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PARTIALLY-COUPLED INTEGRAL EQUATIONS FOR A DYNAMIC FRACTURE PROBLEM IN COUPLED THERMOELASTICITY

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ABSTRACT. Robust asymptotic solution forms reduce a canonical problem of dynamic fracture in a thermoelastic body to a set of partially-coupled integral equations. The set contains both Cauchy and Abel operators but can be solved analytically. The solution shows aspects of the effects of thermoelastic coupling.

1. Introduction. When crack growth is rapid, fracture is a dynamic process and, in a linear thermoelastic solid [**9**], is governed by a fully-coupled system of temperature and linear momentum equations. The growth of a crack of infinite width and semi-infinite length in an unbounded solid is a canonical problem of dynamic fracture in plane strain [**2, 13**], and a simple version suitable for coupled thermoelasticity is sub-critical, steady-state growth driven by forces and heat fluxes applied to opposite faces of the crack as line loads. The line loads lie parallel to the crack edge, and are moved behind it at a fixed distance.

2. Governing equations. In the steady-state, crack growth is at a constant speed, and field variables depend explicitly only on the spatial coordinates $\mathbf{x}$ moving with the crack. If $\mathbf{x}$ is taken as the Cartesian system $\mathbf{x} = (x, y, z)$ affixed so that $(y = 0, x > 0)$ always defines the crack edge and growth is in the negative-x direction, the equations of thermoelasticity become [**4, 6**]

(2.1a)
$$\nabla^2 \mathbf{u} + (m^2 - 1)\nabla\Delta + \chi\nabla\theta - m^2 c^2 \frac{\partial^2 \mathbf{u}}{\partial x^2} = 0, \quad \chi = \chi_0(4 - 3m^2)$$

(2.1b)
$$h\nabla^2\theta - c\frac{\partial}{\partial x}\left(\theta - \frac{m^2\varepsilon}{\chi}\Delta\right) = 0$$

(2.1c)
$$\frac{1}{\mu}\,\sigma = [(m^2 - 2)\Delta + \chi\theta]\,\mathbf{I} + \nabla\mathbf{u} + \mathbf{u}\nabla$$

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The coupled equations (2.1a,b) are for, respectively, linear momentum and temperature, while (2.1c) is a constitutive equation. Here the field variables $(\mathbf{u}, \sigma, \theta)$ are the displacement vector, stress tensor and change in (absolute) temperature from its uniform initial value, $\mathbf{I}$ is the identity tensor and the dilatation $\Delta = \operatorname{tr}(\nabla \mathbf{u})$. In plane strain,

$$\mathbf{u} = (u_x, u_y, 0), \qquad \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right)$$

so that the z-component of (2.1a) is satisfied identically and only the components $(\sigma_{xx}, \sigma_{xy} = \sigma_{yx}, \sigma_{yy}, \sigma_{zz})$ of σ remain. In (2.1) the constants (μ, χ_0) are the shear modulus and thermal expansion coefficient, $(1/m, c)$ are the rotational wave speed and crack speed nondimensionalized with respect to the isothermal dilatational wave speed [**2**], and (ε, h) are the dimensionless thermoelastic coupling constant and thermoelastic characteristic length. In general [**5, 9, 16**]

$$h \approx O(10^{-4})\mu\text{m}, \qquad \varepsilon \approx O(10^{-2}), \quad m > \sqrt{2}. \tag{2.2}$$

Sub-critical steady-state growth requires that crack speed not exceed the long-time thermoelastic Rayleigh speed for the solid, i.e.,

$$0 < c < c_R \tag{2.3}$$

where c_R is the Rayleigh speed nondimensionalized with respect to the isothermal dilatational wave speed. The values $\pm(0, c_R)$ are the roots of the Rayleigh function $R(c)$, where

$$\begin{aligned} R &= 4ab - K^2, \qquad & a &= \sqrt{1 - \frac{c^2}{1+\varepsilon}}, \\ b &= \sqrt{1 - m^2c^2}, \qquad & K &= m^2c^2 - 2 \end{aligned} \tag{2.4}$$

and (2.3) guarantees that (R, a, b) are positive real. The root c_R can be obtained by rationalizing R to a cubic polynomial in c^2 and then extracting [**1**] the appropriate zero, but an expression can also be obtained analytically [**5**] as

$$\begin{aligned} c_R &= \sqrt{2(m^2 - (1/(1+\varepsilon)))}\,\frac{F_0}{m^2}, \\ \ln F_0 &= \frac{1}{\pi}\int_{1/m}^{\sqrt{1+\varepsilon}} \frac{dt}{t} \tan^{-1} \frac{4\sqrt{1+\varepsilon - t^2}\sqrt{m^2t^2 - 1}}{\sqrt{1+\varepsilon}(m^2t^2 - 2)^2}. \end{aligned} \tag{2.5}$$

For boundary conditions, $(\mathbf{u}, \theta)$ must remain finite as $|\mathbf{x}| \to \infty$, while the crack-face loading requires that

$$(2.6)\quad \sigma_{yy} = -P_n\delta(x\text{–}L), \qquad \sigma_{xy} = -P_s\delta(x\text{–}L), \qquad \frac{\partial\theta}{\partial y} = Q_\pm\delta(x\text{–}L)$$

along $(y = 0\pm, x > 0)$. Here the constants $(P_n, P_s, Q_\pm)$ are, respectively, the normal force, shear force and heat fluxes imposed on the upper $(y = 0+)$ and lower $(y = 0-)$ crack faces at a distance L from the moving crack edge $(x, y) = 0$, and the Dirac function $\delta(\ )$ identifies them as line loads in the out-of-plane (z) direction. In addition, $(\mathbf{u}, \theta)$ should be continuous everywhere except perhaps $(y = 0\pm, x > 0)$ and, for finite $|\mathbf{x}|$, be bounded above except perhaps at $(y = 0\pm, x = L)$, and vanish as $L \to \infty$, i.e., the loadings reside an infinite distance from the moving crack edge.

3. Solution approach. In [**6**] it is demonstrated that asymptotic solution candidates of (2.1) valid at so-called large values of $|x|$ are robust because the scaling length is the (quite small) thermoelastic characteristic length h. Use of these candidate forms in (2.6) and the other conditions cited above reduces the crack problem to the integral equation set

$$(3.1\text{a})\quad \frac{R}{2m^2c^2a}\frac{1}{\pi}⨍_0^\infty \frac{\Delta V\,dt}{x-t} = \frac{\chi hK}{m^2ca(1+\varepsilon)}\frac{1}{\pi}\frac{Q_+-Q_-}{L-x} - \frac{P_n}{\mu}\delta(x-L)$$

(3.1b)

$$\frac{R}{2m^2c^2b}\frac{1}{\pi}⨍_0^\infty \frac{\Delta U\,dt}{x-t} + \frac{\chi}{m^2}\frac{\sqrt{h}}{\sqrt{c\,(1+\varepsilon)\pi}}\frac{d}{dx}\int_0^x \frac{\Delta\theta\,dt}{\sqrt{x-t}} = -\frac{P_s}{\mu}\delta(x-L)$$

$$(3.1\text{c})\quad \frac{2\varepsilon}{\chi}\frac{\sqrt{h}}{\sqrt{(c\,(1+\varepsilon)^3\pi}}\frac{d}{dx}\int_0^x \frac{\Delta U\,dt}{\sqrt{x-t}} - \sqrt{\frac{c\,(1+\varepsilon)}{h\pi}}\frac{d}{dx}\int_0^x \frac{\Delta\theta\,dt}{\sqrt{x-t}} = (Q_++Q_-)\delta(x-L)$$

for x in $(0, \infty)$ on the unknown functions $(\Delta U, \Delta V, \Delta\theta)$. Here $⨍$ denotes Cauchy principal value integration and $(\Delta\theta, \Delta U, \Delta V)$ are the discontinuities in θ and the x-derivatives of (u_x, u_y) that occur for a given $x > 0$ when one moves from the lower $(y = 0-)$ to the upper $(y = 0+)$ crack face. That is, integrals with respect to x of $(\Delta U, \Delta V)$ are the slip and separation of the crack faces and, therefore, $(\Delta U, \Delta V, \Delta\theta) \equiv 0$, $x < 0$. For $x > 0$, integrability is expected and continuity, except perhaps at $x = (0, L)$.

4. Solution of integral equation set. Equation (3.1a) defines ΔV, is uncoupled from (3.1b,c) and is of the Fredholm type. Its solution is readily found by Cauchy singular integral methods [**7**] as

$$\frac{\Delta V}{m^2c^2} = \frac{C_1}{\sqrt{\pi x}} + \frac{2a}{\pi R}\frac{P_n}{\mu}\frac{\sqrt{L}}{\sqrt{x}(x-L)} - \frac{\chi h K}{m^2c\,(1+\varepsilon)}(Q_+ - Q_-)\delta(x-L), \quad x > 0 \tag{4.1}$$

where C_1 is an arbitrary constant. The integral equations (3.1b,c) are coupled, but (3.1c) is of the Abel type [**10**] for a linear combination of $(\Delta U, \Delta\theta)$, and can therefore readily be solved to yield the formula

$$\Delta\theta - \frac{2\varepsilon\Delta U}{\chi(1+\varepsilon)} = \frac{C_2}{\sqrt{\pi x}} - \frac{(Q_+ + Q_-)\sqrt{h}}{\sqrt{c\,(1+\varepsilon)(x-L)}}H(x-L), \quad x > 0 \tag{4.2}$$

where C_2 is an arbitrary constant and $H(\ )$ is the Heaviside function.

Linearly combining (3.1b,c) gives the equation

$$\frac{1}{\pi}\int_0^\infty \frac{\Delta U\,dt}{x-t} + \sqrt{\frac{d}{\pi}}\frac{d}{dx}\int_0^x \frac{\Delta U\,dt}{\sqrt{x-t}} = A\delta(x-L) \tag{4.3}$$

on ΔU for x in $(0, \infty)$, where

$$\frac{d}{h} = \left(\frac{4b\varepsilon}{R}\right)^2\left(\frac{c}{1+\varepsilon}\right)^3, \tag{4.4a}$$

$$A = \frac{2m^2c^2b}{R}\left[-\frac{P_s}{\mu} + \frac{\chi h(Q_+ + Q_-)}{m^2c\,(1+\varepsilon)}\right] \tag{4.4b}$$

and d is a crack speed-dependent length proportional to h. The integral equation (4.3) exhibits both Cauchy and Abel operators, and the magnitudes of (ε, h) seen in (2.2) suggest in view of (4.4a) that the Abel operator is a perturbation. However, (2.3)–(2.5) also show in view of (4.4a) that

$$d \approx O(1/c)(c \to 0+), \quad d \to \infty(c \to c_{R^-}).$$

That is, for some crack growth speeds, it is the Cauchy operator that is a perturbation. Therefore, (4.3) is treated directly.

5. Integral equation solution. The homogeneous ($A = 0$) version of (4.3) admits the solution $1/\sqrt{x}$. This form integrates with respect to x to a function that vanishes appropriately as $x \to 0+$, i.e., the crack closes continuously at its edge, but which also becomes unbounded as $x \to \infty$. It is, therefore, discarded, but suggests the trial form

$$\Delta U = \frac{1}{\pi\sqrt{x}} \int_0^\infty \frac{U\sqrt{t}\,dt}{t - x}, \quad x > 0 \tag{5.1}$$

for (4.3) itself, where $U(x)$ is an unknown integrable function that, except perhaps at $x = L$, is bounded above and continuous in $(0, \infty)$. Substitution of (5.1) and use of Cauchy theory then reduces (4.3) to the form

$$U + \sqrt{\frac{d}{\pi}} \frac{d}{dx} \int_x^\infty \frac{U\,dt}{\sqrt{t - x}} = A\delta(x - L) \tag{5.2}$$

on U for x in $(0, \infty)$. Introduction of the quantities

$$\xi = \frac{1}{x}, \qquad \psi(\xi) = \frac{1}{\xi^{3/2}} A\delta\left(\frac{1}{\xi} - L\right), \qquad \omega(\xi) = \frac{1}{\xi^{3/2}} U\left(\frac{1}{\xi}\right) \tag{5.3}$$

in (5.2) yields the equation

$$\omega - \sqrt{\frac{\xi d}{\pi}} \frac{d}{d\xi}\left(\sqrt{\xi} \int_0^\xi \frac{\omega\,d\zeta}{\sqrt{\xi - \zeta}}\right) = \psi(\xi) \tag{5.4}$$

on ω for ξ in $(0, \infty)$. Application of the Laplace transform [**15**]

$$G(p) = \int_0^\infty g(\xi) e^{-p\xi}\,d\xi \tag{5.5}$$

where p is positive real and large enough to ensure existence of the integral, gives, finally, the first-order ordinary differential equation

$$\frac{d\Omega}{dp} + \frac{\Omega}{\sqrt{pd}} = \frac{\Psi}{\sqrt{pd}} \tag{5.6}$$

on $\Omega(p)$. The general solution [**12**] to (5.6) is

$$\Omega = e^{-2\sqrt{p/d}}\left(C_3 + \int_0^p \frac{\Psi}{\sqrt{qd}}\, e^{2\sqrt{q/d}}\, dq\right) \tag{5.7}$$

where C_3 is an arbitrary constant. Boundedness of Ω as $|p| \to \infty$, $\mathrm{Re}\,(p) > 0$ is guaranteed by choosing the branch cut $\mathrm{Im}\,(p) = 0$, $\mathrm{Re}\,(p) < 0$ for $\sqrt{p}$ so that $\mathrm{Re}\,(\sqrt{p}) \geq 0$ in the cut p-plane. This boundedness is required in view of the Abelian theorems [**2**] and (5.4), because $U(x)$ must be bounded as $x \to \infty$.

The use of (5.4), standard inversion tables [**1**] and the Dirac function sifting property [**17**] on (5.7) gives

$$\begin{aligned} \frac{U}{m^2c^2} &= \frac{C_3}{\sqrt{\pi d}}\, e^{-x/d} - \frac{A}{d}\, e^{(L-x)/d} erfc\,(\sqrt{L/d}) \\ &\quad + \frac{A}{\sqrt{d}}\left[\frac{1}{\sqrt{\pi(L-x)}} + \frac{1}{\sqrt{d}}\, e^{(L-x)/d} erfc\,(\sqrt{(L-x)/d})\right] H(L-x) \end{aligned} \tag{5.8}$$

for $x > 0$. For generality, it should be noted that, if the "+" in the Abel operator term in (4.3) is replaced by a "−", the solution becomes

$$\begin{aligned} \frac{U}{m^2c^2} = \frac{A}{\sqrt{d}}\Bigg[\frac{1}{\sqrt{\pi(L-x)}} &+ \frac{1}{\sqrt{d}}\, e^{(L-x)/d}(erfc\,(\sqrt{(L-x)/d}) \\ &- erfc\,(\sqrt{L/d}))\Bigg] H(L-x). \end{aligned} \tag{5.9}$$

6. Solution and observations on coupling effects. Equations (4.1), (4.2), (5.1) and (5.8) constitute the candidate solution for the set (3.1). Invoking boundedness and the condition that $(\mathbf{u}, \theta)$ should vanish as $L \to \infty$ leads to the conclusion that

$$C_1 = C_2 = 0, \qquad C_3 = -\sqrt{\frac{\pi}{d}}A. \tag{6.1}$$

The original crack growth problem is thus essentially solved.

The magnitude of the coupling constant ε exhibited by (2.2) could be, e.g. [**3**], used to justify dropping the ε-term from the temperature equation (2.1b), thereby uncoupling it from the linear momentum equation (2.1a). The asymptotic analytical solutions constructed here show the effect of this: When $\varepsilon = 0$ in (4.2) and (4.3), the fields $(\Delta\theta, \Delta U)$ are uncoupled, $\Delta\theta$ is given by (4.2) and (6.1) alone, and ΔU is, like ΔV, governed by a Cauchy singular integral equation.

This is not surprising: solutions of the classical steady-state temperature boundary-value problem in 2D [**8**] can involve Abel integral equations and forms like the righthand side of (4.2), while solution of the steady-state 2D linear elasticity mixed boundary-value problem, with temperature merely providing a (known) body force, can lead to, in the same manner as its classical elastostatic counterpart [**11, 14**], Cauchy singular integral equations and forms like the righthand side of (4.1).

In conclusion, then, robust asymptotic solutions for the equations of linear coupled thermoelasticity reduce a canonical problem of dynamic steady-state crack growth in plane strain to a set of partially-coupled integral equations. The set exhibits both Cauchy and Abel operators, but can be solved analytically. The solution results demonstrate aspects of thermoelastic coupling.

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