\ &\geq\; f(x) + A^m\, x^{m(\mu+\alpha+\nu-1)/(m-1)},\end{aligned}$$

whence (2.22) follows.

b) *The case when* $b(x)$ *may have no power lower estimate.* We suppose here that $b(x)$ is a nonnegative function satisfying the condition

$$\operatorname{mes}\{x : b(x) = 0,\; x \in (0,d)\} \;=\; 0. \tag{2.23}$$

Theorem 2.2. *Let* $m, \alpha, \eta \in \mathbf{R}$ *be such that*

$$m \,>\, 1, \quad 0 < \alpha \,\leq\, 1, \quad \eta + \alpha + m \,>\, 1, \tag{2.24}$$

and let the nonnegative coefficients $a(x)$, $b(x)$ and the nonnegative kernel $k(x) \in CL^+_{\mathrm{loc}}(0,d)$ admit the lower bounds

$$c(x) \,=\, a(x)b^m(x) \,\geq\, cx^\eta, \quad k(x) \,\geq\, kx^{\alpha-1}, \tag{2.25}$$

with positive constants c, k. Also let $f(x)$ satisfy the conditions (2.4)–(2.5). If equation (1.1) is solvable in $CL^+_{\mathrm{loc}}((0,d),b)$, then its solution $\varphi(x)$ satisfies the estimate

$$\varphi(x) \,\geq\, \frac{Cx^{(\eta+\alpha)/(m-1)}}{b(x)}, \tag{2.26}$$

where

$$C \,=\, \left[ckB\left(\alpha, \frac{\eta+\alpha+m-1}{m-1}\right)\right]^{1/(m-1)}. \tag{2.27}$$

Proof. Taking Remark 2.1 into account, we may reduce equation (1.1) to the equation with only one coefficient:

$$\phi^m(x) \,=\, c(x)\int_0^x k(x-t)\phi(t)\,dt + g(x), \quad 0 < x < d \leq \infty, \tag{2.28}$$

where

$$\phi(x) \,=\, b(x)\varphi(x), \quad c(x) \,=\, a(x)[b(x)]^m, \quad g(x) \,=\, [b(x)]^m f(x)$$

and the solution $\phi(x)$ is sought in the space $CL^+_{\mathrm{loc}}(0,d)$. The statement of the theorem follows immediately by applying Theorem 2.1 to equation (2.28). □

We note a principal difference between Theorems 2.1 and 2.2. In Theorem 2.1 we supposed that both the coefficients $a(x)$ and $b(x)$ have lower power estimates, while in Theorem 2.2 we assumed that the whole product $c(x) = a(x)b^m(x)$ has such a bound. For this reason, the range of applications of Theorems 2.1 and 2.2 is different. In particular, Theorem 2.2 may be applied to the case when one of the coefficients may have exponential decaying at the origin which is supplied by the one with another exponential growth, for example,

$$a(x) = x^{\eta} e^{m/x}, \qquad b(x) = e^{-1/x}, \tag{2.29}$$

in which the coefficient $b(x)$ has no lower power bound so that we cannot apply Theorem 2.1.

3. Equation (1.2) in the case $f(x) \equiv 0$. By $\mathbf{CL}^+_{\mathrm{loc}}(0,d)$ and $\mathbf{C}^+[0,d)$ we denote the subclass of functions $\varphi(x) \in CL^+_{\mathrm{loc}}(0,d)$ or $\varphi(x) \in C^+[0,d)$, respectively, such that $\varphi(x) > 0$ for $x > 0$. Theorem 2.1 on lower estimates of solutions of equation (1.2) is also valid in the case $f(x) \equiv 0$, $x \in (0,d)$, that is, the equation

$$\varphi^m(x) = \int_0^x k(x-t)\varphi(t)\,dt, \quad 0 < x < d \le \infty, \tag{3.1}$$

if we look a priori for solutions in the subclass $\mathbf{CL}^+_{\mathrm{loc}}(0,d)$.

Theorem 3.1. *Let $m > 1$, and let $k(x) \in CL^+_{\mathrm{loc}}(0,d)$ satisfy the condition*

$$k(x) \ge kx^{\alpha-1} \tag{3.2}$$

with $k > 0$ and $0 < \alpha \le 1$. If equation (3.1) *is solvable in $\mathbf{CL}^+_{\mathrm{loc}}(0,d)$, then its solution $\varphi(x)$ satisfies the estimate* (2.9) *with the constant A in* (2.10) *calculated for $\mu = 0$, $\nu = 1$.*

The proof of Theorem 3.1 is in fact the same as that of Theorem 2.1 if we take into account that the condition (2.5) on $f(x)$ was used only to show that $\varphi(x) > 0$ for $x > 0$. Now we have this by definition of the class $\mathbf{CL}^+_{\mathrm{loc}}(0,d)$.

The assumption that the solution $\varphi(x)$ is positive for $x > 0$ is natural, which is seen from the following lemma.

Lemma 3.1. *Let $m > 1$, and let $k(x) \in L^+_{\mathrm{loc}}(0,d)$ be nonzero in a neighborhood of the origin*:

$$k(x) \not\equiv 0, \quad x \in (0,\delta), \quad \textit{for any} \quad \delta > 0. \tag{3.3}$$

If equation (3.1) *is solvable in* $CL^+_{\mathrm{loc}}(0,d)$, *then for the solution* $\varphi(x)$ *only one of the following cases may arise*:

1) $\varphi(x) \equiv 0$ *for* $0 \le x \le d$;

2) $\varphi(x) > 0$ *for* $0 < x \le d$;

3) *there exists* $d_0 \in (0,d)$ *such that* $\varphi(x) \equiv 0$, $0 \le x \le d_0$, *and* $\varphi(x) > 0$, $d_0 < x \le d$.

Proof. Suppose that equation (3.1) has a solution in $CL^+_{\mathrm{loc}}(0,d)$ such that $\varphi(x_0) > 0$. Then $\varphi(x) > 0$ for all $x_0 \le x \le d$. Indeed, we have $\varphi(t) > \varphi(x_0)/2$ for $t \in (x_0, x_0+\delta)$ with a sufficiently small δ. Then

$$\begin{aligned} \varphi^m(x) &= \int_0^x k(x-t)\varphi(t)\,dt \\ &\ge \int_{x_0}^{x_0+\delta} k(x-t)\varphi(t)\,dt \\ &\ge \frac{\varphi(x_0)}{2}\int_0^\delta k(t)\,dt > 0. \end{aligned} \tag{3.4}$$

This gives us the cases 1)–3). We observe that case 3) may be reduced to case 2) by passing to the function $\psi(x)$ in (2.6). The lemma is proved. □

Remark 3.1. In Lemma 3.1, if we assume that $k(x) \equiv 0$ in $(0,\delta_0)$ and $k(x) \not\equiv 0$ in $(\delta_0, \delta_0+\delta)$ for any $\delta > 0$, we can repeat our discussion with passing to the function $\psi(x)$ and obtaining the result as in Lemma 3.1 with shift on δ_0.

4. The upper and lower estimates for averages of solutions. In the case of equation (1.2), we will obtain the upper bound for the "average" of the solution

$$\frac{1}{x}\int_0^x \varphi(t)\,dt$$

for m being an integer $m = 2, 3, 4, \ldots$. To this end, we need two preliminary lemmas.

Lemma 4.1. *Let $a > 0$, $b > 0$, and $m > 1$. The equation $\xi^m = a\xi + b$, $\xi > 0$, has a unique solution ξ_0, so that*

$$\xi^m \le a\xi + b \iff 0 < \xi \le \xi_0$$

and

$$\xi^m \ge a\xi + b \iff \xi \ge \xi_0.$$

In the case $m = 2, 3, \dots$, the following inclusion is also valid

$$\xi^m \le a\xi + b \implies \xi < a^{1/(m-1)} + b^{1/m}. \tag{4.1}$$

Proof. The first statement of Lemma 4.1 is obvious. The proof of the second part is motivated by the first one. Let $g(\xi) = \xi^m - a\xi - b$. By means of the binomial formula, it can be shown that

$$g(a^{1/(m-1)} + b^{1/m}) \ge (m-1)ab^{1/m} > 0. \tag{4.2}$$

Then the first part yields $\xi_0 \le a^{1/(m-1)} + b^{1/m}$. However, we may give the direct rigorous proof. Indeed, if ξ_1 exists such that $\xi_1^m \le a\xi_1 + b$ and $\xi_1 \ge a^{1/(m-1)} + b^{1/m}$. Then

$$\begin{aligned} a\xi_1 + b \ge \xi_1^m \ge (a^{1/(m-1)} + b^{1/m})^m &= a^{m/(m-1)} + mab^{1/m} + \cdots + b \\ &\ge a(a^{1/(m-1)} + mb^{1/m}) + b. \end{aligned}$$

Hence

$$\xi_1 \ge a^{1/(m-1)} + mb^{1/m}. \tag{4.3}$$

Repeating this argument n times, we find

$$\xi_1 \ge a^{1/(m-1)} + m^n b^{1/m} \tag{4.4}$$

for an arbitrary positive integer n, which is impossible. This completes the proof of Lemma 4.1. $\square$

Lemma 4.2. *Let $a > 0$, $b > 0$ and $m > 1$. For the solution $\xi_0 = \xi_0(m)$ of the equation $\xi^m = a\xi + b$, $\xi > 0$, the following assertions are valid*:

1) *In the case* $a+b<1$, *the function* $\xi_0(m)$ *is increasing in* m *and*

$$b^{1/m}(1-a)^{-1/m} \,<\, \xi_0(m) \,<\, (a+b)^{1/m};$$

2) *In the case* $a+b>1$, *the function* $\xi_0(m)$ *is decreasing in* m *and*

$$(a+b)^{1/m} \,<\, \xi_0(m) \,<\, \infty;$$

3) $\xi_0(m)=1$ *for all* $m>1$ *in the case* $a+b=1$.

Proof. The monotonicity of $\xi_0(m)$ is evident geometrically, if one considers the curve $y=\xi^m$ and the lines $y=a\xi+b$, $y=\xi$ and $\xi=1$ in the (ξ,y)-plane. If we take into account the inequalities $\xi_0(m)<1$ in case 1) and $\xi_0(m)>1$ in case 2), then the bounds in 1)–2) easily follow from the equality $\xi_0^m(m)=a\xi_0(m)+b$ and the lemma is proved. □

Theorem 4.1. *Let* $k(x)$, $f(x)\in L^+_{\mathrm{loc}}(0,d)$ *and* $m=2,3,\dots$. *If equation* (1.2) *is solvable in* $CL^+_{\mathrm{loc}}(0,d)$, *then its solution* $\varphi(x)$ *admits the estimates*

$$\begin{aligned}\frac{1}{x}\int_0^x f^{1/m}(t)\,dt \,&\le\, \frac{1}{x}\int_0^x \varphi(t)\,dt \\ &< \left(\int_0^x k(t)\,dt\right)^{1/(m-1)} + \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^{1/m}.\end{aligned} \tag{4.5}$$

Proof. The lefthand inequality in (4.5) is obvious since $\varphi^m(x)\ge f(x)$. To prove the righthand one, we integrate equation (1.2) and obtain

$$\int_0^x \varphi^m(t)\,dt \,\le\, K(x)\Phi(x)+F(x), \tag{4.6}$$

where

$$\Phi(x) \,=\, \int_0^x \varphi(t)\,dt,$$
$$K(x) \,=\, \int_0^x k(t)\,dt$$

and

$$F(x) \,=\, \int_0^x f(t)\,dt.$$

Using the Hölder inequality, we obtain

$$x^{m-1}\int_0^x \varphi^m(t)\,dt \,\geq\, \left(\int_0^x \varphi(t)\,dt\right)^m, \tag{4.7}$$

so that

$$\Phi^m(x) \,\leq\, x^{m-1}K(x)\Phi(x) + x^{m-1}F(x). \tag{4.8}$$

Then the righthand inequality in (4.5) follows from (4.8) by Lemma 4.1 where one should take $\xi = \Phi(x)$, $a = x^{m-1}K(x)$ and $b = x^{m-1}F(x)$. The theorem is proved. $\square$

Hypothesis. *Under the assumptions of Theorem* 4.1 *on $k(x)$ and $f(x)$, the estimate* (4.5) *is probably valid for all $m > 1$.*

Corollary 4.1. *Under the assumptions of Theorem* 4.1, *if $\varphi(x)$ is a solution of equation* (1.2) *in $CL^+_{\mathrm{loc}}(0,d)$, then*

$$\int_0^x \varphi(t)\,dt \,=\, o(x^{1-1/m}), \quad x \to 0.$$

Under the assumption, for the solution to be bounded, we can obtain the upper estimate for the solution itself.

Theorem 4.2. *Let $k(x) \in L^+_{\mathrm{loc}}(0,d)$, $f(x) \in L^+_\infty(0,d)$ and $m = 2,3,\dots$. If equation* (1.2) *is solvable in $L^+_\infty(0,d)$, then*

$$f^{1/m}(x) \,\leq\, \varphi(x) \,\leq\, \left(\int_0^x k(t)\,dt\right)^{1/(m-1)} + \Big(\sup_{0\leq t\leq x} f(x)\Big)^{1/m}. \tag{4.9}$$

Proof. To prove the right inequality in (4.9), we use the estimate

$$\Big(\sup_{0\leq t\leq x}\varphi(t)\Big)^m \,=\, \sup_{0\leq t\leq x}\varphi^m(t) \,\leq\, \Big(\sup_{0\leq t\leq x}\varphi(t)\Big)\int_0^x k(t)\,dt + \sup_{0\leq t\leq x} f(t).$$

Applying Lemma 4.1 with the evident choice of ξ, a and b, we have

$$\sup_{0\le t\le x}\varphi(t)\;\le\;\Big(\int_0^x k(t)\,dt\Big)^{1/(m-1)}+\Big(\sup_{0\le t\le x}f(t)\Big)^{1/m}, \tag{4.10}$$

which yields the right inequality in (4.9) and completes the proof. □

Corollary 4.2. *Let $f(x)\equiv 0$ and $k(x)\in L^+_{\mathrm{loc}}(0,d)$. If equation* (1.2) *is solvable in $L^+_\infty(0,d)$, then for all $m>1$,*

$$\varphi(x)\;\le\;\Big(\int_0^x k(t)\,dt\Big)^{1/(m-1)}. \tag{4.11}$$

A similar estimate was known already under the additional assumption that the kernel $k(t)$ is a continuous increasing function (see, for example, [**26**], [**5**]).

Corollary 4.3. *Let $m>1$, $a>0$, $\alpha>0$ and $f(x)\in C^+[0,d)$. If the integral equation*

$$\varphi^m(x)\;=\;a\int_0^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt+f(x),\quad 0<x<d\le\infty, \tag{4.12}$$

is solvable in $C^+[0,d)$, then its solution satisfies the estimates

$$f^{1/m}(x)\;\le\;\varphi(x)\;\le\;\Big(\frac{a}{\alpha}\Big)^{1/(m-1)}x^{\alpha/(m-1)}+\sup_{0\le t\le x}f^{1/m}(t). \tag{4.13}$$

Theorem 4.3. *Let $k^s(x)\in L^+_{\mathrm{loc}}(0,d)$ for some $s>1$ and $f(x)\in C^+[0,d)$. If equation* (1.2) *with $m>1$ is solvable in $L^+_{\mathrm{loc}}(0,d)$, then its solution $\varphi(x)$ belongs to $C^+[0,d)$.*

Proof. The proof is similar to that in [**5**], where this theorem was given in the case $f(x)\equiv 0$. First of all, we show that

$$\varphi^p(x)\;\in\;L^+_{\mathrm{loc}}(0,d)\quad\text{for any}\quad p\ge 1. \tag{4.14}$$

Indeed, from equation (1.2) we see that the convolution on the righthand side in (1.2) belongs to $L_s(0,d_0)$. Consequently, $\varphi(x) \in L_{ms}(0,d_0)$. Repeating our arguments n times, we obtain that $\varphi(x) \in L_{sm^n}(0,d_0)$ for all n, which yields (4.12). To prove the theorem, it remains to choose n such that $sm^n > s'$, $s' = s/(s-1)$, and take into account that $L_s(0,d_0) * L_{s'}(0,d_0) \subset C[0,d_0]$. $\square$

We note that the statement (4.14) of Theorem 4.3 also holds in the case $s = 1$.

5. Uniqueness of solution of integral equation (1.1). In this section we apply Theorem 2.1 to give some conditions for the uniqueness of the solution $\varphi(x)$ of equation (1.1) in $C^+([0,d),b)$.

Theorem 5.1. *Let $m > 1$, $0 < \alpha \le 1$ and $\mu, \nu \in \mathbf{R}$ be such that $\mu + \alpha + m\nu > 1$. Let l be any number such that $0 < l < d$. Suppose that the functions $a(x), b(x) \in C^+(0,d)$, $k(x) \in L^+_{\mathrm{loc}}(0,d)$ and $f^{1/m}(x) \in C^+([0,d),b)$ satisfy the conditions* (2.5) *and* (2.8). *Let*

$$(5.1) \qquad M := \sup_{x\in(0,l)} \left[x^{1-\mu-\nu-\alpha} a(x) b(x) \int_0^x k(t)\,dt\right] < mA^{m-1},$$

where A is defined in (2.10). *If equation* (1.1) *is solvable in $C^+([0,l),b)$, then its solution $\varphi(x)$ is unique on the interval $[0,l]$.*

Proof. Let $\varphi_1(x)$ and $\varphi_2(x)$ be two solutions of equation (1.1) in $C^+([0,l),b)$. Since $m > 1$, by the mean value theorem we have

$$|[\varphi_1(x)]^m - [\varphi_2(x)]^m| \ge m|\varphi_1(x) - \varphi_2(x)|(\min[\varphi_1(x), \varphi_2(x)])^{m-1}.$$

Hence, in accordance with the lower estimate (2.9) and Remark 2.2, we have

$$|[\varphi_1(x)]^m - [\varphi_2(x)]^m| \ge mA^{m-1}x^{\mu+\alpha+\nu-1}|\varphi_1(x) - \varphi_2(x)|.$$

Then, by (1.1),

$$(5.2) \quad mA^{m-1}x^{\mu+\alpha+\nu-1}|\varphi_1(x) - \varphi_2(x)| \le a(x)\int_0^x k(x-t)b(t)|\varphi_1(t) - \varphi_2(t)|\,dt.$$

Let

$$u(x) = b(x)|\varphi_1(x) - \varphi_2(x)|. \tag{5.3}$$

Then (5.3) can be rewritten as

$$u(x) \le \frac{a(x)b(x)}{mA^{m-1}x^{\mu+\alpha+\nu-1}} \int_0^x k(x-t)u(t)\,dt. \tag{5.4}$$

Let l_0 be a number arbitrarily close to l, $0 < l_0 < l$, and x_0 the maximum point of $u(x)$ on $[0, l_0] : u(x_0) = \max_{0\le x\le l_0} u(x)$. Then

$$\int_0^{x_0} k(x_0 - t)u(t)\,dt \le K(x_0)u(x_0), \tag{5.5}$$

where

$$K(x) = \int_0^x k(t)\,dt.$$

Substituting this into (5.5) and using the notation in (5.2), we arrive at the estimate

$$u(x_0) \le \frac{M}{mA^{m-1}} u(x_0). \tag{5.6}$$

Since $M < mA^{m-1}$, we obtain $u(x_0) = 0$ at the maximum point of $u(x)$ on an arbitrary subinterval $[0, l_0]$. Then $u(x) \equiv 0$ so that $\varphi_1(x) = \varphi_2(x)$, which proves the theorem. □

Corollary 5.1. *Let $m > 1$, $a > 0$, $0 < \alpha \le 1$ and $\mu, \nu \in \mathbf{R}$ be such that*

$$\mu + \alpha + m\nu > 1, \quad \frac{1}{\alpha} < mB\left(\alpha, \frac{\mu+\alpha+m\nu-1}{m-1}\right)$$

and let the function $f^{1/m}(x) \in C^+([0,d), x^{\nu-1})$ satisfy the condition (2.5)*. Then if equation* (2.17) *is solvable in $C^+([0,d), x^{\nu-1})$, its solution $\varphi(x)$ is unique.*

Of course, condition (5.2) of Theorem 5.1 is not necessary for uniqueness, which may be illustrated by the equation

$$\varphi^2(x) = \frac{a}{x^\alpha} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\,dt + f, \quad 0 < x < d, \tag{5.7}$$

where f is a constant, satisfying the condition $f \le (a/[2\alpha])^2$. This equation is solvable in $C^+[0,d)$ and has the unique solution $\varphi(x) = c$, where the constant c is a positive solution of the algebraic equation $\alpha c^2 - ac - \alpha f = 0$, but the condition (5.2) is violated turning into the equality $M = a/\alpha = 2A$.

A slight modification of the assumptions of Theorem 5.1 allows us to extend this theorem to the case when the sign $<$ in (5.2) is replaced by $\le$. We suppose that the coefficients $a(x), b(x)$ and the kernel $k(x)$ satisfy the condition

$$N \,:=\, \sup_{x\in(0,l)} a(x)b^m(x)\int_0^x k(t)\,dt \,<\, \infty. \tag{5.8}$$

Let

$$N_0 \,:=\, \lim_{x\to 0}\, a(x)b^m(x)\int_0^x k(t)\,dt, \tag{5.9}$$

if this limit exists.

Theorem 5.2. *Let $m > 1$, $0 < \alpha \le 1$ and $\mu, \nu \in \mathbf{R}$ be such that $\mu + \alpha + m\nu > 1$. Let l be any number such that $0 < l < d$. Suppose that the functions $a(x), b(x) \in C^+(0,d)$, $k(x) \in L^+_{\text{loc}}(0,d)$ and $f^{1/m}(x) \in C^+([0,d),b)$ satisfy the conditions* (2.8) *and* (2.5) *and* (5.9). *Suppose that*

$$M \,\le\, mA^{m-1}, \tag{5.10}$$

where A is defined in (2.10) *and M is given in* (5.2). *Also let*

$$N_0 \,=\, 0 \quad \textit{in the case when} \quad \lim_{t\to 0} b^m(x)f(x) \,=\, 0. \tag{5.11}$$

If equation (1.1) *is solvable in $C^+([0,l),b)$, then its solution $\varphi(x)$ is unique.*

Proof. 1st step. The proof of this theorem coincides with that of Theorem 5.1 until the inequality (5.5). Starting from there, let us show that $u(0) = 0$. To this end, it is sufficient to establish that

$\lim_{x\to 0} b(x)\varphi_1(x) = \lim_{x\to 0} b(x)\varphi_2(x)$. From equation (1.1) we have, by putting $\psi(x) = b(x)\varphi(x)$ and $g(x) = b^m(x)f(x)$,

$$
\begin{aligned}
\psi^m(x) \;&=\; a(x)b^m(x)\int_0^x k(x-t)[\psi(t)-\psi(x)]\,dt \\
&\quad + \psi(x)a(x)b^m(x)\int_0^x k(t)\,dt + g(x) \\
&=\; I_1 + I_2 + g(x).
\end{aligned}
\tag{5.12}
$$

Evidently, $I_1(x) \le N\sup_{0\le t\le x}|\psi(t)-\psi(x)| \to 0$ as $x \to 0$. Therefore, the limit $\lim_{x\to 0} I_2(x)$ exists, which implies the existence of the limit N_0. This is obvious when $\psi(0) \ne 0$, while the case $\psi(0) = 0$ is possible only when $g(0) = 0$, which is automatic by assumption (5.12).

Thus the value $\lambda = \psi(0)$ always exists under the assumptions of the theorem. To show that this value is unique, it remains to observe that λ is a solution of the equation $\lambda^m - N_0\lambda g(0) = 0$, which follows by passing to the limit in (5.13). In the case $g(0) > 0$, this equation has a unique solution, which is positive, by Lemma 4.1, while in the case $g(0) = 0$ the assumption (5.12) yields this unique solution $\lambda = 0$.

2nd step. Now we can assume that $u(0) = 0$. As in the proof of Theorem 5.1, let l_0 be a number arbitrarily close to l, $0 < l_0 < l$, and x_0 the maximum point of $u(x)$ on $[0, l_0] : u(x_0) = \max_{0\le x\le l_0} u(x)$. If $u(x)$ is a constant, then necessarily $u(x) \equiv 0$ and the theorem is proved. Otherwise, a set $e \subset (0, 1)$ exists such that $\operatorname{mes}\{e\} > 0$ and $\sup_{t\in e} u(t) < u(x_0)$. Then

$$
\int_0^{x_0} k(x_0-t)u(t)\,dt \;<\; u(x_0)\int_0^{x_0} k(t)\,dt. \tag{5.13}
$$

Comparing (5.5) and (5.14) at the point x_0 and taking (5.11) into account, we obtain

$$
u(x_0) \;<\; \frac{M}{mA^{m-1}}\,u(x_0) \;\le\; u(x_0), \tag{5.14}
$$

which is not possible. The theorem is proved. □

Remark 5.1. Under the assumptions of Theorem 5.2, condition (5.9) in some cases may be obtained as a consequence of condition (5.11). For example, if $b(x) \equiv 1$ and $\nu = 1$, from estimate (2.9) for $\varphi(x) \in C^+[0,d)$, it follows that $\mu + \alpha \geq 0$ and the condition (5.11) implies (5.9) with $N_0 = 0$, if $\mu + \alpha > 0$.

Modifying the example in (5.8), we note that the equation

$$\varphi^2(x) \;=\; \frac{a}{|x-1|^\alpha} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt, \quad 0 < x < 2,$$

has the discontinuous solution $\varphi(x) = 0$ if $0 \leq x \leq 1$, and $\varphi = a/\alpha$ if $1 < x \leq 2$. This example shows that Theorem 4.3 is not valid in the general case of equation (1.1) if $a(x)$ has a singularity other than the origin.

In the following theorem we especially single out the particular case of Theorem 5.1 when $k(x) \equiv 1$ and $\alpha = 1$.

Theorem 5.3. *Let $m > 1$ and $\mu, \nu \in \mathbf{R}$ be such that $\mu + m\nu > 0$. Let the functions $a(x), b(x) \in C^+(0,d)$ and $f^{1/m}(x) \in C^+([0,d), b)$ satisfy the conditions* (2.5) *and* (2.8). *Let*

$$M \;:=\; \sup_{x\in(0,d)} x^{1-\mu-\nu} a(x) b(x) \;<\; ab\, \frac{m(m-1)}{m\nu + \mu} \tag{5.15}$$

and equation (1.5) *be solvable in $C^+([0,d), b)$. Then its solution $\varphi(x)$ is unique.*

Corollary 5.2. *Let $a > 0$ and $m, \mu, \nu \in \mathbf{R}$ be such that*

$$m \;>\; 1, \quad 0 \;<\; \mu + m\nu \;<\; m(m-1), \tag{5.16}$$

and let the function $f^{1/m}(x) \in C^+([0,d), x^{\nu-1})$ satisfy condition (2.4). *If the integral equation*

$$\varphi^m(x) \;=\; a x^\mu \int_0^x t^{\nu-1} \varphi(t)\, dt + f(x), \quad 0 < x < d \leq \infty, \tag{5.17}$$

is solvable in the space $C^+([0,d), x^{\nu-1})$, then its solution $\varphi(x)$ is unique.

We note that, in particular, Corollary 5.2 holds for equation (5.17) in the case $\mu = 0$ and $\nu = 1$ if $m > 2$.

Remark 5.2. Corollary 5.2 gives some easily verified conditions when m is given, and we want to know what values of μ and ν are admissible in (5.16) for the uniqueness of the solution. It is easy to check that, inversely, for given values of μ and ν, the possible values of m are described as follows.

Given μ and ν, the conditions (5.16) are satisfied if either of the following is satisfied:

1) In the case $\mu > -(1+\nu)^2/4$, either

(5.18)
$$m > \max\left[1, -\frac{\mu}{\nu}, \frac{1+\nu+\sqrt{D}}{2}\right] \quad \text{or} \quad 1 < m < \frac{1+\nu-\sqrt{D}}{2},$$

when $\nu > 0$, and

$$\max\left[1, \frac{1+\nu+\sqrt{D}}{2}\right] < m < -\frac{\mu}{\nu}, \tag{5.19}$$

when $\nu < 0$ (here $D = (1+\nu)^2 + 4\mu$ and the interval $1 < m < (1+\nu-\sqrt{D})/2$ in (5.18) is nonempty if and only if $1 < \nu < -\mu$);

2) In the case $\mu \le -(1+\nu)^2/4$,

$$m > \max\left[1, -\frac{\mu}{\nu}\right] \quad \text{if} \quad \nu > 0 \tag{5.20}$$

$$1 < m < -\frac{\mu}{\nu} \qquad \text{if} \quad \nu < 0 \tag{5.21}$$

with the value $m = (1+\nu)/2$ excluded in (5.20)–(5.21) in the case $\mu = -(1+\nu)^2/4$;

3) In the remaining case $\nu = 0$, which implies

$$m > \max\left[1, \frac{1+\sqrt{1+4\mu}}{2}\right]. \tag{5.22}$$

In particular, in the case $\nu = 1$

$$m > \max\,[1, -\mu, 1+\sqrt{1+\mu}] \quad \text{if} \quad \mu > -1, \tag{5.23}$$

$$m > \max\,[1, -\mu] \qquad \text{if} \quad \mu \le -1. \tag{5.24}$$

Remark 5.3. If $a(x) \neq 0$ for $x \in [0, d)$ in the equation (1.5), then this equation is equivalent to the following Cauchy problem for the differential equation

$$y'(x) = p(x)y^{1/m}(x) + q'(x), \quad y(0) = q(0), \tag{5.25}$$

where

$$y(x) = \frac{\varphi^m(x)}{a(x)}, \quad p(x) = b(x)a^{1/m}(x), \quad q(x) = \frac{f(x)}{a(x)}.$$

Thus, a result similar to that in Theorem 5.3 is also valid for this problem.

6. The case of the singular coefficient. We return here to the integral equation of type (5.8) with the singular coefficient:

$$\begin{gathered} \varphi^m(x) = \frac{a}{x^\alpha} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt + f(x), \\ 0 < x < d, \ a > 0. \end{gathered} \tag{6.1}$$

First of all we note that from Corollary 5.1 the uniqueness of equation (6.1) is derived. If $m > 1$, $f(x) \in C^+[0, d)$ satisfies the condition (2.5) and equation (6.1) is solvable in $C^+[0, d)$, then its solution $\varphi(x)$ is unique.

We suppose that $f(x) \in C^+[0, d)$ and look for a solution in $L^+_{\text{loc}}(0, d)$. Theorem 4.3 states that in the case when the coefficient of the equation (1.2) is not singular, any solution which is a priori in $L^+_{\text{loc}}(0, d)$ or in fact in $C^+[0, d)$. We show that this is also valid in the case of the singular coefficient, as in (6.1).

Lemma 6.1. *Let* $\alpha > 0$ *and*

$$(K\varphi)(x) = \frac{1}{x^\alpha} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt, \quad 0 < x < d. \tag{6.2}$$

If $1 < p < \infty$, *the operator* K *is bounded in* $L_p(0, d)$ *and maps* $L_1(0, d_0)$ *into* $L_1((0, d_0), b)$, *where* $b = (\log[\gamma/x])^{-\lambda}$, $0 < d_0 < d$, $\gamma > d_0$ *and* $\lambda > 1$.

Proof. In case $p \neq 1$, the statement of the lemma is well known, being a particular case of the Hardy-Littlewood theorem [**10**] on boundedness of integral operators with kernels homogeneous of degree -1:

$$\|K\varphi\|_p \,\leq\, B\biggl(\alpha,\, 1-\frac{1}{p}\biggr)\|\varphi\|_p. \tag{6.3}$$

For $p = 1$ the statement of the lemma may be verified directly. □

Theorem 6.1. *Let $0 < \alpha < 1$ and $f(x) \in C^{+}[0,d)$. If equation* (6.1) *with $m > 1$ is solvable in $L^{+}_{\mathrm{loc}}(0,d)$, then its solution $\varphi(x)$ belongs to $C^{+}[0,d)$.*

Proof. The proof is due to the idea of the proof of Theorem 4.3.

1st step. First of all we show that the statement (4.12) is valid. By the Young theorem, we have $t^{\alpha-1} * \varphi \in L_s(0,d_0)$ for an arbitrary $d_0 \in (0,d)$, where $1 < s < 1/(1-\alpha)$. Since $\alpha + 1/s > 1$ and $0 < \alpha < 1$ we can find s such that

$$\alpha + \frac{1}{s} \,<\, m. \tag{6.4}$$

From equation (6.1), $x^{\alpha}\varphi^{m}(x) \in L_s(0,d_0)$. Moreover, the solution $\varphi(x)$ itself belongs to $L_p(0,1)$ for all p with

$$1 \,<\, p \,<\, \frac{m}{\alpha + 1/s}. \tag{6.5}$$

Indeed, by the Hölder inequality with $\nu = ms/p > 1 + \alpha s > 1$, we have

$$\begin{aligned}\int_0^{d_0} \varphi^p(x)\,dx \,&=\, \int_0^{d_0} \varphi^p(x)\,x^{\alpha s/\nu}x^{-\alpha s/\nu}\,dx \\ &\leq\, \biggl(\int_0^{d_0} (x^{\alpha}\varphi^{m}(x))^s\,dx\biggr)^{1/\nu}\biggl(\int_0^{d_0} x^{-\alpha s\nu'/\nu}\biggr)^{1/\nu'} \\ &<\, \infty.\end{aligned} \tag{6.6}$$

Thus we obtain (4.12) for the values of p in the interval (6.5). Now we show that (4.12) takes place for all $p > 1$. By Lemma 6.1 the integral operator on the righthand side of (6.1) preserves $L_p(0, d_0)$, $p > 1$, so from equation (6.1) we have that $\varphi^m(x) \in L_p(0, d_0)$ or $\varphi(x) \in L_{pm}(0, d_0)$. Repeating the same arguments as in Theorem 4.3, we obtain (4.12).

2nd step. We show that the function $\varphi(x)$ is in reality bounded on $[0, d_0]$. Since it is in $L_p(0, d_0)$ for all $p > 1$, the integral in the righthand side (6.1) is a bounded continuous function. Therefore, to prove boundedness of $\varphi(x)$ we have only to show that $\varphi(x)$ is bounded at a neighborhood of the origin, say on the interval $[0, \delta]$, $\delta = \min[1, d_0]$.

We observe that in the case of the interval $(0, \delta)$, the L_q-norm is an increasing function in q. Therefore, finite or infinite limit $\lim_{q\to\infty} \|\varphi\|_q \le \infty$ exists. It is known (see [**25**, p. 14]) that if this limit is finite, then $\varphi(x) \in L_\infty(0, \delta)$ and

$$\|\varphi\|_\infty = \lim_{q\to\infty} \|\varphi\|_q. \tag{6.7}$$

Therefore it suffices to prove that the sequence $\|\varphi\|_q$ is bounded for all large values of q.

3rd step. Taking the L_q-norm in (6.1) with an arbitrary $q > 1$ and using (6.3), we have $\|\varphi\|_{qm}^m \le aB(\alpha, 1 - 1/q)\|\varphi\|_q + \|f\|_\infty$. We notice that $B(\alpha, 1 - 1/q) \le B(\alpha, 1/2)$ for all $q \ge 2$. Since we may take $q \ge 2$ by (4.12), we have

$$\|\varphi\|_q^m \le aB\left(\alpha, \frac{1}{2}\right)\|\varphi\|_q + \|f\|_\infty, \tag{6.8}$$

where we also have taken into account that $\|\varphi\|_q \le \|\varphi\|_{qm}$ by the monotonicity of the norm. Then $\|\varphi\|_q$ is uniformly bounded with respect to q by Lemma 4.1.

4th step. Since $\varphi(x) \in L_\infty^+(0, \delta)$ we have that $t^{\alpha-1} * \varphi \in C^+[0, \delta]$ and then from equation (6.1) it follows that $\varphi(x) \in C^+(0, \delta]$.

5th step. It remains to prove that $\varphi(x)$ is continuous at the origin. By $\varphi_\pm$ we denote the upper and lower limits of $\varphi(x)$:

$$\varphi_+ = \lim_{x\to 0} \sup_{0\le t\le x} \varphi(t), \qquad \varphi_- = \lim_{x\to 0} \inf_{0\le t\le x} \varphi(t) \tag{6.9}$$

which evidently exist. Taking $\sup_{0\le t\le x}$ on both sides of (6.1), we obtain

$$\sup_{0\le t\le x} \varphi^m(t) \le \frac{a}{\alpha} \sup_{0\le t\le x} \varphi(t) + \sup_{0\le t\le x} f(t). \tag{6.10}$$

Hence

$$\varphi_+^m \le \frac{a}{\alpha}\varphi_+ + f(0).$$

Then by Lemma 4.1

$$\varphi_+ \le \xi_0, \tag{6.11}$$

where ξ_0 is the solution of the equation $\alpha\xi^m - a\xi - \alpha f(0) = 0$ (see Lemma 4.1).

For the lower limits we similarly have

$$\inf_{0\le t\le x} \varphi^m(t) \ge \frac{a}{\alpha} \inf_{0\le t\le x} \varphi(t) + \inf_{0\le t\le x} f(t), \tag{6.12}$$

whence

$$\varphi_-^m \ge \frac{a}{\alpha}\varphi_- + f(0)$$

which yields that

$$\varphi_- \ge \xi_0 \tag{6.13}$$

with the same ξ_0 as in (6.11). Comparing (6.11) and (6.13) we see that necessarily $\varphi_+ = \varphi_-$, so that $\varphi \in C^+[0,1]$ and $\varphi(0) = \xi_0$. The theorem is proved. □

Remark 6.1. Simple examples show that the statement of Theorem 6.1 is not valid if $f(x)$ is only bounded but not a continuous function. For example, for the bounded function $f(x) = 0$, $0 \le x \le d_0$, and $f(x) = 1 - [a/\alpha](x-d_0)^\alpha/x^\alpha$, $d_0 < x \le d$, with $d_0 > 0$ we have the discontinuous bounded solution $\varphi(x) = 0$ if $0 \le x \le d_0$ and $\varphi(x) = 1$ if $d_0 < x \le d$.

Corollary 6.1. *If, in the conditions of Theorem* 6.1*, $f(x)$ is bounded: $f(x) \in L_\infty^+(0,d)$, then any solution $\varphi(x)$ which is a priori in $L_{\mathrm{loc}}^+(0,d)$ is in fact in $L_\infty^+(0,d)$.*

Remark 6.2. We note that it may happen that equation (6.1) is solvable even if $f(x)$ is not integrable near the origin. To illustrate the idea, we take

$$f(x) = \frac{1}{x^{\beta m}} - a\,\frac{B(\alpha, 1-\beta)}{x^\beta},$$

where $\beta < 1$. Then the function $\varphi(x) = 1/x^\beta$ is an integrable solution of equation (6.1) whenever $\beta < 1$ is, although $f(x)$ is not integrable at the origin when $\beta > 1/m$.

Finally we observe that equation (6.1) in the case $m < 1$ may have a nonunique solution. Thus, the equation

$$\sqrt{\varphi(x)} = \frac{a}{x^\alpha} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}\, dt + f, \tag{6.14}$$

when a constant f has a constant solution $\varphi(x) = c$ if c satisfies the equation $ac - \alpha\sqrt{c} + \alpha f = 0$ which has two positive solutions if $4af < \alpha$. In the linear case ($m = 1$) equation (6.1) is an example of linear integral equations with kernels homogeneous of degree -1, the theory of which is well developed (see [**23**] and also the recent survey [**13**]). The solvability of the linear ($m = 1$) equation (6.1) in $C[0, d)$ and the number of its solution were in particular investigated in detail in [**23**].

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