ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 49, Number 8, 2019

THREE RESULTS FOR τ-RIGID MODULES

ZONGZHEN XIE, LIBO ZAN AND XIAOJIN ZHANG

ABSTRACT. τ-rigid modules are essential in the τ-tilting theory introduced by Adachi, Iyama and Reiten. In this paper, we give equivalent conditions for Iwanaga-Gorenstein algebras with self-injective dimension at most one in terms of τ-rigid modules. We show that every indecomposable module over iterated tilted algebras of Dynkin type is τ-rigid. Finally, we give a τ-tilting theorem on homological dimension which is an analog to that of classical tilting modules.

1. Introduction. In 2014, T. Adachi, O. Iyama and I. Reiten [**AIR**] introduced τ-tilting theory to generalize the classical tilting theory. τ-tilting theory is closely related to silting theory [**AI, BZ**] and cluster tilting theory [**KR, IY, BMRRT**] which are popular in the recent years. Therefore, τ-tilting theory has attracted widespread attention. For the latest general results on τ-tilting theory, we refer to [**DIJ, DIRRT, EJR, IJY, IZ1, IZ2, J, K, W**].

Note that τ-rigid modules are important objects and tools in the τ-tilting theory. It is interesting to study the properties of τ-rigid modules and find the indecomposable τ-rigid modules for a given algebra. For the recent development of this topic, we refer to [**A1, A2, DIP, HZ, Mi, Z1, Z2, Zi1, Zi2**]. We also focus on the properties of τ-rigid modules.

For an algebra A, denote by $\operatorname{mod} A$ the category of finitely generated right A-modules. Recall that an algebra A is Iwanaga-Gorenstein, that is, $\operatorname{id}_A A < \infty$ and $\operatorname{id}_{A^{\mathrm{op}}} A < \infty$. In this case, $\operatorname{id}_A A = \operatorname{id}_{A^{\mathrm{op}}} A$. Our first main result gives some new equivalent conditions for an Iwanaga-Gorenstein algebra A with $\operatorname{id}_A A \leq 1$ in terms of τ-rigid modules. We

2010 AMS *Mathematics subject classification.* Primary: 16G10, 16E10.
Keywords and phrases. τ-rigid module, projective dimension, tilted algebra.
Zhang is the corresponding author. The authors are supported by NSFC (No. 11571164 and 11671174).
Received by the editors on August 2, 2018, and in revised form on April 26, 2019.
DOI:10.1216/RMJ-2019-49-8-2791

remark that this result was inspired by Osamu Iyama and Yingying Zhang.

Theorem 1.1 (Theorems 2.6, 2.7)**.** *For an algebra A, the following are equivalent:*

(1) *A is Iwanaga-Gorenstein with* $\mathrm{id}_A A \leq 1$.
(2) *Every classical cotilting module in* $\mathrm{mod}\, A$ *is a classical tilting module.*
(3) $\mathbb{D}A$ *is a τ-rigid module in* $\mathrm{mod}\, A$.
(4) *A is a τ^{-1}-rigid module in* $\mathrm{mod}\, A$.

We are also interested in algebras A satisfying every indecomposable module in $\mathrm{mod}\, A$ is τ-rigid. Easy examples of such algebras are hereditary algebras of Dynkin type. We aim to find more examples in this paper. Recall that Assem and Happel [**AsH**] introduced the following notation of iterated tilted algebras of Dynkin type as a generalization of tilted algebras of Dynkin type [**HR**]. Let Q be a finite, connected, and acyclic quiver. An algebra A_m ($m \geq 1$) is called an iterated tilted algebra of type Q if the following three conditions are satisfied:

(1) $A_0 = KQ$,
(2) T_i is a splitting classical tilting module in $\mathrm{mod}\, A_i$ and
(3) $A_{i+1} = \mathrm{End}_{A_i} T_i$ for $0 \leq i \leq m-1$.

Our second main result is the following:

Theorem 1.2 (Theorem 3.7)**.** *Let B be an iterated tilted algebra of Dynkin type. Then every indecomposable module in* $\mathrm{mod}\, B$ *is τ-rigid.*

For a classical tilting module T in $\mathrm{mod}\, A$ with $B = \mathrm{End}_A T$, by using the tilting theorem of Brenner and Butler [**BB**], one gets that the homological dimension of $N \in \mathrm{Fac}\, T$ gives an upper bound of the homological dimension of $\mathrm{Hom}_A(T, N)$, where $\mathrm{Fac}\, T$ (resp. $\mathrm{Sub}\, T$) is the subcategory of $\mathrm{mod}\, A$ consisting of modules N generated (resp. cogenerated) by T. It is natural to ask: Is there a similar result for τ-tilting modules? We give a positive answer to this question and get our third main result. We should remark that Buan and Zhou have studied the global dimension of 2-term silting complexes in [**BZ**].

Theorem 1.3 (Theorems 4.2, 4.3)**.** *Let A be an algebra, T be a τ-tilting module in* $\operatorname{mod} A$*, $B=\operatorname{End}_A T$ and $C=\operatorname{End}_A \tau T^{\mathrm{op}}$.*

(1) *For any $M\in \operatorname{Fac} T$ with $\operatorname{pd}_A M\leq 1$, $\operatorname{pd}_B \operatorname{Hom}_A(T,M)\leq \operatorname{pd}_A M$ holds.*
(2) *For any $M\in \operatorname{Fac} T$ with $\operatorname{Ext}^i_A(T,M\oplus T)=0$ for any $i\geq 1$, $\operatorname{pd}_B \operatorname{Hom}_A(T,M)\leq \operatorname{pd}_A M$ holds.*
(3) *For any $N\in \operatorname{Sub} \tau T$ with $\operatorname{id}_A N\leq 1$, $\operatorname{pd}_C \operatorname{Hom}_A(N,\tau T)\leq \operatorname{id}_A N$ holds.*
(4) *For any $N\in \operatorname{Sub} \tau T$ with $\operatorname{Ext}^i_A(N\oplus \tau T,\tau T)=0$ for any $i\geq 1$, $\operatorname{pd}_C \operatorname{Hom}_A(N,\tau T)\leq \operatorname{id}_A N$ holds.*

The paper is organized as follows:

In Section 2, we study τ-rigid modules over Iwanaga-Gorenstein algebras and show Theorem 1.1. In Section 3, we study the indecomposable τ-rigid modules over iterated tilted algebras of Dynkin type and show Theorem 1.2. In Section 4, we give the τ-rigid (resp. τ^{-1}-rigid) version of Wakamastu's lemma and then give an upper bound for some special modules over the endomorphism ring of τ-tilting (resp. τ^{-1}-tilting) modules.

Throughout this paper, all algebras are finite dimensional algebras over an algebraically closed field K and $\mathbb{D}=\operatorname{Hom}_K(-,K)$ is the standard duality.

2. Gorenstein algebras and τ-rigid modules. In this section, we aim to study Iwanaga-Gorenstein algebras in terms of τ-rigid modules.

For an algebra A, denote by gl.dim A the global dimension of A. For a right A-module M, denote by $\operatorname{pd}_A M$ (resp. $\operatorname{id}_A M$) the projective dimension (resp. injective dimension) of M, denote by $\mathsf{add}_A M$ the subcategory of direct summands of finite direct sums of M and denote by $|M|$ the number of pairwise nonisomorphic indecomposable summands of M. Firstly, we recall the definition of tilting (resp. cotilting) modules, see [**M**] for details.

Definition 2.1. A module $T\in \operatorname{mod} A$ is called a *tilting* module, if it satisfies

(1) $\operatorname{pd}_A T\leq n$.

(2) $\mathrm{Ext}_A^i(T,T)=0$ for all $i\geq 1$.
(3) There exists an exact sequence $0\to A\to T_0\to T_1\to\cdots\to T_n\to 0$, for all $T_i\in\mathsf{add}\,T$, $0\leq i\leq n$.

In particular, we call T in Definition 2.1 a *classical tilting* module whenever $n=1$. In this case, Definition 2.1(3) is equivalent to $|T|=|A|$. Dually, one can define *cotilting* modules and *classical cotilting* modules.

We also need the following definitions in [**AIR**].

Definition 2.2. (1) We call $T\in\mathrm{mod}\,A$ *τ-rigid* if $\mathrm{Hom}_A(T,\tau T)=0$, where τ is the Auslander-Reiten translation. Moreover, T is called *τ-tilting* if T is τ-rigid and $|T|=|A|$.
(2) We call $T\in\mathrm{mod}\,A$ *τ^{-1}-rigid* if $\mathrm{Hom}_A(\tau^{-1}T,T)=0$. Moreover, T is called *τ^{-1}-tilting* if T is τ^{-1}-rigid and $|T|=|A|$.

Clearly, T is τ^{-1}-rigid (resp. τ^{-1}-tilting) module in $\mathrm{mod}\,A$ if and only if $\mathbb{D}T$ is τ-rigid (resp. τ-tilting) module in $\mathrm{mod}\,A^{\mathrm{op}}$.

Recall that $T\in\mathrm{mod}\,A$ is called *faithful* if the right annihilator of T is zero. Now we can state the following proposition in [**AIR**].

Proposition 2.3. (1) *Any faithful τ-rigid module T in* $\mathrm{mod}\,A$ *is a partial tilting A-module, that is,* $\mathrm{Ext}_A^1(T,T)=0$ *and* $\mathrm{pd}_A\,T\leq 1$.
(2) *Any faithful τ-tilting module in* $\mathrm{mod}\,A$ *is a classical tilting A-module.*

Now we can state the properties of τ-rigid cotilting modules, which is essential in the proof of the main result.

Proposition 2.4. (1) *If a cotilting (resp. tilting) module T in* $\mathrm{mod}\,A$ *is τ-rigid, then T is a classical tilting A-module.*
(2) *If a tilting (resp. cotilting) module T in* $\mathrm{mod}\,A$ *is τ^{-1}-rigid, then T is a classical cotilting A-module.*

Proof. We only prove (1), since the proof of (2) is similar. Because T is cotilting, T is faithful by [**AsSS,** Chapter VI, Lemma 2.2]. By Proposition 2.3, any faithful τ-rigid module in $\mathrm{mod}\,A$ is a partial tilting A-module. Note that $|T|=|A|$, we are done. □

For any $X \in \mathrm{mod}\, A$, denote by $\mathrm{Fac}\, X = \{M \mid X^n \twoheadrightarrow M \text{ for some } n\}$. The following proposition [**AS,** Proposition 5.8] are useful.

Proposition 2.5. *For X and Y in* $\mathrm{mod}\, A$*, we have the following:*

(1) $\mathrm{Hom}_A(X, \tau Y) = 0$ *if and only if* $\mathrm{Ext}^1_A(Y, \mathrm{Fac}\, X) = 0$.
(2) *X is τ-rigid if and only if* $\mathrm{Ext}^1_A(X, \mathrm{Fac}\, X) = 0$.

For an algebra A, denote by $\mathcal{P}(A)$ the subcategory of finitely generated projective right A-modules and denote by $\mathcal{I}(A)$ the subcategory of finitely generated injective right A-modules. Now we recall the following result due to Happel and Unger [**HU,** Lemma 1.3]. We provide a new proof for this result.

Theorem 2.6. *For an algebra A, the following are equivalent:*

(1) *A is Iwanaga-Gorenstein.*
(2) *Every cotilting module in* $\mathrm{mod}\, A$ *is tilting.*
(3) *Every tilting module in* $\mathrm{mod}\, A$ *is cotilting.*
(4) *There exists a tilting-cotilting module in* $\mathrm{mod}\, A$.

Proof. (1) $\Rightarrow$ (2) Assume that A is Iwanaga-Gorenstein and T is a cotilting module in $\mathrm{mod}\, A$. Since T is self-orthogonal and $\mathrm{id}_A T$ is finite, we only need to show that every module in $\mathcal{P}(A)$ has a finite exact coresolution in $\mathsf{add}\, T$.

Denote by ${}^{\perp}T = \{M \in \mathrm{mod}\, A \mid \mathrm{Ext}^i_A(M, T) = 0 \text{ for } i \geq 1\}$ and $X_T = \{X \mid 0 \to X \to T_0 \xrightarrow{f_0} T_1 \to \cdots \xrightarrow{f_n} T_{n+1} \to \cdots, T_i \in \mathsf{add}\, T, \mathrm{Im}\, f_n \in {}^{\perp} T, n \geq 0\}$. Since T is a cotilting A-module, we have that $\mathcal{P}(A)$ is contained in $\mathcal{X}_T = {}^{\perp}T$.

Then there exists an exact sequence

$$0 \to P \to T_0 \xrightarrow{f_0} T_1 \to \cdots \to T_n \xrightarrow{f_n} T_{n+1} \to \cdots$$

for all A-module $P \in \mathcal{P}(A)$, where $T_i \in \mathsf{add}\, T$ and $X_i = \mathrm{Im}\, f_i$ is in $\mathcal{X}_T$ for all $i \geq 0$. Let $\mathrm{id}_A \mathcal{P}(A) \leq r$. Then $\mathrm{Ext}^1_A(X_r, X_{r-1}) = \mathrm{Ext}^2_A(X_r, X_{r-2}) = \cdots = \mathrm{Ext}^{r+1}_A(X_r, P) = 0$, hence the exact sequence $0 \to X_{r-1} \to T_r \to X_r \to 0$ splits, such that $X_{r-1} \in \mathsf{add}\, T$. This implies that T is a tilting A-module.

(2) $\Rightarrow$ (1) Assume that a module T in $\operatorname{mod} A$ is a cotilting-tilting module. By the definitions of cotilting and tilting modules every module in $\mathcal{I}(A)$ has a finite exact resolution in $\mathsf{add}\, T$ and every module in $\mathcal{P}(A)$ has a finite exact coresolution in $\mathsf{add}\, T$. Since $\operatorname{id}_A T$ and $\operatorname{pd}_A T$ both are finite, it follows immediately that both $\operatorname{pd}_A \mathcal{I}(A)$ and $\operatorname{id}_A \mathcal{P}(A)$ are finite. Therefore A is Gorenstein.

Similarly, one can prove the equivalence of (1) and (3).

In the following we show the equivalence of (1) and (4).

(1) $\Rightarrow$ (4) Assume that A is Gorenstein. Then A is a tilting-cotilting module.

(4) $\Rightarrow$ (1) is similar to (2) $\Rightarrow$ (1). □

Now we are in a position to show the main result in this section.

Theorem 2.7. *For an algebra A, the following are equivalent:*

(1) *A is Iwanaga-Gorenstein with* $\operatorname{id}_A A \leq 1$.
(2) *$\mathbb{D}A$ is a τ-rigid module in* $\operatorname{mod} A$.
(3) *A is a τ^{-1}-rigid module in* $\operatorname{mod} A$.

Proof. We show the equivalence of (1) and (2). Similarly, one can show the equivalence of (1) and (3).

(1) $\Rightarrow$ (2) For any $M \in \operatorname{Fac} \mathbb{D}A$, there exists a short exact sequence

$$0 \to N \to \mathbb{D}A^n \to M \to 0 \tag{2.1}$$

Applying the functor $\operatorname{Hom}_A(\mathbb{D}A, -)$ to the short exact sequence (2.1) yields the following long exact sequence $0 \to \operatorname{Hom}_A(\mathbb{D}A, N) \to \operatorname{Hom}_A(\mathbb{D}A, \mathbb{D}A^n) \to \operatorname{Hom}_A(\mathbb{D}A, M) \to \operatorname{Ext}^1_A(\mathbb{D}A, N) \to \operatorname{Ext}^1_A(\mathbb{D}A, \mathbb{D}A^n) \to \operatorname{Ext}^1_A(\mathbb{D}A, M) \to \operatorname{Ext}^2_A(\mathbb{D}A, N) \to \operatorname{Ext}^2_A(\mathbb{D}A, \mathbb{D}A^n) \to \cdots$ Then $\operatorname{Ext}^1_A(\mathbb{D}A, M) \simeq \operatorname{Ext}^2_A(\mathbb{D}A, N)$ since $\mathbb{D}A$ is an injective A-module, and $\operatorname{pd}_A \mathbb{D}A \leq 1$ since $\operatorname{id}_A A \leq 1$. Thus $\operatorname{Ext}^1_A(\mathbb{D}A, M) \simeq \operatorname{Ext}^2_A(\mathbb{D}A, N) = 0$. We have $\operatorname{Ext}^1_A(\mathbb{D}A, \operatorname{Fac} \mathbb{D}A) = 0$, therefore $\mathbb{D}A$ is a τ-rigid A-module by Proposition 2.5. Since $|\mathbb{D}A| = |A|$, one gets $\mathbb{D}A$ is a τ-tilting A-module.

(2) $\Rightarrow$ (1) Since $\mathbb{D}A$ is τ-rigid and $|\mathbb{D}A| = |A|$, $\mathbb{D}A$ is a τ-tilting A-module. By [**AsSS,** Chapter VI, Lemma 2.2], $\mathbb{D}A$ is faithful. Then $\mathbb{D}A$ is a classical tilting A-module by Proposition 2.3(2), and hence $\operatorname{pd}_A \mathbb{D}A \leq 1$.

Thus $\mathrm{id}_{A^{\mathrm{op}}} A \leq 1$. Note that $\mathrm{id}_A A \leq 1$ if and only if $\mathrm{id}_{A^{\mathrm{op}}} A \leq 1$, then $\mathrm{id}_A A \leq 1$. □

The following corollary is immediate.

Corollary 2.8. *For an algebra A, if one of the following conditions is satisfied:*

(1) *every τ-tilting A-module is a τ^{-1}-tilting A-module;*
(2) *every τ^{-1}-tilting A-module is a τ-tilting A-module,*

then A is Iwanaga-Gorenstein with $\mathrm{id}_A A \leq 1$.

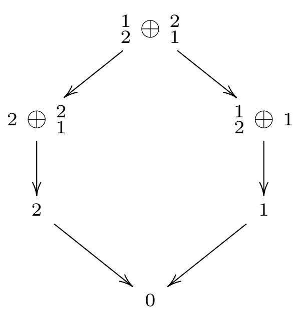
Proof. We only prove (1) since the proof of (2) is similar. By assumption we have that the τ-tilting module A is a τ^{-1}-tilting module. Then A is a Gorenstein algebra with $\mathrm{id}_A A \leq 1$ by Theorem 2.7(3). □

We should remark that the converse of Corollary 2.8 is not true in general.

Example 2.9. Let A be the algebra given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with relations $\alpha\beta = \beta\alpha = 0$. Then the support τ-tilting quiver of A is the following:



One can show that $2 \oplus \begin{smallmatrix}2\\1\end{smallmatrix}$ is τ-tilting but not τ^{-1}-tilting.

At the end of this section, we give an example to show that the existence of τ-tilting-τ^{-1}-tilting modules (even τ-rigid classical cotilting modules) is not equivalent to 1-Gorensteiness in general.

Example 2.10. Let A be the algebra given by the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ with the relation $\alpha\beta = 0$. Then $T = {}^{1}_{2} \oplus {}^{2}_{3} \oplus 2$ is a τ-tilting-τ^{-1}-tilting module in $\operatorname{mod} A$ (actually a classical tilting-cotilting module) but $\operatorname{gl.dim} A = 2$.

3. Iterated tilted algebras and τ-rigid modules. In this section, we focus on the τ-rigid modules over iterated tilted algebras and show every indecomposable module over an iterated tilted algebra of Dynkin type is τ-rigid. Throughout this section, all tilting modules are classical tilting modules.

Firstly, we need the notion of torsion pairs.

Definition 3.1. Let A be an algebra. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\operatorname{mod} A$ is called a *torsion pair* if the following conditions are satisfied:

(1) $\operatorname{Hom}_A(M, N) = 0$ for all $M \in \mathcal{T}, N \in \mathcal{F}$.
(2) $\operatorname{Hom}_A(M, -)|_{\mathcal{F}} = 0$ implies $M \in \mathcal{T}$.
(3) $\operatorname{Hom}_A(-, N)|_{\mathcal{T}} = 0$ implies $N \in \mathcal{F}$.

To introduce the tilting theorem due to Brenner and Bulter, we also need the following:

Definition 3.2. Let A be an algebra. Any tilting module T in $\operatorname{mod} A$ induces *torsion pairs* $(\mathcal{T}(T), \mathcal{F}(T))$ in $\operatorname{mod} A$ and $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ with $B = \operatorname{End}_A T$, where

$$\mathcal{T}(T) = \{M_A \mid \operatorname{Ext}^1_A(T, M) = 0\},$$
$$\mathcal{F}(T) = \{M_A \mid \operatorname{Hom}_A(T, M) = 0\},$$
$$\mathcal{X}(T) = \{X_B \mid \operatorname{Hom}_B(X, \mathbb{D}T) = 0\} = \{X_B \mid X \otimes_B T = 0\},$$
$$\mathcal{Y}(T) = \{Y_B \mid \operatorname{Ext}^1_B(Y, \mathbb{D}T) = 0\} = \{Y_B \mid \operatorname{Tor}^B_1(Y, T) = 0\}.$$

Now we can state the tilting theorem of Brenner and Bulter **[BB]** as follows:

Theorem 3.3. *Let A be an algebra, T be a tilting module in* $\operatorname{mod} A$ *and* $B = \operatorname{End}_A T$. *Let* $(\mathcal{T}(T), \mathcal{F}(T))$ *and* $(\mathcal{X}(T), \mathcal{Y}(T))$ *be the induced torsion pairs in* $\operatorname{mod} A$ *and* $\operatorname{mod} B$, *respectively. Then T has the following properties:*

(1) *${}_BT$ is a tilting B-module, and the canonical K-algebra homomorphism $A \to \operatorname{End}_B T^{\text{op}}$ defined by $a \mapsto (t \mapsto ta)$ is an isomorphism.*
(2) *The functors* $\operatorname{Hom}_A(T, -)$ *and* $- \otimes_B T$ *induce quasi-inverse equivalences between* $\mathcal{T}(T)$ *and* $\mathcal{Y}(T)$.
(3) *The functors* $\operatorname{Ext}^1_A(T, -)$ *and* $\operatorname{Tor}^B_1(-, T)$ *induce quasi-inverse equivalences between* $\mathcal{F}(T)$ *and* $\mathcal{X}(T)$.

Recall that a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\operatorname{mod} A$ is called *splitting* if for any indecomposable $M \in \operatorname{mod} A$ either $M \in \mathcal{T}$ or $M \in \mathcal{F}$ holds. For a tilting module T_A with $B = \operatorname{End}_A T$, T_A is said to be *splitting* if the induced torsion pair $(\mathcal{X}(T), \mathcal{Y}(T))$ in $\operatorname{mod} B$ is splitting. The following propositions in [**AsSS**] are critical in the proof of the main result in this section.

Proposition 3.4. [**AsSS,** VI, Corollary 5.7] *For an algebra A, if* $\operatorname{gl.dim} A \leq 1$, *then every tilting module in* $\operatorname{mod} A$ *is splitting.*

Proposition 3.5. [**AsSS,** VI, Proposition 5.2] *Let A be an algebra, T be a splitting tilting module in* $\operatorname{mod} A$, *and* $B = \operatorname{End}_A T$. *Then any almost split sequence in* $\operatorname{mod} B$ *lies entirely in either* $\mathcal{X}(T)$ *or* $\mathcal{Y}(T)$, *or else it is of the form*

$$0 \to \operatorname{Hom}_A(T, I) \to \operatorname{Hom}_A(T, I/\operatorname{soc} I) \oplus \operatorname{Ext}^1_A(T, \operatorname{rad} P) \to \operatorname{Ext}^1_A(T, P) \to 0,$$

where P is an indecomposable projective A-module not lying in $\mathsf{add}\, T$ *and I is the indecomposable injective A-module such that* $P/\operatorname{rad} P \cong \operatorname{soc} I$.

Keeping the symbols as above, we can recall the following proposition.

Proposition 3.6. [**AsSS,** VI, Lemma 5.3] *Let* $0 \to L \to M \to N \to 0$ *be an almost split sequence in* $\operatorname{mod} B$.

(1) *If* $L, M, N \in \mathcal{Y}(T)$, *then* $0 \to L \otimes_B T \to M \otimes_B T \to N \otimes_B T \to 0$ *is almost split in* $\mathcal{T}(T)$.

(2) *If* $L, M, N \in \mathcal{X}(T)$, *then* $0 \to \mathrm{Tor}_1^B(L,T) \to \mathrm{Tor}_1^B(M,T) \to \mathrm{Tor}_1^B(N,T) \to 0$ *is almost split in* $\mathcal{F}(T)$.

Let Q be a finite, connected, and acyclic quiver. Recall that an algebra B is called an *iterated tilted algebra of type* Q if there is a series of algebras $A_0 = KQ, A_1, \dots, A_m = B$ such that T_i is a splitting classical tilting module over A_i and $A_{i+1} = \mathrm{End}_{A_i} T_i$ for $0 \le i \le m-1$. Now we are in a position to show the main result of this section.

Theorem 3.7. *Let* B *be an iterated tilted algebra of Dynkin type* Q. *Then every indecomposable module in* $\mathrm{mod}\, B$ *is* τ*-rigid.*

Proof. Assume that $B = A_m$ is the iterated tilted algebra of Dynkin type Q with the corresponding splitting tilting modules T_i for $0 \le i \le m-1$. We prove the assertion by induction on m.

If $m = 1$, then $B = A_1 = \mathrm{End}_{A_0} T_0$ is a tilted algebra of Dynkin type.

Let N be any indecomposable module in $\mathrm{mod}\, B$. By Proposition 3.4, N is either in $\mathcal{X}(T)$ or in $\mathcal{Y}(T)$.

If N is projective, then there is nothing to show. Now assume that N is not projective. Then there is an almost split sequence $0 \to L \to M \to N \to 0$. By Proposition 3.5, the exact sequence is either in $\mathcal{Y}(T)$, $\mathcal{X}(T)$ or a connecting sequence.

(1) If $0 \to L \to M \to N \to 0$ in $\mathcal{Y}(T)$, then $0 \to L \otimes_B T \to M \otimes_B T \to N \otimes_B T \to 0$ is Auslander-Reiten sequence in $\mathrm{mod}\, A_0$ by Proposition 3.6. Since A_0 is the path algebra of a Dynkin quiver, A_0 is a representation-finite hereditary algebra. This implies every indecomposable module in $\mathrm{mod}\, A_0$ is directing and thus τ-rigid. By Theorem 3.3, $\mathrm{Hom}_B(N, L) \simeq \mathrm{Hom}_{A_0}(N \otimes_B T, L \otimes_B T) = 0$, hence N is τ-rigid.

(2) If $0 \to L \to M \to N \to 0$ in $\mathcal{X}(T)$, then $0 \to \mathrm{Tor}_1^B(L,T) \to \mathrm{Tor}_1^B(M,T) \to \mathrm{Tor}_1^B(N,T) \to 0$ is Auslander-Reiten sequence in $\mathrm{mod}\, A_0$ by Proposition 3.6. As we showed in (1) every indecomposable module in $\mathrm{mod}\, A_0$ is τ-rigid. By Theorem 3.3, $\mathrm{Hom}_B(N, L) \simeq \mathrm{Hom}_{A_0}(\mathrm{Tor}_1^B(N,T), \mathrm{Tor}_1^B(L,T)) = 0$, hence N is τ-rigid.

(3) If $0 \to L \to M \to N \to 0$ is a connecting sequence, then $N \simeq \mathrm{Ext}^1_{A_0}(T, P(a)) \in \mathcal{X}(T)$, $L \simeq \mathrm{Hom}_{A_0}(T, I(a)) \in \mathcal{Y}(T)$ by Proposition 3.5. Thus, $\mathrm{Hom}_B(N, \tau N) = \mathrm{Hom}_B(N, L) = 0$, N is τ-rigid.

Now assume the assertion holds for $B = A_m$. In the following we show the assertion holds for $B = A_{m+1}$.

By induction assumption, every indecomposable module in $\operatorname{mod} A_m$ is τ-rigid. For any indecomposable module $N \in \operatorname{mod} B$, if N is projective, then there is nothing to show. We assume that N is not projective. Since T_m is splitting, then N is either in $\mathcal{Y}(T_m)$ or in $\mathcal{X}(T_m)$. Putting $T = T_m$ in the proof of the case $m = 1$, one gets the desired result. $\square$

Example 3.8. Let $A_0 = KQ$ be the algebra given by the quiver $Q : 1 \to 2 \to 3 \to 4$ and let T_0 be the tilting module

$$\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 4$$

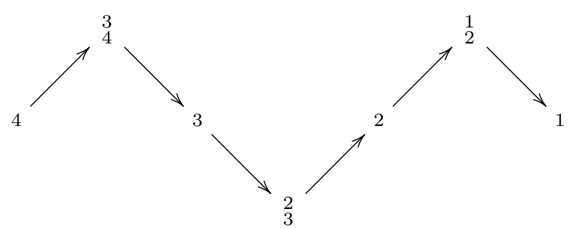
in $\operatorname{mod} A_0$. Then

(1) $A_1 = \operatorname{End}_{A_0} T_0$ is given by the quiver $Q' : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4$ with the relation $\alpha_2\alpha_3 = 0$ and $\operatorname{gl.dim} A_1 = 2$.

(2)
$$T_1 = \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}3\\4\end{smallmatrix} \oplus 4$$
in $\operatorname{mod} A_1$ is a classical tilting module and $A_2 = \operatorname{End}_{A_1} T_1$ is given by the quiver $Q'' : 1 \xrightarrow{\beta_1} 2 \xrightarrow{\beta_2} 3 \xrightarrow{\beta_3} 4$ with relations $\beta_1\beta_2 = 0$ and $\beta_2\beta_3 = 0$.

(3) $\operatorname{gl.dim} A_2 = 3$ implies that A_2 is iterated tilted but not tilted.

(4) The Auslander-Reiten quiver of A_2 is as follows:



One can show that every indecomposable module in $\operatorname{mod} A_2$ is τ-rigid.

4. τ-tilting modules and homological dimension. In this section, we give the relationship between τ-tilting modules and homological dimension, which is an analog of that of classical tilting modules (see [**AsSS,** Lemma 4.1] for details).

For an A-module M, denote by $M^{\perp_0}$ (resp. ${}^{\perp_0}M$) the subcategory consisting of N such that $\mathrm{Hom}_A(M,N)=0$ (resp. $\mathrm{Hom}_A(N,M)=0$). Firstly, we introduce the following lemma known as Wakamastu's Lemma.

Lemma 4.1. (1) *Let* θ*:* $0\to Y\to T'\xrightarrow{g} X$ *be an exact sequence in* $\mathrm{mod}\,A$*, where* T *is* τ*-rigid, and* g*:* $T'\to X$ *is a right* $\mathsf{add}\,T$*-approximation. Then we have* $Y\in{}^{\perp_0}(\tau T)$*.*

(2) *Let* ϑ*:* $Y\xrightarrow{f} U\to Z\to 0$ *be an exact sequence in* $\mathrm{mod}\,A$*, where* T *is* τ*-rigid,* $U\in\mathsf{add}\,\tau T$*, and* f*:* $Y\to U$ *is a left* $(\mathsf{add}\,\tau T)$*-approximation. Then we have* $Z\in T^{\perp_0}$*.*

Proof. (1) is given by Adachi, Iyama and Reiten in [**AIR**]. We only prove (2).

Replacing Y by $\mathrm{Ker}\,f$, we can assume that f is an injective. We apply $\mathrm{Hom}_A(T,-)$ to ϑ and get the exact sequence

$$0=\mathrm{Hom}_A(T,U)\to\mathrm{Hom}_A(T,Z)\to\mathrm{Ext}^1_A(T,Y)\xrightarrow{\mathrm{Ext}^1_A(T,f)}\mathrm{Ext}^1_A(T,U)$$

where we have $\mathrm{Hom}_A(T,U)=0$ because $U\in\mathsf{add}\,\tau T$. Since f: $Y\to U$ is a left $(\mathsf{add}\,\tau T)$-approximation, the induced map $(f,\tau T)$: $\mathrm{Hom}_A(U,\tau T)\to\mathrm{Hom}_A(Y,\tau T)$ is surjective. Then the induced map $\overline{\mathrm{Hom}}_A(U,\tau T)\to\overline{\mathrm{Hom}}_A(Y,\tau T)$ of the maps modulo injectives is surjective. By the Auslander-Reiten duality, the map $\mathrm{Ext}^1_A(T,f)$: $\mathrm{Ext}^1_A(T,Y)\to\mathrm{Ext}^1_A(T,U)$ is injective. It follows that $\mathrm{Hom}_A(T,Z)=0$. □

Dually, one can show Wakamastu's Lemma in terms of τ^{-1}-rigid modules.

Recall from [**AsSS,** Chapter VI, Lemma 4.1], for an algebra A, T a classical tilting module in $\mathrm{mod}\,A$ and $B=\mathrm{End}_A\,T$, if $M\in\mathrm{Fac}\,T$, then $\mathrm{pd}_B\,\mathrm{Hom}_A(T,M)\le\mathrm{pd}_A\,M$ holds. We prove an analog result in terms of τ-tilting modules as follows.

Theorem 4.2. *Let* A *be an algebra,* T *be a* τ*-tilting module in* $\mathrm{mod}\,A$ *and* $B=\mathrm{End}_A\,T$*. For any* $M\in\mathrm{Fac}\,T$*, we have*

(1) *If* $\mathrm{pd}_A\,M\le 1$ *holds, then* $\mathrm{pd}_B\,\mathrm{Hom}_A(T,M)\le\mathrm{pd}_A\,M$ *holds.*

(2) *If* $\operatorname{Ext}^i_A(T, M \oplus T) = 0$ *holds for any* $i \geq 1$*, then*

$$\operatorname{pd}_B \operatorname{Hom}_A(T, M) \leq \operatorname{pd}_A M$$

holds.

Proof. (1) If $\operatorname{pd}_A M = 0$, then $M \in \operatorname{Fac} T$ implies $M \in \mathsf{add}\, T$. One gets $\operatorname{Hom}_A(T, M)$ is a projective module in $\operatorname{mod} B$ since $\operatorname{Hom}_A(T, -)$ induces an equivalence between $\mathsf{add}\, T$ and $\mathsf{add}\, B$.

Now, assume $\operatorname{pd}_A M = 1$. Since $M \in \operatorname{Fac} T$, by Lemma 4.1 we get a short exact sequence

$$0 \to L \to T_0 \to M \to 0 \tag{4.1}$$

with $L \in {}^{\perp_0}(\tau T) = \operatorname{Fac} T$.

Recall that $L \in \mathcal{C} \subseteq \operatorname{mod} A$ is Ext-projective if $\operatorname{Ext}^1_A(L, \mathcal{C}) = 0$. In the following we show $L \in \mathsf{add}\, T$, that is, L is Ext-projective in $\operatorname{Fac} T$.

For any $N \in \operatorname{Fac} T$, applying the functor $\operatorname{Hom}(-, N)$ to the exact sequence (4.1), we get a long exact sequence $\operatorname{Ext}^1_A(M, N) \to \operatorname{Ext}^1_A(T_0, N) \to \operatorname{Ext}^1_A(L, N) \to \operatorname{Ext}^2_A(M, N)$. Hence $\operatorname{Ext}^1_A(L, N) = 0$ holds because of $\operatorname{pd}_A M = 1$ and $N \in \operatorname{Fac} T$. We are done.

Applying the functor $\operatorname{Hom}_A(T, -)$ to the sequence (4.1) again, we get the assertion since $\operatorname{Hom}(T, -)$ is an equivalence between $\mathsf{add}\, T$ and $\mathsf{add}\, B$.

(2) If $\operatorname{pd}_A M = \infty$, then there is nothing to show.

Now we can assume that $\operatorname{pd}_A M = t < \infty$. Since $M \in \operatorname{Fac} T$, by Lemma 4.1 we get a short exact sequence $0 \to L \to T_0 \to M \to 0$ with $L \in {}^{\perp_0}(\tau T) = \operatorname{Fac} T$, so $\operatorname{Ext}^1_A(T, L) = 0$. Applying the functor $\operatorname{Hom}_A(T, -)$ to the sequence (4.1), one gets $\operatorname{Ext}^{i+1}_A(T, L) \simeq \operatorname{Ext}^i_A(T, M) = 0$ for any $i \geq 1$ by assumption, and hence $\operatorname{Ext}^i_A(T, L) = 0$ for any $i \geq 1$. Continuing the similar process, we get the following long exact sequence

$$\cdots \to T_n \xrightarrow{f_n} T_{n-1} \to \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0 \tag{4.2}$$

with $T_i \in \mathsf{add}\, T$ and $L_{i+1} = \operatorname{Ker} f_i \in {}^{\perp_0}(\tau T) = \operatorname{Fac} T$ for $i \geq 0$ and $\operatorname{Ext}^j_A(T, L_{i+1}) = 0$ for $j \geq 1$ and $i \geq 0$.

Next we show that the exact sequence $0 \to L_{t+1} \to T_t \to L_t \to 0$ splits.

Since $\operatorname{pd}_A M = t < \infty$, then $\operatorname{Ext}_A^{t+1}(M, L_{t+1}) = 0$. On the other hand, applying the functor $\operatorname{Hom}_A(-, L_{t+1})$ to the sequence (4.2), one gets $0 = \operatorname{Ext}_A^{t+1}(M, L_{t+1}) \simeq \operatorname{Ext}_A^{t}(L_1, L_{t+1}) \simeq \cdots \simeq \operatorname{Ext}_A^{1}(L_t, L_{t+1})$ since $\operatorname{Ext}_A^i(T, M) = 0$ and $L_i \in \operatorname{Fac} T$ hold for any $i \geq 1$. Hence we have a long exact sequence.

$$0 \to T_t \xrightarrow{f_t} T_{t-1} \to \cdots \to T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0. \tag{4.3}$$

Applying the functor $\operatorname{Hom}_A(T, -)$ to the exact sequence (4.3), we have $0 \to \operatorname{Hom}_A(T, T_t) \to \operatorname{Hom}_A(T, T_{t-1}) \to \cdots \to \operatorname{Hom}_A(T, T_1) \to \operatorname{Hom}_A(T, T_0) \to \operatorname{Hom}_A(T, M) \to 0$ and hence $\operatorname{pd}_B \operatorname{Hom}_A(T, M) \leq t = \operatorname{pd}_A M$. □

For a module T in $\operatorname{mod} A$, we denote by $\operatorname{Sub} T = \{N \mid N \rightarrowtail T^n \text{ for some integer } n\}$. Then we have the following on the injective dimensions:

Theorem 4.3. *Let A be an algebra, T be a τ-tilting module in* $\operatorname{mod} A$ *and $C = \operatorname{End}_A \tau T^{\mathrm{op}}$. For any $N \in \operatorname{Sub} \tau T$, we have*

(1) *If* $\operatorname{id}_A N \leq 1$ *holds, then* $\operatorname{pd}_C \operatorname{Hom}_A(N, \tau T) \leq \operatorname{id}_A N$ *holds.*
(2) *If* $\operatorname{Ext}_A^i(\tau T \oplus N, \tau T) = 0$ *holds for any $i \geq 1$, then*

$$\operatorname{pd}_C \operatorname{Hom}_A(N, \tau T) \leq \operatorname{id}_A N$$

holds.

Proof. Throughout the proof, we denote by $U = \tau T$.

(1) If $\operatorname{id}_A N = 0$, then $N \in \operatorname{Sub} U$ implies $N \in \operatorname{\mathsf{add}} U$. One gets $\operatorname{Hom}_A(N, U)$ is a projective C-module since $\operatorname{Hom}_A(-, U)$ induces a duality between $\operatorname{\mathsf{add}} U$ and $\operatorname{\mathsf{add}} C$.

Assume $\operatorname{id}_A N = 1$. Since $N \in \operatorname{Sub} U$, by Lemma 4.1 we get a short exact sequence

$$0 \to N \to U_0 \to L \to 0 \tag{4.4}$$

where $L \in T^{\perp_0} = \operatorname{Sub} U$. In the following we show $L \in \operatorname{\mathsf{add}} U$, that is, $\operatorname{Ext}_A^1(N', L) = 0$ holds for any $N' \in \operatorname{Sub} U$. Applying the functor $\operatorname{Hom}_A(N', -)$ to the exact sequence (4.4), one gets the exact sequence $\operatorname{Ext}_A^1(N', U) \to \operatorname{Ext}_A^1(N', L) \to \operatorname{Ext}_A^2(N', N)$. The assertion follows from the facts U is Ext-injective and $\operatorname{id}_A N = 1$.

(2) If $\mathrm{id}_A N = \infty$, then there is nothing to show. So we can assume that $\mathrm{id}_A N = s < \infty$.

By Lemma 4.1 we get the following exact sequence

$$0 \to N \xrightarrow{f_0} U_0 \xrightarrow{f_1} U_1 \cdots \xrightarrow{f_s} U_s \to \cdots \tag{4.5}$$

with f_i the minimal left $\mathsf{add}\, U$-approximation. Denote by $L_i = \mathrm{Im}\, f_i$, then one gets $\mathrm{Ext}_A^k(L_i, U) = 0$ for any $k \geq 1$ and $i \geq 0$.

In the following we show the exact sequence $0 \to L_s \to U_s \to L_{s+1} \to 0$ splits. Applying the functor $\mathrm{Hom}_A(L_{s+1}, -)$ to the exact sequence (4.5), one gets $0 = \mathrm{Ext}_A^{s+1}(L_{s+1}, N) \simeq \mathrm{Ext}_A^s(L_{s+1}, L_1) \simeq \cdots \simeq \mathrm{Ext}_A^1(L_{s+1}, L_s)$ since $\mathrm{id}_A N \leq n$. Hence we have the following exact sequence

$$0 \to N \xrightarrow{f_0} U_0 \xrightarrow{f_1} \cdots \xrightarrow{f_s} U_s \to 0$$

Applying the functor $\mathrm{Hom}_A(-, U)$, one gets the assertion since

$$\mathrm{Ext}_A^k(L_i, U) = 0$$

holds for any $k, i \geq 1$. □

At the end of this section, we give an example to show our main results.

Example 4.4. Let A be the algebra given by the quiver

$$Q:\ 1 \underset{\beta_2}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_1}{\overset{\alpha_2}{\rightleftarrows}} 3$$

with relations $\alpha_1\beta_2 = 0$ and $\alpha_2\beta_1 = \beta_2\alpha_1$. Then

(1) A is an Auslander algebra and

$$T = {}^{1}{}_{2}{}_{3} \oplus {}^{\ \ 2}_{1\ \ 3}{}_{\ \ 2}{}_{\ \ \ 3} \oplus {}_{1}{}^{2}$$

is a τ-tilting module in $\mathrm{mod}\, A$.

(2) $B = \mathrm{End}_A T$ is given by the quiver

$$Q':\ 3 \xrightarrow{\gamma_3} 2 \underset{\gamma_1}{\overset{\gamma_2}{\rightleftarrows}} 1$$

with the relation $\gamma_1\gamma_2 = 0$ and $\mathrm{gl.dim}\, B = 2$.

(3) One can show $M = {}^{2}{}_{3} \in \operatorname{Fac} T$ with $\operatorname{pd}_A M = 1$, $\operatorname{Hom}_A(T, M) = S(2)$ in $\operatorname{mod} B$, and $\operatorname{pd}_B \operatorname{Hom}_A(T, M) \leq \operatorname{pd}_A M$.

Acknowledgements. The first author wants to thank Professor Zhaoyong Huang for supervision and continuous encouragement. The third author would like to thank Professor Osamu Iyama and Professor Bin Zhu for hospitality and useful discussion. All authors would like to thank the referee for useful suggestions to improve this paper.

REFERENCES

A1. T. Adachi, *The classification of τ-tilting modules over Nakayama algebras*, J. Algebra **452** (2016), 227–262.

A2. T. Adachi, *Characterizing τ-tilting modules over algebras with radical square zero*, Proc. Amer. Math. Soc. **144** (2016), no. 11, 4673–4685.

AI. T. Aihara and O. Iyama, *Silting mutation in triangulated categories*, J. London. Math. Soc. **85** (2012), no. 3, 633–668.

AIR. T. Adachi, O. Iyama and I. Reiten, *τ-tilting theory*, Compos. Math. **150** (2014), no. 3, 415–452.

AS. M. Auslander and S.O. Smalø, *Addendum to "Almost split sequences in subcategories"*, J. Algebra **69** (1981), 426–454. Addendum to *J. Algebra* **71** *(1981), 592–594*.

AsH. I. Assem and D. Happel, *Generalized tilted algebra of type An*, Comm. Algebra **20** (1981), 2101–2125.

AsSS. I. Assem, D. Simson and A. Skowroński, *Elements of the representation theory of associative algebras I: techniques of reperesentation theory*, London Math. Soc. Student Texts **65**, Cambridge Univ. Press, Cambridge (2006).

BB. S. Brenner and M.C.R. Bulter, *Generalization of the Bernstein-Gelfand-Ponomarev reflection functors*, 103–169 in *Representation theory II: Proc. Second Internat. Conf.* (Carleton University, Ottawa, Ontario, 1979), Lecture Notes in Math. **832** Springer-Verlag, 103–169 (1980).

BMRRT. A.B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. **204** (2006), no. 2, 572–618.

BZ. A.B. Buan and Y. Zhou, *Endomorphism algebras of 2-term silting complexes* Algebr. Represent. Th. **21** (2018), no. 1, 181–194.

DIJ. L. Demonet, O. Iyama and G. Jasso, *τ-tilting finite algebras and g-vectors*, Int. Math. Res. Not. IMRN **3** (2019), 852–892.

DIP. L. Demonet, O. Iyama and Y. Palu, *τ-tilting theory for Schurian algebras*, in preparation.

DIRRT. L. Demonet, O. Iyama, N. Reading, I. Reiten and H. Thomas, *Lattice theory of torsion class*, arXiv:1711.01785.

EJR. F. Eisele, G. Janssens and T. Raedschelders, *A reduction theorem for τ-rigid modules*, Math. Zeit. **290** (2018), no. 3-4, 1377–1413.

HR. D. Happel and C.M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), no. 2, 399–443.

HU. D. Happel and L. Unger, *Modules of finite projective dimension and cocovers*, Math. Ann. **306** (1996), no. 1, 445–457.

HZ. Z. Huang and Y. Zhang, *G-stable support τ-tilting modules*, Front. Math. China **11** (2016), no. 4, 1057–1077.

IJY. O. Iyama, P. Jorgensen and D. Yang, *Intermediate co-t-structures, two-term silting objects, τ-tilting modules and torsion classes*, Algebra Number Theory **8** (2014), no. 10, 2413–2431.

IY. O. Iyama and Y. Yoshino, *Mutations in triangulated categories and rigid Cohen-Macaulay modules*, Invent. Math. **172** (2008), no. 1, 117–168.

IZ1. O. Iyama and X. Zhang, *Classifying τ-tilting modules over the Auslander algebra of $K[X]/(x^n)$*, to appear in *J. Math. Soc. Japan*, arXiv: 1602.05037.

IZ2. O. Iyama and X. Zhang, *Tilting modules over Auslander-Gorenstein algebra*, Pacific J. Math. **298** (2019), no. 2, 399–416.

J. G. Jasso, *Reduction of τ-tilting modules and torsion pairs*, Int. Math. Res. Not. IMRN **16** (2015), 7190–7237.

K. R. Kase, *Weak orders on symmetric groups and posets of support τ-tilting modules*, Int. J. Algebr. Comput. **27** (2017), no. 5, 501–546.

KR. B. Keller and I. Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi-Yau*, Adv. Math. **211** (2007), no. 1, 123–151.

M. Y. Miyashita, *Tilting modules of finite projective dimension*, Math. Zeit. **193** (1986), no. 1, 113–146.

Mi. Y. Mizuno, *Classifying τ-tilting modules over preprojective algebras of Dynkin type*, Math. Zeit. **277** (2014), no. 3, 665–690.

W. J. Wei, *τ-tilting theory and $*$-modules*, J. Algebra **414** (2014), 1–5.

Z1. X. Zhang, *τ-rigid modules for algebras with radical square zero*, to appear in Algebr. Colloq., arXiv:1211.5622.

Z2. X. Zhang, *τ-rigid modules over Auslander algebras*, Taiwanese J. Math. **21** (2017), no. 4, 327–338.

Zi1. S. Zito, *τ-rigid modules from tilted to cluster-tilted algebras*, Comm. Algebra. **47** (2019), no. 9, 3716–3734.

Zi2. S. Zito, *τ-tilting finite tilted and cluster-tilted algebras*, arXiv:1902.05866v1.

Department of Mathematics, Nanjing University, Nanjing, China
Email address: **xiezongzhen3@163.com**

School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing, China
Email address: **zanlibo@nuist.edu.cn**

School of Mathematics and Statistics, Jiangsu Normal University, Xuzhou, China
Email address: **xjzhangmaths@163.com**