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EXISTENCE OF POSITIVE SOLUTION FOR A SEMI POSITONE RADIAL p-LAPLACIAN SYSTEM

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ABSTRACT. In this paper, we prove, for λ and μ large, the existence of a positive solution for the semi-positone elliptic system

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = \lambda\omega(x)f(v) & \text{in } \Omega, \\ -\Delta_q v = \mu\rho(x)g(u) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = B_1(0) = \{x \in \mathbb{R}^N : |x| \leq 1\}$, and, for $m > 1$, Δ_m denotes the m-Laplacian operator $p, q > 1$. The *weight* functions ω, $\rho\colon \overline{\Omega} \to \mathbb{R}$ are radial, continuous, nonnegative and not identically null, and the non-linearities $f, g\colon [0,\infty) \to \mathbb{R}$ are continuous functions such that $f(t)$, $g(t) \geq -\sigma$. The result presented extends, for the radial case, some results in the literature **[9, 10]**. In particular, we do not impose any monotonic condition on f or g. The result is obtained as an application of the Schauder fixed point theorem and the maximum principle.

1. The system studied.

Consider the boundary value problem

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = \lambda\omega(x)f(v) & \text{in } \Omega, \\ -\Delta_q v = \mu\rho(x)g(u) & \text{in } \Omega, \\ (u,v) = (0,0) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\}$, Δ_m, $m > 1$, denotes the m-Laplacian operator $p, q > 1$. The *weight* functions ω, $\rho\colon \overline{\Omega} \to \mathbb{R}$ are continuous, nonnegative and not identically nulls, and the non-linearities $f, g\colon [0,\infty) \to \mathbb{R}$ are continuous functions such that

(H0) there exists a $\sigma > 0$ such that $f(v), g(u) \geq -\sigma$;

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(H1) $\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty$;

(H2) $\displaystyle\lim_{t\to\infty} \frac{f^{1/(p-1)}(Cg(t)^{1/(q-1)})}{t} = 0$ for every $C > 0$.

An example of functions which satisfy (H0), (H1) and (H2) is given by $f(t) = t^\alpha - \sigma$ and $g(t) = t^\beta - \sigma$, where $\sigma > 0$ and $\alpha\beta < (p-1)(q-1)$.

For a given non negative continuous function, and not identically null $h : \Omega \to \mathbb{R}$ and $m > 1$, let $\phi_{m,h}$ be the only solution of

$$\begin{cases} -\Delta_m u = h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $m > 1$. Observe that, by the maximum principle, $\phi_{m,h}(x) > 0$ for all $x \in \Omega$.

With the above hypothesis, we establish, for λ and μ large, the following result.

Theorem 1.1. *Suppose that*

(i) $f, g : [0, +\infty) \to \mathbb{R}$ *are continuous nonlinearities satisfying* (H0), (H1) *and* (H2);
(ii) *the weight functions* w, ρ: $\overline{\Omega} \to \mathbb{R}$ *are radial, continuous, nonnegative and not identically nulls in* Ω.

Then, problem (P) *has at least a nontrivial and positive solution* $(u, v) \in C^{1,\alpha}(\Omega) \times C^{1,\alpha}(\Omega)$, $0 < \alpha < 1$, *for* λ *and* μ *large. In addition,*

$$\lambda^{1/(p-1)} \frac{L^{1/(p-1)}}{2} \|\phi_{p,\omega}\|_\infty \le \|u\|_\infty \le C_\lambda \|\phi_{p,\omega}\|_\infty$$

and

$$\mu^{1/(q-1)} \frac{L^{1/(q-1)}}{2} \phi_{q,\rho} \le \|v\|_\infty \le \mu^{1/(q-1)} \widetilde{g}(C_\lambda \|\phi_{p,\omega}\|_\infty)^{1/(q-1)} \phi_{q,\rho},$$

where C_λ *is a large constant which depends only upon* λ, *and* L *is a constant which depends upon* p, q, ω, ρ *and* Ω.

We do not impose sign conditions in $f(0)$ or $g(0)$, and f and g are not necessarily monotonic. The semi-positone case is considered to be a very challenging problem for partial differential equations (for more details regarding semi-positone problems, see [**2**] and the references therein). The main result is obtained as an application of the Schauder fixed point theorem. For completeness, we present the following theorem [**5,** Proof].

Theorem 1.2 (Schauder fixed point theorem)**.** *Let $T : X \to X$ be a completely continuous operator, where X is a Banach space. If $K \subset X$ is a nonempty convex, bounded, closed set, and $T(K) \subset K$, then T has at least one fixed point in K.*

The main motivation for this paper was the research of Dalmasso [**4**] and Hai and Shivaji [**9**]. In [**4**], problem (P) was considered, where Ω is a bounded and smooth domain, under the assumptions that $p = q = 2$, $\omega = \rho \equiv 1$ and $\lambda = \mu$. The nonlinearities f and g are non negatives, at least continuous if $N = 1$, and locally Holder continuous with exponent $\beta \in (0; 1]$ if $N \geq 2$, as well as non-decreasing functions. The main strategy used was a representation formula via the Green function and the Schauder fixed point theorem. Hai and Shivaji, where $p = q = 2$ [**9**], extended the study of [**4**] to the semi positone case without assuming monotonic conditions on f and g. In [**10**], Hai and Shivaji considered system (P) when $p = q$, $\omega = \rho$ and $\lambda = \mu$, and f and g are also considered to be continuously differentiable and satisfy (H1) in addition to

$$\lim_{x\to+\infty} \frac{f(Mg(x)^{1/(p-1)})}{x^{p-1}} = 0 \quad \text{for every } M > 0. \tag{1.2}$$

The authors have dealt with the semi positone case; however, f and g should be monotonic functions. Their approach was based on the sub and supersolution methods. In [**6**], Hai dealt with system (P) when $p,\ q > 1$, $\omega = \rho = 1$, and the nonlinearities f and g are positives, that is, the semi-positone case is not considered, continuous, and nondecreasing in $[0, +\infty)$, $g(x) > 0$ for $x > 0$, and

$$\limsup_{x\to 0^+} \frac{f^{1/(p-1)}(cg^{1/(q-1)}(x))}{x} = \infty$$
$$\liminf_{x\to\infty} \frac{f^{1/(p-1)}(cg^{1/(q-1)}(x))}{x} = 0.$$

Maximum principle and fixed point arguments were applied to guarantee the existence of the solution when the nonlinearities are possibly singular at 0.

Chhetri, Hai and Shivaji **[3]** proved the existence of a radial solution when $p = q$, $\lambda = \mu$ large and Ω is an annulus. For a general bounded region Ω, a non-existence result was presented for the case where $f(0) < 0$, $g(0) < 0$ and small λ.

The case $\lambda = \mu = 1$, f and g non-negatives, was also considered by Martins and Ferreira **[12]** when f and g have local behavior. The result was obtained when there exist positive constants $0 < \delta < M$ such that

(a) $0 \leq f(v) \leq k_1 M^{p-1}$, $0 \leq g(u) \leq k_1 M^{q-1}$ if $0 \leq u, v \leq M$;
(b) $f(v) \geq k_2 v^{p-1}$ if $0 \leq v \leq \delta$;

where constants k_1 and k_2 depend only upon ω, ρ and Ω. No conditions at ∞ were imposed, but the semi-positone case was not considered.

In this paper, besides considering the semi-positone case, we do not impose any monotonic conditions on f and g. Our strategy is to apply the Schauder fixed point theorem.

2. Existence result proof. In this section, we present the proof of Theorem 1.1 for the radial case:

$$\Omega = B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\},$$

where the weight functions are radials. In this manner, an existence result is proven for the system

$$(\mathrm{P}_B) \qquad \begin{cases} -(r^{N-1}\psi_p(u'(r)))' = \lambda r^{N-1}\omega(r) f(v(r)) & \text{in } B_1(0), \\ -(r^{N-1}\psi_q(v(r)))' = \mu r^{N-1}\rho(r) g(u(r)) & \text{in } B_1(0), \\ (u,v) = (0,0) & \text{on } \partial B_1(0), \end{cases}$$

where $\psi_m(t) = |t|^{m-2}t$.

For $m > 1$, let m' be the conjugate of m, that is, $1/m + 1/m' = 1$. It is easy to see that (u, v) is a pair of radial solutions of (P_B) if, and only if, (u, v) is a fixed point of

$$T : C(B_1, \mathbb{R}) \longrightarrow C(B_1, \mathbb{R}),$$

given by

$$T(u,v) = (T_1(u,v), T_2(u,v)), \tag{2.1}$$

where

$$T_1(u(r),v(r)) = \lambda^{1/(p-1)} \int_r^1 \psi_{p'}\bigg(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f(v(s))\, ds \bigg) d\theta,$$

and

$$T_2\big(u(r),v(r)\big) = \mu^{1/(q-1)} \int_r^1 \psi_{q'}\bigg(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \rho(s) g(u(s))\, ds \bigg) d\theta.$$

It is well known that, in the radial case, the function that solves (1.1) is given by

$$\phi_{m,h}(r) = \int_r^1 \psi_{m'}\bigg(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} h(s)\, ds \bigg) d\theta.$$

Observe that $\phi_{m,h}$ is positive and decreasing for $r \in [0,1]$.

For $m > 1$ and a continuous, non negative and radial function $h : B_1(0) \to \mathbb{R}$, let $\tau_{m,h} \in (0,1)$ be chosen such that

$$\int_{\tau_{m,h}/2}^{\tau_{m,h}} \psi_{m'}\bigg(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} h(s)\, ds \bigg) d\theta > \phi_{m,h}(\tau_{m,h}) > 0. \tag{2.2}$$

We also define

(2.3)

$$L_{m,h} = \left(\frac{2\psi_{m'}(\sigma)\phi_{m,\omega}(\tau_{m,h})}{\int_{\tau_{m,h}/2}^{\tau_{m,h}} \psi_{m'}(\int_0^\theta (s/\theta)^{N-1} h(s)\, ds)\, d\theta - \phi_{m,h}(\tau_{m,h})} \right)^{m-1} > 0,$$

$$L = \max\{L_{p,\omega}, L_{q,\rho}\} \tag{2.4}$$

and

(2.5)

$$(\underline{u}(r), \underline{v}(r)) = \bigg(\lambda^{1/(p-1)} \frac{L^{1/(p-1)}}{2} \phi_{p,\omega}(r),\ \mu^{1/(q-1)} \frac{L^{1/(q-1)}}{2} \phi_{q,\rho}(r) \bigg).$$

The next lemma is required for the proof of Theorem 1.1 in the radial case.

Lemma 2.1. *Let G and H be such that*

$$G(v)(r) = \int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \omega(s) f(v(s))\,ds\right) d\theta - \frac{L^{1/(p-1)}}{2}\int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \omega(s)\,ds\right) d\theta$$

and

$$H(u)(r) = \int_r^1 \psi_{q'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \rho(s) g(u(s))\,ds\right) d\theta - \frac{L^{1/(q-1)}}{2}\int_r^1 \psi_{q'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} \rho(s)\,ds\right) d\theta.$$

Then, there exists a pair $(\lambda^*, \mu^*) > (0,0)$ such that

$$G(u,v)(r), H(u,v)(r) \geq 0,$$

for every $(u,v) \geq (\underline{u}, \underline{v})$ and $(\lambda, \mu) \geq (\lambda^*, \mu^*)$.

Proof. We present the proof for G, although the proof of H may be obtained using the same method. In order to simplify notation, we denote $\tau_{p,\omega}$ by τ, and divide the proof into two cases.

Case (i). $r \in [0, \tau/2)$. In this case, we have

$$\begin{aligned}
G(v)(r) &= \int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f(v(s))\,ds\right) d\theta \\
&\quad - \frac{L^{1/(p-1)}}{2}\int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta \\
&= \int_r^\tau \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f(v(s))\,ds\right) \\
&\quad - \frac{L^{1/(p-1)}}{2}\psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta \\
&\quad + \int_\tau^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f(v(s))\,ds\right) \\
&\quad - \frac{L^{1/(p-1)}}{2}\psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right) d\theta.
\end{aligned} \tag{2.6}$$

Since $\phi_{m,h}$ is a positive and decreasing function, it follows that, for all $r \in [0, \tau/2)$, we have

$$\underline{v}(r) = \mu^{1/(q-1)} \frac{L^{1/(q-1)}}{2} \phi_{q,\rho}(r) \geq \mu^{1/(q-1)} \frac{L^{1/(q-1)}}{2} \phi_{q,\rho}\left(\frac{\tau}{2}\right) > 0,$$

that is,

$$v(r) > \underline{v}(r) \geq \mu^{1/(q-1)} \frac{L^{1/(q-1)}}{2} \phi_{q,\rho}(\tau/2) > 0.$$

By (H1), there exists a $\mu_1 > 0$ large enough such that $\mu > \mu_1$ implies that $f(v(s)) \geq L$ for all $s \in [0, \tau/2)$. Then, in (2.6), we can write

$$\begin{aligned}(2.7) \qquad G(v)(r) \geq \int_r^\tau & L^{1/(p-1)} \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\, ds\right) \\ & - \frac{L^{1/(p-1)}}{2} \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\, ds\right) d\theta \\ & + \int_\tau^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f(v(s))\, ds\right) \\ & - \frac{L^{1/(p-1)}}{2} \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\, ds\right) d\theta.\end{aligned}$$

Thus,

$$\begin{aligned}(2.8) \qquad G(v)(r) \geq & \frac{L^{1/(p-1)}}{2} \int_r^\tau \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\, ds\right) d\theta \\ & + \int_\tau^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f(v(s))\, ds\right) d\theta \\ & - \frac{L^{1/(p-1)}}{2} \phi_{p,\omega}(\tau).\end{aligned}$$

Since $r \in [0, \tau/2)$, we can write

$$\begin{aligned}(2.9) \quad & \frac{L^{1/(p-1)}}{2} \int_r^\tau \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\, ds\right) d\theta \\ & \qquad \geq \frac{L^{1/(p-1)}}{2} \int_{\tau/2}^\tau \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\, ds\right) d\theta.\end{aligned}$$

On the other hand, by (H0), it follows that

$$\int_\tau^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)f(v(s))\,ds\right)d\theta \geq -\psi_{p'}(\sigma)\int_\tau^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right)d\theta. \tag{2.10}$$

Applying (2.9) and (2.10) to (2.8), we have

$$\begin{aligned} G(v)(r) \geq \frac{L^{1/(p-1)}}{2}\int_{\tau/2}^\tau \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right)d\theta \\ - \psi_{p'}(\sigma)\int_\tau^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right)d\theta \\ - \frac{L^{1/(p-1)}}{2}\phi_{p,\omega}(\tau); \end{aligned}$$

this allows us to write

$$\begin{aligned} G(v)(r) \geq \frac{L^{1/(p-1)}}{2}\left(\int_{\tau/2}^\tau \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\,ds\right)d\theta - \phi_{p,\omega}(\tau)\right) \\ - \psi_{p'}(\sigma)\phi_{p,\omega}(\tau) = 0. \end{aligned}$$

Case (ii). $r \in [\tau/2, 1]$. Since

$$\begin{aligned} \int_r^1 \psi_{p'}\left(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)f(v(s))\,ds\right)d\theta \\ = \int_r^1 \theta^{(1-N)/(p-1)}\psi_{p'}\left(\int_0^\theta s^{N-1}\omega(s)f(v(s))\,ds\right)d\theta, \end{aligned}$$

we can rewrite G as

$$\begin{aligned} G(v)(r) = \int_r^1 \theta^{(1-N)/(p-1)}\psi_{p'}\left(\int_0^\theta s^{N-1}\omega(s)f(v(s))\,ds\right)d\theta \\ - \frac{L^{1/(p-1)}}{2}\phi_{p,\omega}(r) \end{aligned}$$

or

(2.11)

$$G(v)(r) = \int_r^1 \theta^{(1-N)/(p-1)}\psi_{p'} \cdot \left(\left(\int_0^{\tau/2} s^{N-1}\omega(s)f(v(s))\,ds + \int_{\tau/2}^{\theta} s^{N-1}\omega(s)f(v(s))\,ds\right)\right) d\theta - \frac{L^{1/(p-1)}}{2}\phi_{p,\omega}(r).$$

Define

$$C := \frac{\sigma \int_{\tau/2}^1 s^{N-1}\omega(s)\,ds + (L/2^{p-1}) \int_0^1 s^{N-1}\omega(s)\,ds}{\int_0^{\tau/2} s^{N-1}\omega(s)\,ds} > 0. \tag{2.12}$$

As in Case (i), by (H1), there exists a $\mu_2 > 0$ such that $f(v(s)) \geq C$ for every $\mu > \mu_2$ and $s \in [0, \tau/2]$. Then,

$$\int_0^{\tau/2} s^{N-1}\omega(s)f(v(s))\,ds \geq C \int_0^{\tau/2} s^{N-1}\omega(s)\,ds. \tag{2.13}$$

By (H0), (2.11) and (2.13), we have

$$G(v)(r) \geq \int_r^1 \Bigg(\theta^{(1-N)/(p-1)}\psi_{p'} \cdot \left(\int_0^{\tau/2} s^{N-1}\omega(s)C\,ds - \sigma \int_{\tau/2}^{\theta} s^{N-1}\omega(s)\,ds\right)\Bigg) d\theta - \frac{L^{1/(p-1)}}{2}\phi_{p,\omega}(r) \tag{2.14}$$

On the other hand, since

$$\int_{\tau/2}^{\theta} s^{N-1}\omega(s)\,ds \leq \int_{\tau/2}^{1} s^{N-1}\omega(s)\,ds$$

and

$$\phi_{p,\omega}(r) = \int_r^1 \theta^{(1-N)/(p-1)}\psi_{p'}\left(\int_0^{\tau/2} s^{N-1}\omega(s)\,ds + \int_{\tau/2}^{\theta} s^{N-1}\omega(s)\,ds\right) d\theta,$$

we have

$$\phi_{p,\omega}(r) \leq \int_r^1 \theta^{(1-N)/(p-1)} \psi_{p'}\left(\int_0^1 s^{N-1}\omega(s)\, ds \right) d\theta \tag{2.15}$$

Then, using (2.11), (2.15) and (H0), it follows that

$$\begin{aligned} G(v)(r) \geq & \int_r^1 \theta^{(1-N)/(p-1)} \psi_{p'}\left(\left(\int_0^{\tau/2} s^{N-1}\omega(s) C\, ds - \sigma \int_{\tau/2}^1 s^{N-1}\omega(s)\, ds \right) \right) d\theta \\ & - \frac{L^{1/(p-1)}}{2} \int_r^1 \theta^{(1-N)/(p-1)} \psi_{p'}\left(\int_0^1 s^{N-1}\omega(s)\, ds \right) d\theta. \end{aligned}$$

Then, from (2.12), we have $G(v)(r) \geq 0$ for all $r \in [\tau/2, 1]$. □

Proof of Theorem 1.2. According to Lemma 2.1, it follows that

$$(T_1(u,v)(r), T_2(u,v)(r)) \geq (\underline{u}(r), \underline{v}(r)),$$

for every $r \in [0,1]$ and $(\lambda, \mu) \geq (\lambda^*, \mu^*)$. Define $\widetilde{g}(s) = \sup_{t \leq s} g(t)$, and let $(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v})$ be

$$(\overline{u}, \overline{v})(r) = (C_\lambda \phi_{p,\omega}(r), \mu^{1/(q-1)} \widetilde{g}(C_\lambda \|\phi_{p,\omega}\|_\infty)^{1/(q-1)} \phi_{q,\rho}(r)), \tag{2.16}$$

where C_λ is a constant to be chosen.

We claim that, if $(u,v) \leq (\overline{u}, \overline{v})$, then $T(u,v) \leq (\overline{u}, \overline{v})$. In fact, since $\widetilde{g}$ is an increasing function and $u \leq \overline{u}$, we have $\widetilde{g}(u) \leq \widetilde{g}(\overline{u}) = \widetilde{g}(C_\lambda \phi_{p,\omega})$. Thus,

$$\begin{aligned} T_2(u,v)(r) &= \mu^{1/(q-1)} \int_r^1 \psi_{q'}\left(\int_0^\theta \left(\frac{s}{\theta} \right)^{N-1} \rho(s) g(u(s))\, ds \right) d\theta \\ &\leq \mu^{1/(q-1)} \int_r^1 \psi_{q'}\left(\int_0^\theta \left(\frac{s}{\theta} \right)^{N-1} \rho(s) \widetilde{g}(u(s))\, ds \right) d\theta \\ &\leq \mu^{1/(q-1)} \int_r^1 \psi_{q'}\left(\int_0^\theta \left(\frac{s}{\theta} \right)^{N-1} \rho(s) \widetilde{g}(C_\lambda \phi_{p,\omega})\, ds \right) d\theta \\ &\leq \mu^{1/(q-1)} \widetilde{g}(C_\lambda \|\phi_{p,\omega}\|_\infty)^{1/(q-1)} \int_r^1 \psi_{q'}\left(\int_0^\theta \left(\frac{s}{\theta} \right)^{N-1} \rho(s)\, ds \right) d\theta \\ &= \mu^{1/(q-1)} \widetilde{g}(C_\lambda \|\phi_{p,\omega}\|_\infty)^{1/(q-1)} \phi_{q,\rho}(r) = \overline{v}(r). \end{aligned}$$

On the other hand, $v(s) \le \overline{v}(s) = \mu^{1/(q-1)}\widetilde{g}(C_\lambda \|\phi_{p,\omega}\|_\infty/)^{1/(q-1)}\phi_{q,\rho}(r)$ implies that

$$\widetilde{f}(v(s)) \le \widetilde{f}(\mu^{1/(q-1)}\widetilde{g}(C_\lambda \|\phi_{p,\omega}\|_\infty)^{1/(q-1)}\phi_{q,\rho}(r)).$$

Thus,

$$\begin{aligned}
&T_1(u,v)(r)\\
&\quad = \lambda^{1/(p-1)} \int_r^1 \psi_{p'}\bigg(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) f(v(s))\, ds\bigg)\, d\theta\\
&\quad \le \lambda^{1/(p-1)} \int_r^1 \psi_{p'}\bigg(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s) \widetilde{f}(v(s))\, ds\bigg)\, d\theta\\
&\quad \le \lambda^{1/(p-1)} \int_r^1 \psi_{p'}\bigg(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\\
&\qquad\qquad \cdot \widetilde{f}\bigg(\mu^{1/(q-1)}\widetilde{g}\bigg(C_\lambda \|\phi_{p,\omega}\|_\infty\bigg)^{1/(q-1)}\phi_{q,\rho}\bigg)\, ds\bigg)\, d\theta\\
&\quad \le \lambda^{1/(p-1)} \widetilde{f}^{1/(p-1)}(\mu^{1/(q-1)}\|\phi_{q,\rho}\|_\infty \widetilde{g}(C_\lambda \|\phi_{p,\omega}\|_\infty)^{1/(q-1)})\\
&\qquad \cdot \int_r^1 \psi_{p'}\bigg(\int_0^\theta \left(\frac{s}{\theta}\right)^{N-1} w(s)\, ds\bigg)\, d\theta\\
&\quad \le \lambda^{1/(p-1)} \widetilde{f}^{\,1/(p-1)}(\mu^{1/(q-1)}\|\phi_{q,\rho}\|_\infty \widetilde{g}(C_\lambda \|\phi_{p,\omega}\|_\infty)^{1/(q-1)})\phi_{p,\omega}(r).
\end{aligned}$$

According to (H2), if C_λ is large enough, it is possible to obtain $T_1(u,v) \le C_\lambda \phi_{p,\omega}(r)$. Then, $[(\underline{u},\underline{v});(\overline{u},\overline{v})]$ is invariant by T. Since this set is bounded, closed, convex, and T is completely continuous, it follows that T has a fixed point which is a solution of (P). □

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