ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 49, Number 4, 2019

ON k-RESTRICTED OVERPARTITIONS

UHA ISNAINI

ABSTRACT. We introduce k-restricted overpartitions, which are generalizations of overpartitions. In such partitions, among those parts of the same magnitude, one of the first k occurrences may be overlined. We first give the generating function and establish the 5-dissections of k-restricted overpartitions. Then we provide a combinatorial interpretation for certain Ramanujan type congruences modulo 5. Finally, we pose some problems for future work.

1. Introduction. The object in this study is the idea of a k-restricted overpartition. A k-restricted overpartition of n is a partition of n in which one of the first k occurrences of a part may be overlined. For example, all 2-restricted overpartitions of 4 are

$$4,\quad \bar{4},\quad 3+1,\quad \bar{3}+1,\quad 3+\bar{1},\quad \bar{3}+\bar{1},\quad 2+2,\quad \bar{2}+2,\quad 2+\bar{2},$$
$$2+1+1,\quad \bar{2}+1+1,\quad 2+\bar{1}+1,\quad \bar{2}+\bar{1}+1,\quad 2+1+\bar{1},\quad \bar{2}+1+\bar{1},$$
$$1+1+1+1,\quad \bar{1}+1+1+1,\quad 1+\bar{1}+1+1.$$

The total number of k-restricted overpartitions of n is denoted by $\bar{p}_k(n)$. Hence, $\bar{p}_2(4) = 18$. We note that when $k = 1$, we have ordinary overpartitions [**4**] denoted by $\bar{p}(n)$. Thus, $\bar{p}_1(n) = \bar{p}(n)$. While these objects have not appeared in the literature before, they are related to ordered pairs of partitions [**5**].

We adopt the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \tag{1}$$

$$(a_1, a_2, \ldots, a_n; q)_\infty := (a_1;q)_\infty (a_2;q)_\infty \cdots (a_n;q)_\infty, \tag{2}$$

2010 AMS *Mathematics subject classification.* 11P83, 05A17.
Keywords and phrases. Partitions, overpartitions, congruences.
The author was supported by the National Institute of Education (Singapore) Ph.D. scholarship.
Received by the editors on September 9, 2018, and in revised form on October 31, 2018.
DOI:10.1216/RMJ-2019-49-4-1207

$$\begin{pmatrix} a_1, a_2, \dots, a_n \\ b_1, b_2, \dots, b_m \end{pmatrix}; q \Big)_\infty := \frac{(a_1, a_2, \dots, a_n; q)_\infty}{(b_1, b_2, \dots, b_m; q)_\infty}. \tag{3}$$

Suppose $p_{-k}(n)$ enumerates the coefficient of q^n in $(q;q)_\infty^{-k}$ for fixed k. It is well-known that

$$p_{-2}(5n+a) \equiv 0 \pmod 5, \quad a = 2, 3, 4, \tag{4}$$

and a combinatorial interpretation for (4) is given in [**5**]. This motivates us to find all Ramanujan type congruences modulo 5 for k-restricted overpartitions and their combinatorial interpretations.

Our article is structured as follows. In the next section, we recall some preliminary results. In Section 3, we show that for every positive integer k the generating function for k-restricted overpartitions is given by

$$\sum_{n=0}^{\infty} \bar{p}_k(n) q^n = \frac{(q^{k+1}; q^{k+1})_\infty}{(q;q)_\infty^2}. \tag{5}$$

We also prove 5-dissections for every positive integer k. The desired 5-dissections are stated in the following five theorems.

Theorem 1.1. *For every positive integer k,*

$$\sum_{n=0}^{\infty} \bar{p}_{5k}(5n) q^n \equiv \frac{(q^{5k+1}; q^{5k+1})_\infty}{(q, q^4; q^5)_\infty (q^{5k+1}, q^{4(5k+1)}; q^{5(5k+1)})_\infty^2} \pmod 5, \tag{6}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k}(5n+1) q^n \equiv 2\frac{(q^{5k+1}; q^{5k+1})_\infty}{(q^2, q^3; q^5)_\infty (q^{5k+1}, q^{4(5k+1)}; q^{5(5k+1)})_\infty^2} + 4q^k \frac{(q^{5(5k+1)}; q^{5(5k+1)})_\infty}{(q, q^4; q^5)_\infty} \pmod 5, \tag{7}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k}(5n+2) q^n \equiv 3q^k \frac{(q^{5(5k+1)}; q^{5(5k+1)})_\infty}{(q^2, q^3; q^5)_\infty} + 4q^{2k} \frac{(q^{5k+1}; q^{5k+1})_\infty}{(q, q^4; q^5)_\infty (q^{2(5k+1)}, q^{3(5k+1)}; q^{5(5k+1)})_\infty^2} \pmod 5, \tag{8}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k}(5n+3) q^n \equiv 3q^{2k} \frac{(q^{5k+1}; q^{5k+1})_\infty}{(q^2, q^3; q^5)_\infty (q^{2(5k+1)}, q^{3(5k+1)}; q^{5(5k+1)})_\infty^2} \pmod 5, \tag{9}$$

$$(10)\quad \sum_{n=0}^{\infty} \bar{p}_{5k}(5n+4)q^n \equiv 0 \pmod 5.$$

Theorem 1.2. *For every positive integer k,*

$$(11)\quad \sum_{n=0}^{\infty} \bar{p}_{5k-1}(5n)q^n \equiv \frac{1}{(q,q^4;q^5)_\infty}\Bigg(\frac{(q^{5k};q^{5k})_\infty}{(q^{5k},q^{4(5k)};q^{5(5k)})_\infty^2} + 4q^k(q^{5(5k)};q^{5(5k)})_\infty + 4q^{2k}\frac{(q^{5k};q^{5k})_\infty}{(q^{2(5k)},q^{3(5k)};q^{5(5k)})_\infty^2}\Bigg) \pmod 5,$$

$$(12)\quad \sum_{n=0}^{\infty} \bar{p}_{5k-1}(5n+1)q^n \equiv \frac{2}{(q^2,q^3;q^5)_\infty}\Bigg(\frac{(q^{5k};q^{5k})_\infty}{(q^{5k},q^{4(5k)};q^{5(5k)})_\infty^2} + 4q^k(q^{5(5k)};q^{5(5k)})_\infty + 4q^{2k}\frac{(q^{5k};q^{5k})_\infty}{(q^{2(5k)},q^{3(5k)};q^{5(5k)})_\infty^2}\Bigg) \pmod 5,$$

$$(13)\quad \sum_{n=0}^{\infty} \bar{p}_{5k-1}(5n+a)q^n \equiv 0 \pmod 5, \quad a = 2,3,4.$$

Theorem 1.3. *For every positive integer k,*

$$(14)\quad \sum_{n=0}^{\infty} \bar{p}_{5k-2}(5n)q^n \equiv \frac{(q^{5k-1};q^{5k-1})_\infty}{(q,q^4;q^5)_\infty(q^{5k-1},q^{4(5k-1)};q^{5(5k-1)})_\infty^2} + 3q^k\frac{(q^{5(5k-1)};q^{5(5k-1)})_\infty}{(q^2,q^3;q^5)_\infty} \pmod 5,$$

$$(15)\quad \sum_{n=0}^{\infty} \bar{p}_{5k-2}(5n+1)q^n \equiv 2\frac{(q^{5k-1};q^{5k-1})_\infty}{(q^2,q^3;q^5)_\infty(q^{5k-1},q^{4(5k-1)};q^{5(5k-1)})_\infty^2} \pmod 5,$$

$$(16)\quad \sum_{n=0}^{\infty} \bar{p}_{5k-2}(5n+2)q^n \equiv 0 \pmod 5,$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k-2}(5n+3)q^n \equiv 4q^{2k} \frac{(q^{5k-1};q^{5k-1})_\infty}{(q,q^4;q^5)_\infty (q^{2(5k-1)},q^{3(5k-1)};q^{5(5k-1)})_\infty^2} \pmod 5, \tag{17}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k-2}(5n+4)q^n \equiv 4q^{k} \frac{(q^{5(5k-1)};q^{5(5k-1)})_\infty}{(q,q^4;q^5)_\infty} + 3q^{2k} \frac{(q^{5k-1};q^{5k-1})_\infty}{(q^2,q^3;q^5)_\infty (q^{2(5k-1)},q^{3(5k-1)};q^{5(5k-1)})_\infty^2} \pmod 5. \tag{18}$$

Theorem 1.4. *For every positive integer k,*

$$\sum_{n=0}^{\infty} \bar{p}_{5k-3}(5n)q^n \equiv \frac{(q^{5k-2};q^{5k-2})_\infty}{(q,q^4;q^5)_\infty (q^{5k-2},q^{4(5k-2)};q^{5(5k-2)})_\infty^2} \pmod 5, \tag{19}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k-3}(5n+1)q^n \equiv 2 \frac{(q^{5k-2};q^{5k-2})_\infty}{(q^2,q^3;q^5)_\infty (q^{5k-2},q^{4(5k-2)};q^{5(5k-2)})_\infty^2} + 4q^{2k} \frac{(q^{5k-2};q^{5k-2})_\infty}{(q,q^4;q^5)_\infty (q^{2(5k-2)},q^{3(5k-2)};q^{5(5k-2)})_\infty^2} \pmod 5, \tag{20}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k-3}(5n+2)q^n \equiv 3q^{2k} \frac{(q^{5k-2};q^{5k-2})_\infty}{(q^2,q^3;q^5)_\infty (q^{2(5k-2)},q^{3(5k-2)};q^{5(5k-2)})_\infty^2} \pmod 5, \tag{21}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k-3}(5n+3)q^n \equiv 4q^{k} \frac{(q^{5(5k-2)};q^{5(5k-2)})_\infty}{(q,q^4;q^5)_\infty} \pmod 5, \tag{22}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k-3}(5n+4)q^n \equiv 3q^{k} \frac{(q^{5(5k-2)};q^{5(5k-2)})_\infty}{(q^2,q^3;q^5)_\infty} \pmod 5. \tag{23}$$

Theorem 1.5. *For every positive integer k,*

$$\sum_{n=0}^{\infty} \bar{p}_{5k-4}(5n)q^n \equiv \frac{(q^{5k-3};q^{5k-3})_\infty}{(q,q^4;q^5)_\infty (q^{5k-3},q^{4(5k-3)};q^{5(5k-3)})_\infty^2} + 3q^{2k} \frac{(q^{5k-3};q^{5k-3})_\infty}{(q^2,q^3;q^5)_\infty (q^{2(5k-3)},q^{3(5k-3)};q^{5(5k-3)})_\infty^2} \pmod 5, \tag{24}$$

$$\sum_{n=0}^{\infty}\bar{p}_{5k-4}(5n+1)q^n \equiv 2\frac{(q^{5k-3};q^{5k-3})_\infty}{(q^2,q^3;q^5)_\infty(q^{5k-3},q^{4(5k-3)};q^{5(5k-3)})^2_\infty} \pmod 5, \tag{25}$$

$$\sum_{n=0}^{\infty}\bar{p}_{5k-4}(5n+2)q^n \equiv 4q^k\frac{(q^{5(5k-3)};q^{5(5k-3)})_\infty}{(q,q^4;q^5)_\infty} \pmod 5, \tag{26}$$

$$\sum_{n=0}^{\infty}\bar{p}_{5k-4}(5n+3)q^n \equiv 3q^k\frac{(q^{5(5k-3)};q^{5(5k-3)})_\infty}{(q^2,q^3;q^5)_\infty} \pmod 5, \tag{27}$$

$$\sum_{n=0}^{\infty}\bar{p}_{5k-4}(5n+4)q^n \equiv 4q^{2k}\frac{(q^{5k-3};q^{5k-3})_\infty}{(q,q^4;q^5)_\infty(q^{2(5k-3)},q^{3(5k-3)};q^{5(5k-3)})^2_\infty} \pmod 5. \tag{28}$$

We remark that (10), (13) and (16) are Ramanujan type congruences and (13) is a trivial consequence of (4).

In Section 4, we give some additional congruences modulo 5. The combinatorial interpretations for (10), (13) and (16) are provided in Section 5. Finally, we pose some problems for future work in Section 6.

2. Preliminary results.

In this section, we recall some results from the literature. Since the results are well-known, for convenience we give references in [**8**].

Theorem 2.1 (Euler [**8,** (1.6.2)])**.** *The following identity holds:*

$$(q;q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}.$$

Theorem 2.2 (Jacobi [**8,** (1.7.1)])**.** *The following identity holds:*

$$(q;q)^3_\infty = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2}.$$

The following lemma was stated by Ramanujan [**2,** Entry 25, p. 40].

Lemma 2.3. *The following* 2*-dissections hold:*

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}, \tag{29}$$

$$\frac{1}{(q;q)_\infty^4} = \frac{(q^4;q^4)_\infty^{14}}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty^4} + 4q\frac{(q^4;q^4)_\infty^2(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^{10}}. \tag{30}$$

We note that (29) and (30) are essentially the 2-dissections of $\phi(q)$ and $\phi(q)^2$ in [**8,** (1.9.4)] and [**8,** (1.10.1)], respectively.

The following lemma was proved by Hirschhorn and Sellers [**7**].

Lemma 2.4. *The following* 2*-dissection holds:*

$$\frac{(q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q^8;q^8)_\infty(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2(q^{40};q^{40})_\infty} + q\frac{(q^4;q^4)_\infty^3(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3(q^8;q^8)_\infty(q^{20};q^{20})_\infty}.$$

The next lemma can be obtained from the quintuple product identity; see [**2,** p. 80], [**3**], [**6**] or [**8,** (8.1.1)].

Lemma 2.5. *We have*

$$(q;q)_\infty = (q^{25};q^{25})_\infty(R(q^5)^{-1} - q - q^2R(q^5)),$$

where

$$R(q^5) = \left(\begin{matrix} q^5, q^{20} \\ q^{10}, q^{15} \end{matrix}; q^{25}\right)_\infty.$$

The following lemma can also be found in [**8,** (3.6.4)].

Lemma 2.6 (Atkin and Swinnerton-Dyer [**1**])**.** *We have*

$$(q;q)_\infty^3 \equiv F(q^5) - 3qG(q^5) \pmod 5,$$

where

$$F(q^5) = \sum_{k=-\infty}^{\infty} (-1)^k q^{(25k^2-5k)/2} = (q^{10}, q^{15}, q^{25}; q^{25})_\infty,$$

$$G(q^5) = \sum_{k=-\infty}^{\infty} (-1)^k q^{(25k^2-15k)/2} = (q^5, q^{20}, q^{25}; q^{25})_\infty.$$

Finally, the lemma below is derived from [**8,** (12.3.1) and (12.3.3)].

Lemma 2.7. *Suppose ζ is a 5th root of unity other than 1. Then*

$$\frac{1}{(\zeta q;q)_\infty(\zeta^{-1}q;q)_\infty} = \frac{1}{(q^5,q^{20};q^{25})_\infty} + q\frac{\zeta+\zeta^{-1}}{(q^{10},q^{15};q^{25})_\infty}.$$

We also use the following consequence of the binomial theorem. For any positive integer k and prime p,

$$(q^k;q^k)_\infty^p \equiv (q^{pk};q^{pk})_\infty \pmod p. \tag{31}$$

3. The generating function and proofs of Theorems 1.1–1.5. In this section, we show that the generating function of k-restricted overpartitions is given by (5).

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty}\bar{p}_k(n)q^n = \frac{(q^{k+1};q^{k+1})_\infty}{(q;q)_\infty^2}.$$

Proof. We see immediately that

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_k(n)q^n &= \prod_{n=1}^{\infty} 1+2q^n+3q^{2n}+\cdots+(k+1)q^{kn}+(k+1)q^{(k+1)n}+\cdots\\ &= \prod_{n=1}^{\infty}\left((1+q^n+q^{2n}+\cdots+q^{kn})\sum_{i=0}^{\infty}q^{in}\right)\\ &= \prod_{n=1}^{\infty}\frac{(1+q^n+q^{2n}+\cdots+q^{kn})}{(1-q^n)} = \prod_{n=1}^{\infty}\frac{(1-q^{(k+1)n})}{(1-q^n)^2}\\ &= \frac{(q^{k+1};q^{k+1})_\infty}{(q;q)_\infty^2},\end{aligned}$$

as claimed. □

Now we prove Theorems 1.1–1.5.

Proofs of Theorems 1.1–1.5. From (31), Lemma 2.5 and Lemma 2.6,

$$\begin{aligned}\frac{(q^m;q^m)_\infty}{(q;q)_\infty^2} &= (q;q)_\infty^3 (q^m;q^m)_\infty \frac{1}{(q;q)_\infty^5}\\ &\equiv \Big(F(q^5)-3qG(q^5)\Big)(R(q^{5m})^{-1}-q^m-q^{2m}R(q^{5m}))\\ &\qquad\qquad \times \frac{(q^{25m};q^{25m})_\infty}{(q^5;q^5)_\infty} \pmod 5\\ &\equiv \Big(F(q^5)R(q^{5m})^{-1}-q^mF(q^5)-q^{2m}F(q^5)R(q^{5m})\\ &\quad -3qG(q^5)R(q^{5m})^{-1}+3q^{m+1}G(q^5)+3q^{2m+1}G(q^5)R(q^{5m})\Big)\\ &\qquad\qquad \times \frac{(q^{25m};q^{25m})_\infty}{(q^5;q^5)_\infty} \pmod 5.\end{aligned}$$

The desired results can be obtained by substituting $m=5k+a$, $0\le a\le 4$. As an illustration, we give one example for $m=5k+1$. We get

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_{5k}(5n)q^n &\equiv (q^2,q^3,q^5;q^5)_\infty \left(\begin{matrix} q^{2(5k+1)},q^{3(5k+1)} \\ q^{5k+1},q^{4(5k+1)}\end{matrix};q^{5(5k+1)}\right)_\infty\\ &\qquad\qquad \times \frac{(q^{5(5k+1)};q^{5(5k+1)})_\infty}{(q;q)_\infty} \pmod 5,\\ \sum_{n=0}^{\infty}\bar{p}_{5k}(5n+1)q^n &\\ \equiv \Big(-q^k(q^2,q^3,q^5;q^5)_\infty &-3(q,q^4,q^5;q^5)_\infty\\ \times\left(\begin{matrix} q^{2(5k+1)},q^{3(5k+1)} \\ q^{5k+1},q^{4(5k+1)}\end{matrix};q^{5(5k+1)}\right)_\infty\Big) &\frac{(q^{5(5k+1)};q^{5(5k+1)})_\infty}{(q;q)_\infty} \pmod 5,\\ \sum_{n=0}^{\infty}\bar{p}_{5k}(5n+2)q^n &\\ \equiv \Big(-q^{2k}(q^2,q^3,q^5;q^5)_\infty &\left(\begin{matrix} q^{5k+1},q^{4(5k+1)} \\ q^{2(5k+1)},q^{3(5k+1)}\end{matrix};q^{5(5k+1)}\right)_\infty\\ +3q^k(q,q^4,q^5;q^5)_\infty\Big) &\frac{(q^{5(5k+1)};q^{5(5k+1)})_\infty}{(q;q)_\infty} \pmod 5,\end{aligned}$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k}(5n+3)q^n \equiv 3q^{2k}(q,q^4,q^5;q^5)_\infty \left(\begin{matrix} q^{5k+1}, q^{4(5k+1)} \\ q^{2(5k+1)}, q^{3(5k+1)} \end{matrix} ; q^{5(5k+1)} \right)_\infty \times \frac{(q^{5(5k+1)};q^{5(5k+1)})_\infty}{(q;q)_\infty} \pmod 5,$$

$$\sum_{n=0}^{\infty} \bar{p}_{5k}(5n+4)q^n \equiv 0 \pmod 5.$$

After some simplification, we arrive at Theorem 1.1. □

4. Other congruences modulo 5. In this section, we provide some congruences modulo 5 satisfied by $\bar{p}_k(n)$.

Theorem 4.1. *For every positive integer m, if $8m+1$ is not a square then*

$$\bar{p}_4(m) \equiv 0 \pmod 5.$$

Proof. From Theorem 2.2 and (31), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \bar{p}_4(n)q^n &= \frac{(q^5;q^5)_\infty}{(q;q)_\infty^2} \\ &\equiv (q;q)_\infty^3 \pmod 5 \\ &\equiv \sum_{k=0}^{\infty} (-1)^k(2k+1)q^{(k^2+k)/2} \pmod 5 \\ &\equiv \sum_{k=0}^{\infty} (-1)^k(2k+1)q^{((2k+1)^2-1)/8} \pmod 5. \end{aligned}$$

Equating the coefficients of q^m where $8m+1$ is not a square gives us the desired result. □

Theorem 4.2. *We have*

$$\bar{p}_9(4n+2) \equiv 0 \pmod 5, \tag{32}$$

$$\bar{p}_9(4n+3) \equiv 0 \pmod 5. \tag{33}$$

Proof. From (29), we get

$$\sum_{n=0}^{\infty}\bar{p}_9(n)q^n = \frac{(q^{10};q^{10})_\infty}{(q;q)_\infty^2}$$
$$= \frac{(q^8;q^8)_\infty^5(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}.$$

This yields

$$\sum_{n=0}^{\infty}\bar{p}_9(2n)q^n = \frac{(q^4;q^4)_\infty^5(q^5;q^5)_\infty}{(q;q)_\infty^5(q^8;q^8)_\infty^2} \tag{34}$$

and

$$\sum_{n=0}^{\infty}\bar{p}_9(2n+1)q^n = 2\frac{(q^2;q^2)_\infty^2(q^5;q^5)_\infty(q^8;q^8)_\infty^2}{(q;q)_\infty^5(q^4;q^4)_\infty}. \tag{35}$$

Applying Lemma 2.4 and (30) to (34) gives

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_9(2n)q^n &= \left(\frac{(q^4;q^4)_\infty^5}{(q^8;q^8)_\infty^2}\right)\left(\frac{1}{(q;q)_\infty^4}\right)\left(\frac{(q^5;q^5)_\infty}{(q;q)_\infty}\right)\\
&= \frac{(q^4;q^4)_\infty^{19}(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^{16}(q^8;q^8)_\infty^5(q^{40};q^{40})_\infty}\\
&\quad + q\frac{(q^4;q^4)_\infty^{22}(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^{17}(q^8;q^8)_\infty^7(q^{10};q^{10})_\infty}\\
&\quad + 4q\frac{(q^4;q^4)_\infty^7(q^8;q^8)_\infty^3(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^2(q^{10};q^{10})_\infty^2(q^{40};q^{40})_\infty}\\
&\quad + 4q^2\frac{(q^4;q^4)_\infty^{19}(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^{13}(q^{20};q^{20})_\infty}\\
&\equiv \frac{(q^4;q^4)_\infty^4(q^{20};q^{20})_\infty^5}{(q^2;q^2)_\infty(q^{10};q^{10})_\infty^3(q^{40};q^{40})_\infty^2}\\
&\quad + 4q^2\frac{(q^4;q^4)_\infty^4(q^{20};q^{20})_\infty^2(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^3(q^{10};q^{10})_\infty} \pmod 5,\end{aligned}$$

where the last congruence comes from (31). Equating coefficients of q^{2n+1} provides us (32). Similarly, applying Lemma 2.4, (30) and (31) to (35) gives:

$$\begin{aligned}\sum_{n=0}^{\infty}\bar{p}_9(2n+1)q^n &= 2\left(\frac{(q^5;q^5)_\infty}{(q;q)_\infty}\right)\left(\frac{1}{(q;q)_\infty^4}\right)\left(\frac{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\right)\\
&= 2\frac{(q^4;q^4)_\infty^{13}(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^{14}(q^8;q^8)_\infty(q^{40};q^{40})_\infty}\\
&\quad + 2q\frac{(q^4;q^4)_\infty^{16}(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^{15}(q^8;q^8)_\infty^3(q^{20};q^{20})_\infty}\\
&\quad + 8q\frac{(q^4;q^4)_\infty(q^8;q^8)_\infty^7(q^{20};q^{20})_\infty^2}{(q^2;q^2)_\infty^{10}(q^{40};q^{40})_\infty}\\
&\quad + 8q^2\frac{(q^4;q^4)_\infty^4(q^8;q^8)_\infty^5(q^{10};q^{10})_\infty(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty^{11}(q^{20};q^{20})_\infty}\\
&\equiv 2\frac{(q^4;q^4)_\infty^3(q^{20};q^{20})_\infty^4}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty(q^{10};q^{10})_\infty^2(q^{40};q^{40})_\infty}\\
&\quad + 3q^2\frac{(q^4;q^4)_\infty^4(q^8;q^8)_\infty^5(q^{40};q^{40})_\infty}{(q^2;q^2)_\infty(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty} \pmod 5.\end{aligned}$$

Extracting terms involving q^{2n+1} proves (33). □

5. Combinatorial interpretations. In this section, we provide combinatorial interpretations for (10), (13) and (16). We start with the following theorem.

Theorem 5.1. *There is a bijection δ between the set of k-restricted overpartitions of n and the set of pairs of partitions (α, β) with $|\alpha|+|\beta|=n$, where α is an ordinary partition and β is a partition with no parts divisible by $k+1$.*

Proof. Let λ be a k-restricted overpartition of n. We wish to construct a pair of partitions $(\alpha,\beta)=\delta(\lambda)$ such that $|\alpha|+|\beta|=n$, where α is an ordinary partition and β is a partition with no parts divisible by $k+1$. Suppose that t is a part of λ and t appears m_t times with the i-th part being overlined (the 0th part being overlined means no overlined part). We use the following procedure.

- Move $(m_t - i)$ parts equal to t in λ to α.
- Split each of the remaining i parts equal to t in λ into $k+1$ equal parts. Repeat until no parts are divisible by $k+1$.
- Move the results into β.

The above procedure is reversible. Hence δ is a bijection. This completes the proof. □

As an example, we have that $\bar{9}+9+3+\bar{3}$ is a 2-restricted overpartition of 24. We have the following procedure:

$$\bar{9}+9+3+\bar{3} \mapsto (9, 9+3+3) \mapsto (9, 3+3+3+\underbrace{1+\cdots+1}_{\text{6-times}}) \mapsto (9, \underbrace{1+\cdots+1}_{\text{15-times}}),$$

where we split every part divisible by 3 in β. The inverse is given by

$$(9, \underbrace{1+\cdots+1}_{\text{15-times}}) \mapsto (9, \underbrace{3+3+3}_{\text{merge them}}+3+3) \mapsto (9, 9+3+3) \mapsto \bar{9}+9+3+\bar{3},$$

where we merge every three equal parts in β.

Now we introduce a birank of a k-restricted overpartition.

Definition 5.2. Let λ be a k-restricted overpartition and $(\alpha, \beta) = \delta(\lambda)$. Then the birank of λ, denoted $r_k(\lambda)$, is defined by

$$r_k(\lambda) = \#(\alpha) - \#(\beta),$$

where $\#(\alpha)$ is the number of parts of α and $\#(\beta)$ is the number of parts of β. For examples, see Table 1.

We define $R_k(m, n)$ to be the number of k-restricted overpartitions of n having birank m, and $R_k(r, m, n)$ to be the number of k-restricted overpartitions of n having birank congruent to r mod m.

The following theorem is a direct consequence of Theorem 2.1 in [**5**].

Theorem 5.3. *We have*

$$R_{5k-1}(0, 5, 5n+j) = R_{5k-1}(1, 5, 5n+j) = R_{5k-1}(2, 5, 5n+j), \quad j = 2, 3, 4.$$

(We are unable to find the relations between $R_{5k-1}(a, 5, 5n+b)$ where $a = 0, 1, 2$ and $b = 0, 1$.)

Theorem 5.4. *We have*

$$R_{5k-2}(0, 5, 5n+2) = R_{5k-2}(1, 5, 5n+2) = R_{5k-2}(2, 5, 5n+2), \tag{36}$$

$$R_{5k}(0, 5, 5n+4) = R_{5k}(1, 5, 5n+4) = R_{5k}(2, 5, 5n+4). \tag{37}$$

TABLE 1. The case for $k=5$ and $n=4$.

λ	$(\alpha,\beta)=\delta(\lambda)$	$r_5(\lambda)\pmod 5$
4	$(4,\varnothing)$	1
$\bar{4}$	$(\varnothing,4)$	4
$3+1$	$(3+1,\varnothing)$	2
$3+\bar{1}$	$(3,1)$	0
$\bar{3}+1$	$(1,3)$	0
$\bar{3}+\bar{1}$	$(\varnothing,3+1)$	3
$2+2$	$(2+2,\varnothing)$	2
$2+\bar{2}$	$(\varnothing,2+2)$	3
$\bar{2}+2$	$(2,2)$	0
$2+1+1$	$(2+1+1,\varnothing)$	3
$2+1+\bar{1}$	$(2,1+1)$	4
$2+\bar{1}+1$	$(2+1,1)$	1
$\bar{2}+1+1$	$(1+1,2)$	1
$\bar{2}+1+\bar{1}$	$(\varnothing,2+1+1)$	2
$\bar{2}+\bar{1}+1$	$(1,2+1)$	4
$1+1+1+1$	$(1+1+1+1,\varnothing)$	4
$1+1+1+\bar{1}$	$(\varnothing,1+1+1+1)$	1
$1+1+\bar{1}+1$	$(1,1+1+1)$	3
$1+\bar{1}+1+1$	$(1+1,1+1)$	0
$\bar{1}+1+1+1$	$(1+1+1,1)$	2

Proof. We can see that the generating function for $R_k(m,n)$ is given by

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}R_k(m,n)z^mq^n=\frac{(q^{k+1};q^{k+1})_\infty}{(zq;q)_\infty(z^{-1}q;q)_\infty}. \tag{38}$$

Applying Lemma 2.7 by substituting $z=\zeta=e^{2\pi i/5}$ to (38) gives

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty}R_k(m,n)\zeta^mq^n=\frac{(q^{k+1};q^{k+1})_\infty}{(q^5,q^{20};q^{25})_\infty}+q\frac{(\zeta+\zeta^{-1})(q^{k+1};q^{k+1})_\infty}{(q^{10},q^{15};q^{25})_\infty}.$$

To simplify notation, we define

$$R_x:=\sum_{n=0}^{\infty}R(x,5,n)q^n.$$

We have

$$\sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} R_k(m,n)\zeta^n q^n = R_0 + \zeta R_1 + \zeta^2 R_2 + \zeta^3 R_3 + \zeta^4 R_4.$$

Since $\zeta^2 = 1-\zeta-\zeta^3-\zeta^4$, $R_1 = R_4$ and $R_2 = R_3$, we get

$$R_0 - R_2 + (\zeta+\zeta^4)(R_1 - R_2) = \frac{(q^{k+1};q^{k+1})_\infty}{(q^5,q^{20};q^{25})_\infty} + q\frac{(\zeta+\zeta^{-1})(q^{k+1};q^{k+1})_\infty}{(q^{10},q^{15};q^{25})_\infty}. \tag{39}$$

For some non-negative integer k', (36) and (37) are the cases when $k=5k'+3$ and $k=5k'$, respectively. Using Theorem 2.1 and substituting $k=5k'+3$, the right-hand side of (39) becomes

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{(5k'+4)(3n^2-n)/2} \times\left(\frac{1}{(q^5,q^{20};q^{25})_\infty} + (\zeta+\zeta^{-1})\frac{q}{(q^{10},q^{15};q^{25})_\infty}\right), \tag{40}$$

Note that all possibilities of $(5k'+1)(3n^2-n)/2$ are $0,3,4 \pmod 5$. Substituting $k=5k'+3$ to the left-hand side of (39) then extracting coefficients of q^{5m+2} in (40) gives us (36). With a similar argument, we get (37). This completes the proof. $\square$

6. Closing remarks. By analyzing some values of $\bar{p}_k(n)$ via Maple, we find a Ramanujan type congruence modulo 7 for $\bar{p}_4(n)$.

Conjecture 6.1. The following congruence holds:

$$\bar{p}_4(7n+6) \equiv 0 \pmod 7.$$

Thus, it will be interesting to introduce a new birank that could provide a combinatorial interpretation of $\bar{p}_4(7n+6) \equiv 0 \pmod 7$.

Acknowledgements. The author would like to acknowledge his gratitude to the anonymous referee for his/her comments, which improved this paper a great deal. He also thanks Dr. Toh Pee Choon for his guidance in this work.

REFERENCES

1. A.O.L. Atkin and H.P.F. Swinnerton-Dyer, *Some properties of partitions*, Proc. Lond. Math. Soc. **4** (1954), 84–106.

2. B.C. Berndt, *Ramanujan's notebooks part III*, Springer (1991).

3. S. Cooper, *The quintuple product identity*, Int. J. Number Theory **2** (2006), 115–161.

4. S. Corteel and J. Lovejoy, *Overpartitions*, Trans. Amer. Math. Soc. **356** (2004), 1623–1635.

5. P. Hammond and R. Lewis, *Congruences in ordered pairs of partitions*, Int. J. Math. Math. Sci. **7** (2004), 2509–2512.

6. M.D. Hirschhorn, *A generalisation of the quintuple product identity*, J. Austral. Math. Soc. Ser. A. **44** (1988), 42–45.

7. M.D. Hirschhorn and J.A. Sellers, *Elementary proofs of parity results for 5-regular partitions*, Bull. Austral. Math. Soc. **81** (2010), 58–63.

8. M.D. Hirschhorn, *The power of q: a personal journey*, Developments in Math. **49**, Springer (2017).

Mathematics & Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore

Department of Mathematics, Universitas Gadjah Mada, Indonesia
Email address: **uhaisnaini@yahoo.co.id, isnainiuha@ugm.ac.id**