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A NOTE ON SINGULAR POINTS OF BUNDLE HOMOMORPHISMS FROM A TANGENT DISTRIBUTION INTO A VECTOR BUNDLE OF THE SAME RANK

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ABSTRACT. We consider bundle homomorphisms between tangent distributions and vector bundles of the same rank. We study the conditions for fundamental singularities when the bundle homomorphism is induced from a Morin map. When the tangent distribution is the contact structure, we characterize singularities of the bundle homomorphism by using the Hamilton vector fields.

1. Introduction. In **[8, 10]**, the notion of a coherent tangent bundle is introduced. It is a bundle homomorphism between the tangent bundle and a vector bundle with the same rank with a kind of metric. This is a generalization of fronts and C^∞-maps between the same dimensional manifolds. Singular points of bundle homomorphisms $\phi\colon TM \to E$ are points, where $\phi_p\colon T_pM \to E_p$ is not a bijection. In **[8, 10]**, differential geometric invariants of singularities of bundle homomorphisms are defined and investigated. On the other hand, in **[11]**, topological properties of singular sets of bundle homomorphisms without metric are studied. See **[3]** for another type of application of the coherent tangent bundle.

In this paper, we consider rank r, $r < m$, tangent distributions instead of tangent bundles of m-dimensional manifolds. Since $r < m$, the singularities appearing on the bundle homomorphisms are slightly different from the case $\phi\colon TM \to E$, where $\dim M = \operatorname{rank} E = m$.

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Let D_1 be a rank r tangent distribution on an m-dimensional manifold M. Let N be an r dimensional manifold and $f\colon M \to N$ a map. Then, a bundle homomorphism $\phi = df\colon D_1 \to f^*TN$ is induced from f. Singularities of ϕ should be related to D_1 and f. In this paper, we consider the low-dimensional case, we study the relationships when f is a Morin map and D_1 is a foliation or a contact structure when $m = 3$, $r = 2$.

2. Bundle homomorphisms and their singular point.

2.1. Singular points of bundle homomorphisms. With the terminology of [**10**], we give a definition of singular points of bundle homomorphisms. Let M be an m-dimensional manifold, and let D_1 be a rank r, $r < m$, tangent distribution of M, namely, a subbundle of TM. Let D_2 be a rank r vector bundle over M and $\phi\colon D_1 \to D_2$ a bundle homomorphism. If the rank of the linear map $\phi_p : (D_1)_p \to (D_2)_p$ is less than r, then $p \in M$ is called a *singular point* of ϕ. We denote by S the set of singular points of ϕ. If the rank of ϕ_p is $r-1$, then p is called a *corank one singular point*. Let $p \in M$ and U be a sufficiently small neighborhood of p. Taking frames $\{e_1, \dots, e_r\}$ and $\{g_1, \dots, g_r\}$ of D_1 and D_2 on U, respectively, then ϕ can be considered as a matrix-valued function M_ϕ by

$$(\phi(e_1), \dots, \phi(e_r)) = (g_1, \dots, g_r)M_\phi. \tag{2.1}$$

Lemma 2.1. *If p is a corank one singular point of ϕ, then there exist a neighborhood U of p and a section $\eta_\phi \in \Gamma(D_1)$ such that, if $q \in S \cap U$, then $(\eta_\phi)_q$ is a generator of the kernel of ϕ_q.*

Proof. By taking frames of D_1 and D_2 near p, we take M_ϕ as in (2.1). Since $\operatorname{rank} M_\phi(p) = r-1$, only one eigenvalue of $M_\phi(p)$ is zero, and the others are not zero. Thus, the eigenvalue having minimum absolute value among the eigenvalues of M_ϕ is uniquely determined and is a real-valued C^∞ function near p. Hence, corresponding eigenvector η_ϕ is also well defined. We have the desired section identifying η_ϕ as a section. $\square$

We call η_ϕ the *null section* of ϕ. We say that $p \in S$ is *non-degenerate* if $d\lambda_\phi(p) \neq 0$, where λ_ϕ is defined by

$$\lambda_\phi = \det M_\phi, \tag{2.2}$$

near p and M_ϕ is as in (2.1).

The notions of null section and non-degeneracy are introduced in [**4**].

Lemma 2.2. *Non-degenerate singular points are of corank one.*

Proof. Let p be a non-degenerate singular point. We assume that $\operatorname{rank} M_\phi(p) < r-1$. Then, any $r-1$ columns of $M_\phi(p)$ are linearly dependent. Let $\vec{m}_j$ denote the jth column of M_ϕ, namely, $M_\phi = (\vec{m}_1, \dots, \vec{m}_r)$. Since any $r-1$ columns of $M_\phi(p)$ are linearly dependent,

$$(\det M_\phi)_{u_i}(p) = \sum_{j=1}^{r} \det\bigl(\vec{m}_1, \dots, \vec{m}_{j-1}, (\vec{m}_j)_{u_i}, \vec{m}_{j+1}, \dots, \vec{m}_r\bigr)(p) = 0$$

holds for any $1 \le i \le m$, where $(u_1, \dots, u_m)$ is a coordinate system near p, and $(\ )_{u_i} = \partial/\partial u_i$. This contradicts the non-degeneracy. $\square$

Since $S = \{\lambda_\phi(p) = 0\}$, S is a codimension one submanifold near a non-degenerate singular point. With the terminology of [**9**], we give the following definition.

Definition 2.3. We say that a non-degenerate singular point $p \in S$ is an *A_k-like singular point* ($k \le m$) if $\eta_\phi\lambda_\phi(p) = \cdots = \eta_\phi^{k-1}\lambda_\phi(p) = 0$, $\eta_\phi^k\lambda_\phi(p) \neq 0$, and the rank of the differential of the map

$$(\lambda_\phi, \eta_\phi\lambda_\phi, \dots, \eta_\phi^{k-1}\lambda_\phi) \colon U \longrightarrow \boldsymbol{R}^k$$

is k at p, where U is a neighborhood of p. An A_1-like singular point is also called a *fold-like singular point.* Furthermore, an A_2-like (respectively, A_3-like) singular point is also called a *cusp-like* (respectively, *swallowtail-like*) *singular point.*

We remark that, if $k = 1$, then the above condition is reduced to $\eta_\phi\lambda_\phi(p) \neq 0$. If $k = 2$, then it is reduced to $\eta_\phi\lambda_\phi(p) = 0$ and $\eta_\phi^2\lambda_\phi(p) \neq 0$. Here, ξf stands for the directional derivative of a function f by the vector field ξ, and $\xi^i f$ stands for the i times directional derivative by ξ. See [**5**] for other characterizations of these singularities.

Lemma 2.4. *The definitions of A_k-like singular points* $(1 \leq k \leq m)$ *do not depend on the choice of the frames of D_1, D_2, nor on the choice of the null section.*

Proof. Let p be a singular point, and let U be a neighborhood of p. We change the frames of D_1 by a $GL(r, \boldsymbol{R})$-valued function $C_1 : U \to GL(r, \boldsymbol{R})$ and change the frames of D_2 by a $GL(r, \boldsymbol{R})$-valued function C_2. Then, M_ϕ is changed to $C_2^{-1} M_\phi C_1$. Thus, the independence of the choice of frames is clear. We show the independence of the choice of the null section, and the case of fold-like singular points is also clear since $\eta_\phi \lambda_\phi(p)$ is a directional derivative of $(\eta_\phi)_p$. Furthermore, the independence of the non-degeneracy is also clear. We assume that p is a non-degenerate singular point, and $\eta_\phi \lambda_\phi(p) = 0$. We set $\widetilde{\eta} = a\eta_\phi + \widetilde{b}$, where a is a non-zero function, and $\widetilde{b}$ is a vector field which vanishes on S. Since $d\lambda_\phi(p) \neq 0$, and $\widetilde{b}$ vanishes on $S = \lambda_\phi^{-1}(0)$, there exists a vector field b, such that $\widetilde{b} = \lambda_\phi b$. Thus, $\widetilde{\eta} = a\eta_\phi + \lambda_\phi b$. Then, we have

$$
\begin{aligned}
\widetilde{\eta}^2 \lambda_\phi = {} & (a(\eta_\phi b \lambda_\phi) + (ba)(\eta_\phi \lambda_\phi) + a(b\eta_\phi \lambda_\phi) + (b\lambda_\phi)^2 + \lambda_\phi(b^2 \lambda_\phi))\lambda_\phi \\
& + a(\eta_\phi a + b\lambda_\phi)\eta\lambda_\phi + a^2 \eta_\phi^2 \lambda_\phi.
\end{aligned} \tag{2.3}
$$

We show, for any $i \leq k$, that there exist functions $\alpha_{i,0}, \alpha_{i,1}, \dots, \alpha_{i,i-1}$ such that

$$
\widetilde{\eta}^i \lambda_\phi = \sum_{j=0}^{i-1} \alpha_{i,j} \eta_\phi^j \lambda_\phi + a^i \eta_\phi^i \lambda_\phi, \tag{2.4}
$$

where $\eta_\phi^0 \lambda_\phi = \lambda_\phi$ by induction. By (2.3), the case $i = 2$ is true. By (2.4),

$$
\begin{aligned}
\widetilde{\eta}^{i+1} \lambda_\phi = a\Bigg(& \sum_{j=0}^{i-1} \big((\eta_\phi \alpha_{i,j})\eta_\phi^j \lambda_\phi + \alpha_{i,j} \eta_\phi^{j+1} \lambda_\phi\big) + (\eta_\phi a^i)\eta_\phi^i \lambda_\phi + a^i \eta_\phi^{i+1} \lambda_\phi \Bigg) \\
& + \lambda\big(b(\widetilde{\eta}^i \lambda_\phi)\big)
\end{aligned}
$$

holds, and this shows the assertion. Then, it is seen that, for any $i \leq k$,

$$
\lambda_\phi = \eta_\phi \lambda_\phi = \cdots = \eta_\phi^i \lambda_\phi = 0, \eta_\phi^{i+1} \lambda_\phi \neq 0
$$

is equivalent to

$$\lambda_\phi = \widetilde{\eta}\lambda_\phi = \cdots = \widetilde{\eta}^i\lambda_\phi = 0, \widetilde{\eta}^{i+1}\lambda_\phi \neq 0.$$

Next, we assume that $d\lambda_\phi(p) \neq 0$ and $\lambda_\phi = \eta_\phi\lambda_\phi = \cdots = \eta_\phi^i\lambda_\phi = 0$ for $i \leq k$. We choose a frame $\{e_1, \dots, e_{m-1}, \eta_\phi\}$ of TM. Then, $\{e_1, \dots, e_{m-1}, \widetilde{\eta}\}$ is also a frame of TM. By (2.4), we see that, for $1 \leq l \leq m-1$ and $0 \leq q \leq i$,

$$e_l\widetilde{\eta}^q\lambda_\phi = \sum_{j=0}^{q-1}\Big((e_l\alpha_{q,j})\eta_\phi^j\lambda_\phi + \alpha_{q,j}(e_l\eta_\phi^j\lambda_\phi)\Big) + (e_la^q)\eta_\phi^q\lambda_\phi, +a^q(e_l\eta_\phi^q\lambda_\phi)$$

and

$$e_l\widetilde{\eta}^q\lambda_\phi = \sum_{j=0}^{q-1}\alpha_{q,j}(e_l\eta_\phi^j\lambda_\phi) + a^q(e_l\eta_\phi^q\lambda_\phi)$$

at p. By this formula and by the elementary row operations, we see that the ranks of the matrices

$$\begin{pmatrix} e_1\lambda_\phi & \cdots & e_{m-1}\lambda_\phi & \eta_\phi\lambda_\phi \\ e_1\eta_\phi\lambda_\phi & \cdots & e_{m-1}\eta_\phi\lambda_\phi & \eta_\phi^2\lambda_\phi \\ \vdots & \vdots & \vdots & \vdots \\ e_1\eta_\phi^i\lambda_\phi & \cdots & e_{m-1}\eta_\phi^i\lambda_\phi & \eta_\phi^{i+1}\lambda_\phi \end{pmatrix}$$

and

$$\begin{pmatrix} e_1\lambda_\phi & \cdots & e_{m-1}\lambda_\phi & \widetilde{\eta}\lambda_\phi \\ e_1\widetilde{\eta}\lambda_\phi & \cdots & e_{m-1}\widetilde{\eta}\lambda_\phi & \widetilde{\eta}^2\lambda_\phi \\ \vdots & \vdots & \vdots & \vdots \\ e_1\widetilde{\eta}^i\lambda_\phi & \cdots & e_{m-1}\widetilde{\eta}^i\lambda_\phi & \widetilde{\eta}^{i+1}\lambda_\phi \end{pmatrix}$$

are the same at p. This proves the lemma. $\square$

We have the following characterization in regards to the relation with the singular set and the null section. (This is analogous to the standard A_k singular point for wave front and Morin map. See **[9]** for details.) Let p be an A_k-like singular point and U a neighborhood of p. We set $S_1 = S$, and $S_i = \{q \in U \mid \lambda_\phi(q) = \eta_\phi\lambda_\phi(q) = \cdots = \eta_\phi^{i-1}\lambda_\phi(q) = 0\}$, $i \leq k$.

Proposition 2.5. *If $p \in S$ is an A_k-like singular point, then S_i, $i = 1, \dots, k$, are submanifolds with codimension i, and*

$$S_i = \{q \in S_{i-1} \mid \eta_\phi(q) \in T_q S_{i-1}\}.$$

Moreover, $\eta_\phi(p) \notin T_p S_k$.

Proof. By the definition of S_i and the property that the map $(\lambda_\phi, \eta_\phi\lambda_\phi, \dots, \eta_\phi^{k-1}\lambda_\phi)$ is a submersion, S_i is a codimension i. Since

$$\begin{aligned} S_i &= \{q \in U \mid \lambda_\phi(q) = \eta_\phi\lambda_\phi(q) = \cdots = \eta_\phi^{i-1}\lambda_\phi(q) = 0\} \\ &= \{q \in S_{i-1} \mid \eta_\phi^{i-1}\lambda_\phi(q) = 0\} \\ &= \{q \in S_{i-1} \mid \eta_\phi(q) \in T_q S_{i-1}\}, \end{aligned}$$

the last assertions are obvious. □

2.2. Geometric interpretations of singularities. We give geometric interpretations of singularities of bundle homomorphisms assuming $(m, r) = (2, 1), (3, 1)$ and $(3, 2)$. We set $r = 1$. Then, D_1 is generated by a vector field e, and η_ϕ on S can be chosen as e. Thus, the configuration of S and η_ϕ is the same as the standard A_k-singularities of wave fronts **[9]**. The pictures of S and D_1 are drawn in Figure 1 (case $m = 2$) and Figure 2 (case $m = 3$).

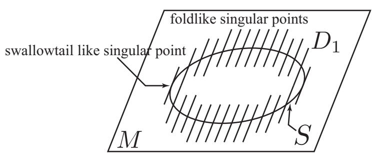

FIGURE 1. S and D_1 of fold-like singular point and of cusp-like singular point in the case of $m = 2$.

We set $m = 3$ and $r = 2$. If p is a fold-like singular point, then $(\eta_\phi)_p \notin T_p S$; thus, $(D_1)_p \neq T_p S$. Let p be a cusp-like singular point. If $e_1\lambda_\phi = e_2\lambda_\phi = 0$ at p, then $(D_1)_p = T_p S$, where $\{e_1, e_2\}$ is a frame of D_1. In this case, we call p a *cusp-like singular point of tangent type.* If $(e_1\lambda_\phi, e_2\lambda_\phi) \neq (0, 0)$ at p, then $(D_1)_p$ is transversal to $T_p S$. In this case, we call p a *cusp-like singular point of transverse type.* The picture of S and D_1 is drawn in Figure 3.

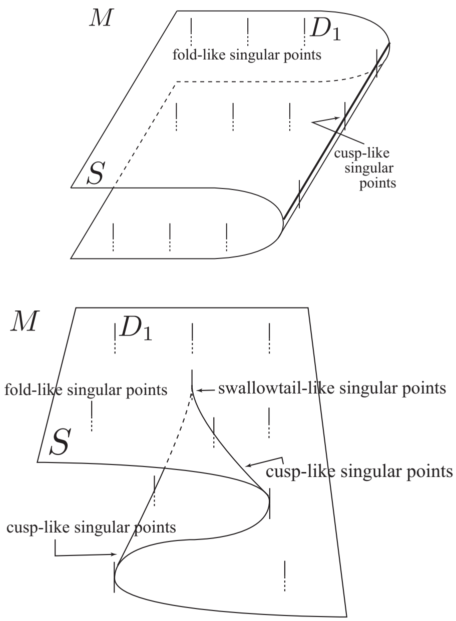

FIGURE 2. S and D_1 of fold-like, cusp-like and swallowtail-like singular points in the case of $m = 3$.

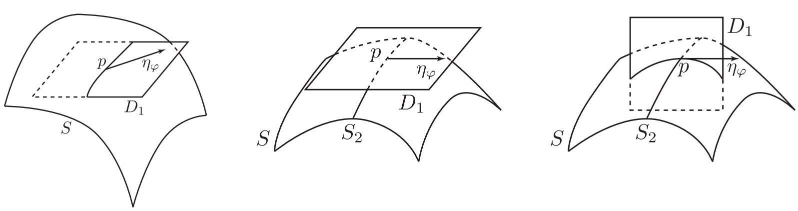

FIGURE 3. S and D_1 of fold-like singular and cusp-like singular points of tangent and transverse types.

If $p \in S$ is a swallowtail-like singular point, then S_2 is a one-dimensional submanifold of S. Let (u, v) be a coordinate system of S and (u, v, w) a coordinate system of M near p. For a function $\widetilde{c}(u, v)$ which vanishes on S_2, let us set a coordinate expression of a null section

η_ϕ associated to (u,v) near p as

$$\eta_\phi = \widetilde{a}(u,v)\partial_u + \widetilde{b}(u,v)\partial_v + \widetilde{c}(u,v)\partial_w.$$

We take a parametrization of S_2 with respect to (u,v) as $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ ($\gamma(0) = p$) and set the functions as

$$a(t) = \widetilde{a}(\gamma_1(t), \gamma_2(t)), \qquad b(t) = \widetilde{b}(\gamma_1(t), \gamma_2(t)).$$

Then, $\eta_{\gamma(t)} = a(t)\partial_u + b(t)\partial_v$ is the restriction of a null section η_ϕ on $\gamma(t)$. We have the following proposition.

Proposition 2.6. *Let $p \in S$ be a swallowtail-like singular point. We set*

$$\mu(t) = \det\begin{pmatrix} \gamma_1'(t) & a(t) \\ \gamma_2'(t) & b(t) \end{pmatrix}.$$

Under the above notation, it holds that $\mu(0) = 0$ and $\mu'(0) \neq 0$.

Proof. Let (u,v) be a coordinate system of S satisfying $\partial_v = \eta_p$. We assume that $(\eta_\phi\lambda_\phi)_u(p) = 0$. Then, since $S = \{\lambda_\phi = 0\}$, it holds that $(\lambda_\phi)_u = (\lambda_\phi)_v = 0$ at p. Moreover, since $\partial_v = \eta_p$, it holds that $(\eta_\phi\lambda_\phi)_v(p) = (\lambda_\phi)_{vv}(p) = 0$. Thus, $((\lambda_\phi)_u, (\lambda_\phi)_v, (\lambda_\phi)_w)(p)$ and $((\eta_\phi\lambda_\phi)_u, (\eta_\phi\lambda_\phi)_v, (\eta_\phi\lambda_\phi)_w)(p)$ are linearly dependent, where (u,v,w) is a coordinate system on M. This contradicts the condition rank $d(\lambda_\phi, \eta_\phi\lambda_\phi, \eta_\phi^2\lambda_\phi)(p) = 3$. Thus, $(\eta_\phi\lambda_\phi)_u(p) \neq 0$. Since $(\eta_\phi\lambda_\phi)_u(p) \neq 0$, we have a parametrization of γ as $\gamma(t) = (\gamma_1(t), t)$. Since $\eta_\phi^2\lambda_\phi(p) = 0$, we have $\gamma_1'(0) = 0$. On the other hand, we may take $\eta_{\gamma(t)} = a(t)\partial_u + \partial_v$ ($a(0) = 0$). Then, $\mu(t) = \gamma_1'(t) - a(t)$, and thus, $\mu(0) = 0$.

Since $\eta_\phi^2\lambda_\phi(p) = 0$, it holds that $(\lambda_\phi)_{vv}(p) = 0$ by $(\lambda_\phi)_u = (\lambda_\phi)_v = \widetilde{a} = \widetilde{c} = \widetilde{c}_v = \widetilde{b} - 1 = 0$ at p. Then, we see that $\eta_\phi^3\lambda_\phi(p) \neq 0$ is equivalent to

$$(\lambda_\phi)_w(p)\big(\widetilde{a}_v(p)\widetilde{c}_u(p) + \widetilde{c}_{vv}(p)\big) + 3\widetilde{a}_v(p)(\lambda_\phi)_{uv}(p) + (\lambda_\phi)_{vvv}(p) \neq 0. \tag{2.5}$$

On the other hand, since $\eta_\phi(\lambda_\phi)_\phi(\gamma_1(v), v) = 0$, we have

$$\gamma_1''(0) = -\frac{2\widetilde{a}_v(p)(\lambda_\phi)_{uv}(p) + \widetilde{c}_{vv}(p)(\lambda_\phi)_w(p) + (\lambda_\phi)_{vvv}(p)}{\widetilde{c}_u(p)(\lambda_\phi)_w(p) + (\lambda_\phi)_{uv}(p)}. \tag{2.6}$$

By $a'(0) = \widetilde{a}_v(p)$, (2.5) and (2.6), we have

$$-\mu'(0) = -\gamma_1''(0) + a'(0)$$
$$= \frac{\widetilde{a}_v\widetilde{c}_u(\lambda_\phi)_w + 3\widetilde{a}_v(\lambda_\phi)_{uv} + \widetilde{c}_{vv}(\lambda_\phi)_w + (\lambda_\phi)_{vvv}}{\widetilde{c}_u(\lambda_\phi)_w + (\lambda_\phi)_{uv}}(p) \neq 0.$$

□

As in the case of a cusp-like singular point, a swallowtail-like singular point has tangent and transverse types. Let $\{e_1, e_2\}$ be a frame of D_1. If $e_1\lambda_\phi = e_2\lambda_\phi = 0$ at p, then $(D_1)_p = T_pS$. In this case, we call p a *swallowtail-like singular point of tangent type*. If $(e_1\lambda_\phi, e_2\lambda_\phi) \neq (0,0)$ at p, then $(D_1)_p$ is transversal to T_pS. In this case, we call p a *swallowtail-like singular point of transverse type* (Figure 4). Ignoring arrangements of D_1, the relationship of S, S_2 and η_ϕ is similar to that of the Morin singularities of $(\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^3, 0)$ **[9]**.

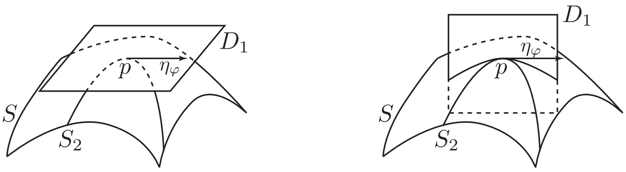

FIGURE 4. S and D_1 of swallowtail-like singular points of tangent and transverse types.

3. Generic singularities. We show that, if $m = 3$ and $r = 2$, then the generic singularities of ϕ are fold-like, transversal cusp-like, tangent cusp-like and transversal swallowtail-like singular points. The bundle homomorphism ϕ can be regarded as a section of the homomorphism bundle $\hom(D_1, D_2)$. We set $E = \hom(D_1, D_2)$. Since the set of sections $\Gamma(E)$ is a subset of $C^\infty(M, E)$, we consider $\Gamma(E)$ as a topological subspace of $C^\infty(M, E)$ with the Whitney C^∞ topology.

Proposition 3.1. *Let M be a smooth 3-manifold, D_1 a* rank 2 *distribution on M and D_2 a* rank 2 *vector bundle on M. Let $\phi : D_1 \to D_2$ be a bundle homomorphism. If ϕ admits only corank one singularities,* *then the set*

$$\{\phi \in \Gamma(E) \mid \textit{any } p \in S \textit{ is fold-like, transversal cusp-like,} \\ \textit{tangent cusp-like or transversal swallowtail-like}\}$$

is dense in $\Gamma(E)$*.*

For the proof of Proposition 3.1, we need the jet transversality theorem for vector bundle sections. Let $J^k(\Gamma(E))$ be the subbundle of $J^k(M,E)$ consisting of all k-jets of sections. Let $j^k\colon M \to J^k(\Gamma(E))$ be the jet-extension.

Proposition 3.2. *Let M be a manifold, and let K be a submanifold of $J^k(\Gamma(E))$. Then, the set*

$$\{f \in \Gamma(E) \mid j^k f \textit{ is transverse to } K\}$$

is residual in $\Gamma(E)$*, and open dense if K is closed.*

This is shown [**12,** Theorem 2.6] for sections of the tangent bundle. However, the proof uses the local triviality of the tangent bundle; thus, the same proof works for the case of interchanging the tangent bundle to a general vector bundle E.

Proof of Proposition 3.1*.* We set

$$Z = \{j^3\phi(p) \in J^3(M,E) \mid \phi(p) = O\}.$$

Then, Z is independent of the choice of frames, a closed submanifold of codimension 4, and $J^3(M,E) \setminus Z$ is an open submanifold. Next, we set

$$D = \{j^3\phi(p) \in J^3(M,E) \mid \lambda(p) = 0, d\lambda(p) = (0,0,0)\},$$

where $\lambda = \lambda_\phi$ is the function defined in (2.2). Then, D is independent of the choice of frames and is a closed submanifold of $J^3(M,E) \setminus Z$ of codimension 4.

Next, we consider

$$W_1 = \{j^3\phi(p) \in J^3(M,E) \mid \lambda(p) = 0,\ \eta\lambda(p) = 0, \\ \eta^2\lambda(p) = 0,\ \eta^3\lambda(p) = 0\},$$

$$W_2 = \{j^3\phi(p) \in J^3(M,E) \mid \lambda(p) = 0,\ \eta\lambda(p) = 0,\\ \eta^2\lambda(p) = 0,\ \operatorname{rank} d(\lambda, \eta\lambda)(p) = 1\},$$

$$W_3 = \{j^3\phi(p) \in J^3(M,E) \mid \lambda(p) = 0,\ \eta\lambda(p) = 0,\\ \eta^2\lambda(p) = 0,\ e\lambda(p) = 0\},$$

where $\{e, \eta\}$ is a local frame of D_1. The sets W_1, W_2, W_3 are independent of the choice of the frames. If we show that W_1, W_2, W_3 are closed submanifolds of $J^3(M,E) \setminus (Z \cup D)$ of codimension 4, then, by Proposition 3.2,

$$\mathcal{O} = \{\phi \in \Gamma(E) \mid j^3\phi \text{ is transverse to } Z,\ D,\ W_1,\ W_2 \text{ and } W_3\}$$

is a residual subset of $\Gamma(E)$, and so, is dense. On the other hand, since $\dim M = 3$, $j^3\phi$ is transverse to Z, D, W_1 and W_2 are equivalent to $j^3\phi(M) \cap (Z \cup D \cup W_1 \cup W_2) = \emptyset$. Thus, any $\phi \in \mathcal{O}$ has only fold-like, transversal cusp-like, tangent cusp-like and transversal swallowtail-like singular points as singular points. Hence, the proof is reduced to showing the next lemma. $\square$

Lemma 3.3. *The set W_1, W_2, W_3 are closed submanifolds of $J^3(M,E) \setminus (Z \cup D)$ of codimension 4.*

Proof. Let $p \in M$, and take a coordinate neighborhood U near p. It is sufficient to show that W_1, W_2, W_3 are closed submanifolds in $J^3(U, E|_U) \setminus (Z \cup D)$. Since W_1, W_2, W_3 are independent of the choice of coordinate systems, we choose a coordinate system (u, v, w) on U satisfying $\partial_v = e$ and $\partial_w = \eta$. Let

$$j^3\phi(p) = \begin{pmatrix} j^3a(p) & j^3b(p) \\ j^3c(p) & j^3d(p) \end{pmatrix},$$

where a, b, c, d are functions. Then, in $J^3(U, E|_U) \setminus (Z \cup D)$,

$$W_1 = \left\{ \begin{pmatrix} j^3a(p) & j^3b(p) \\ j^3c(p) & j^3d(p) \end{pmatrix} \middle|\ p \in U,\ h_1(p) = h_2(p) = h_3(p) = h_4(p) = 0 \right\},$$

$$W_2 = \left\{ \begin{pmatrix} j^3a(p) & j^3b(p) \\ j^3c(p) & j^3d(p) \end{pmatrix} \middle|\ p \in U,\ h_1(p) = h_2(p) = h_3(p) = h_5(p) = 0 \right\},$$

$$W_3 = \left\{ \begin{pmatrix} j^3a(p) & j^3b(p) \\ j^3c(p) & j^3d(p) \end{pmatrix} \,\middle|\, p \in U,\ h_1(p)=h_2(p)=h_3(p)=h_6(p)=0 \right\},$$

where $h_1 = ad - bc$, $h_2 = (ad - bc)_w$, $h_3 = (ad - bc)_{ww}$, $h_4 = (ad - bc)_{www}$,

$$h_5 = (ad - bc)_u (ad - bc)_{vw} - (ad - bc)_v (ad - bc)_{uw}$$

and $h_6 = (ad - bc)_v$. We define three functions

$$H_i : J^3(U, E|_U) \setminus (Z \cup D) \longrightarrow \boldsymbol{R}^4, \quad i = 1, 2, 3,$$

by $H_1 = (h_1, h_2, h_3, h_4)$, $H_2 = (h_1, h_2, h_3, h_5)$, $H_3 = (h_1, h_2, h_3, h_6)$. Then, it is sufficient to show that $(0, 0, 0, 0)$ is a regular value of each H_1, H_2 and H_3. We calculate the derivative of H_1 with respect to the 16 coordinates of $J^3(U, E|_U)$ corresponding to the zero, first, second and third derivatives by ∂_w of a, b, c, d. The matrix representation of them is

$$\left(\begin{array}{c|cccc} \mathcal{M} & & & & \\ \hline * & d & -c & -b & a \end{array} \right),$$

$$\mathcal{M} = \begin{pmatrix} d & -c & -b & a & & & & & & & & \\ * & * & * & * & d & -c & -b & a & & & & \\ * & * & * & * & * & * & * & * & d & -c & -b & a \end{pmatrix},$$

where the blank entries are zero. Since $(a, b, c, d) \neq (0, 0, 0, 0)$, we have the assertion for H_1. We calculate the derivative of H_2 with respect to the 20 coordinates of $J^3(U, E|_U)$ corresponding to the zero, first and second derivatives by ∂_w of a, b, c, d and corresponding to the derivatives by ∂_u, ∂_w, and ∂_v, ∂_w of a, b, c, d. The matrix representation of them is

$$\left(\begin{array}{c|c} \mathcal{M} & \\ \hline * & \mathcal{X} \end{array} \right),$$

$$\mathcal{X} = (d(h_1)_v, -c(h_1)_v, -b(h_1)_v, a(h_1)_v, d(h_1)_u, -c(h_1)_u, -b(h_1)_u, (h_1)_u).$$

Since $(a, b, c, d) \neq (0, 0, 0, 0)$, and $(h_1)_u(p) = (h_1)_v(p) = h_2(p) = 0$ means that $d \det \phi(p) = (0, 0, 0)$, we have the assertion for H_2.

Next, we calculate the derivative of H_3 with respect to the 16 coordinates of $J^3(U, E|_U)$ corresponding to the zero, first and second derivatives by ∂_w of a, b, c, d and corresponding to the derivatives by ∂_v

of a, b, c, d. The matrix representation of them is

$$\left(\begin{array}{c|cccc} \mathcal{M} & & & & \\ \hline * & d & -c & -b & a \end{array}\right).$$

Since $(a, b, c, d) \neq (0, 0, 0, 0)$, we have the assertion for H_3. □

Using the same method, we have the following:

Proposition 3.4. *Let M be a smooth 3-manifold, D_1 a* rank 1 *distribution on M and D_2 a* rank 1 *vector bundle on M. Let $\phi : D_1 \to D_2$ be a bundle homomorphism. If ϕ admits only corank one singularities, then the set*

$$\{\phi \in \Gamma(E) \mid \textit{any } p \in S \textit{ is fold-like, cusp-like or swallowtail-like}\}$$

is dense.

Proposition 3.5. *Let M be a smooth 2-manifold, D_1 a* rank 1 *distribution on M and D_2 a* rank 1 *vector bundle on M. Let $\phi : D_1 \to D_2$ be a bundle homomorphism. If ϕ admits only corank one singularities, then the set*

$$\{\phi \in \Gamma(E) \mid \textit{any } p \in S \textit{ is fold-like or cusp-like}\}$$

is dense.

Propositions 3.4 and 3.5 can be shown by the same method as Proposition 3.1. We show the subsets of the jet spaces by applying Proposition 3.2.

Proof of Proposition 3.4. We set

$$W_1 = \{j^3\phi(p) \in J^3(M, E) \mid \lambda(p) = 0,\ d\lambda(p) = (0, 0, 0)\},$$
$$W_2 = \{j^3\phi(p) \in J^3(M, E) \mid \lambda(p) = e\lambda(p) = e^2\lambda(p) = e^3\lambda(p) = 0\},$$

where e is a frame of D_1. Then, W_1 and W_2 are closed submanifolds of codimension 4. □

Proof of Proposition 3.5. We set

$$W_1 = \{j^2\phi(p) \in J^2(M, E) \mid \lambda(p) = 0,\ d\lambda(p) = (0, 0)\},$$
$$W_2 = \{j^2\phi(p) \in J^2(M, E) \mid \lambda(p) = e\lambda(p) = e^2\lambda(p) = 0\}.$$

Then, W_1 and W_2 are closed submanifolds of codimension 3. □

4. Morin singularities from a manifold with tangent distribution. Let D_1 be a rank r tangent distribution on M, let N be an r-dimensional manifold and $f\colon M \to N$ a map. Setting $D_2 = f^*TN$ and $\phi\colon D_1 \to D_2$ by

$$\phi(v) = df(v),$$

we obtain a bundle homomorphism between D_1 and D_2. We call the above ϕ a *bundle homomorphism induced by* f. In this section, assuming f is a Morin singularity, we consider the relationships of ϕ, D_1 and f in the case of $m = 3$, $r = 2$. Moreover, we assume that M is an open neighborhood of 0 in $\boldsymbol{R}^3$, N is an open neighborhood of 0 in $\boldsymbol{R}^2$ and $f\colon (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^2, 0)$.

4.1. Morin singularities. Here, a brief review is given on the Morin singularities of $(\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^2, 0)$. The map-germ $f\colon (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^2, 0)$ is called a *definite fold* (respectively, an *indefinite fold*) if it is $\mathcal{A}$-equivalent to the map-germ $(u, v, w) \mapsto (u, v^2 + w^2)$ (respectively, $(u, v^2 - w^2)$) at 0. Two map-germs $f, g\colon (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^n, 0)$ are $\mathcal{A}$-*equivalent* if there exist diffeomorphism-germs $\sigma\colon (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ and $\tau\colon (\boldsymbol{R}^n, 0) \to (\boldsymbol{R}^n, 0)$ such that $\tau \circ f \circ \sigma^{-1} = g$. The map-germ $f\colon (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^2, 0)$ is called a *cusp* if it is $\mathcal{A}$-equivalent to the map-germ $(u, v, w) \mapsto (u, v^2 + w^3 + uw)$. Definite folds, indefinite folds and cusps are called Morin singularities, and it is known that generic singularities appearing on maps from a 3-manifold to a 2-manifold are only Morin singularities. A characterization of Morin singularities is given as follows: let $f\colon (\boldsymbol{R}^3, 0) \to (\boldsymbol{R}^2, 0)$ be a map-germ, and let the origin 0 be a rank one singular point of f, namely, $\operatorname{rank} df_0 = 1$. Then, there exists a triple of vector fields $\{\xi, \eta_1, \eta_2\}$ such that

$$\langle \xi(0), \eta_1(0), \eta_2(0)\rangle = T_0\boldsymbol{R}^3, \qquad \langle \eta_1, \eta_2\rangle = \ker df_p, \quad p \in S(f),$$

where $S(f)$ is the set of singular points of f. We set

$$\lambda_1 = \det(\xi f, \eta_1 f),$$

$$\lambda_2 = \det(\xi f, \eta_2 f),$$
$$H = \begin{pmatrix}\eta_1\lambda_1 & \eta_2\lambda_1\end{pmatrix}.$$

Then, f at 0 is a definite fold (respectively, indefinite fold) if and only if $\det H(0) > 0$ (respectively, $\det H(0) < 0$). We assume that $\operatorname{rank} H(0) = 1$. Then, there exists a vector field $\theta = a_1\eta_1 + a_2\eta_2$ on $S(f)$ such that $\langle\theta_0\rangle = \ker H(0)$. Then, f at 0 is a cusp if and only if $\theta H(0) \neq 0$. See [**7**] for details.

4.2. Conditions for singularities. We take a local frame $\{e_1, e_2\}$ of D_1. We regard e_1 and e_2 as vector fields. We consider the conditions of singular points of fold-like, cusp-like and swallowtail-like under the assumption that f is regular, fold and cusp, since these are generic singular points.

When f is regular at 0, and $D_1 \not\supset \ker df_0$, then ϕ is non-singular. When f is singular at 0, and $D_1 \subset \ker df_0$, then ϕ is of rank zero at 0. We assume that $D_1 \cap \ker df_0$ is one-dimensional. By changing the frame, we may assume that $e_1 f(0) \neq 0$. The bundle homomorphism ϕ can be represented by the matrix

$$(e_1 f,\ e_2 f)$$

by using $\{e_1, e_2\}$ and the trivial frame on $\boldsymbol{R}^2$. Since $\operatorname{rank}\phi = 1$ at 0, we take a null section η_ϕ, and set

$$\lambda_\phi = \det(e_1 f, e_2 f) = \det(e_1 f, \eta_\phi f).$$

The following proposition holds.

Proposition 4.1. *We assume that $D_1 \cap \ker df_0$ is one-dimensional. The singular point p of ϕ is a fold-like singular point if and only if $\det(e_1 f, \eta_\phi^2 f) \neq 0$ at p. A non-degenerate singular point p is cusp-like singular point (respectively, swallowtail-like singular point) if and only if $\det(e_1 f, \eta_\phi^2 f) = 0$, and $\det(e_1 f, \eta_\phi^3 f) \neq 0$ at p, respectively, $\det(e_1 f, \eta_\phi^2 f) = \det(e_1 f, \eta_\phi^3 f) = 0$, $\det(e_1 f, \eta_\phi^4 f) \neq 0$, and the map*

$$\big(\det(e_1 f, \eta_\phi f), \det(e_1 f, \eta_\phi^2 f), \det(e_1 f, \eta_\phi^3 f)\big)$$

is a submersion at p.

Proof. Since $\eta_\phi f(p) = 0$, it is obvious that the assertion for the fold-like singular point. Let p be a non-degenerate singular point and $\eta_\phi \lambda_\phi(p) = 0$. Since $\eta_\phi f = 0$ on $S = \{\lambda_\phi = 0\}$, and p is non-degenerate, there exists a vector-valued function g such that $\eta_\phi f = \lambda_\phi g$. Then, by the assumption $\eta_\phi \lambda_\phi(p) = 0$, $\eta_\phi^2 f(p) = 0$. Hence, $\eta_\phi^2 \lambda_\phi(p) \neq 0$ is equivalent to $\det(e_1 f, \eta_\phi^3 f)(p) \neq 0$. This proves the assertion for the cusp-like singular point. Next, we assume that p is a non-degenerate singular point, and $\eta_\phi \lambda_\phi(p) = \eta_\phi^2 \lambda_\phi(p) = 0$. Then, by the same reasoning as above, we have $\eta_\phi^2 f(p) = \eta_\phi^3 f(p) = 0$. Thus, we see that $\eta_\phi^3 \lambda_\phi(p) \neq 0$ is equivalent to $\det(e_1 f, \eta_\phi^4 f)(p) \neq 0$, and $\det(\lambda_\phi, \eta_\phi \lambda_\phi, \eta_\phi^2 \lambda_\phi)(p) \neq 0$ is equivalent to $\det d\big(\det(e_1 f, \eta_\phi f)$, $\det(e_1 f, \eta_\phi^2 f)$ and $\det(e_1 f, \eta_\phi^3 f)\big)(p) \neq 0$. This proves the assertion. $\square$

4.3. Restriction of singularities of ϕ by singular types of f. We are still taking a local frame $\{e_1, e_2\}$ of D_1 and regarding e_1 and e_2 as vector fields. We assume that f at 0 is a definite fold singular point. Then, $\operatorname{rank}(e_1 f, e_2 f, e_3 f) = 1$ on $S(f)$, where $\{e_1, e_2, e_3\}$ is a frame of $T\boldsymbol{R}^3$. Thus, there exist functions k_1, k_2 such that $e_2 f = k_1 e_1 f, e_3 f = k_2 e_1 f$ on $S(f)$. Taking extensions of k_1 and k_2 on $\boldsymbol{R}^3$, we set

$$\eta_2 = -k_1 e_1 + e_2, \qquad \eta_3 = -k_2 e_1 + e_3,$$

and also set

$$\lambda_2 = \det(e_1 f, e_2 f) = \det(e_1 f, \eta_2 f),$$
$$\lambda_3 = \det(e_1 f, e_3 f) = \det(e_1 f, \eta_3 f).$$

Then, we see that η_2 is a null section of ϕ, and λ_2 is the same as λ_ϕ. Since f is a definite fold,

$$H = \det \begin{pmatrix} \eta_2\lambda_2 & \eta_3\lambda_2 \\ \eta_2\lambda_3 & \eta_3\lambda_3 \end{pmatrix} > 0.$$

In particular, $\eta_2\lambda_2 \neq 0$. Thus, ϕ is fold-like at 0 if $\operatorname{rank}\phi(0) = 1$.

Next, we assume that f at 0 is a cusp singular point. Then, we take k_1, k_2, η_2, η_3 and λ_2, λ_3 as above. We assume that ϕ is not fold-like. Since η_2 is a null section of ϕ, and λ_2 is the same as λ_ϕ, it holds that $\eta_2\lambda_2(0) = 0$. Then, since f is cusp,

$$H(0) = \det \begin{pmatrix} \eta_2\lambda_2 & \eta_3\lambda_2 \\ \eta_2\lambda_3 & \eta_3\lambda_3 \end{pmatrix}(0) = 0.$$

Since $\eta_3\lambda_2(0) = \eta_2\lambda_3(0)$, it holds that $\eta_3\lambda_2(0) = 0$. Hence, the kernel of H is $\theta = \eta_2$ at 0. Then, f is cusp if and only if

$$\eta_2^2\lambda_2(0)\ \eta_3\lambda_3(0) \neq 0.$$

Thus, ϕ is non-degenerate and not fold-like at 0, then ϕ is cusp-like at zero.

4.4. The case D_1 is a foliation. In this section, we assume that D_1 is a foliation. By taking a coordinate system (x, y, z) on $\boldsymbol{R}^3$, we may assume $D_1 = \langle e_1, e_2\rangle = \langle \partial_x, \partial_y\rangle$. Let L be the leaf which contains the origin, namely, $f|_L = f(x, y, 0)$. We have the following.

Proposition 4.2. *Let $f : ((\mathbb{R}^3, D_1), 0) \to (\mathbb{R}^2, 0)$ be a map-germ from a trivially foliated manifold with trivial leaf D_1 to a plane around the origin. Let ϕ be an induced bundle homomorphism of f. If ϕ has a corank one singular point at p, and $D_1 \cap \ker df_0$ is one-dimensional, then the following hold*:

(i) *ϕ is fold-like if and only if $f|_L$ is a fold*;

(ii) *if ϕ is non-degenerate, then ϕ is cusp-like if and only if $f|_L$ is a cusp*;

(iii) *if ϕ satisfies that* $\operatorname{rank} d(\lambda_\phi, \eta_\phi\lambda_\phi)(0) = 2$, *then ϕ is swallowtail-like if and only if $f|_L$ is a swallowtail.*

A map-germ $g\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ is called a *fold* if g is $\mathcal{A}$-equivalent to the map-germ $(u, v) \mapsto (u, v^2)$ at 0. Furthermore, a map-germ $g\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ is called a *cusp* (respectively, *swallowtail*) if g is $\mathcal{A}$-equivalent to $(u, v) \mapsto (u, v^3 + uv)$ at 0 (respectively, $(u, v) \mapsto (u, v^4 + uv)$ at 0). Criteria for these singularities are obtained as follows: Let $g\colon (\boldsymbol{R}^2, 0) \to (\boldsymbol{R}^2, 0)$ be a map-germ. We set $\lambda = \det J$, where J is the Jacobian matrix of g. A singular point $p \in S(g)$ is *non-degenerate* if $d\lambda(p) \neq 0$. A *null vector field* is a never-vanishing vector field such that it generates $\ker dg_q$ at $q \in S(g)$. The following fact is known.

Fact 4.3 ([**6, 9, 13**]). *Let $g\colon (\boldsymbol{R}^2, p) \to (\boldsymbol{R}^2, 0)$ be a map-germ and p a non-degenerate singular point. Then, we can take a null vector field η. Let λ be the determinant of the Jacobi matrix of g. Then, g at p is a fold if and only if $\eta\lambda(p) \neq 0$. Moreover, p is a cusp* (*respectively,* *a swallowtail) if and only if* $\eta\lambda(p) = 0$ *and* $\eta^2\lambda(p) \neq 0$ *(respectively,* $\eta\lambda(p) = \eta^2\lambda(p) = 0$ *and* $\eta^3\lambda(p) \neq 0$*).*

Proof of Proposition 4.2. We may assume that $f_x(0,0,0) \neq 0$. Then, there exists a function $k_1(x,y,z)$ such that, if $p \in S$, then

$$f_y(p) = k_1(p) f_x(p).$$

Taking an extension of k_1 on a neighborhood U of p, we obtain a null vector field

$$\eta_\phi = -k_1 e_1 + e_2.$$

On the other hand, there exists a function $l(x,y)$ such that, if $q \in S(f|_L)$, then

$$f_y(q) = l(q) f_x(q).$$

Taking an extension of l on $U \cap \{z = 0\}$, we take a null section of $f|_L$

$$\eta_L = -l e_1 + e_2.$$

Set $\lambda_L(x,y) = \det(e_1 f(x,y,0), \eta_L f(x,y,0))$. Then, since $\lambda_\phi(x,y,0) = \lambda_L(x,y)$, and $\eta_\phi(x,y,0) = \eta_L(x,y)$, we see

$$\eta_\phi^i \lambda(0) = \eta_L^i \lambda_L(0), \quad i = 1,2,3. \tag{4.1}$$

The assertion is obvious by Fact 4.3 and (4.1). □

4.5. The case D_1 is a contact structure. In this section, we assume that D_1 is a contact structure. Let M be a three-dimensional manifold and D a rank 2 distribution. We take a 1-form α such that $D = \alpha^{-1}(0)$ at each point. Then, D is called a *contact structure* if $\alpha \wedge d\alpha$ never vanish. Let $H : M \to \boldsymbol{R}$ be a function. It is known that there is a unique vector field X_H such that the interior product $i_{X_H}\alpha$ is equal to H. This vector field is called the *Hamiltonian vector field* of H. See [**1**], for example.

We denote by X the Hamiltonian vector field of λ_ϕ. Since X is contained in D_1 on S, we consider the relationship with the behavior of X and the singularities of ϕ. We may assume that $D_1 = \langle e_1, e_2 \rangle = \langle \partial_x, \partial_y - x\partial_z \rangle$, without loss of generality. Since ϕ can be expressed by $(f_x, f_y - x f_z)$,

$$\lambda := \lambda_\phi = \det(f_x, f_y - x f_z).$$

The Hamiltonian vector field X of λ_ϕ is

$$X = (\lambda_y - x\lambda_z)\partial_x - \lambda_x\partial_y - (\lambda - x\lambda_x)\partial_z = (\lambda_y - x\lambda_z)e_1 - \lambda_x e_2 - \lambda\partial_z.$$

Since $S = \{\lambda_\phi = 0\}$ holds, $X_p \in D_1$ is equivalent to $p \in S$. We have the following.

Theorem 4.4. *Let $f : ((\mathbb{R}^3, D_1), 0) \to (\mathbb{R}^2, 0)$ be a map-germ from a contact manifold with contact structure D_1 to a plane around the origin. Let ϕ be an induced bundle homomorphism of f. If ϕ has a corank one singular point at p, and $D_1 \cap \ker df_0$ is one-dimensional, then $p \in S$ is fold-like if and only if*

$$X_p \quad \text{and} \quad (\eta_\phi)_p$$

are linearly independent, where η_ϕ is a null section of ϕ.

Proof. Since ϕ is a corank one singular point at p, there exist functions k_1, k_2 on S such that $(k_1, k_2) \neq (0, 0)$ and $k_1 e_1 f + k_2 e_2 f = 0$. Expanding k_1, k_2 to a neighborhood of p, we can take a null section $\eta_\phi = k_1 e_1 + k_2 e_2$. Then,

$$\begin{aligned}\eta_\phi\lambda &= k_1 e_1\lambda + k_2 e_2\lambda = k_1\lambda_x + k_2(\lambda_y - x\lambda_z)\\ &= \det\begin{pmatrix} k_1 & -(\lambda_y - x\lambda_z) \\ k_2 & \lambda_x \end{pmatrix}\end{aligned}$$

shows the assertion. □

By Theorem 4.4, on the set of non-fold-like singular points, X is parallel to the null section. By Propositions 2.5 and 2.6, we have the following.

Corollary 4.5. *If $p \in S$ is a cusp-like singular point, then $X_p \notin T_pS_2$. If $p \in S$ is a swallowtail-like singular point, then*

$$\widetilde{\mu}(0) = 0, \qquad \widetilde{\mu}'(0) \neq 0,$$

where $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ $(\gamma(0) = p)$ is a parametrization of S_2, $\eta_{\gamma(t)} = a(t)\partial_u + b(t)\partial_v$ and

$$\widetilde{\mu}(t) = \begin{pmatrix} \gamma_1'(t) & a(t) \\ \gamma_2'(t) & b(t) \end{pmatrix}.$$

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