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ON A BOUNDARY INTEGRAL METHOD FOR THE SOLUTION OF THE HEAT EQUATION IN UNBOUNDED DOMAINS WITH A NONSMOOTH BOUNDARY

OLAF HANSEN

ABSTRACT. We study a boundary integral method for the solution of the heat equation in an unbounded domain D in $\mathbf{R}^2$. It is assumed that the boundary of D is a polygon $\Gamma = \partial D$ and that $\mathbf{R}^2 \setminus D$ is a simply connected domain. We use a method which was proposed by Chapko and Kress [**2**] for the case of a smooth bounded domain D and analyze this method in the presence of a boundary with corners.

1. Introduction. In this paper we study the numerical solution of the following initial value problem

$$(1.1)\qquad \begin{cases} u_t(x,t) = c\Delta u(x,t), & (x,t) \in D \times (0,T], \\ u(x,t) = F(x,t), & (x,t) \in \Gamma \times [0,T], \\ u(x,t) \overset{|x|\to\infty}{\longrightarrow} 0, & t \in [0,T], \\ u(x,0) = 0, & x \in D. \end{cases}$$

Here $D \subset \mathbf{R}^2$ is an unbounded domain and the boundary $\Gamma := \partial D$ is a polygon. We further assume that the constants c and T are greater than zero. The function F on the boundary should be sufficiently smooth (see Section 2) and should fulfill certain conditions at time $t = 0$, especially

$$(1.2)\qquad F(\cdot, 0) \equiv 0.$$

There are several ways to approximate the solution of (1.1) with the help of a boundary integral equation. One way would be to use a single

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or double layer potential on $\Gamma \times [0, T]$ and to approximate the solution of the resulting integral equation of the first or second kind on $\Gamma \times [0, T]$ with a Galerkin method. The corresponding integral operators and the Galerkin method were studied by Costabel in [**4**].

On the other hand one can first discretize the Volterra part of the integral equation on $\Gamma \times [0, T]$ and then apply known numerical schemes for the approximate solution of the resulting integral equations on Γ. This approach was used by Lubich and Schneider in [**10**] and [**11**] for the case of smooth boundaries.

In this paper we will follow the approach of Chapko and Kress [**2**]. They use Rothe's method for the time discretization of (1.1) and then use a special sequence of "fundamental solutions" for the solution of the resulting sequence of boundary value problems. The advantage of their method lies in the fact that they don't have to calculate integrals over the unbounded domain D. These integrals appear if one solves an inhomogeneous boundary value problem with the boundary layer method in the usual way.

In the present situation we have to study the resulting properties of the boundary integral operators on polygons and also the smoothness of the solution u in time is not obvious (see [**8**]). In Section 2 we will study the time discretization of (1.1) and the convergence of the Rothe method under sufficiently strong assumptions on F. In the next section we study the system of boundary integral equations on Γ and we prove the stability of the collocation method for the single layer approach. Here we use results of Elschner and Graham [**6**]. In Section 4 we derive our numerical algorithm and in Section 5 we present some numerical results which show that our method is applicable and the necessity of a special parametrization for Γ near the corners.

2. The time discretization and the boundary integral equations. In the following we assume that

$$\Gamma = \bigcup_{j=1}^{M} \Gamma_j \subset B_R(0), \tag{2.1}$$

$\Gamma_j = [\xi_{j-1}, \xi_j]$, $j = 1(1)M$, $\xi_0 = \xi_M$, ξ_j, $j = 1(1)M$, are the corners of the polygon Γ and $R > 0$. Let ω_j be the angle at corner j. Define $\overline{\omega}_j$

to be the greater of the two angles at corner j

$$\overline{\omega}_j := \max\{\omega_j, 2\pi - \omega_j\}. \tag{2.2}$$

We will always assume $\overline{\omega}_j \in (\pi, 2\pi)$ and denote by $\tilde{A}$ the symmetric operator

$$\tilde{A}u := -c\Delta u, \quad u \in D_{\tilde{A}} := C^2_{0,\mathrm{comp}}(\overline{D}),$$

$C^2_{0,\mathrm{comp}}(\overline{D}) := \{u \in C^2(\overline{D}) \,|\, u|_\Gamma \equiv 0, u \text{ has compact support in } \overline{D}\}$, $c > 0$ a constant. $\tilde{A}$ is a positive operator,

$$(\tilde{A}u, u)_{L^2(D)} \geq 0,$$

and the energy space for $\tilde{A}$ is $H^1_0(D)$. By A we denote the Friedrich extension of $\tilde{A}$ and A has the following properties [**15**]

$$A \quad \text{is self-adjoint,} \tag{2.3}$$

$$\sigma(A) \subset [0, \infty), \tag{2.4}$$

and

$$D_A \subset H^1_0(D). \tag{2.5}$$

To formulate (1.1) as an evolution equation in a standard form we need the following two results.

Lemma 2.1. *There is a continuous extension operator*

$$\gamma^- : C^4(\Gamma, \mathbf{R}) \longrightarrow C^4_0(\overline{B_{2R}(0)}),$$

where we denote by $C^4(\Gamma, \mathbf{R})$ the functions which are continuous on Γ and four times continuous differentiable on every closed line $[\xi_{j-1}, \xi_j]$, $j = 1(1)M$. The functions in $C^4_0(\overline{B_{2R}(0)})$ are four times continuous differentiable in $\overline{B_{2R}(0)}$ and the extension by zero is four times continuous differentiable on $\mathbf{R}^2$.

Proof. With the help of a partition of unity of Γ we can reduce the problem to the following two cases:

a. $f \in C^4(\Gamma, \mathbf{R})$, $f|_{U_\varepsilon(\xi_j)} \equiv 0$, $j = 0(1)M-1$,

b. $f \in C^4(\Gamma, \mathbf{R})$, $\operatorname{supp}(f) \subset \cup_{j=0}^{M-1} U_{2\varepsilon}(\xi_j)$,

where $\varepsilon := \min_{i,j=1(1)M, i\neq j} |\xi_i - \xi_j|/8$.

In case a) it is very simple to extend the function to a function in $C_0^4(B_{2R}(0))$. In case b) we only have to consider the following special case:

$$\tilde{\Gamma} \;:=\; \{(x,0)\,|\,x \geq 0\} \,\cup\, \{y(\alpha,\beta)\,|\,y \geq 0\},$$

$\alpha^2 + \beta^2 = 1$, $\beta \neq 0$.

We define

$$\tilde{\Gamma}_{\mathrm{Ref}} \;:=\; \{(x,0)\,|\,x \geq 0\} \,\cup\, \{(0,y)\,|\,y \geq 0\}$$

and the linear invertible mapping $L : \tilde{\Gamma}_{\mathrm{Ref}} \xrightarrow{1:1} \tilde{\Gamma}$ by

$$L\begin{pmatrix} x \\ y \end{pmatrix} \;:=\; \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

On the reference configuration we define the function f_{Ref} by

$$f_{\mathrm{Ref}} \;:=\; f \circ L \in C^4(\Gamma_{\mathrm{Ref}}, \mathbf{R})$$

and the restriction of f_{Ref} to the lines

$$\begin{aligned} f_1(x) \;&:=\; f_{\mathrm{Ref}}(x,0) \in C^4([0,\infty), \mathbf{R}), \\ f_2(x) \;&:=\; f_{\mathrm{Ref}}(0,x) \in C^4([0,\infty), \mathbf{R}). \end{aligned}$$

At the next step we define a function $g_T \in C^\infty(\mathbf{R}^2, \mathbf{R})$. The Taylor coefficients of g_T are equal to the coefficients of f_1 and f_2 at zero.

$$\begin{aligned} g_T(x,y) \;:=\;& f_{\mathrm{Ref}}(0,0) + (f_1'(0), f_2'(0))\begin{pmatrix} x \\ y \end{pmatrix} \\ &+ \frac{1}{2}\,(x,y)\begin{pmatrix} f_1''(0) & 0 \\ 0 & f_2''(0) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \\ &+ \frac{1}{6}\left(f_1^{(3)}(0)x^3 + f_2^{(3)}(0)y^3\right) \\ &+ \frac{1}{24}\left(f_1^{(4)}(0)x^3 + f_2^{(4)}(0)y^4\right), \\ f_{1,T}(x) \;:=\;& f_1(x) - \sum_{j=0}^{4} \frac{f_1^{(j)}(0)}{j!}x^j, \\ f_{2,T}(x) \;:=\;& f_2(x) - \sum_{j=0}^{4} \frac{f_2^{(j)}(0)}{j!}y^j. \end{aligned}$$

Now it is clear that

$$g(x,y) := f_{1,T}(|x|) + f_{4,T}(|y|) \in C^4(\mathbf{R}^2, \mathbf{R}),$$

and

$$g + g_T|_{\tilde{\Gamma}_{\mathrm{Ref}}} = f_{\mathrm{Ref}}.$$

Further on the mapping $f_{\mathrm{Ref}} \to g + g_T$ is linear.

Now let $\varphi \in C_0^\infty(\mathbf{R}^2, \mathbf{R})$,

$$\begin{aligned} \varphi|_{U_{2\varepsilon}(0)} &\equiv 1, \\ \varphi(x) &\equiv 0, \quad |x| \geq 3\varepsilon. \end{aligned}$$

We define $\gamma^- f$ by

$$(\gamma^- f)(x,y) := \varphi(x,y)((g + g_T) \circ L^{-1})(x,y).$$

This mapping is linear, continuous (we only need the derivatives of f up to the order of 4) and $\operatorname{supp}(\gamma^- f) \subset \overline{B_{3\varepsilon}(0)}$. We use this construction for every corner of Γ and get the extension result in case b). This proves our lemma. $\square$

Definition 2.2. a. If X is a Banach space we denote by $C^{k,\vartheta}([0,T], X)$ the Banach space of k-times continuously differentiable functions, for which the kth derivative is ϑ-Hölder continuous.

b. By $C^{4,(1,1)}(\Gamma \times [0,T], \mathbf{R})$ we denote the continuous functions $G \in C(\Gamma \times [0,T], \mathbf{R})$, for which we have

$$G \in C^{1,1}([0,T], C^4(\Gamma, \mathbf{R})), \ j = 1(1)M.$$

Corollary 2.3. *Let $G \in C^{4,(1,1)}(\Gamma \times [0,T], \mathbf{R})$, then we define*

$$\tilde{G}(x,t) := (\gamma^- G(\cdot, t))(x), \quad (x,t) \in \mathbf{R}^2 \times [0,T].$$

It follows

$$\tilde{G} \in C^{1,1}([0,T], C_0^4(\overline{B_{2R}(0)})).$$

Proof. As a natural candidate for $D\tilde{G}$ we have

$$D\tilde{G}(x,t) := (\gamma^- DG)(x,t),$$

where $(DG)(x,t) := G_t(x,t),\ (x,t) \in \Gamma \times [0,T]$. For $t,\ t+h \in [0,T]$, $h \neq 0$, we get by Lemma 2.1

$$\begin{aligned}
&\|\tilde{G}(t+h) - \tilde{G}(t) - h\, D\tilde{G}(t)\|_{C^4(\overline{B_{2R}(0)})} \\
&\quad = \|\gamma^- \Big(G(t+h) - G(t) - h\, DG(\cdot,t)\Big)\|_{C^4(\overline{B_{2R}(0)})} \\
&\quad \leq \text{const}\ \|G(\cdot,t+h) - G(\cdot,t) - h\, DG(\cdot,t)\|_{C^4(\Gamma)} \\
&\quad = O(h),
\end{aligned}$$

by the assumption. Together with the Hölder-continuity of $D\tilde{G}$ this proves the corollary. □

The mapping $G \to \tilde{G}$ will also be denoted by γ^-. In the following we will always assume that the function F (see (1.1)) fulfills

$$F \in C^{4,(1,1)}(\Gamma \times [0,T], \mathbf{R}) \tag{2.6}$$

and

$$\Big(\frac{d}{dt}\,(\gamma^- F)\Big)|_{t=0} \equiv 0. \tag{2.7}$$

Using standard results for evolution equations we get the following result.

Theorem 2.4. *Let* (1.2), (2.6) *and* (2.7) *be fulfilled. Then the initial value problem* (1.1) *has a solution* $u \in C^{1,1}([0,T], L^2(D))$, $\Delta u \in C^{0,1}([0,T], L^2(D))$.

Proof. We define

$$H(t) := c\Delta(\gamma^- F)(t) - \frac{d}{dt}(\gamma^- F)(t), \quad t \in [0,T].$$

Corollary 2.3 and the continuity of $\Delta : C_0^2(\overline{B_{2R}(0)}) \to C^0(\overline{B_{2R}(0)})$, show that

$$H \in C^{0,1}([0,T], C^0(\overline{B_{2R}(0)})).$$

By restriction to D we get $H \in C^{0,1}([0,T], L^2(D))$. We consider the following initial value problem in $L^2(D)$

$$\begin{cases} \tilde{u}_t = -A\tilde{u} + H, & t \in (0,T], \\ \tilde{u}(0) = 0. \end{cases} \tag{2.8}$$

Theorem 4.3.5 in Pazy [**14**] shows that (2.8) has a unique solution $\tilde{u} : [0,T] \to D_A$, $A\tilde{u}$, $\tilde{u} \in C^{0,1}([0,T], L^2(D))$. The solution of (1.1) is given by

$$u := \tilde{u} + \gamma^- F.$$

This proves the theorem. $\square$

Now we know that our problem (1.1) has a unique solution u and we can try to approximate it. For the time discretization we choose a number $N \in \mathbf{N}$ and define the time step $h := T/N$. The application of Rothe's method to the initial value problem (1.1) gives us the following sequence of boundary value problems

$$\begin{cases} \dfrac{v_n(x) - v_{n-1}(x)}{h} = c\Delta v_n(x), & x \in D,\ n = 1(1)N, \\ v_0(x) = 0, & x \in D, \\ v_n|_\Gamma = F(\cdot, nh), & n = 1(1)N, \\ v_n(x) \overset{\|x\|\to\infty}{\longrightarrow} 0, & n = 1(1)N. \end{cases} \tag{2.9}$$

The functions v_n are approximations for $u(\cdot, nh)$. The first equation of (2.9) is equivalent to

$$-\Delta v_n + \gamma^2 v_n = \gamma^2 v_{n-1}, \quad n = 1(1)N, \tag{2.10}$$

where the constant $\gamma = \gamma(N)$ is defined by

$$\gamma := \sqrt{\frac{1}{ch}} = \sqrt{\frac{N}{cT}}.$$

Our discretization (2.9)/(2.10) can be interpreted as an implicit Euler scheme and we will write this scheme in an explicit form

$$\begin{aligned}(2.9) \;\Leftrightarrow\quad && v_n - hc\Delta v_n \;&=\; v_{n-1} \\ \Leftrightarrow\; && \big(v_n-(\gamma^- F)(nh)\big)-hc\Delta\big(v_n-(\gamma^- F)(nh)\big) \;&=\; v_{n-1}-(\gamma^- F)(nh) \\ && &\quad + hc\Delta(\gamma^- F(nh))\end{aligned}$$

Define $\hat{v}_n := v_n - (\gamma^- F)(nh)$. Then the above equation together with the boundary condition in (2.9) is equivalent to

$$\begin{aligned}\hat{v}_n + hA\,\Delta\hat{v}_n \;&=\; v_{n-1} - (\gamma^- F)(nh) + hc\Delta(\gamma^- F)(nh) \\ \hat{v}_n|_\Gamma \;&=\; 0.\end{aligned}$$

By (2.5) we get

$$\hat{v}_n \;=\; (I + hA)^{-1}\big(v_{n-1} - (\gamma^- F)(nh) + hA(\gamma^- F)(nh)\big).$$

This implies

$$\begin{aligned}v_n \;&=\; \hat{v}_n + (\gamma^- F)(nh) \\ &=\; v_{n-1} + h\phi((n-1)h, v_{n-1}, h), \quad n = 1(1)N,\end{aligned} \tag{2.12}$$

where the function ϕ is given by
(2.13)

$$\begin{aligned}\phi(t, v, h) \;=\; \frac{1}{h}\Big((\gamma^- F)(t{+}h) - v + (I{+}hA)^{-1}\big(v - (\gamma^- F)(t{+}h) \\ + ch\Delta(\gamma^- F)(t{+}h)\big)\Big)\end{aligned}$$

Lemma 2.5. *The implicit Euler scheme* (2.12)/(2.13) *is consistent for* (1.1) *and has order* 1 (*F has to satisfy* (1.2), (2.6) *and* (2.7)).

Proof. Let $s, s+h \in [0,T]$, $h \neq 0$. We get

$$\begin{aligned}
&\frac{u(s+h)-u(s)}{h} - \phi(s,u(s),h) \\
&\quad = \frac{1}{h}\Big(u(s+h)-u(s)+u(s)-(\gamma^- F)(s+h) \\
&\qquad\quad - (I+hA)^{-1}\Big[u(s)-(\gamma^- F)(s+h)+ch\Delta((\gamma^- F)(s+h))\Big]\Big) \\
&\quad = \frac{(I+hA)^{-1}}{h}\Big((I+hA)\tilde{u}(s+h)-\Big[\dots\Big]\Big) \\
&\quad = \frac{(I+hA)^{-1}}{h}\Big(\tilde{u}(s+h)+h(H(s+h)-\tilde{u}_t(s+h))-\Big[\dots\Big]\Big)
\end{aligned}$$

The functions $\tilde{u}$ and H were defined in the proof of Theorem 2.4. The definition of $\tilde{u}$ and H implies

$$\begin{aligned}
&\frac{u(s+h)-u(s)}{h} - \phi(s,u(s),h) \\
&\quad = \frac{(I+hA)^{-1}}{h}\Big(u(s+h)-(\gamma^- F)(s+h) \\
&\qquad\qquad + ch\Delta((\gamma^- F)(s+h)) - h\frac{d}{dt}(\gamma^- F)(s+h) \\
&\qquad\qquad - h\,u_t(s+h) + h\frac{d}{dt}(\gamma^- F)(s+h) - \Big[\dots\Big]\Big) \\
&\quad = \frac{(I+hA)^{-1}}{h}\Big(u(s+h)-u(s)-h\,u_t(s+h)\Big)
\end{aligned}$$

By (2.4) and Theorem 2.4 we get

$$\begin{aligned}
&\left\|\frac{u(s+h)-u(s)}{h} - \phi(s,u(s),h)\right\|_{L^2(D)} \\
&\qquad \le \|(I+hA)^{-1}\|\,\frac{1}{|h|}\left|\int_s^{s+h}\|u_t(\nu)-u_t(s+h)\|\,d\nu\right| \\
&\qquad \le \text{const}\,\frac{1}{|h|}\left|\int_s^{s+h}(s+h-\nu)\,d\nu\right| \\
&\qquad = \frac{\text{const}}{2}\,|h|.
\end{aligned}$$

This proves our lemma. $\square$

Now we define

$$\hat{v}_{n+1} = u(nh) + h\phi(nh, u(nh), h), \quad n = 0(1)N-1. \tag{2.14}$$

The definition of ϕ shows

$$\begin{aligned} \|\hat{v}_{n+1} - v_{n+1}\|_{L^2(D)} &= \|(I+hA)^{-1}(u(nh) - v_n)\|_{L^2(D)} \\ &\le \|u(nh) - v_n\|_{L^2(D)}, \end{aligned} \tag{2.15}$$

because $\sigma(I + hA) \subset [1, \infty)$. Equation (2.15), Lemma 2.5 and Section 2.5 in Dekker and Verwer [**5**] show the following theorem.

Theorem 2.6. *Let the assumptions of Theorem* 2.4 *be fulfilled and define* $v_{n,N}$, $n = 1(1)N$, $n \in \mathbf{N}$, *by* (2.9). *Then there exists a constant* C, *which does not depend on* N *such that*

$$\max_{n=1(1)N} \left\| u\left(\frac{n}{N}\right) - v_{n,N} \right\|_{L^2(D)} \le \frac{C}{N}.$$

This means that our implicit Euler scheme, which is given by Rothe's method, is convergent with order one.

To solve the sequence of boundary value problems (2.9) numerically we use a boundary integral method, which we study in the following section.

Now we study the regularity of the sequence $(v_n)_{n=1(1)N}$. First we introduce some notations and repeat a theorem of Hammoudi [**8**].

For $\omega \in (0, 2\pi)$ we denote by K_ω the angle

$$K_\omega := \left\{ r \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \mid 0 \le r < \infty,\ \theta \in [0, \omega] \right\}; \tag{2.16}$$

further we consider the following special functions on K_ω.

$$\begin{cases} \sigma_{\nu,l,j}(r,\theta) := r^{l\nu+2j}\sin(l\nu\theta), \\ S_{\nu,l,j}(r,\theta) := r^{l\nu+2j}(\log(r)\sin(l\nu\theta) + \theta\cos(l\nu\theta)), \\ \qquad l \in \mathbf{N},\ j \in \mathbf{N}_0,\ \nu \in \left(\frac{1}{2}, \infty\right), \end{cases} \tag{2.17}$$

and two sets of integers

$$\begin{cases} M_{\nu,s} \,:=\, \{l \in \mathbf{N} \,|\, l\nu \in \mathbf{N},\ 0 < l\nu < s\}, \\ L_{\nu,s} \,:=\, \{l \in \mathbf{N} \,|\, l\nu \notin \mathbf{N},\ 0 < l\nu < s\}. \end{cases} \tag{2.18}$$

Hammoudi proved in his thesis [**8**].

Theorem 2.7. *Let $s \in \mathbf{R}$, $s \geq 1$, $s \notin \{l(\pi/\omega_j) \,|\, j = 1(1)M,\ l \in \mathbf{N}\}$, and $f \in H^{s-1}(D)$. By $u \in H_0^1(D)$ we denote the solution of the equation*

$$\begin{cases} (-\Delta + \gamma^2)u(x) \,=\, f(x), \quad x \in D, \\ u|_\Gamma \,=\, 0. \end{cases} \tag{2.19}$$

For an arbitrary corner ξ_m, $m \in \{1, \dots, M\}$, of Γ we assume without restriction that $\xi_m = 0$ and D should coincide with K_{ω_m} in a neighborhood $B_{2\varepsilon}(0)$, $\varepsilon > 0$. By $\varphi \in C_0^\infty(\mathbf{R}^2)$ we denote a cut-off function

$$\varphi(x) = 1, \quad x \in B_\varepsilon(0), \qquad \varphi(x) = 0, \quad x \in \mathbf{R}^2 \setminus B_{2\varepsilon}(0). \tag{2.20}$$

Then we get

$$\begin{gathered} \varphi\, u \,=\, u_R + \Bigg(\sum_{l \in L_{\nu,s}} \sum_{0 < l\nu + 2j < s} C_{l,j} \sigma_{\nu,l,j} + \sum_{l \in M_{\nu,s}} \sum_{0 < l\nu + 2j < s} D_{l,j} S_{\nu,l,j} \Bigg) \\ u_R \in H^{s+1}(K_{\omega_j}),\ \nu = (\pi/\omega_m),\ C_{j,l},\ D_{j,l} \in \mathbf{R},\ \text{for all } j, l. \end{gathered} \tag{2.21}$$

Lemma 2.8. *Let $s \geq 1$, $\omega \in (0, 2\pi)$, $\nu := \pi/\omega$ and*

$$\begin{aligned} f(r,\theta) \,:=\, \varphi(r,\theta) \Bigg(& \sum_{l \in L_{\nu,s}} \sum_{0 < l\nu + 2j < s} C_{l,j} \sigma_{\nu,l,j}(r,\theta) \\ & + \sum_{l \in M_{\nu,s}} \sum_{0 < l\nu + 2j < s} D_{l,j} S_{\nu,l,j}(r,\theta) \Bigg), \end{aligned} \tag{2.22}$$

where φ is a cut-off function which fulfills (2.20), $C_{l,j}, D_{l,j} \in \mathbf{R}$. For $t \geq s$ a function $w(r,\theta)$ exists of the following form

$$w(r,\theta) \,=\, \varphi(r,\theta) \Bigg(\sum_{l \in L_{\nu,s}} \sum_{j=1}^{k_l'} C_{l,j}' \sigma_{\nu,l,j}(r,\theta) + \sum_{l \in M_{\nu,s}} \sum_{j=1}^{k_l''} D_{l,j}' S_{\nu,l,j}(r,\theta) \Bigg) \tag{2.23}$$

k'_l, $k''_l \in \mathbf{N}_0$, $C'_{l,j}$, $D'_{l,j} \in \mathbf{R}$, *such that*

$$(-\Delta+\gamma^2)w(r,\theta) \;=\; f(r,\theta)+w_R(r,\theta),$$

$w_R \in H^t(K_\omega)$.

Proof. For the proof it is sufficient to consider only the case

$$f(r,\theta) \;=\; \varphi(r,\theta)\sigma_{\nu,l,j_0}(r,\theta)$$

or

$$f(r,\theta) \;=\; \varphi(r,\theta)S_{\nu,l,j_0}(r,\theta).$$

For simplicity we will only treat the first case. We make the ansatz

$$\tilde{w}(r,\theta) \;=\; \sum_{j=j_0+1}^{m} \alpha_j \sigma_{\nu,l,j}(r,\theta),$$

$m \geq \max\{\big[(t-l\nu)/2\big]+1, j_0+1\}$, this implies $\varphi\sigma_{\nu,l,m} \in H^t(K_\omega)$, see [**8**], [**9**]. We get

$$(-\Delta+\gamma^2)\sigma_{\nu,l,j} \;=\; -((l\nu+2j)^2-(l\nu)^2)\sigma_{\nu,l,j-1}+\gamma^2\sigma_{\nu,l,j}.$$

This implies

$$\begin{aligned}(-\Delta+\gamma^2)\tilde{w} \;&=\; -\sum_{j=j_0+1}^{m}\alpha_j((l\nu{+}2j)^2-(l\nu)^2)\sigma_{\nu,l,j-1}+\gamma^2\sum_{j=j_0+1}^{m}\alpha_j\sigma_{\nu,l,j}\\ &=\; -\alpha_{j_0+1}((l\nu+2(j_0+1))^2-(l\nu)^2)\sigma_{\nu,l,j_0}\\ &\quad+\sum_{j=j_0+2}^{m}\big(-\alpha_j((l\nu+2j)^2-(l\nu)^2)+\alpha_{j-1}\gamma^2\big)\sigma_{\nu,l,j-1}\\ &\quad+\alpha_m\,\gamma^2\,\sigma_{\nu,l,m}.\end{aligned}$$

Now we choose

$$\begin{aligned}\alpha_{j_0+1} \;&:=\; -\frac{1}{(l\nu+2j_0)^2-(l\nu)^2},\\ \alpha_j \;&:=\; \frac{\alpha_{j-1}\gamma^2}{(l\nu+2j)^2-(l\nu)^2\big)}, \quad j=j_0+2(1)m,\end{aligned}$$

this implies

$$(-\Delta+\gamma^2)\,\tilde{w}(r,\theta) \,=\, \sigma_{\nu,l,j_0}(r,\theta)+\alpha_m\gamma^2\sigma_{\nu,l,m}.$$

Defining

$$w(r,\theta) \,:=\, \varphi(r,\theta)\,\tilde{w}(r,\theta)$$

leads to

$$(-\Delta+\gamma^2)w \,=\, \varphi\,\sigma_{\nu,l,j_0}+\alpha_m\gamma^2\varphi\,\sigma_{\nu,l,m}-(\Delta\varphi)\tilde{w}-(\nabla\varphi)(\nabla\tilde{w})$$

and we have proven our result, because $\Delta\varphi$ and $\nabla\varphi$ are equal to 0 in $B_\varepsilon(0)$. □

In the next lemma we combine the last two results in order to describe the behavior of the functions v_n near the corners.

Lemma 2.9. *There is an* $\overline{s}\in(3.5,4]$ *such that all* v_n, $n\in\{1,\dots N\}$ (*see* (2.9), (2.10)) *have the following representation near the corner* ξ_m, $m\in\{1,\dots,M\}$,

$$\varphi v_n \,=\, v_{n,R}+\varphi\bigg(\sum_{l\in L_{\nu,\overline{s}}}\;\sum_{0<l\nu+2j<\overline{s}-1} C_{l,j}\sigma_{\nu,l,j} + \sum_{l\in M_{\nu,\overline{s}}}\;\sum_{0<l\nu+2j<\overline{s}-1} D_{l,k}S_{\nu,l,k}\bigg), \tag{2.24}$$

$v_{n,R}\in H^{\overline{s}}(D)$, $\nu=(\pi/\omega_m)$, $C_{l,j},D_{l,j}\in\mathbf{R}$. *Here we have again assumed* $\xi_m=0$ *and* $D\cap B_{2\varepsilon}(0)=K_{\omega_m}\cap B_{2\varepsilon}(0)$. *The coefficients* $C_{l,j}$ *and* $D_{l,j}$ *depend on* n *and* m, *but we don't indicate this dependence. The function* φ *is again a cut-off function* (*see* (2.20)).

Proof. Let $s\in(2.5,3]$ such that

$$s\notin\left\{l\frac{\pi}{\omega_j}\,\Big|\, j=1(1)M,\; l\in\mathbf{N}\right\}.$$

Define $\overline{s}:=s+1$. We will prove our assertion by induction.

$n=1$: v_1 is the solution of

$$(-\Delta+\gamma^2)v_1 \;=\; 0,$$
$$v_1|_\Gamma \;=\; F(\cdot,h).$$

Let $\tilde v_1$ be the solution of

$$(-\Delta+\gamma^2)\tilde v_1 \;=\; -(-\Delta+\gamma^2)(\gamma^- F(\cdot,h)),$$
$$\tilde v_1|_\Gamma \;=\; 0.$$

By Lemma 2.1 we know $\gamma^- F(\cdot,h)\in H^4(\mathbf{R}^2)$ and therefore

$$(-\Delta+\gamma^2)(\gamma^- F(\cdot,h)) \;\in\; H^2(\mathbf{R}^2) \;\subset\; H^{s-1}(D).$$

Theorem 2.7 implies that $\tilde v_1$ has a representation of the form (2.24), but

$$v_1 \;=\; \tilde v_1+\gamma^- F(\cdot,h)$$

and so the assertion is proved for $n=1$.

$n\to n+1$: v_{n+1} solves

$$(-\Delta+\gamma^2)v_{n+1}=\gamma^2 v_n,$$
$$v_{n+1}|_\Gamma = F(\cdot,(n+1)h).$$

Again we first look for the regularity of $\tilde v_{n+1}$ which solves

$$(-\Delta+\gamma^2)\tilde v_{n+1} \;=\; \gamma^2 v_n-(-\Delta+\gamma^2)(\gamma^- F(\cdot,(n+1)h),$$
$$\tilde v_{n+1} \;=\; 0.$$

By induction we know

$$\begin{aligned}&\varphi(\gamma^2 v_n-(-\Delta+\gamma^2)(\gamma^- F(\cdot,(n+1)h)))\\ &\;=\; v_{n,R}+\varphi\biggl(\sum_{l\in L_{\nu,\overline{s}}}\;\sum_{0<l\nu+2j<\overline{s}-1} C_{l,j}\sigma_{\nu,l,j}+\sum_{l\in L_{\nu,\overline{s}}}\;\sum_{0<l\nu+2j<\overline{s}-1} D_{l,j}S_{\nu,l,j}\biggr)\\ &\quad-\varphi(-\Delta+\gamma^2)\gamma^- F(\cdot,(n+1)h).\end{aligned}$$

Lemma 2.8, for example with $t=s+1$, shows the existence of $w(r,\theta)$,

(2.25)

$$w(r,\theta) \;=\; \varphi(r,\theta)\biggl(\sum_{l\in L_{\nu,\overline{s}}}\sum_{j=1}^{k_l'} C'_{l,j}\sigma_{\nu,l,j}(r,\theta)+\sum_{l\in M_{\nu,\overline{s}}}\sum_{j=0}^{k_l''} D'_{l,j}S_{\nu,l,j}(r,\theta)\biggr)$$

such that

$$\begin{aligned}(-\Delta+\gamma^2)(\varphi\tilde v_{n+1}-w) \;&=\; \varphi(-\Delta+\gamma^2)\tilde v_{n+1}-(\nabla\varphi)(\nabla\tilde v_{n+1})\\ &\quad -(\Delta\varphi)\tilde v_{n+1}+(-\Delta+\gamma^2)w\\ &=\; v_{n,R}-\varphi(-\Delta+\gamma^2)\gamma^-F(\cdot,(n+1)h)+w_R\\ &\quad -(\nabla\varphi)(\nabla\tilde v_{n+1})-(\Delta\varphi)\tilde v_{n+1}\end{aligned}$$

Here $v_{n,R}\in H^{s+1}(D)$, $\varphi\gamma^-F(\cdot,(n+1)h)\in H^4(\mathbf{R}^2)$, $(\nabla\varphi)(\nabla\tilde v_{n+1})$, $(\Delta\varphi)\tilde v_{n+1}\in H^3(D)$ because $\Delta\varphi$ and $\nabla\varphi$ are zero near the origin and $\tilde v_{n+1}\in H^4(D\setminus U(C))$, where $U(C)$ is a neighborhood of the corners of Γ. $w_R\in H^{s+1}(D)$ by Lemma 2.8 and therefore the right-hand side belongs to $H^{s-1}(D)$ and Theorem 2.7 implies

$$\begin{aligned}\varphi\tilde v_{n+1}-w \;=\; \tilde v_{n+1,R}+\varphi\Big(&\sum_{l\in L_{\nu,\overline{s}}}\;\sum_{0<l\nu+2j<\overline{s}-1} C'_{l,j}\sigma_{\nu,l,j}\\ &+\sum_{l\in M_{\nu,\overline{s}}}\;\sum_{0<l\nu+2j<\overline{s}-1} D'_{l,j}S_{\nu,l,j}\Big),\end{aligned}$$

$\tilde v_{n+1,R}\in H^{s+1}(D)$. Now (2.25) shows

$$\begin{aligned}\varphi\tilde v_{n+1} \;=\; \tilde v'_{n+1,R}+\varphi\Big(&\sum_{l\in L_{\nu,\overline{s}}}\;\sum_{0<l\nu+2j<\overline{s}-1} C''_{l,j}\sigma_{\nu,l,j}\\ &+\sum_{l\in M_{\nu,\overline{s}}}\;\sum_{0<l\nu+2j<\overline{s}-1} D''_{j,l}S_{\nu,l,j}\Big),\end{aligned}$$

$\tilde v'_{n+1}\in H^{s+1}(D)$. The equation

$$v_{n+1} \;=\; \tilde v_{n+1}+\gamma^-F(\cdot,(n+1)h)$$

now proves the assertion for $n+1$. $\square$

Remark. Lemma 2.9 shows that the kind of singularity of $(v_n)_{n=1(1)N}$ near the corners stays essentially the same during the evolution.

3. The boundary integral operator and its properties. We denote by P the operator which appeared in equation (2.10)

$$Pu := -\Delta u + \gamma^2 u \tag{3.1}$$

The fundamental solution Φ_0 for P is given by

$$\Phi_0(x,y) = K_0(\gamma|x-y|), \quad x \neq y,\ x,y \in \mathbf{R}^2. \tag{3.2}$$

By the same letter Φ_0 we will also denote Green's operator for P with kernel Φ_0. The modified Bessel functions K_0 and I_0 are defined in the following way [**1**]

$$\begin{aligned} K_0(z) &= -\left(\ln\left(\frac{z}{2}\right) + \gamma_{\mathrm{Eul}}\right) I_0(z) \\ &\quad + \sum_{j=1}^{\infty} \left(\frac{1}{(j!)^2}\left(\frac{z^2}{4}\right)^j \sum_{k=1}^{j} \frac{1}{k}\right) \quad z \in \mathbf{C} \setminus \{y|y \le 0\}, \end{aligned} \tag{3.3}$$

$$I_0(z) = \sum_{j=0}^{\infty} \frac{1}{(j!)^2}\left(\frac{z^2}{4}\right)^2, \quad z \in \mathbf{C}. \tag{3.4}$$

Chapko and Kress used a special modification of the above given fundamental solution in order to avoid the calculation of integrals over the unbounded domain D. They introduced the functions

$$\begin{aligned} \Phi_n(x,y) := K_0(\gamma|x-y|)\, p_n(|x-y|) - K_0'(\gamma|x-y|)\, q_n(|x-y|), \\ n \in \mathbf{N}_0. \end{aligned} \tag{3.5}$$

The functions p_n are even polynomials, $p_n(0) = 1$, $n \in \mathbf{N}_0$, and $\deg(p_n) \in \{n-1, n\}$. The functions q_n are odd polynomials with $\deg(q_n) \in \{n-1, n\}$ and we have $p_0(x) \equiv 1$, $q_0(x) \equiv 0$. The construction of p_n and q_n is given explicitly in their article [**2**].

The functions Φ_n have the following property

$$P_y\, \Phi_n(x,y) = \gamma^2\, \Phi_{n-1}(x,y), \quad x \neq y,\ n \in \mathbf{N}, \tag{3.6}$$

and $\Phi_n(x,y)$ has the same kind of singularity at $x = y$ as the fundamental solution K_0.

In this paper we will only study a single layer approach for the solution of (2.9). We make the following ansatz

$$\begin{aligned} \tilde{v}_n(x) = -\frac{1}{\pi} \sum_{m=1}^{n} \int_{\Gamma} \Phi_{n-m}(x,y)\varphi_m(y)\, ds_y, \\ x \in \mathbf{R}^2 \setminus \Gamma,\ n = 1(1)N, \end{aligned} \tag{3.7}$$

where the functions $\varphi_m \in L^2(\Gamma)$, $m = 1(1)N$, are unknown. Now the equations (3.6) and (3.7) imply the following lemma.

Lemma 3.1. *We have $\tilde{v}_n = v_n$, $n = 1(1)N$, if and only if the sequence $(\varphi_n)_{n=1(1)N}$ solves the following system of integral equations*

$$(L\varphi_n)(x) = F(x, nh) + \frac{1}{\pi} \sum_{m=1}^{n-1} \int_\Gamma \Phi_{n-m}(x,y)\varphi_m(y)\,ds_y, \qquad x \in \Gamma,\ n = 1(1)N, \tag{3.8}$$

where the integral operator L is given by

$$(L\varphi)(x) := -\frac{1}{\pi} \int_\Gamma \Phi_0(x,y)\varphi(y)\,ds_y. \tag{3.9}$$

For the study of the mapping properties of L we follow the article of Costabel [**3**]. The first step is to define $\partial_\nu u|_\Gamma$, where ν is the outer normal of Γ for a sufficiently large function space. We introduce the following Hilbert spaces

$$\begin{cases} H^1_P(D) := \{u \in H^1(D) \mid Pu \in L^2(D)\}, \\ H^1_P(D^c) := \{u \in H^1(D^c) \mid Pu \in L^2(D^c)\}, \end{cases} \tag{3.10}$$

where $D^c := \mathbf{R}^2 \setminus \overline{D}$ and the norm is given by

$$\begin{cases} \|u\|^2_{H^1_P(D)} := \|u\|^2_{H^1} + \|Pu\|^2_{L^2(D)}, \\ \|u\|^2_{H^1_P(D^c)} := \|u\|^2_{H^1} + \|Pu\|^2_{L^2(D^c)}. \end{cases} \tag{3.11}$$

By γ_0 we denote the trace operator

$$\gamma_0 u := u|_\Gamma. \tag{3.12}$$

Lemma 3.2 [**3**, Lemma 3.6]. *The operator*

$$\gamma_0 : H^s_{\mathrm{loc}}(\mathbf{R}^2) \longrightarrow H^{s-(1/2)}(\Gamma)$$

is continuous for $s \in [(1/2),(3/2)]$, *and there is a continuous right inverse* γ_0^- *of* γ_0

$$\gamma_0^- \,:\, H^{s-(1/2)}(\Gamma) \longrightarrow H^s_{\mathrm{comp}}(\mathbf{R}^2),$$

$s \in [(1/2),1]$.

For $\varphi \in H^s(\Gamma)$, $s \ge -1$, the function

$$(L_0\varphi)(x) \,:=\, -\frac{1}{\pi}\int_\Gamma \Phi_0(x,y)\varphi(y)\,ds_y, \quad x \in \mathbf{R}^2 \setminus \Gamma, \tag{3.13}$$

is well defined, because $\Phi_0(x,\cdot) \in H^1(\Gamma)$. This definition implies

$$L \,=\, \gamma_0 \circ L_0. \tag{3.14}$$

In the next lemma we extend the definition of ∂_ν to functions in $H^1_P(D^c)$ and $H^1_P(D)$, respectively, and collect some properties of the mapping $u \to \partial_\nu u|_\Gamma$. We omit the proof, because it is very similar to the proofs in [**3**].

Lemma 3.3. a. *For* $u \in H^1_P(D^c)$ *we define the mapping*

$$\begin{aligned}\varphi &\longrightarrow \langle \gamma_1^c u, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)},\\ &:= \int_{D^c} \nabla u \nabla(\gamma_0^-\varphi) + \gamma^2 u(\gamma_0^-\varphi)\,dx - \int_{D^c}(Pu)(\gamma_0^-\varphi)\,dx,\\ &\qquad \varphi \in H^{1/2}(\Gamma).\end{aligned} \tag{3.15}$$

$\gamma_1^c u$ *is a continuous mapping* $H^{1/2}(\Gamma) \to \mathbf{R}$ *and* $\gamma_1^c u$ *depends continuously on* u. *If* $u \in H^2(D^c)$, *then the mapping can be written in the following form*

$$\langle \gamma_1^c u, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \,=\, \int_\Gamma (\partial_\nu u)\varphi\,ds_y. \tag{3.16}$$

b. *For* $u \in H^1_P(D)$ *we define*

$$\begin{aligned}\varphi &\longrightarrow \langle \gamma_1 u, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}, \quad \varphi \in H^{1/2}(\Gamma),\\ &:= -\int_D \nabla u \nabla(\gamma_0^-\varphi) + \gamma^2 u(\gamma_0^-\varphi)\,dx + \int_D (Pu)\,(\gamma_0^-\varphi)\,dx.\end{aligned} \tag{3.17}$$

$\gamma_1 u$ is also continuous on $H^{1/2}(\Gamma)$ and γ_1 is continuous as a mapping from $H^1_P(D)$ into $H^{-1/2}(\Gamma)$. If $u \in H^2(D)$ we have

$$\langle \gamma_1 u, \varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} = \int_\Gamma \partial_\nu u\, \varphi\, ds_y. \tag{3.18}$$

With the help of γ_1 and γ_1^c we can prove the first and second Green formula for functions in $H^1_P(D)$ and $H^1_P(D^c)$, respectively,

(3.19)
$$\int_D (Pu)v\,dx \;=\; \int_D (\nabla u)(\nabla v) + \gamma^2 uv\,dx + \langle \gamma_1 u, \gamma_0 v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)},$$

(3.20)
$$\begin{aligned}\int_D (uPv - vPu)\,dx \;=\; &\langle \gamma_1 v, \gamma_0 u\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\\ &- \langle \gamma_1 u, \gamma_0 v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}, \quad u, v \in H^1_P(D),\end{aligned}$$

and

$$\begin{aligned}\int_{D^c} (Pu)v\,dx \;=\; &\int_{D^c} (\nabla u)(\nabla v) + \gamma^2 uv\,dx\\ &- \langle \gamma_1^c u, \gamma_0 v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)},\end{aligned} \tag{3.21}$$

(3.22)
$$\begin{aligned}\int_{D^c} (uPv - vPu)\,dx \;=\; &\langle \gamma_1^c u, \gamma_0 v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\\ &- \langle \gamma_1^c v, \gamma_0 u\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}, \quad u, v \in H^1_P(D^c).\end{aligned}$$

Applying Green's formula with $v(y) = \Phi_0(x, y)$, $u \in H^2(\mathbf{R})$, and using the density of the H^2 functions in $H^1_P(D)$ and $H^1_P(D^c)$, respectively, we get the following results

Lemma 3.4. *Let $u \in L^2(\mathbf{R}^2)$, $u|_D \in H^1_P(D)$, $u|_{D^c} \in H^1_P(D^c)$. Then we have the following representation for u*

$$\begin{aligned}u(x) \;=\; &(\Phi_0 Pu)(x) + \langle \partial_\nu \Phi_0(x,\cdot), [u]\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\\ &- \langle [\gamma_1 u], \Phi_0(x,\cdot)\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.\end{aligned}$$

Remark. Here we have used the notation

$$[u] \;=\; \gamma_0 u|_D - \gamma_0 u|_{D^c} \tag{3.23}$$

$$[\gamma_1 u] \;=\; \gamma_1 u|_D - \gamma_1^c u|_{D^c}. \tag{3.24}$$

Now we can prove two results, which are a little bit stronger than the corresponding results in Theorem 1 (iii) and Theorem 2 in [**3**], because the operator P which we study here is much simpler than the operators which were studied by Costabel.

Theorem 3.5. a. *$L : H^{(-1/2)+\sigma}(\Gamma) \to H^{(1/2)+\sigma}(\Gamma)$ is continuous, $\sigma \in (-1/2, 1/2)$.*

b. *There is a constant $C_L > 0$ such that*

$$\langle \varphi, L\varphi \rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)} \;\geq\; C_L \|\varphi\|^2_{H^{-1/2}(\Gamma)}, \quad \forall\, \varphi \in H^{-1/2}(\Gamma).$$

Proof. a. We have

$$L_0 \;=\; \Phi_0 \circ \gamma_0'.$$

By Lemma 3.2 we know

$$\gamma_0 \;:\; H^s(\mathbf{R}^2) \;\longrightarrow\; H^{s-(1/2)}(\Gamma), \quad s \in \left(-\frac{1}{2}, \frac{1}{2}\right),$$

which implies

$$\gamma_0' \;:\; H^{(1/2)-s}(\Gamma) \;\longrightarrow\; H^{-s}_{\mathrm{comp}}(\Gamma).$$

Because of $\gamma^2 > 0$ we have

$$\Phi_0 \;:\; H^s(\mathbf{R}^2) \;\longrightarrow\; H^{s+2}(\mathbf{R}^2), \quad \forall\, s \in \mathbf{R}^2.$$

These formulas show

$$L_0 \;:\; H^{(1/2)-s}(\Gamma) \;\longrightarrow\; H^{2-s}(\mathbf{R}^2), \quad s \in \left(\frac{1}{2}, \frac{3}{2}\right).$$

Denoting $1-s$ by σ this implies that L_0 is a continuous operator from $H^{\sigma-(1/2)}(\Gamma)$ into $H^{1+\sigma}(\mathbf{R}^2)$. Now we have

$$L \;=\; \gamma_0 \circ \Phi_0 \circ \gamma_0' \;:\; H^{\sigma-1/2}(\Gamma) \;\longrightarrow\; H^{\sigma+(1/2)}(\Gamma),$$

by Lemma 3.2. This proves a).

b. First we note the jump relations for the single and double layer potentials

$$[\gamma_0 L_0\varphi] = 0, \quad [\gamma_1 L_0\varphi] = -\varphi, \quad \varphi \in H^{-1/2}(\Gamma),$$

see [**3**].

For $\varphi \in H^{-(1/2)}(\Gamma)$ we have $L_0\varphi \in H^{3/2}(\mathbf{R}^2)$ and because Φ_0 is the fundamental solution for P we have

$$L_0\varphi \in H_P^1(D), \quad \text{resp. } L_0\varphi \in H_P^1(D^c).$$

The first Green formula gives

$$\begin{aligned}\int_{D^c} |\nabla L_0\varphi|^2 + \gamma^2 |L_0\varphi|^2\, dx &= \langle \gamma_1^c L_0\varphi, \gamma_0 L_0\varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\\ \int_{D} |\nabla L_0\varphi|^2 + \gamma^2 |L_0\varphi|^2\, dx &= -\langle \gamma_1 L_0\varphi, \gamma_0 L_0\varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.\end{aligned}$$

By addition we get

$$\begin{aligned}\int_{\mathbf{R}^2} |\nabla L_0\varphi|^2 + \gamma^2 |L_0\varphi|^2\, dx &= -\langle [\gamma_1 L_0\varphi], \gamma_0 L_0\varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\\ &= -\langle [\gamma_1 L_0\varphi], L\varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\\ &= \langle \varphi, L\varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.\end{aligned}$$

Here we have also used the jump relations from above. Now we put our results together and get

$$\begin{aligned}\|\varphi\|^2_{H^{-1/2}(\Gamma)} &= \|[\gamma_1 L_0\varphi]\|^2_{H^{-1/2}(\Gamma)}\\ &\overset{\text{Lemma 3.3}}{\leq} C\|L_0\varphi\|^2_{H^1(\mathbf{R}^2)}\\ &\leq C\max\{1, 1/\gamma^2\}\int_{\mathbf{R}^2} |\nabla L_0\varphi|^2 + \gamma^2|L_0\varphi|^2\, dx\\ &= C\max\{1, 1/\gamma^2\}\langle \varphi, L\varphi\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}.\end{aligned}$$

This proves b). $\square$

For our further studies we introduce parametrizations of Γ and investigate the properties of the resulting integral operators on $[0, 2\pi]$. By

$$\alpha : [0, 2\pi] \longrightarrow \mathbf{R}^2, \tag{3.25}$$

we denote a parametrization of Γ, which is piecewise linear

$$\alpha|_{[\tau_j,\tau_{j+1}]}(\tau) = \xi_j + \frac{(\tau - \tau_j)}{\tau_{j+1} - \tau_j}(\xi_{j+1} - \xi_j), \quad j = 0(1)M-1, \tag{3.26}$$

$0 = \tau_0 < \tau_1 < \ldots < \tau_M$. To take care of the corner singularities of the solutions of (3.8) we introduce a further parametrization α_q, $q = (q_0, \ldots, q_{M-1})$, $q_j \geq 1$, $j = 0(1)M-1$,

$$\alpha_q := \alpha \circ \tilde{\alpha}_q, \tag{3.27}$$

where $\tilde{\alpha}_q : [0, 2\pi] \xrightarrow{1:1} [0, 2\pi]$ is monotone increasing and has the following properties

$$\begin{cases} \tilde{\alpha}_q \in C^3([0, 2\pi]), \\ \tilde{\alpha}_q'(\tau) > 0, \quad \tau \in [0, 2\pi] \setminus \{\tau_0, \ldots, \tau_M\}, \\ \tilde{\alpha}_q(\tau) = \tau_j + c_j \operatorname{sgn}(\tau - \tau_j)|\tau - \tau_j|^{q_j}, \\ \qquad |\tau - \tau_j| \text{ sufficiently small}, \ c_j > 0, \ j = 0(1)M-1. \end{cases} \tag{3.28}$$

An explicit construction of $\tilde{\alpha}_q$, given q, can be found for example in the article [**6**]. This construction gives also

$$\tilde{\alpha}_{(1,1,\ldots,1)}(\tau) \equiv \tau. \tag{3.29}$$

Equation (3.8) can be written in the following form

$$\begin{aligned} &-\frac{1}{\pi}\int_0^{2\pi} \Phi_0(\alpha_q(s), \alpha_q(\tau))\varphi_n(\alpha_q(\tau))|\alpha_q'(\tau)|\, d\tau \\ &\quad = F(\alpha_q(s), nh) \\ &\qquad + \frac{1}{\pi}\sum_{m=1}^{n-1}\int_0^{2\pi} \Phi_{n-m}(\alpha_q(s), \alpha_q(\tau))\varphi_m(\alpha_q(\tau))|\alpha_q'(\tau)|\, d\tau. \end{aligned} \tag{3.30}$$

We define $\tilde{\varphi}_{q,m}(\tau) := \varphi_m(\alpha_q(\tau))|\alpha_q'(\tau)|$. This leads us to the study of the following system of integral equations on $[0, 2\pi]$:

(3.31)

$$\begin{aligned}(A_q\tilde{\varphi}_{n,q})(s) & \\ &= F(\alpha_q(s), nh) + \frac{1}{\pi}\sum_{m=1}^{n-1}\int_0^{2\pi}\Phi_{n-m}(\alpha_q(s),\alpha_q(\tau))\tilde{\varphi}_{q,m}(\tau)\,d\tau, \\ & \qquad n = 1(1)N,\end{aligned}$$

where the operator A_q is given by

(3.32)

$$\begin{aligned}(A_q\psi)(s) &:= -\frac{1}{\pi}\int_0^{2\pi}\Phi_0(\alpha_q(s),\alpha_q(\tau))\psi(\tau)\,d\tau \\ &= \frac{1}{\pi}\int_0^{2\pi}\ln|\alpha_q(s)-\alpha_q(\tau)|\,\psi(\tau)\,d\tau \\ &\quad + \int_0^{2\pi}\ln|\alpha_q(s)-\alpha_q(\tau)|^2\,k_1(|\alpha_q(s)-\alpha_q(\tau)|^2)\,\psi(\tau)\,d\tau \\ &\quad + \int_0^{2\pi}k_2(|\alpha_q(s)-\alpha_q(\tau)|^2)\psi(\tau)\,d\tau \\ &=: (A_q^{(1)}\psi)(s) + (A_q^{(2)}\psi)(s) + (A_q^{(3)}\psi)(s).\end{aligned}$$

The functions k_1 and k_2 are holomorphic functions on $\mathbf{C}$ and have the following form

$$\begin{aligned}k_1(z) &= \sum_{j=1}^{\infty}a_j z^j, \quad a_j \in \mathbf{R},\ j = 1(1)\infty, \\ k_2(z) &= \sum_{j=0}^{\infty}b_j z^j, \quad b_j \in \mathbf{R},\ j = 0(1)\infty.\end{aligned}$$

The following result was proved by Graham and Elschner.

Theorem 3.6 [**6**, Theorem 2]. *Let* $q = (q_0, \dots, q_{M-1})$, $q_j \geq 1$, $j = 0(1)M-1$.

$$A_q^{(1)} : L^2([0,2\pi]) \longrightarrow H^1([0,2\pi])$$

is continuous and has a bounded inverse.

A direct calculation shows that the functions

$$\ln(|\alpha_q(s)-\alpha_q(\tau)|^2)k_1(|\alpha_q(s)-\alpha_q(\tau)|^2) \quad \text{and} \quad k_2(|\alpha_q(s)-\alpha_q(\tau)|^2)$$

are two times differentiable for $s \neq t$ and both derivatives are bounded on $[0,2\pi]^2$. This implies

$$A_q^{(2)},\, A_q^{(3)} \,:\, L^2([0,2\pi]) \longrightarrow H^2([0,2\pi]) \tag{3.33}$$

are continuous. Theorem 3.6, equation (3.33) and the compact inclusion of $H^2([0,2\pi])$ into $H^1([0,2\pi])$ imply

Corollary 3.7. *Let $q=(q_0,\dots,q_{M-1})$, $q_j \geq 1$, $j=0(1)M-1$.*

$$A_q \,:\, L^2([0,2\pi]) \longrightarrow H^1([0,2\pi])$$

is a Fredholm operator with index 0.

Now we can use the same arguments as Elschner and Graham in [**6**] to prove the following theorem.

Theorem 3.8. *Let $q=(q_0,\dots,q_{M-1})$, $q_j \geq 1$, $j=0(1)M-1$.*

$$A_q \,:\, L^2([0,2\pi]) \longrightarrow H^1([0,2\pi])$$

is continuous and has a bounded inverse.

Proof. Because of Corollary 3.7 we only have to show that

$$A_q u \,=\, 0, \quad u \in L^2([0,2\pi])$$

implies $u=0$.

Let $\tilde{q}=(1,\dots,1)$. Then $\tilde{u}(\tau) := (u\circ\alpha_q^{-1})(\tau)(\alpha_q^{-1})'(\tau)$ solves

$$A_{\tilde{q}}\tilde{u} \,=\, 0.$$

Our aim is to show that $\tilde{u} \in L^p([0,2\pi])$, $p>1$. Near τ_j, $j \in \{0,\dots,M-1\}$, we have

$$\alpha_q^{-1}(\nu) \,=\, \tau_j + \frac{1}{c_j^{1/(q_j)}}\,\operatorname{sgn}(\nu-\tau_j)\,|\nu-\tau_j|^{1/q_j},$$

$$(\alpha_q^{-1})'(\nu) \,=\, \frac{1}{c_j^{1/q_j}q_j}\,|\nu-\tau_j|^{(1-q_j)/q_j}.$$

For some $\varepsilon > 0$ and $p \in (1, 2)$ we have

$$
\begin{aligned}
\int_{s_j}^{s_j+\varepsilon} |\tilde{u}(\tau)|^p \, d\tau \; &= \; \tilde{c}_j \int_{\tau_j}^{\tau_j+\varepsilon} \left| u(\tilde{\alpha}_q^{-1}(\tau))(\tau - \tau_j)^{((1-q_j)/q_j)} \right|^p d\tau \\
&= \; \hat{c}_j \int_{\tau_j}^{\tau_j+\tilde{\varepsilon}} |u(s)^p| \, |s - \tau_j|^{p(1-q_j)} |s - \tau_j|^{q_j - 1} \, ds \\
&\leq \; \hat{c}_j \left(\int_{\tau_j}^{\tau_j+\tilde{\varepsilon}} |u(s)|^2 \, ds \right)^{p/2} \\
&\quad \cdot \left(\int_{\tau_j}^{\tau_j+\tilde{\varepsilon}} (\tau - \tau_j)^{(2(p-1)(1-q_j))/(2-p)} \, d\tau \right)^{(2-p)/2} \\
&< \; \infty,
\end{aligned}
$$

if $p - 1$ is sufficiently small. Here $\tilde{\varepsilon} := (\varepsilon / c_j)^{1/q_j}$. This implies $\tilde{u} \in L^p([0, 2\pi])$. But then

$$
\tilde{u} \circ \alpha^{-1} \; \in \; L^p(\Gamma) \; \subset \; H^{-1/2}(\Gamma)
$$

lies in the kernel of the operator L. Theorem 3.5 implies $\tilde{u} \circ \alpha^{-1} = 0$. This shows the injectivity of A_q and together with Corollary 3.7 we have proved our theorem. $\square$

At least we will study the regularity of the solutions $(\varphi_n)_{n=1(1)N}$. This will give us the right exponents q_j near corners in order to get a quadratic convergence rate of our linear approximations.

Lemma 3.9. *Let $(\varphi_n)_{n=1(1)N}$ be the solution of the sequence of integral equations* (3.8). *For arbitrary $n \in \{1, \ldots, N\}$ and $m \in \{1, \ldots, M\}$ we find $\delta > 0$ such that φ_n has the following representation near corner m (we assume $\tau_m = 0$)*

$$
(\varphi_n \circ \alpha)(\tau) \; = \; \varphi_R(\tau) + \varphi_S(\tau), \quad \tau \in U_\delta(\tau_m), \tag{3.34}
$$

where $\varphi_R|_{[0,\delta]} \in H^2([0, \delta])$, $\varphi_R|_{[-\delta,0]} \in H^2([-\delta, 0])$ and the function φ_S belongs to C^∞ outside $\{0\}$. Near the origin the growth of φ_S is estimated by

$$
|\varphi_S(\tau)| \; \leq \; C|\tau - \tau_m|^{\frac{\pi}{\omega_m} - 1} |\log|\tau - \tau_m||, \tag{3.35}
$$

$C > 0$.

Proof. We first define

$$w_n(x) := -\frac{1}{\pi}\int_\Gamma \Phi_0(x,y)\varphi_n(y)\,ds_y, \quad x \in \mathbf{R}^2, \tag{3.36}$$

$$\begin{aligned} R_n(x) &:= \frac{1}{\pi}\sum_{j=1}^{n-1}\int_\Gamma \Phi_{n-j}(x,y)\varphi_j(y)\,ds_y \\ &= -\frac{1}{\pi}\sum_{j=1}^{n-1} w_j(x) + \frac{1}{\pi}\sum_{j=1}^{n-1}\int_\Gamma (\Phi_{n-j}-\Phi_0)(x,y)\varphi_j(y)\,ds_y \\ &\qquad x \in \mathbf{R}^2. \end{aligned} \tag{3.37}$$

We remark here that the kernel $\Phi_{n-j} - \Phi_0$ fulfills

$$|\Phi_{n-j}(x,y) - \Phi_0(x,y)| \le C|x-y|\log(|x-y|) \tag{3.38}$$

in the neighborhood of the diagonal. The jump relations of the single layer potential (see [**3**]) imply

$$[\gamma_1 w_n] = -\varphi_n \tag{3.39}$$

$$[\gamma_1 R_n] = \sum_{j=1}^{n-1}\varphi_j \tag{3.40}$$

by (3.38). Formula (3.39) shows that we can deduce the regularity of φ_n from the regularity of w_n. The function w_n fulfills the following differential differential equation in $\mathbf{R}^2 \setminus \Gamma$

$$\begin{aligned} (-\Delta + \gamma^2)w_n(x) &= 0, \quad x \in \mathbf{R}^2 \setminus \Gamma, \\ w_n|_\Gamma &= F(\cdot, nh) + R_n|_\Gamma. \end{aligned}$$

Now we can repeat the arguments of the proof of Lemma 2.9. Again the regularity of w_n follows from the regularity of $\tilde{w}_n$ which solves

$$\begin{aligned} (-\Delta + \gamma^2)\tilde{w}_n(x) &= -(-\Delta + \gamma^2)(\gamma^- F(\cdot, nh) + R_n)(x), \quad x \in \mathbf{R}^2 \setminus \Gamma, \\ &= -(-\Delta + \gamma^2)(\gamma^- F(\cdot, nh))(x) - \gamma^2 v_{n-1}(x), \\ \tilde{w}_n|_\Gamma &= 0. \end{aligned}$$

where we have used (3.6). If we assume that $\xi_m = 0$, $B_{2\varepsilon}(0) \cap D = B_{2\varepsilon}(0) \cap K_{\omega_m}$, $\varepsilon > 0$, then it follows (using the same arguments as in the proof of Lemma 2.9)

$$\begin{aligned}\varphi \tilde{w}_n|_{B_\varepsilon(0)\cap D} \,=\, \tilde{v}_R + \varphi\Bigg(&\sum_{l\in L_{\nu,\overline{s}}} \sum_{0<l\nu+2j<\overline{s}-1} C_{l,j}\sigma_{\nu,l,j} \\ &+ \sum_{l\in M_{\nu,\overline{s}}} \sum_{0<l\nu+2j<\overline{s}-1} D_{l,j}S_{\nu,l,j}\Bigg),\end{aligned}$$

$v_R \in H^{\overline{s}}(D)$, $\nu = (\pi/\omega_m)$, $\overline{s} \in (3.5, 4]$. Here we assume in contrast to Theorem 2.7

$$\overline{s} \,\notin\, \left\{ l\,\frac{\pi}{\omega_j},\; l\,\frac{\pi}{2\pi - \omega_j} \,\Big|\, l \in \mathbf{N} \right\},$$

but this is clearly possible.

An analogous formula holds for $\varphi \tilde{w}_n|_{B_\varepsilon(0)\cap D^c}$, where ν has to be replaced by $(\pi/(2\pi - \omega_m))$ and $\tilde{v}_r$ by $\tilde{v}_{1,R} \in H^2(D^c)$.

The relation

$$w_n \,=\, \tilde{w}_n + \gamma^- F(\cdot, nh) + R_n$$

proves a similar representation for w_n around each corner because $\gamma^- F(\cdot, nh) \in H^4(\mathbf{R}^2)$. Now (3.39) and (3.40) show

$$\begin{aligned}-\varphi_n \,&=\, [\gamma_1 w_n] \\ &=\, [\gamma_1 \tilde{w}_n] + [\gamma_1(\gamma^- F(.,\cdot, nh))] + [\gamma_1 R_n] \\ &=\, [\gamma_1 \tilde{w}_n] + \sum_{j=1}^{n-1} \varphi_j.\end{aligned}$$

Near the corner ξ_m we get

$$\begin{aligned}-\varphi_n \,=\, \Bigg[\frac{\partial}{\partial n}\Bigg(&\sum_{l\in L_{\nu,\overline{s}}} \sum_{0<l\nu+2j<\overline{s}-1} C_{l,j}\sigma_{\nu,l,j} + \sum_{l\in M_{\nu,\overline{s}}} \sum_{0<l\nu+2j<\overline{s}-1} D_{l,j}S_{\nu,l,j} \\ &+ \sum_{l\in L_{\nu_1,\overline{s}}} \sum_{0<l\nu_1+2j<\overline{s}-1} C'_{l,j}\sigma'_{\nu_1,l,j} + \sum_{l\in M_{\nu_1,\overline{s}}} \sum_{0<l\nu_1+2j<\overline{s}-1} D'_{l,j}S'_{\nu_1,l,j}\Bigg)\Bigg] \\ &+ \sum_{j=1}^{n-1} \varphi_j + \frac{\partial}{\partial n}\,(\tilde{v}_R - \tilde{v}_{1,R})|_\Gamma.\end{aligned}$$

Here $\sigma'_{\nu_1,l,j}$, $S'_{\nu_1,l,j}$ have the same form as $\sigma_{\nu,l,j}$ and $S_{\nu,l,j}$, but on the complementary angle (see (2.17)). We have denoted the outer normal by n because ν has another meaning here. Now an induction and the formulas (2.17) for the functions $\sigma_{\nu,l,j}$ and $S_{\nu,l,j}$ prove the lemma. □

Lemma 3.10. *Let*

$$q_j > \frac{5}{2}\,\frac{\overline{\omega}_j}{\pi}, \quad j = 1(1)M. \tag{3.41}$$

Then the solution $\varphi_{n,q}$, $n = 1(1)N$, *of the sequence* (3.31) *belongs to* $H^2([0,2\pi])$.

Proof. By (3.30) and (3.31) we know

$$\varphi_{n,q}(\tau) = \varphi_n(\alpha_q(\tau))|\alpha'(\tau)|.$$

Only the smoothness of $\varphi_{n,q}$ near the points τ_m, $m = 1(1)M$, has to be considered, because away from the corners the operator L is a pseudo differential operator of order -1 and the right-hand side of (3.30) is in $H^4([\tau_j,\tau_{j+1}])$, $j = 0(1)M-1$.

But near the corners we have the representation of Lemma 3.9. This implies

$$\varphi_{n,q}(\tau) = \varphi_{n,R}(\alpha_q(\tau))|\alpha'(\tau)| + \varphi_{n,S}(\alpha_q(\tau))|\alpha'(\tau)|,\\ \tau \in U_\delta(\tau_m).$$

In the neighborhood of τ_m we have

$$|\varphi_{n,S}(\alpha_q(\tau))\alpha_q'(\tau)| \\ \le C|\tau-\tau_m|^{q_m[(\pi/\overline{\omega}_m)-1]}\,|\log|\tau-\tau_m||\,|\tau-\tau_m|^{q_m-1}$$

and we note

$$q_m\left(\frac{\pi}{\overline{\omega}_m}-1\right) + q_m - 1 = q_m\,\frac{\pi}{\overline{\omega}_m} - 1 \\ \ge \frac{3}{2},$$

if (3.41) is fulfilled. This shows that the first part is in H^2. The second term on the right-hand side is also in H^2 on both sides of τ_m and the function and the first derivative is zero at τ_m. This shows that this factor is in $H^2(U_\delta(\tau_m))$. Now Lemma 3.10 is proven. □

4. The numerical algorithm. In order to approximate v_n, $n = 1(1)N$, see (2.9), numerically, we use the ansatz (3.7) and calculate approximations for $\tilde{\varphi}_{q,m}$, $m = 1(1)N$, see (3.30).

We use a collocation method, where the trial space consists of periodic, continuous and piecewise linear functions on a uniform grid of $[0, 2\pi]$. We denote the trial space by

(4.1)
$$\mathcal{T}_K := \{u \in C_{\text{per}}([0,2\pi]) | u|_{[(j/K),((j+1)/K)]} \text{ linear } j = 0(1)K-1\},$$

and we will always assume that

$$\tau_{j,K} := \tau_j K \in \{0, \dots, K\}, \quad j = 0(1)M. \tag{4.2}$$

The solution $\hat{\varphi}_{q,n,K} \in \mathcal{T}_K$ of the collocation method is defined by

$$\begin{aligned}(A_q\hat{\varphi}_{q,n,K})\left(\frac{j}{K}\right) &= F\left(\alpha_q\left(\frac{j}{K}\right), nh\right)\\ &\quad + \frac{1}{\pi}\sum_{m=1}^{n-1}\int_0^{2\pi}\Phi_{n-m}\left(\alpha_q\left(\frac{j}{K}\right), \alpha_q(\tau)\right)\hat{\varphi}_{q,m,K}(\tau)\,d\tau,\\ &\quad j = 0(1)K-1,\ n = 1(1)N.\end{aligned}$$

As usual, see [**6**], we have to introduce a modification of the above method in order to prove the stability of the collocation method. Let $i^* \in \mathbf{N}_0$ be fixed. For K sufficiently large we denote by $\hat{\varphi}^*_{q,n,K}$ the solution of

$$\begin{aligned}(A_q\hat{\varphi}^*_{q,n,K})\left(\frac{j}{K}\right) &= F\left(\alpha_q\left(\frac{j}{K}\right), nh\right)\\ &\quad + \frac{1}{\pi}\sum_{m=1}^{n-1}\int_0^{2\pi}\Phi_{n-m}\left(\alpha_q\left(\frac{j}{K}\right), \alpha_q(\tau)\right)\hat{\varphi}^*_{q,m,K}(\tau)\,d\tau, \qquad (4.4)\\ &\quad j = 0(1)K-1,\ |j-\tau_{j,K}| \geq i^*,\ n = 1(1)N,\end{aligned}$$

$$\hat{\varphi}^*_{q,n,K} \in \mathcal{T}^*_K := \left\{u \in \mathcal{T}_K \middle| u\left(\frac{j}{K}\right) = 0,\ |j-\tau_{j,K}| < i^*\right\}. \tag{4.5}$$

Theorem 4.1. *Let* (1.2), (2.6), (2.7) *and* (3.38) *be fulfilled,* $N \in \mathbf{N}$ *fixed. If* i^* *is chosen sufficiently large, there exists a* $K_0 \in \mathbf{N}$ *such that for every* $K \geq K_0$ *the solution of* (4.4) *is uniquely determined and we get*

$$\|\hat{\varphi}_{q,m} - \hat{\varphi}^*_{q,m,K}\|L^2[0,2\pi] \leq \frac{C}{K^2}, \quad m = 1(1)N, \tag{4.6}$$

where the constant C *does not depend on* K *or* m.

Proof. The approximation order of the spaces $\mathcal{T}_K$ for the solutions $\tilde{\varphi}_{q,m}$ is clear by Corollary 3.10 and (3.38). The stability of the collocation method for sufficiently large i^* is proved in [**6**] for the operator with a logarithmic kernel. But by Corollary 3.7 we know that A_q is only a compact perturbation of the logarithmic operator $A_q^{(1)}$. Theorem 3.8 shows that A_q has a bounded inverse and Theorem II.3.1. of [**7**] now proves the theorem. □

A possible numerical strategy for the approximate solution of (1.1) would be as follows (here we will indicate the dependence of our solutions on the time discretization parameter N which was fixed until now, but we will omit the i^*-modification, which is rarely needed in practice):

Given an $\varepsilon > 0$ choose $N \in \mathbf{N}$ and calculate $\hat{\varphi}^{(N)}_{q,n,K_1}$, $n = 1(1)N$, q chosen according to (3.41), such that

$$\|\hat{\varphi}^{(N)}_{q,n,K_1} - \hat{\varphi}^{(N)}_{q,n,2K_1}\|_{L^2([0,2\pi])} \leq \frac{3}{8}\,\varepsilon, \quad n = 1(1)N.$$

Then repeat the calculation with N replaced by $2N$. This gives $\hat{\varphi}^{(2N)}_{q,n,K_2}$, $n = 1(1)2N$, with the property

$$\|\hat{\varphi}^{(2N)}_{q,n,K_2} - \hat{\varphi}^{(2N)}_{q,n,2K_2}\|_{L^2([0,2\pi])} \leq \frac{3}{8}\,\varepsilon, \quad n = 1(1)2N.$$

If

$$\|\hat{\varphi}^{(N)}_{q,n,K_1} - \hat{\varphi}^{(2N)}_{q,2n,K_2}\|_{L^2([0,2\pi])} \leq \frac{1}{4}\,\varepsilon, \quad n = 1(1)N,$$

then accept $\hat{\varphi}^{(N)}_{q,n,K_1}$ as an approximation for $\hat{\varphi}^{(N)}_{q,n}$, $n = 1(1)N$. Otherwise start again with N replaced by $2N$.

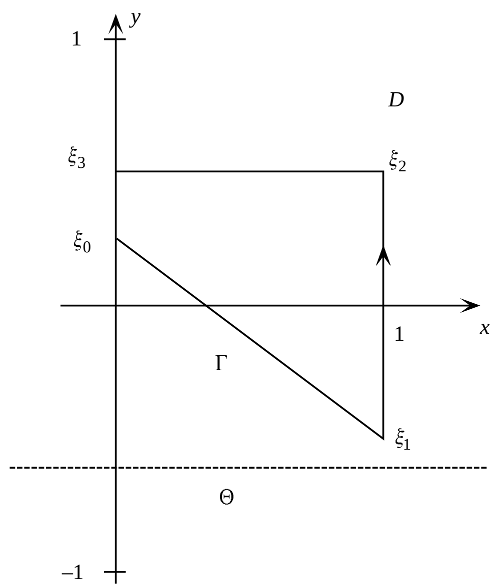

FIGURE 1. The domain D and its boundary Γ.

The approximations $\hat{v}^{(N)}_{q,n,K_1}$ for v_n, $n = 1(1)N$, are calculated by

(4.7)
$$\hat{v}^{(N)}_{q,n,K_1}(x) := -\frac{1}{\pi}\sum_{m=1}^{n}\int_\Gamma \Phi^{(N)}_{n-m}(x,y)\hat{\varphi}^{(N)}_{q,m,K_1}(y)\, ds_y, \quad n = 1(1)N.$$

Here we have also indicated the dependence of the kernels Φ on the time discretization.

5. A numerical example. To illustrate the convergence results of Theorem 2.6 and Theorem 4.1 we solve numerically equation (1.1) for the following case.

The polygon Γ has the corners $\xi_0 = (0, 0.25)$, $\xi_1 = (1, -.5)$, $\xi_2 = (1, 0.5)$, $\xi_3 = (0, 0.5)$ and $\xi_4 = \xi_0$ (see Figure 1). The piecewise linear parametrization $\alpha : [0, 2\pi] \to \Gamma$ is defined by $\tau_i = \pi/2 \times i$, $i = 0(1)4$ (see (3.25) and (3.26)).

The right-hand side F of the equation (1.1) is defined by

(5.1) $$F(x,t) := t^2(1 + \sin(\alpha^{-1}(x) + t)), \quad (x,t) \in \Gamma \times [0,1],$$

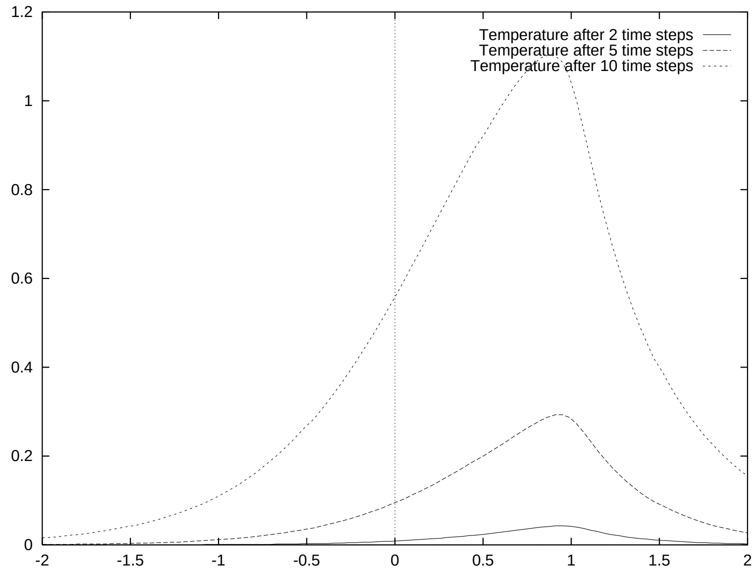

FIGURE 2. The temperature development at line Θ.

so (1.2), (2.6) and (2.7) are fulfilled and we have $T = 1$.

For the numerical computations we use two grading vectors (see (3.27)),

$$q_1 = (1,1,1,1) \quad \text{and} \quad q_2 = (4.5, 4.5, 4, 4). \tag{5.2}$$

The entries of vector q_2 are about 0.25 greater than the requirement in (3.41). Besides the function $\hat{\varphi}^{(N)}_{q,n,K}$ we will also compute the function $\hat{v}^{(N)}_{q,n,K}$ (see (4.7)), at 201 equidistributed points on the line $\Theta = [-2,2] \times \{-0.6\}$. This will allow us to estimate the rate of convergence of $\hat{v}^{(N)}_{q,n,K}$ in $L^2(\Theta)$. To calculate the integrals over Γ we use a singularity subtraction technique and calculate the integrals over the logarithmic singularity explicitly and use 3 point Gauss formulas for the regular part.

In Figure 2 the development of the temperature along the line Θ is shown. For the calculation of the data we have used $N = 10$, 64 partitions of Γ and $q = (4.5, 4.5, 4, 4)$.

TABLE 1. Fixed time discretization, several space discretizations, grading vector q_1.

K	$\|\hat{\varphi}^{(4)}_{q_1,4,K} - \hat{\varphi}^{(4)}_{q_1,4,256}\|_{L^2(\Gamma)}$	EOC	$\|\hat{v}^{(4)}_{q_1,4,K} - \hat{v}^{(4)}_{q_2,4,256}\|_{L^2(\Theta)}$	EOC
8	0.71227		0.08339	
16	0.56359	0.34	0.04420	0.92
32	0.45004	0.33	0.02182	1.02
64	0.34708	0.38	0.01024	1.09
128	0.20804	0.74	0.00467	1.13

TABLE 2. Fixed time discretization, several space discretizations, grading vector q_2.

K	$\|\hat{\varphi}^{(4)}_{q_2,4,K} - \hat{\varphi}^{(4)}_{q_2,4,256}\|_{L^2(\Gamma)}$	EOC	$\|\hat{v}^{(4)}_{q_2,4,K} - \hat{v}^{(4)}_{q_2,4,256}\|_{L^2(\Theta)}$	EOC
8	1.37605		0.26335	
16	0.51794	1.41	0.02445	3.43
32	0.16422	1.65	0.00284	3.11
64	0.03382	2.27	0.00032	3.15
128	0.00722	2.22	0.00009	3.59

In the first example we choose a time discretization with $N = 4$ and study the influence of the grading on the convergence. Therefore we calculate $\|\hat{\varphi}^{(4)}_{q_1,4,K} - \hat{\varphi}^{(4)}_{q_1,4,256}\|_{L^2(\Gamma)}$, $\|\hat{\varphi}^{(4)}_{q_2,4,K} - \hat{\varphi}^{(4)}_{q_2,4,256}\|_{L^2(\Gamma)}$, $\|\hat{v}^{(4)}_{q_1,4,K} - \hat{v}^{(4)}_{q_2,4,256}\|_{L^2(\Gamma)}$ and $\|\hat{v}^{(4)}_{q_2,4,K} - \hat{v}^{(4)}_{q_2,4,256}\|_{L^2(\Gamma)}$, $K = 8, 16, 32, 64, 128$. This means that we use $\hat{\varphi}^{(4)}_{q_1,4,256}$ and $\hat{\varphi}^{(4)}_{q_2,4,256}$, respectively, as a reference solution. We also estimate the order of convergence (EOC).

The results show that one has to use the parametrization with a slow velocity near the corners in order to get the convergence rate of 2 which one expects for the piecewise linear approximations (see Theorem 4.2). For the functions $\hat{v}$ one gets a better convergence in both cases but here also the grading vector q_2 shows much better results.

In the next step we only use the vector q_2 and fix a space discretization with 64 partitions of Γ. The above result shows that this space

TABLE 3. Several time discretizations, one space discretization, grading vector q_2.

N	$\lVert\hat{\varphi}^{(N)}_{q_2,N,64} - \hat{\varphi}^{(64)}_{q_2,64,64}\rVert_{L^2(\Gamma)}$	EOC	$\lVert\hat{v}^{(N)}_{q_2,N,64} - \hat{v}^{(64)}_{q_2,64,64}\rVert_{L^2(\Theta)}$	EOC
2	4.68448		0.18635	
4	2.62000	1.41	0.09233	1.01
8	1.28590	1.03	0.04368	1.08
16	0.56238	1.19	0.01894	1.21
32	0.18920	1.57	0.00642	1.56

discretization causes an error of about 0.001. Then we calculate the approximations for the time $t = 1$ with several time discretizations and again estimate the order of convergence.

Both EOC columns show clearly the linear convergence rate of the implicit Euler scheme which was predicted by Theorem 2.6 (the EOC of 1.5 as well as the higher estimated orders in the last rows of the previous examples are caused by the effect that we compare the results with a calculated reference solution).

The above results show that the method of Chapko and Kress can be used to calculate the solution of the heat equation (1.1) if the boundary of D has corners. The only drawback of this method is the large computing time which is needed to calculate the right-hand side of the collocation equation (4.4). But the effort for this calculation corresponds to the effort which would arise if one solves the Volterra integral equation on the boundary (see [**3**]) to approximate the solutions of (1.1).

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Olaf Hansen, Fachbereich Mathematik, Johannes Gutenberg-Universität Mainz, Saarstr. 21, 55099 Mainz, Germany
E-mail address: hansen@mathematik.uni-mainz.de