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MAY MODULES OF COUNTABLE RANK

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ABSTRACT. In a 1990 paper, W. May studied the question of when isomorphisms of the endomorphism rings of mixed modules are necessarily induced by isomorphisms of the underlying modules. In so doing he introduced a class of mixed modules over a complete discrete valuation domain; we later renamed these modules after their inventor. The class of May modules of countable torsion-free rank is particularly important. A decomposition theorem is established for such modules. The modules in this class are characterized in several ways. Finally, an example is constructed showing that several of these ideas do not extend to May modules of uncountable torsion-free rank.

1. Introduction. Throughout, all modules will be over a fixed complete discrete valuation domain $\mathbf{R}$, and $p \in \mathbf{R}$ will be a prime. Our terminology and notation will generally follow that found in [**2**].

Suppose B is a submodule of M. We say B is

(a) *full-rank* if M/B is torsion;

(b) an *NT-submodule* if B is nice in M and M/B is totally projective; and

(c) an *NFT-submodule* if it is free and an NT-submodule.

A module M is a **May module** if every full-rank free submodule $F \subseteq M$ contains an NFT-submodule $B \subseteq F$. This class was studied in [**4**] in the context of endomorphism rings of mixed modules. In [**3**], these results were clarified and the class was named after its inventor. With a few exceptions that were specifically described in [**3**], it was shown that if M is a May module and N is any other module, then any ring isomorphism of their endomorphism rings, $E(M) \to E(N)$, will be induced by an isomorphism of the underlying modules, $M \to N$.

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Whenever we use the term *rank* we will always mean torsion-free rank. May modules of finite rank were studied in **[1]**. In this paper we are primarily concerned with May modules of countable rank. We begin by reviewing the following statement, which we have slightly reworded using our terminology.

Corollary 1.1 ([**4**], Corollary B)**.** *Suppose M is a reduced module of countable rank and torsion T.*

(1) M is a May module if and only if it has an NFT-submodule.

(2) If T is totally projective, then M is a May module.

Our objective is to investigate this class more thoroughly. First, we will show that each such module has a natural decomposition (Theorem 3.5). In addition, we will provide two distinct characterizations of the modules in the class (Theorems 4.1 and 4.4).

In proving these results we will employ the language of *valuated modules* and *valuated vector spaces* which we briefly review (see, for example, [**6**]). A *valuation* on a module V is a function $|\ \ |$ from V to the ordinals (with ∞ adjoined) such that for every ordinal α, $V(\alpha) := \{x \in V : |x| \geq \alpha\}$ is a submodule of V and $pV(\alpha) \subseteq V(\alpha+1)$. For simplicity, we will assume all valuated modules are *reduced* in the sense that $V(\infty) = 0$. If $pV = 0$, then V is a valuated *vector space* (over the residue field $\mathbf{R}/p\mathbf{R}$). When we are only concerned about module properties, we will often emphasize this by including the word *algebraic*. So, for example, we can talk about valuated modules that are algebraically free. By an *isometry* of valuated modules we mean an algebraic isomorphism that preserves valuations.

Suppose V is a submodule of the reduced module M. We denote the height function on M by h_M. If for all $x \in V$ we let $|x| = h_M(x)$, then we clearly obtain a valuation on V. In this case we call M a *realization* of V. If, in addition, V is an NT-submodule of M, then we say M is an NT-realization of V. It is well-known that any valuated module has an NT-realization (see [**6**], Theorem 1).

If V is a valuated module and $x \in V$, then the *value sequence* of x is given by $(|x|, |px|, |p^2x|, \ldots)$. We say the value sequence has a *gap* at $|p^kx|$ if $|p^kx| + 1 < |p^{k+1}x|$. We say V is:

(a) *non-gapped* if every torsion-free element of V has no gaps in its value sequence.

(b) *finitely-gapped* if every torsion-free element of V has only a finite number of gaps in its value sequence.

(c) *infinitely-gapped* if every torsion-free element of V has an infinite number of gaps in its value sequence.

(d) α-*limiting* if every torsion-free element of $x \in V$ satisfies $\sup_{k<\omega}\{|p^k x|\} = \alpha$. Similarly, we say V is *limiting* if it is α-limiting for some α of countable cofinality.

If M is a realization of V with M/V torsion, it is elementary that M (under the height valuation) will satisfy one of (b),(c) or (d) if and only if the same can be said of V.

If V is a valuated module, then an algebraic direct sum decomposition $V = \oplus_{i\in I} W_i$ is *valuated* if the value of any element of V is the minimum of the values of its components. A basis $\{b_i\}_{i\in I}$ of an algebraically free valuated module B is called a *decomposition basis* if the corresponding algebraic decomposition is actually valuated. The module M is a **Warfield module** if it is an NT-realization of a valuated module with a decomposition basis.

A valuated module B will be said to be a *valuated-projective* if it has a decomposition basis and is non-gapped. The module M is **balanced-projective** if it is an NT-realization of a valuated-projective module. Clearly, a balanced-projective module is a (particularly simple) Warfield module. Using different terminology, these classes were introduced in **[9]** and **[10]**. Both classes can be completely described using cardinal invariants. It is easy to see that a Warfield module will be a May module (see **[4]**, Corollary B(3)); so in particular, a balanced-projective module will be a May module.

Summarizing the contents of the paper, Section 2 contains a review of some background material, together with a remark or two about its application to the specific case of May modules.

In Section 3 we describe a natural way to decompose an arbitrary May module of countable rank. We first establish a useful result on valuated decompositions (Lemma 3.2). Suppose P is an algebraically free valuated module of countable rank, $P(\nu) = 0$ and $I = \{\alpha \leq \nu : \alpha$ is a limit ordinal of countable cofinality$\}$. Then there is a full-rank nice submodule $E \subseteq P$ with a valuated decomposition $E = \oplus_{\alpha \in I} E_\alpha$ where each E_α is α-limiting.

This valuated decomposition determines an algebraic decomposition of May modules. We say the modules M_1 and M_2 are *H-isomorphic* if there are totally projective modules H_1 and H_2 such that $M_1 \oplus H_1 \cong M_2 \oplus H_2$; in this case we write $M_1 \cong_H M_2$. If M_1 and M_2 have isometric NFT-submodules, then $M_1 \cong_H M_2$ (Lemma 3.3). If M is a May module of countable rank, then $M \cong_H \bigoplus_{\alpha \in I} M_\alpha$, where each M_α is α-limiting (Theorem 3.5). In fact, the factors M_α will be unique up to H-isomorphism.

We then consider the individual terms in this decomposition. We show that each M_α is isomorphic to a sum $N_\alpha \oplus L_\alpha$ where N_α is balanced-projective and L_α is infinitely gapped. Putting these together for all $\alpha \in I$ shows that $M \cong_H N \oplus L$, where N is balanced-projective and L is infinitely gapped (Corollary 3.12).

In particular, we characterize the two extreme cases in this analysis. First, a finitely-gapped module M of countable rank is a May module if and only if it is balanced-projective (Theorem 3.8). On the other hand, a reduced infinitely-gapped module M of countable rank is a May module if and only if its torsion submodule is totally projective (Theorem 3.11). This is a partial converse of Corollary 1.1(2).

In Section 4 we deepen the connection between countable rank May modules and balanced-projective modules. We verify that a reduced module M of countable rank is a May module if and only if it has a balanced-projective submodule N such that M/N is countably generated (Theorem 4.1). Interestingly, it is not necessary to assume that N has any particular properties as a submodule of M (such as being isotype or nice). Since the balanced-projectives form a particularly tractable class of modules, this characterization gives a fairly concrete view of May modules of countable rank. It tells us, for example, that a reduced module that is an extension of a May module of countable rank by a quotient that is countably generated will also be a May module (Corollary 4.3).

We give a second characterization of countable rank May-modules (Theorem 4.4). In this result no reference is made to NFT-submodules at all. In fact, the characterization is completely expressed using two important classes of torsion-modules, namely the totally projective modules and their generalization, the S-modules.

In Section 5 we present some results on May modules of uncountable rank. Suppose M is a May module of arbitrary rank with torsion T. We identify two different situations in which T must be totally projective: first, if M is limiting and infinitely-gapped (Theorem 3.10); and second, if the length of M is countable (Theorem 5.1).

On the negative side, with the help of a little set theory, we construct an example of a finitely-gapped May module of rank ω_1 that is not balanced-projective (Theorem 5.3). In other words, Theorem 3.8, which was central to our discussion of May modules of countable rank, cannot be extended to May modules of uncountable rank.

We conclude with a few open questions.

2. Preliminaries. In **[3]** it was verified that the torsion May modules are precisely the totally projective modules; the torsion-free May module are precisely the free modules; and if α is an ordinal, then M is a May module if and only if both $p^\alpha M$ and $M/p^\alpha M$ are May modules. If B is an NFT-submodule of the May module M, then since both B and M/B are reduced, so is M.

We mention a couple of important consequences of the fact that we are assuming $\mathbf{R}$ is a *complete* discrete valuation domain. First, it is well known that any countable rank torsion-free module will be isomorphic to a direct sum $F \oplus D$, where F is free and D is divisible. Also, if M is a module, then any finitely generated submodule of M will actually be nice in M.

The following is the cornerstone of this investigation:

Lemma 2.1 ([**4**], Lemma 11)**.** *Suppose M is a reduced module of countable rank and F is a free full-rank submodule with basis $\{b_n\}_{n<\omega}$. Then there is a sequence $\{m_n\}_{n<\omega}$ of non-negative integers such that $F' := \langle p^{m_n} b_n : n < \omega\rangle$ is a nice submodule of M.*

When applying Lemma 2.1, if $F \subseteq M$ is either non-gapped, finitely-gapped, infinitely-gapped or α-limiting, then the nice submodule $F' \subseteq F$ will likewise be, respectively, non-gapped, finitely-gapped, infinitely-gapped or α-limiting. In addition, for any valuated decomposition of F, by choosing bases for each term separately, we may assume that the decomposition of F leads to a corresponding decomposition of F'. So, for example, if $\{b_n\}_{n<\omega}$ is a decomposition basis for F, then $\{p^{m_n}b_n\}_{n<\omega}$ is a decomposition basis for F'.

In addition, if V is an algebraically free valuated module of countable rank, then by considering an NT-realization M of V, if $F \subseteq V$ is full-rank, then we can find a submodule $F' \subseteq F$ that is full-rank and nice in V.

We next recall a useful result of Wallace.

Theorem 2.2 ([**7**], Theorem 1)**.** *Suppose G is a reduced torsion module with a totally projective submodule H. If G/H is countably generated, then G is also totally projective.*

Because of the centrality of Corollary 1.1 to our discussions, we include a proof based upon Lemma 2.1 and Theorem 2.2.

Proof. (Corollary 1.1) Regarding the first statement, it is obvious that if M is a May module, then it has an NFT-submodule.

Conversely, suppose M is a countable rank module with an NFT-submodule C. If $F \subseteq M$ is any free full-rank submodule of M, then we need to find an NFT-submodule contained in F. Replacing F by $F \cap C$, we may assume that $F \subseteq C$.

By Lemma 2.1 there is a full-rank nice submodule $B \subseteq F$. Observe that C/B will be a countably generated nice submodule of M/B and $(M/B)/(C/B) \cong M/C$ is totally projective. There is a nice composition series for C/B consisting of finitely generated submodules, and using a nice composition series for M/C, extendible to a nice composition series for M/B. Therefore, M/B is totally projective, so that B is an NFT-submodule, completing the argument.

Regarding the second statement, suppose now that T is totally projective. If $F \subseteq M$ is free and full-rank, then by Lemma 2.1, we can find a nice, full-rank submodule $B \subseteq F$. Since M is reduced and B is nice in M, we can conclude that M/B is also reduced. The map $x \mapsto x + B$ gives an embedding of T into M/B. The cokernel of this map is isomorphic to $M/(T + B)$, which is the epimorphic image of M/T. Therefore, the cokernel is countably generated, and Theorem 2.2

implies that M/B is totally projective; i.e., B is an NFT-submodule of M.

So by our first statement, M is a May module. $\square$

We repeat for emphasis the important point contained in Corollary 1.1(1): If a countable rank module has a single NFT-submodule, then it is a May module, so that in fact, it has many such submodules. We next sharpen this observation slightly.

Proposition 2.3. *Suppose M is a May module of countable rank. Then a submodule $B \subseteq M$ is an NFT-submodule if and only if it is nice, free and full-rank.*

Proof. Necessity being obvious, suppose B is nice, free and full-rank; we need to show M/B is totally projective. Let $A \subseteq B$ be an NFT-submodule of M. Using an element of a nice system for M/A, there is a countably generated nice submodule $C/A \subseteq M/A$ containing B/A such that $(M/A)/(C/A) \cong M/C$ is totally projective. Clearly M/C has a nice composition series, and since B is nice and C/B is countably generated, C/B also has a nice composition series. Fitting these together gives a nice composition series for M/B, completing the proof. $\square$

We make note of the following elementary consequence.

Proposition 2.4. *Suppose M is a May module of countable rank. Then any summand of M is also a May module.*

Proof. Suppose we have a decomposition $M = K \oplus L$. By Lemma 2.1, there are free, full-rank nice submodule $B \subseteq K$ and $C \subseteq L$. By Proposition 2.3, $B \oplus C$ is an NFT-submodule of M. As $(K/B) \oplus (L/C) \cong M/(B \oplus C)$, K/B and L/C are totally projective. Hence by Corollary 1.1(1), K and L are May modules. $\square$

We now review some well-known results regarding valuated vector spaces. Suppose V is a λ-bounded valuated vector space (i.e., $V(\lambda) = 0$). A subset $U \subseteq V$ will be said to be $< \lambda$-*bounded* if there is a $\beta < \lambda$ such that $U(\beta) = 0$.

If λ is a limit ordinal of countable cofinality, we say V is λ-*summable* if it is the ascending union of a sequence $\{U_n\}_{n<\omega}$ of $< \lambda$-bounded submodules. If V is the valuated direct sum of a collection of $< \lambda$-bounded modules, then it is easy to verify that V is λ-summable. In particular, if T is a torsion-module that is the direct sum of modules of length strictly less than λ, then the socle $T[p]$ is λ-summable. Therefore, by classical results, if H is a p^λ-bounded totally projective module, then its socle $H[p]$ is λ-summable.

If λ is a limit ordinal, then a torsion module T is said to be a C_λ-*module* if for every $\nu < \lambda$, $T/p^\nu T$ is totally projective. For example, any totally projective module is a C_λ-module for any λ. Clearly, any torsion module is a C_ω-module.

We mention another useful result, due to Wallace, restated slightly using our terminology. It was a generalization of an earlier result of Megibben for countable limit ordinals ([**5**], Theorem A). This, in turn, was a generalization of the classical "Kulikov criterion" for direct sums of cyclic modules ([**2**], Theorem 3.5.1).

Proposition 2.5 ([**8**], Proposition 2.5)**.** *Let λ be a limit ordinal of countable cofinality and T be a p^λ-bounded C_λ-module. Then T is totally-projective if and only if its socle $T[p]$ is λ-summable.*

The following application is valid for May modules of arbitrary rank.

Lemma 2.6. *Suppose M is a May module whose length λ is a limit ordinal of countable cofinality and T is the torsion submodule of M. If T is a C_λ-module, then it is totally projective.*

Proof. Let B be an NFT-submodule of M; so M/B will be totally projective. Since $p^\lambda M = 0$, M/B will be p^λ-bounded, so that $(M/B)[p]$ is λ-summable. Let $(M/B)[p]$ be the ascending union of U_n, where for each $n < \omega$, $U_n(\gamma_n) = 0$ for some $\gamma_n < \lambda$. If for each $n < \omega$ we let $W_n = \{x \in T[p] : x + B \in U_n\}$, then it follows that $T[p]$ will be the ascending union of the W_ns. If $x \in W_n(\gamma_n)$, then $x + B \in U_n(\gamma_n) = 0$; i.e., $W_n(\gamma_n) \subseteq T[p] \cap B = 0$. Therefore, $T[p]$ is also λ-summable, so that T is totally projective by Proposition 2.5. □

We will now apply summability to another collection of modules. The torsion module T is said to be an S-*module* if it is isomorphic to the torsion submodule of a balanced-projective module. These were introduced by Warfield in **[9]** and **[10]** where they were completely classified by cardinal invariants. An S-module decomposes into a direct sum $H \oplus (\oplus E_i)$, where H is totally projective and, for each i, there is a limit ordinal λ_i of uncountable cofinality and a rank-one balanced-projective module N_i with torsion E_i such that $p^{\lambda_i} N_i \cong \mathbf{R}$; so $p^{\lambda_i} E_i = 0$.

The S-modules are closed under direct sums and summands, and if α is any ordinal, then T is an S-module if and only if $p^\alpha T$ and $T/p^\alpha T$ are S-modules. The following observation is a direct consequence of the above discussion:

Lemma 2.7. *If λ is a limit ordinal of countable cofinality, then any p^λ-bounded S-module has a λ-summable socle.*

We make one more observation about λ-summable valuated vector spaces. Suppose V is λ-bounded, $x \in V$ and $U \subseteq V$ is a $< \lambda$-bounded subspace. Choose $\beta < \lambda$ so that $U(\beta) = 0$. Consider $U' = U + \langle x \rangle$. Either U' is also β-bounded, or we can find a $u \in U$ such that $\beta < |x+u| := \gamma < \lambda$. In the latter case it easily follows that $U'(\gamma+1) = 0$. In either case U' is also $< \lambda$-bounded.

By induction, the above implies that if $F \subseteq V$ is finitely generated and U is $< \lambda$-bounded, then $U + F$ is also $< \lambda$-bounded. From this, we can readily conclude the following:

Lemma 2.8. *Suppose λ is a limit ordinal of countable cofinality, V is a λ-bounded valuated vector space and $W \subseteq V$ is subspace such that V/W is countably generated. If W is λ-summable, then so is V.*

3. Decomposing May modules of countable rank.

We now present a natural way to decompose May modules of countable rank. If V is a valuated module and $X \subseteq V$, let $X^* = X \setminus \{0\}$ and $|X| = \{|x| : x \in X\}$. So V is α-limiting if and only if for every torsion-free $x \in V$, $\sup(|\langle x \rangle^*|) = \alpha$. And recall that V is limiting if and only if it is α-limiting for some α.

Lemma 3.1. *If B is an algebraically free valuated module of finite rank, then B is limiting if and only if $|B^*|$ has order type ω (i.e., it can be thought of as a strictly increasing sequence of ordinals).*

Proof. It being obvious that if $|B^*|$ has order type ω, then B is limiting, we assume that B is α-limiting. If $\beta < \alpha$, then we need to show that $\beta \cap |B^*|$ is a finite set. Suppose first that $B(\beta)$ has smaller rank than B. This means that $B/B(\beta)$ is not a torsion module. So we can find a $y \in B$ such that $p^k y \notin B(\beta)$ for all $k < \omega$. This, however, contradicts that B is α-limiting.

So we may assume $B(\beta)$ has the same rank as B for all $\beta < \alpha$. It follows that each such $B/B(\beta)$ is a finitely generated torsion module. Therefore, since any composition series for $B/B(\beta)$ must be finite, there are only a finite number of distinct submodules of the form $B(\gamma) \subseteq B$ for some $\gamma < \beta$. As these submodules correspond to the elements of $\beta \cap |B^*|$, this set must be finite. □

Recall that if $C \subseteq B$ are valuated modules and C is nice in B, then every coset $b + C$ has an element b_p of maximal value (i.e., it is *proper* with respect to C). Setting $|b + C| = |b_p|$ makes B/C into a valuated module. It is easy to check that $|B| = |C| \cup |B/C|$.

The next result is the key to our analysis of countable rank May modules. It will allow us to break them apart into submodules that are limiting.

Lemma 3.2. *Suppose P is an algebraically free valuated module of countable rank with $P(\nu) = 0$ and $I = \{\alpha \leq \nu : \alpha$ is a limit ordinal of countable cofinality$\}$. There is a nice full-rank submodule $E \subseteq P$ with a valuated decomposition $E = \oplus_{\alpha \in I} E_\alpha$ such that each E_α is α-limiting, and $|E^*_\alpha| \cap |E^*_\beta| = \varnothing$ for $\alpha \neq \beta \in I$.*

Proof. Let $\{b_j\}_{j<\omega}$ be a basis for P. By inducting on $j < \omega$, we construct, for each $\alpha \in I$, an α-limiting submodule $E_{\alpha,j}$ of P satisfying the following:

(1) $E_{\alpha,0} \subseteq E_{\alpha,1} \subseteq E_{\alpha,2} \subseteq \cdots \subseteq E_{\alpha,j}$.

(2) $\oplus_{\alpha \in I} E_{\alpha,j}$ is a full-rank submodule of $P_j := \langle b_0, b_1, \ldots, b_j \rangle$.

(3) If $\alpha \neq \beta \in I$, then $|E^*_{\alpha,j}| \cap |E^*_{\beta,j}| = \varnothing$.

For a given $j < \omega$, since P_j has finite rank, (2) implies that $E_{\alpha,j} = 0$ for all but finitely many $\alpha \in I$. In addition, (3) implies that the sum in (2) is valuated.

Assume for a moment that we have constructed these submodules. For each $\alpha \in I$ we let $E'_\alpha := \cup_{j<\omega} E_{\alpha,j}$ and $E' := \oplus_{\alpha\in I} E'_\alpha$. Applying Lemma 2.1, we may assume $E_\alpha \subseteq E'_\alpha$ and $E := \oplus_{\alpha\in I} E_\alpha \subseteq E'$ is nice and full-rank in P. It is straightforward to verify that our requirements are satisfied.

Again, we proceed by induction on j. If $j = 0$, then let $\alpha := \sup |\langle b_0\rangle^*| \in I$. We just let $E_{\alpha,0} = \langle b_0\rangle$ and $E_{\beta,0} = 0$ when $\alpha \neq \beta \in I$. Clearly, all our conditions will hold.

So suppose we have defined each $E_{\alpha,j-1}$ for all $\alpha \in I$; we will show how to construct all the $E_{\alpha,j}$.

Let $U := \oplus_{\alpha\in I} E_{\alpha,j-1} \subseteq P_{j-1}$ and $V := \langle b_j\rangle + U$. Since U is finitely generated, it will be nice in V. Set $\gamma := \sup(|(V/U)^*|) \in I$.

Recall $E_{\alpha,j-1} = 0$ for all but finitely many $\alpha \in I$; and even if $\alpha \neq \gamma$ and $E_{\alpha,j-1} \neq 0$, by Lemma 3.1, $|E^*_{\alpha,j-1}|$ has order type ω and supremum $\alpha \neq \gamma$. This mean that, after possibly replacing b_j with $p^m b_j$ for some $m < \omega$, we may assume that $\kappa := |b_j + U|_{V/U}$ has the property that $\gamma \neq \alpha \in I$ implies that $|E^*_{\alpha,j-1}| \cap [\kappa, \gamma) = \varnothing$.

Let $b \in b_j + U$ be proper with respect to U; so $|b| = \kappa$. We define $E_{\gamma,j} := E_{\gamma,j-1} + \langle b\rangle$, and if $\gamma \neq \alpha \in I$, we set $E_{\alpha,j} := E_{\alpha,j-1}$.

Claim: $|E^*_{\gamma,j}| \subseteq |E^*_{\gamma,j-1}| \cup [\kappa, \gamma)$.

For any $m < \omega$ it is easy to see that

$$\kappa \leq |p^m b| \leq |p^m b + E_{\gamma,j-1}|_{E_{\gamma,j}/E_{\gamma,j-1}} \leq |p^m b + U|_{V/U} < \gamma.$$

So that

$$|E^*_{\gamma,j}| = |E^*_{\gamma,j-1}| \cup |(E_{\gamma,j}/E_{\gamma,j-1})^*| \subseteq |E^*_{\gamma,j-1}| \cup [\kappa, \gamma),$$

which gives the Claim.

We now verify the three conditions in our induction. Condition (1) is obvious for j, and condition (2) follows from the observation that we have increased the rank of the sum by 1, so that it must remain full-rank in P_j. So we need to verify condition (3).

Let $\alpha, \beta \in I$ with $\alpha \neq \beta$. If $\alpha \neq \gamma \neq \beta$, then it follows that

$$|E^*_{\alpha,j}| \cap |E^*_{\beta,j}| = |E^*_{\alpha,j-1}| \cap |E^*_{\beta,j-1}| = \varnothing.$$

On the other hand, suppose $\alpha \neq \gamma = \beta$. Since $E_{\alpha,j} = E_{\alpha,j-1}$ and $|E^*_{\alpha,j-1}| \cap [\kappa,\gamma) = \varnothing$, the Claim implies

$$\begin{aligned} |E^*_{\alpha,j}| \cap |E^*_{\beta,j}| &\subseteq |E^*_{\alpha,j-1}| \cap (|E^*_{\gamma,j-1}| \cup [\kappa,\gamma)) \\ &= (|E^*_{\alpha,j-1}| \cap |E^*_{\gamma,j-1}|) \cup (|E^*_{\alpha,j-1}| \cap [\kappa,\gamma)) = \varnothing, \end{aligned}$$

which gives (3).

Finally, we need to verify that each $E_{\alpha,j}$ remains α-limiting. If $\alpha \neq \gamma$, this follows because $E_{\alpha,j} = E_{\alpha,j-1}$. So consider $\alpha = \gamma$.

As noted above, by (3) for this j, we know the sum in (2) is valuated. Therefore, V/U is isometric to $E_{\gamma,j}/E_{\gamma,j-1}$, so that

$$|E^*_{\gamma,j}| = |E^*_{\gamma,j-1}| \cup |(E_{\gamma,j}/E_{\gamma,j-1})^*| = |E^*_{\gamma,j-1}| \cup |(V/U)^*|$$

has order type ω and supremum γ, completing the proof. □

The following is our primary tool for building H-isomorphisms.

Lemma 3.3. *(a) If M_1 and M_2 are NT-realizations of isometric submodules $C_1 \subseteq M_1$ and $C_2 \subseteq M_2$, then $M_1 \cong_H M_2$.*

(b) If $M_1 \cong_H M_2$, then M_1 is a May module if and only if M_2 is a May module.

Proof. (a): Suppose $\phi : C_1 \to C_2$ is an isometry. We can clearly find totally projective modules H_1 and H_2 such that the relative Ulm invariants of C_1 in $M_1 \oplus H_1$ agree with the relative Ulm invariants of C_2 in $M_2 \oplus H_2$. This implies $M_1 \oplus H_1 \cong M_2 \oplus H_2$, completing the proof of (a).

(b): Suppose H is totally projective. It is easy to check M is a May module if and only if $M \oplus H$ is a May module. Therefore M_1 is a May module if and only if $M_1 \oplus H_1 \cong M_2 \oplus H_2$ is a May module if and only if M_2 is a May module. □

Combining Lemma 3.3(b) with Proposition 2.4 gives the following:

Corollary 3.4. *If M_1 is a countable rank May module, $M_1 \cong_H M_2$ and N is a summand of M_2, then N is a May module.*

This brings us to our main decomposition theorem.

Theorem 3.5. *Suppose M is a May module of countable rank, $p^\nu M=0$ and $I=\{\alpha\leq\nu:\alpha$ is a limit ordinal of countable cofinality$\}$. Then $M\cong_H \bigoplus_{\alpha\in I} M_\alpha$, where each M_α is an α-limiting May module. Furthermore, this decomposition is unique in the sense that if $M\cong_H \bigoplus_{\alpha\in I} M'_\alpha$ is another such decomposition, then for all $\beta\in I$, $M_\beta\cong_H M'_\beta$.*

Proof. We first establish the existence of this decomposition. Let $P\subseteq M$ be an NFT-submodule. Find $E=\oplus_{\alpha\in I}E_\alpha\subseteq P$ as in Lemma 3.2. By Proposition 2.3, E is also an NTF-submodule of M. If for each $\alpha\in I$ we let M_α be some NT-realization of E_α, then $\bigoplus_{\alpha\in I} M_\alpha$ will be another NT-realization of E. By Lemma 3.3(a), $M\cong_H\bigoplus_{\alpha\in I} M_\alpha$. By Corollary 3.4, each M_α is a May module.

Turning now to uniqueness, suppose we have totally projective modules H,H' and an isomorphism $H\oplus\bigoplus_{\alpha\in I} M_\alpha\to H'\oplus\bigoplus_{\alpha\in I} M'_\alpha$. Absorbing the totally projective terms into one of the others in these decompositions, we may assume $H=H'=0$ and the isomorphism is actually an equality. Call the resulting module M.

Fix some $\beta\in I$; we want to show $M_\beta\cong_H M'_\beta$. For each $\alpha\in I$, let $E_\alpha\subseteq M_\alpha$ be an NFT-submodule; so $E=\oplus_{\alpha\in I}E_\alpha$ is an NFT-submodule of M. Let $J=\{\alpha\in I:\alpha>\beta\}$. Note that for each $\alpha\in J$, $E_\alpha(\beta)$ is full-rank in M_α; replacing E_α by $E_\alpha(\beta)$, we may assume $E(\beta)=\oplus_{\alpha\in J}E_\alpha$.

Similarly, for each $\alpha\in I$, let $E'_\alpha\subseteq M'_\alpha$ be an NFT-submodule. Using Lemma 2.1, we may clearly assume $E':=\oplus_{\alpha\in I}E'_\alpha\subseteq E$ and $E'(\beta)=\oplus_{\alpha\in J}E'_\alpha$.

Note that if $x\in E$, then $x\in E_\beta\oplus E(\beta)$ if and only if $\sup_{n<\omega}|p^n x|\geq\beta$; similarly for $x\in E'$. It follows that $E'_\beta\oplus E'(\beta)$ will be full-rank in $E_\beta\oplus E(\beta)$.

Therefore, under the obvious map $M\to M/p^\beta M$, E'_β maps isometrically to a nice full-rank submodule of the image of E_β. It follows that we can consider E'_β to be an NFT-submodule of both M_β and M'_β. So by Lemma 3.3(a), $M_\beta\cong_H M'_\beta$. □

We now analyze some of the terms in the above decomposition a bit more. We noted before that if M is an α-limiting module and α is not of the form $\beta+\omega$, then M must be infinitely gapped. The next observation shows that if α *is* of this form, then we can split M into a part that is valuated-projective and another part that is infinitely-gapped.

Lemma 3.6. *Suppose $\alpha = \beta + \omega$ and M is an α-limiting May module of countable rank. Then there is a decomposition $M = N \oplus L$, where N is balanced-projective and L is infinitely gapped.*

Proof. Suppose first that $\beta = 0$. If T is the torsion submodule of M, then $M/T = N' \oplus D$, where N' is free (and hence balanced-projective) and D is divisible. It follows that $M = N \oplus L$, where $N \cong N'$ and $L/T = D$. If L were not infinitely-gapped, it would be possible to find a torsion-free $x \in L$ such that $\langle x \rangle$ is pure in L. But since $\mathbf{R}$ is complete, this cyclic submodule would be a summand, which contradicts that D is divisible. Therefore, L must be infinitely-gapped.

Suppose now that $\beta > 0$. So as above, there is a decomposition $p^\beta M \cong N_\beta \oplus L_\beta$. Note that $M/p^\beta M$ will be a torsion May module, i.e., it is totally projective. It easily follows from the theory of totally projective modules that the decomposition $p^\beta M = N_\beta \oplus L_\beta$ extends to a decomposition $M = N \oplus L$, as required. $\square$

Suppose E_α is one of the terms in Lemma 3.2. If α is not of the form $\beta + \omega$, we already know that E_α is infinitely gapped. Suppose then that α is of this form. Consider an NT-realization M_α of E_α; Lemma 3.6 implies that E_α has a full-rank nice submodule of the form $B_\alpha \oplus C_\alpha$, where B_α is valuated-projective and C_α is infinitely gapped. Splicing this all together for all α gives the following:

Corollary 3.7. *If P is an algebraically free valuated module of countable rank, then there is a nice full-rank submodule $E \subseteq P$ with a valuated decomposition $E = B \oplus C$, where B is valuated-projective and C is infinitely-gapped.*

We note in passing that Corollary 3.7 depends heavily on the assumption that $\mathbf{R}$ is a *complete* discrete valuation ring. To see why this is true, suppose for example that R is the integers localized as some prime. We think of R as a subring of the p-adic integers, $\mathbf{R}$. Let P be any submodule of $\mathbf{R}$ of (R-)rank 2, where for each $y \in P$, we let $|y| = h_{\mathbf{R}}(y)$. Since $\mathbf{R}$ is torsion-free, it is immediate that P

is non-gapped. If $E \subseteq P$ satisfied Corollary 3.7, we would have to have $E = B$ is valuated-projective and rank 2. On the other hand, this contradicts the fact that for any $n < \omega$, $E(n)/E(n+1)$ embeds in $\mathbf{R}(n)/\mathbf{R}(n+1) \cong \mathbf{R}/p\mathbf{R} \cong R/pR$.

This brings us to an important step in our discussions. At a later point we will show that it does not generalize to modules of uncountable rank.

Theorem 3.8. *Suppose M is a module of countable rank. Then M is balanced-projective if and only if it is a finitely-gapped May module.*

Proof. If M is balanced-projective, then it is clearly a finitely-gapped May module.

Conversely, suppose M is a finitely-gapped May module of countable rank. Let P be some NFT-submodule of M. Now, let $B \oplus C \subseteq P$ be as in Corollary 3.7. Since M is finitely-gapped, we can conclude that $C = 0$. By Proposition 2.3, M is an NT-realization of B, so that it is balanced-projective. $\square$

If M is a module and $B \subseteq M$ is a submodule, then the *purification* of B is the submodule N such that N/B is the torsion submodule of M/B. To avoid repetition, we include the following argument as a separate statement.

Lemma 3.9. *Suppose M is a module, P is a free, full-rank submodule of M with a valuated decomposition $P = B \oplus C$, where C is infinitely-gapped. If N is the purification of B and λ is an ordinal, then the natural map $N/p^\lambda N \to M/p^\lambda M$ restricts to an isomorphism on their torsion submodules.*

Proof. Since M/N is torsion-free, N is an isotype submodule of M. This easily implies that the map is injective.

Suppose $x + p^\lambda M$ is a torsion element of $M/p^\lambda M$. It follows that we can find $k < \omega$ such that $p^k x \in p^\lambda M$. Since P is full-rank, after possibly increasing k, we may assume that $p^k x = b + c \in P(\lambda) = B(\lambda) \oplus C(\lambda)$.

Since C is infinitely-gapped, after again possibly increasing k, we may assume $|c| \geq \lambda + k$. Let $y \in p^\lambda M$ satisfy $p^k y = c$.

Clearly, $x+p^\lambda M=x-y+p^\lambda M$. And since $p^k(x-y)=(b+c)-c=b\in B$, we can conclude that $x-y\in N$. This implies that our map is also surjective on the torsion and completes the argument. $\square$

Interestingly, the next result is actually valid for May modules of arbitrary rank.

Theorem 3.10. *If M is a limiting and infinitely-gapped May module with torsion T, then T is totally projective.*

Proof. Let C be an α-limiting NFT-submodule of M. Note that $p^\alpha M\cap C=0$, so that $p^\alpha M$ will be a torsion May module, i.e., $p^\alpha M=p^\alpha T$ is totally projective. Therefore, T will be totally projective if and only if $T/p^\alpha T$ is totally projective. And since $T/p^\alpha T$ is the torsion submodule of the May module $M/p^\alpha M$, we may assume $p^\alpha M=0$, i.e., M is p^α-bounded.

Suppose $\lambda<\alpha$. Since M is α-limiting it follows that $C(\lambda)$ is full-rank in M. In particular, this implies that $M/p^\lambda M$ is torsion. Therefore, by Lemma 3.9 (with $B=0$), it follows that $T/p^\lambda T\cong M/p^\lambda M$ is a torsion May module, i.e., it is totally projective. Therefore, T is a C_α-module, and by Lemma 2.6, T is totally projective. $\square$

Theorem 3.8 applies when $C=0$ in Corollary 3.7. On the other hand, the following partial converse to Corollary 1.1(2) applies when $B=0$.

Theorem 3.11. *If M is a reduced infinitely-gapped module of countable rank with torsion T, then M is a May module if and only if T is totally projective.*

Proof. If T is totally projective, then by Corollary 1.1(2), M is a May module.

Conversely, assume M is a May module. As in Theorem 3.5, there is a totally projective module H and an isomorphism $M\oplus H\cong\bigoplus_{\alpha\in I}M_\alpha$, where each M_α is α-limiting.

Since M is infinitely-gapped, so is each M_α. By Theorem 3.10, the torsion submodule of each M_α is totally projective. Therefore, $T\oplus H$ is totally projective, implying that T is totally projective, as well. $\square$

By considering NT-realizations of the terms in Corollary 3.7, we can conclude:

Corollary 3.12. *If M is a May module of countable rank, then $M \cong_H N \oplus L$, where N is a finitely-gapped May module (i.e., a balanced-projective) and L is an infinitely-gapped May module (i.e., its torsion submodule is totally projective).*

The following shows that even for Warfield modules, in results like Theorem 3.5 or Corollary 3.12, though a given May module of countable rank will be H-isomorphic to such a direct sum, it may not actually be isomorphic to one.

Example 3.13. *There is a Warfield module of rank 2 that is not the direct sum of a balanced-projective module and an infinitely-gapped May module.*

Proof. Let $\langle x\rangle$ be a valuated-projective module such that $|x| = \omega$ and let $\langle y\rangle$ be an ω-limiting infinitely-gapped valuated module. Let P be the valuated direct sum $\langle x\rangle \oplus \langle y\rangle$. It is a standard construction in the theory of Warfield modules that there is an NT-realization M of P that is not isomorphic to the direct sum of two modules of rank 1. $\square$

The problem in Example 3.13 is that there are not enough non-zero relative Ulm invariants of P in M for such a decomposition to occur.

4. Characterizations of May modules of countable rank. In this section we present two such characterizations. We begin with:

Theorem 4.1. *Let M be a reduced module of countable rank. Then M is a May module if and only if it has a balanced-projective submodule $N \subseteq M$ such that M/N is countably generated.*

Proof. Suppose first that we are given the balanced-projective submodule $N \subseteq M$. Using the height valuation on N, let $B \subseteq N$ be a valuated-projective NFT-submodule. We can extend this to a full-rank free submodule $P := B \oplus C \subseteq M$. It should be noted that if N is not isotype in M, then on B the height valuation from N may be different than the height valuation from M. In addition, using the height valuation from M, we are *not* assuming this sum is valuated.

Applying Lemma 2.1, we may assume P is nice in M and B will remain a valuated-projective NFT-submodule of N.

Since M is reduced and $P \subseteq M$ is nice, it follows that M/P is reduced. In addition, the totally projective module N/B naturally embeds in M/P. Since the cokernel of this embedding is isomorphic to $M/(P+N)$, which is the epimorphic image of M/N, it is necessarily countably generated. So by Theorem 2.2, M/P is totally projective.

It follows that P is an NFT-submodule of M, so that by Corollary 1.1(1), M will be a May module.

Conversely, suppose M is a May module. As usual, find an NFT-submodule $P \subseteq M$ which is a valuated sum $P = B \oplus C$, where B is valuated-projective and C is infinitely gapped. Let N be the purification of B.

Since M/N is torsion-free, it follows that N is an isotype submodule of M and B is nice and valuated-projective in N. Since C maps to an essential submodule of M/N, we can conclude that M/N is countably generated.

Clearly, $P/B \cong C$ is an infinitely-gapped nice, free submodule of M/B. Since $(M/B)/(P/B) \cong M/P$ is totally projective, by Corollary 1.1(1), M/B is a May module. By Theorem 3.10, the torsion of M/B will necessarily be totally projective. Since this torsion is, in fact, N/B, it follows that B is an NFT-submodule of N.

Therefore, N will be balanced-projective, completing the proof. $\square$

For future reference, we note one point made in the last result.

Corollary 4.2. *Suppose M is a reduced module of countable rank, $P \subseteq M$ is a free, nice, full-rank submodule with a valuated decomposition $P = B \oplus C$ where B is valuated-projective and C is infinitely gapped. If N is the purification of B, then M is a May module if and only if N is balanced-projective.*

It is perhaps surprising that in Theorem 4.1 the submodule N is not assumed to have any special properties, such as being nice or isotype in M.

Corollary 4.3. *Suppose L is a reduced module of countable rank with a May submodule M. If L/M is countably generated, then L is also a May module.*

Proof. Suppose $N \subseteq M$ is as in Theorem 4.1. Since L/M and M/N are countably generated, so is L/N. So again using Theorem 4.1, L is a May module. $\square$

Observe the parallel between Corollary 4.3 and Wallace's Theorem (Theorem 2.2). In fact, Wallace's Theorem is simply the torsion case of Corollary 4.3.

We now present a second characterization of May modules of countable rank that does not involve NFT-submodules or balanced-projective modules, but only certain torsion modules.

We first point out a simple idea. If P is a countably generated full-rank submodule of M and λ is a limit ordinal of uncountable cofinality, then there is an ordinal $\beta < \lambda$ such that $P(\beta) = P(\lambda)$. From this it is easy to see that $P(\lambda)$ is full-rank in $p^\beta M$, and in particular, that $p^\beta M/p^\lambda M$ will be torsion.

This brings us to our second characterization of countable rank May modules.

Theorem 4.4. *Suppose M is a reduced module of countable rank with torsion T. Then M is a May module if and only if (a) T is an S-module; and (b) for every limit ordinal λ of uncountable cofinality, there is an ordinal $\beta < \lambda$ such that $p^\beta M/p^\lambda M$ is totally projective.*

Proof. Suppose $P = B \oplus C$ and N are as in Corollary 4.2. so M is a May module if and only if N is a balanced-projective.

We claim that M satisfies (a) and (b) if and only if N does. Note that N and M have the same torsion, namely T, so the equivalence of (a) for the two modules is trivial.

Considering (b), if λ is a limit ordinal of uncountable cofinality, then we can find a $\beta < \lambda$ such that $p^\beta M/p^\lambda M$ and $p^\beta N/p^\lambda N$ are torsion. By

Lemma 3.9, $N/p^\lambda N \to M/p^\lambda M$ is an isomorphism on their torsions, so that $p^\beta N/p^\lambda N \cong p^\beta M/p^\lambda M$. Therefore, condition (b) is also equivalent for the two modules.

Therefore, there is no loss of generality in assuming that $C=0$ and $M=N$ is finitely-gapped.

Suppose first that M is balanced-projective. It follows by definition that T is an S-module.

Turning now to condition (b), suppose λ is a limit ordinal of uncountable cofinality. Find $\beta < \lambda$ such that $p^\beta M/p^\lambda M$ is torsion. Since this quotient will also be balanced-projective, it must be totally projective.

Conversely, suppose (a) and (b) hold. We prove that M is balanced-projective by induction on the length of T, which we denote by μ. So assume the result holds for every finitely-gapped reduced module of countable rank satisfying (a) and (b) whose torsion submodule has strictly smaller length.

Claim 1: If $\nu < \mu$ is an ordinal, then $M/p^\nu M$ is balanced-projective.

Observe that $p^\nu M/p^\nu T = X \oplus D$, where X is free and D is divisible and torsion-free. Therefore, there is a decomposition $M/p^\nu T = M' \oplus D$. The torsion submodule of M' is $T/p^\nu T$, and so it is an S-module of length $\nu < \mu$.

In addition, $p^\nu M' \cong X$ and $M'/p^\nu M' \cong M/p^\nu M$. Therefore, if $\lambda \leq \nu$ has uncountable cofinality, then there is a $\beta < \lambda$ such that

$$\frac{p^\beta M'}{p^\lambda M'} \cong \frac{p^\beta (M'/p^\nu M')}{p^\lambda (M'/p^\nu M')} \cong \frac{p^\beta (M/p^\nu M)}{p^\lambda (M/p^\nu M)} \cong \frac{p^\beta M}{p^\lambda M}$$

is totally projective.

It follows from induction that M' is balanced-projective. Therefore, $M/p^\nu M \cong M'/p^\nu M'$ is also balanced-projective.

Suppose that $0 < \gamma \leq \omega$ and $\nu + \gamma = \mu$. It follows that $p^\nu T$ is a separable S-module, i.e., it is a direct sum of cyclics. So by Corollary 1.1(2), $p^\nu M$ will be a finitely-gapped May module, i.e., a balanced-projective. So, by Claim 1, M will be one, as well.

Next, if μ is a limit ordinal of uncountable cofinality, by hypothesis, we can find $\beta<\mu$ so that $p^\beta M/p^\mu M$ is totally projective. Since $T\cap p^\mu M=0$, $p^\mu M$ will be reduced, countable-rank and torsion-free, i.e., it is a free module. This implies that $p^\beta M$ is a balanced-projective module. And, again by Claim 1, M will be one, as well.

Therefore, it suffices to consider the case where μ is a limit ordinal of countable cofinality that is not of the form $\mu=\nu+\omega$ (i.e., it is a limit of limit ordinals).

Again, we may assume B is a nice, full-rank, valuated-projective submodule of M. Replacing B by $B+p^\mu M$, there is no loss of generality in assuming $p^\mu M\subseteq B$, so that M/B is p^μ-bounded; we need to show that it is totally projective.

Claim 2: M/B is a C_μ-module.

Let $\nu<\mu$ be a limit ordinal. We must show that

$$\begin{aligned}(M/B)/p^\nu(M/B) &= (M/B)/([B+p^\nu M]/B)\\ &\cong M/[B+p^\nu M]\cong (M/p^\nu M)/([B+p^\nu M]/p^\nu M)\end{aligned}$$

is totally projective.

By Claim 1, $M/p^\nu M$ is balanced-projective. There is clearly a valuated decomposition $B=B_1\oplus B_2$ where $B_1(\nu)=0$ and $B_2\subseteq p^\nu M$. Therefore, $\overline{B}:=[B+p^\nu M]/p^\nu M\cong B_1$ and is nice, free and full-rank in $M/p^\nu M$. By Proposition 2.3, $\overline{B}$ is an NFT-submodule of $M/p^\nu M$. So our quotient is totally projective, completing the verification of Claim 2.

Claim 3: The homomorphism $T[p]\to M/B$ given by $x\to x+B$ preserves values.

If this failed, we could find $x\in T[p]$, $y\in B$ such that $|x|=|y|<|x+y|$. It follows that $|py|=|p(x+y)|\geq|x+y|+1>|y|+1=|py|$, which is a contradiction.

We now complete the proof. By Claim 3, T can be viewed as an isotype submodule of M/B. Since T is an S-module, by Lemma 2.7, $T[p]$ will be μ-summable. In addition, it is clear that $(M/B)/T$ is countably generated. By Lemma 2.8, this implies that $(M/B)[p]$ is μ-summable. Therefore, by Claim 2 and Proposition 2.5, M/B is totally projective, as required. $\square$

As an application, it is possible to use Theorem 4.4 to give a different proof of Proposition 2.4.

5. May modules of uncountable rank. May modules of uncountable ranks are clearly much more complicated than those of countable rank. Theorem 3.10 and the following do give us some information regarding their torsion submodules. The proof uses the famous result, due to Hill, that any isotype submodule of a totally projective module of countable length is also totally projective.

Theorem 5.1. *Suppose M is a May module of arbitrary rank with torsion T. If M has countable length, then T is totally projective.*

Proof. We induct on the length of M, which we denote by λ.

If $\lambda = \alpha + 1$, it follows that $p^\alpha M = p^\alpha T$ is torsion. And by induction, $M/p^\alpha M$ is also a May module of length $\alpha < \lambda$. Therefore, its torsion $T/p^\alpha T$ is totally projective, so the same holds for T.

So assume λ is a limit ordinal. Whenever $\alpha < \lambda$, $M/p^\alpha M$ is a May module, so its torsion $\check{T}_\alpha$ is totally projective. Since $T/p^\alpha T$ is isomorphic to an isotype submodule of $\check{T}_\alpha$, by Hill's Theorem we can conclude that $T/p^\alpha T$ is also totally projective.

Therefore, T is a C_λ-module. So by Lemma 2.6, T is totally projective, as required. □

Corollary 5.2. *If M is a May module of arbitrary rank with torsion T, then T is a C_{ω_1}-module.*

Proof. For every countable $\lambda < \omega_1$, $M/p^\lambda M$ will be a May module, so that by Theorem 5.1, its torsion, $\check{T}_\lambda$, will be totally projective. Since $T/p^\lambda T$ can be viewed as an isotype submodule of $\check{T}_\lambda$, $T/p^\lambda T$ will also be totally projective. □

In the main result of this section we will use a little bit of set theory – namely the prediction principle $\diamondsuit(\omega_1)$. There are many equivalent ways to express $\diamondsuit(\omega_1)$; the following will be convenient for our purposes:

$\diamondsuit(\omega_1)$ - *If $\mathcal{L} \subseteq \omega_1$ is the collection of countable limit ordinals, then there is a set of functions $\{g_\xi : \xi \to \omega : \xi \in \mathcal{L}\}$ with the property that for any* *possible function $g : \omega_1 \to \omega$, there is an $\xi \in \mathcal{L}$ such that $g(\beta) = g_\xi(\beta)$ for all $\beta < \xi$ (in other words, g restricts to g_ξ on ξ).*

This statement is true, for example, in any model of the constructible universe, but is independent of the standard axioms of set theory.

Theorem 3.8 clearly had a central position in our investigation of May modules of countable rank. The following shows that it cannot be proven to hold for May modules of uncountable rank.

Theorem 5.3. *Assuming $\diamondsuit(\omega_1)$, there is a finitely-gapped May module M with $p^\omega M = 0$ and rank ω_1 that is not balanced-projective.*

Proof. Fundamental to our construction is the following idea, in which we introduce some convenient, albeit non-standard, terminology.

Definition 5.4. *The valuated module B will be said to be* **2-closed** *if it is algebraically free, finitely-gapped and every non-zero $y \in B$ satisfies the following:*

(2a) $|py| \le |y| + 2$;

(2b) $h_B(y) = 0$ implies $|y| = 0$.

Clearly, being 2-closed is inductive in the sense that if B is an algebraically free valuated module that is the ascending union of a collection of 2-closed submodules, then B itself will be 2-closed. We note the following consequence:

(2c) If B is 2-closed and $y \in B$ is non-zero, then $|y| \le 2h_B(y) < \omega$.

Suppose $h_B(y) = m$; it follows that $y = p^m y'$, where $h_B(y') = 0$. So by (2b) we have $|y'| = 0$. Therefore, by (2a), $|y| = |p^m y'| \le |y'| + 2m = 2h_B(y)$.

Here is the key step in proving Theorem 5.3:

Lemma 5.5. *Assuming $\diamondsuit(\omega_1)$, there is a 2-closed valuated module B_{ω_1} of rank ω_1 such that no full-rank submodule of B_{ω_1} is non-gapped, i.e., whenever S is a full-rank submodule of B_{ω_1}, there is a non-zero $y \in S$ such that $|py| > |y| + 1$.*

Before verifying this lemma, we show how it implies Theorem 5.3. Let M be an NT-realization of B_{ω_1}; so M is finitely-gapped. Since $B_{\omega_1}(\omega)=0$, we can also view $M/p^\omega M$ as an NT-realization of B_{ω_1}, so there is no loss of generality in assuming that $p^\omega M=0$. We need to show that M is a May module.

Let $\{b_i\}_{i<\omega_1}$ be an algebraic basis for B_{ω_1}. If f is any function from ω_1 to ω, then let $P_f=\langle p^{f(i)}b_i : i\in\omega_1\rangle\subseteq B_{\omega_1}$. We claim that any such P_f must be nice in B_{ω_1}, and hence nice in M. To that end, suppose $x\in B_{\omega_1}$ is not an element of P_f. Since B_{ω_1}/P_f is algebraically the direct sum of bounded cyclic modules, it follows from (2c) that

$$\sup\{|x+y| : y\in P_f\}\le \sup\{2h_{B_{\omega_1}}(x+y) : y\in P_f\}=2h_{B_{\omega_1}/P_f}(x+P_f),$$

which must be finite. Therefore, the first collection of values must have a maximal element, so $x+P_f$ must have a proper element with respect to $|\ |$.

To show M is a May module, let F be any full-rank submodule of M. Clearly, there is a function f_0 from ω_1 to ω such that $P_{f_0}\subseteq F$. We know that P_{f_0} is nice in M, so it will suffice to show that M/P_{f_0} is totally projective.

There is clearly an ordinal λ and a collection of functions $\{f_\nu\}_{\nu\le\lambda}$ such that $\{P_{f_\nu}\}_{\nu\le\lambda}$ is a smoothly increasing chain starting with $P_{f_0}\subseteq F$, ending with $P_{f_\lambda}=B_{\omega_1}$, and satisfying $P_{f_{\nu+1}}/P_{f_\nu}\cong \mathbf{R}/p\mathbf{R}$ whenever $\nu<\lambda$. It follows that for each $\nu\le\lambda$, P_{f_ν} is nice in B_{ω_1} (and so it is nice in M), so that P_{f_ν}/P_{f_0} is nice in B_{ω_1}/P_{f_0} (and so it is nice in M/P_{f_0}). Therefore, $\{P_{f_\nu}/P_{f_0}\}_{\nu\le\lambda}$ gives a nice composition series for B_{ω_1}/P_{f_0}. Extending this using a nice composition series for M/B_{ω_1} gives a nice composition series for M/P_{f_0}, showing that it is totally projective, as required. Therefore, M is a May module.

We now show that M is not a balanced-projective module. If, in fact, M were balanced-projective, then it would be an NT-realization of a full-rank non-gapped submodule Q. However, letting $S=Q\cap B_{\omega_1}$ would contradict the statement of Lemma 5.5.

We now turn to proving Lemma 5.5. We will build up B_{ω_1} inductively using $\diamondsuit(\omega_1)$. As we do so, the following is the key step:

Lemma 5.6. *Suppose B is a 2-closed valuated module of countable rank and X is a full-rank submodule of B that is valuated-projective; so* *there is an isometry*

$$X \cong \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \langle x_3 \rangle \oplus \cdots$$

which we will assume is an equality. Let $\check{B}$ *be an algebraically free module of the form* $\langle x_0 \rangle \oplus B$*, which contains the full-rank free submodule* $\check{X} := \langle x_0 \rangle \oplus X$*. There is a valuation on* $\check{B}$ *extending the valuation on* B *such that*

(A) $\check{B}$ *is also 2-closed;*

(B) For every $j < \omega$ *there is an element* $x \in X$ *such that*

$$|p(p^j x_0 + x)| > |p^j x_0 + x| + 1,$$

that is, $p^j x_0 + x$ *has a gap in its value sequence.*

As will be seen, the algebraic decompositions $\check{X} = \langle x_0 \rangle \oplus X$ and $\check{B} = \langle x_0 \rangle \oplus B$ will not be valuated; in fact, this would make it impossible to satisfy condition (B).

Before we get into the technical details involved in proving Lemma 5.6, we show how it leads to a proof of Lemma 5.5, and so to a proof of Theorem 5.3.

Fix an algebraic decomposition $B_{\omega_1} := \oplus_{i<\omega_1} \langle b_i \rangle$. If $\lambda \leq \omega_1$, let $B_\lambda = \oplus_{i<\lambda} \langle b_i \rangle \subseteq B_{\omega_1}$, so $\{B_\lambda\}_{\lambda \leq \omega_1}$ is a smoothly ascending chain. If f is any function from ω_1 to ω, then we have already defined $P_f := \langle p^{f(i)} b_i : i \in \omega_1 \rangle \subseteq B_{\omega_1}$. Similarly, if f_ξ is a function from $\xi < \omega_1$ to ω, then we let $P_{f_\xi} := \langle p^{f_\xi(i)} b_i : i \in \xi \rangle \subseteq B_\xi$.

We start with a collection of functions $\{g_\xi\}_{\xi \in \mathcal{L}}$ as in $\Diamond(\omega_1)$. Our strategy will be to inductively define a valuation $|\ |$ on B_λ, where in each case B_λ will be 2-closed. Certainly, if $\lambda = 0$, then $B_0 = 0$ has only one valuation. Next, suppose $\lambda > 0$ and this has been done for all $\xi < \lambda$. First, if λ is a limit ordinal, then the valuation on B_λ will be uniquely determined by the equation $B_\lambda = \cup_{\xi<\lambda} B_\xi$; since 2-closure is inductive, this B_λ will also be 2-closed.

So now suppose $\lambda = \xi + 1$ is a successor ordinal.

Case 1: ξ is 0 or a successor ordinal.

In this situation, we just give the height valuation to $\langle b_\xi\rangle$ and let B_λ be the valuated sum $\langle b_\xi\rangle \oplus B_\xi$. It is routine to show this B_ξ will also be 2-closed.

Case 2: ξ is a limit ordinal.

Since $P_{g_\xi} \subseteq B_\xi$ is a finitely-gapped valuated module of countable rank, it follows from Corollary 3.6 that there is a full-rank submodule $X \subseteq P_{g_\xi}$ that is valuated-projective. Therefore, X is isometric to a valuated direct sum $\bigoplus_{k=1}^{\omega}\langle x_k\rangle$. If we use $x_0 = b_\xi$, it follows from Lemma 5.6 (with $B = B_\xi$) that we can extend $|\ |$ to a valuation on $\check{B} := B_\lambda = \langle b_\xi\rangle \oplus B_\xi = \langle x_0\rangle \oplus B_\xi$ making it 2-closed and with the property that for every $j < \omega$ there is a $x \in X \subseteq P_{g_\xi}$ such that $p^j b_\xi + x = p^j x_0 + x$ has a gap in its value sequence.

Continuing this construction for all $\lambda < \omega_1$, we arrive at our $B_{\omega_1} = \cup_{\lambda<\omega_1} B_\lambda$, which will once again be 2-closed.

We now verify that Lemma 5.5 holds, which will imply Theorem 5.3. Let S be a full-rank submodule of B_{ω_1}. It is easy to construct a function $g : \omega_1 \to \omega$ such that $P_g \subseteq S$. By $\diamondsuit(\omega_1)$, there is a limit ordinal ξ such that $g(\beta) = g_\xi(\beta)$ for all $\beta < \xi$. Therefore, we are in Case 2 of our construction and we let $X \subseteq P_{g_\xi}$ be as in that argument. If we let $j = g(\xi) < \omega$, then $p^j x_0 = p^j b_\xi \in S$, $x \in X \subseteq P_{g_\xi} \subseteq P_g \subseteq S$ and $p^j x_0 + x \in S$ has a gap in its value sequence. This verifies Lemma 5.5 and completes the proof of Theorem 5.3.

So we now need to complete the construction in Lemma 5.6. This is where things get a bit technical. First, after possibly replacing each x_k ($k = 1, 2, \dots$) by $p^{m_k} x_k$ for some m_k, there is clearly no loss of generality in assuming that there is a sequence of positive integers $0 < j_1 < j_2 < \dots$, such that $|x_k| = 2j_k$; we also set $j_0 = 0$.

We will define a function v from $\check{B}$ to the (finite) ordinals that agrees with $|\ |$ on B and then show that it is a valuation with the required properties.

We need to identify some important elements $y_j \in \check{X} \subseteq \check{B}$. For $j < \omega$, let

$$y_j = \sum_{\{k : j_k \le j\}} p^{j-j_k} x_k = p^j x_0 + x, \quad \text{where} \quad x \in X \subseteq B. \tag{$\dagger$}$$

Our approach to defining v is to set $v(y_j) = 2j$ and to demand that each y_j be a proper element of $y_j + B = p^j x_0 + B$. Clearly, if $y \in \check{B} \setminus B$, then $y = \alpha y_j + w$, where $j < \omega$, $\alpha \in \mathbf{R}$ is a unit and $w \in B$; in fact, j, α and w are uniquely determined by y. So we are defining v by setting

$$v(y) = \min\{v(y_j), |w|\} = \min\{2j, |w|\}.$$

Having defined our extension, we need to verify that it is, in fact, a valuation on $\check{B}$ and that (A) and (B) hold.

We start with a straightforward observation. If $j < \omega$, then let $k < \omega$ be the smallest value such that $j < j_k$. Consider $py_j = y_{j+1} - (y_{j+1} - py_j)$. The second term is in B and equals

$$y_{j+1} - py_j = \begin{cases} 0, & \text{if } j+1 \neq j_k, \\ x_k, & \text{if } j+1 = j_k. \end{cases}$$

In either case, $|y_{j+1} - py_j| \geq 2(j+1) = 2j+2$ and $v(py_j) = 2j+2 = v(y_{j+1})$.

More generally, if $m < \omega$ we will have

$$\begin{aligned} y_{j+m} - p^m y_j =& (y_{j+m} - py_{j+m-1}) + p(y_{j+m-1} - py_{j+m-2}) \\ &+ \cdots + p^{m-1}(y_{j+1} - py_j), \end{aligned}$$

which again implies that $|y_{j+m} - p^m y_j| \geq 2j + 2$.

To show that v is a valuation, we first show that $\check{B}[n] := \{y \in \check{B} : v(y) \geq n\}$ is a submodule for all $n < \omega$. Suppose j is the smallest integer such that $v(y_j) = 2j \geq n$. We verify that $\check{B}[n]$ agrees with the submodule $\langle y_j \rangle + B(n)$, where $B(n) = \{z \in B : |z| \geq n\}$.

Clearly, if $y \in B$, then $v(y) \geq n$ if and only if $y \in B(n)$. So suppose $y \in \check{B} \setminus B$. Then $v(y) \geq n$ if and only if for some $m < \omega$ and unit $\alpha \in \mathbf{R}$ we have $y \in \alpha y_{j+m} + B(n)$. Since $|y_{j+m} - p^m y_j| \geq 2j + 2 \geq n$, $y_{j+m} - p^m y_j \in B(n)$. Therefore, $v(y) \geq n$ if and only if for some $m < \omega$ and unit $\alpha \in \mathbf{R}$ we have $y \in \alpha p^m y_j + B(n)$. And clearly, this is equivalent to $y \in \langle y_j \rangle + B(n)$.

To complete our verification that v is a valuation we must show that $p\check{B}[n] \subseteq \check{B}[n+1]$. Continuing the above notation, we know that $pB(n) \subseteq B(n+1) \subseteq \check{B}[n+1]$, and $py_j = y_{j+1} - (y_{j+1} - py_j) \in \langle y_{j+1} \rangle + B(2j+2) \subseteq \check{B}[n+1]$, so that $p\check{B}[n] \subseteq \check{B}[n+1]$, as required.

Having shown that v is, in fact, a valuation, we will write $|y| = v(y)$ for all $y \in \check{B}$.

We now turn to showing (A), i.e., that $\check{B}$ is 2-closed. First, we show that $\check{B}$ is finitely-gapped. If $0 < k < \omega$, consider the submodule

$$Z_k = \langle p^{j_k} x_0 \rangle \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \cdots \oplus \langle x_{k-1} \rangle \subseteq \check{X}.$$

It is easily seen that for every $y \in \check{B}$ there is an $m < \omega$ and a $0 < k < \omega$ such that $p^m y \in Z_k$. Therefore, to show that $\check{B}$ is finitely-gapped, it suffices to show that Z_k is non-gapped.

Let y be in Z_k. Since X is non-gapped, we may assume $y = \alpha y_j + w$, where $j \geq j_k$, α is a unit and $w \in X$. In our decomposition $\check{X} = \bigoplus_{i=0}^{\omega} \langle x_i \rangle$, the x_k component of y is zero (since $y \in Z_k$). And if we look at the expression for y_j in (†), the x_k component of αy_j is $\alpha p^{j-j_k} x_k$. This implies that the x_k component of w will be $-\alpha p^{j-j_k} x_k$. Next,

$$|-\alpha p^{j-j_k} x_k| = (j - j_k) + 2j_k = j + j_k \leq 2j,$$

which implies that $|w| \leq 2j$. This means $|y| = \min\{2j, |w|\} = |w|$. And since $w \in X$, we have $|pw| = |w| + 1 \leq 2j + 1 < 2j + 2 = |py_j|$. Therefore,

$$|py| = |p\alpha y_j + pw| = |pw| = |w| + 1 = |y| + 1,$$

so that $\check{B}$ is finitely-gapped, as stated.

We now consider (2a) in Definition 5.4; so suppose $0 \neq y \in \check{B}$. If $y \in B$, we already know that $|py| \leq |y| + 2$. So we may suppose $y = \alpha y_j + w$, where α is a unit and $w \in B$. Since $|p\alpha y_j| = 2j + 2 = |\alpha y_{j+1}|$, both $\alpha p y_j$ and αy_{j+1} are proper elements of $py + B$. Therefore,

$$\begin{aligned} |py| = |\alpha p y_j + pw| &= \min\{2j + 2, |pw|\} \leq \min\{|\alpha y_j| + 2, |w| + 2\} \\ &= \min\{|\alpha y_j|, |w|\} + 2 = |y| + 2. \end{aligned}$$

Next, considering condition (2b), suppose we have a non-zero $y \in \check{B}$ such that $h_{\check{B}}(y) = 0$; we need to show $|y| = 0$. If $y \in B$, then since B is 2-closed, this follows directly. And if $y \notin B$, then express $y = p^j \alpha x_0 + z \in \langle x_0 \rangle \oplus B$ with $j \in \omega$, $\alpha \in \mathbf{R}$ is a unit and $z \in B$. Since $h_{\check{B}}(y) = 0$, either $j = 0$ or $h_B(z) = 0$. First, if $j = 0$, then since $y_0 = x_0$, we would have

$$|y| = |\alpha x_0 + z| = |\alpha y_0 + z| = \min\{0, |z|\} = 0.$$

So suppose $j > 0$ and $h_B(z) = 0$. Since $z \in B$, we can conclude $|z| = 0$. In addition, $|p^j \alpha x_0| \geq j > 0 = |z|$; so $|y| = |p^j \alpha x_0 + z| = 0$. Therefore, (2b) follows.

Finally, turning to (B), given $j < \omega$, consider $y_j = p^j x_0 + x$ where $x \in X$, as in (†). Since we know $|y_j| = 2j$ and $|py_j| = 2j + 2$, (B) is immediate. This completes the proof of Lemma 5.6, and hence of Theorem 5.3. □

In fact, M is a p^ω-bounded balanced-projective module if and only if it is a direct sum of cyclics; so what we have constructed is a finitely-gapped May module of length ω that fails to be a direct sum of cyclics.

To construct examples of May modules of uncountable rank, one easy technique is to look at sufficiently large Warfield modules. Similarly, any uncountable direct sum of countable rank May modules will also be a May module. It is easy to see that a finitely gapped module in either of these classes would have to be balanced-projective. In particular, the module M constructed in Theorem 5.3 is neither a Warfield module, nor a direct sum of countable rank May modules.

The module M is also a counter-example to the natural generalization of Theorem 4.1. To see this, suppose $N \subseteq M$ is balanced-projective and M/N is countably generated. Since $p^\omega N \subseteq p^\omega M = 0$, N must be a direct sum of cyclics. This, in turn, would imply that $M = N_1 \oplus C$, where N_1 is a direct sum of cyclics and C is countably generated.

Since C is countably generated and reduced, it follows from Corollary 1.1(2) that it is a May module. Since it is clearly finitely-gapped, C, and hence $N_1 \oplus C = M$, is balanced-projective, contrary to its construction.

6. Questions. We conclude with a few open questions. Let M be a May module of arbitrary rank with torsion T.

Question 1: Can we conclude that T is an S-module? (cf., Theorem 4.4)

Question 2: If N is a summand of M, can we conclude that N is a May module? (cf., Proposition 2.4)

Question 3: Is $M \cong_H N \oplus L$, where N is a finitely-gapped May module and L is an infinitely-gapped May module? (cf., Corollary 3.12)

Now suppose M and N are May modules of countable rank. The following are variations on famous questions posed by Kaplansky.

Question 4: If M is H-isomorphic to a summand of N and N is H-isomorphic to a summand of M, can we conclude that $M \cong_H N$?

Question 5: If $M \oplus M \cong_H N \oplus N$, can we conclude that $M \cong_H N$?

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