

EXISTENCE AND ROUGHNESS OF NONUNIFORM (h, k, μ, ν) -TRICHOTOMY FOR NONAUTONOMOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. The objective of this paper is to explore the existence and roughness of the nonuniform (h, k, μ, ν) -trichotomy for nonautonomous differential equations. We first propose a more general notion of trichotomies called the nonuniform (h, k, μ, ν) -trichotomy for linear nonautonomous differential equations. Then, we give a complete characterization of the notion of nonuniform (h, k, μ, ν) -trichotomy for linear nonautonomous differential equations and prove that any linear nonautonomous differential equation admits a nonuniform (h, k, μ, ν) -trichotomy if it has an (H, K, L) Lyapunov exponent with different signs in a finite-dimensional space. Finally, we establish the roughness of nonuniform (h, k, μ, ν) -trichotomies in a very concise manner, which implies that the nonuniform (h, k, μ, ν) -trichotomy persists under sufficiently small linear perturbations. This study exhibits some new interesting findings in trichotomy that extend the corresponding results for uniform and nonuniform trichotomies.

1. Introduction. Trichotomies describe stable, unstable, and neutral subspaces of solutions of linear nonautonomous differential equations and have been proved to be powerful in characterizing the dynamics of nonautonomous differential equations. In 1976, Sacker and Sell [18] first introduced the notion of exponential trichotomies and established the existence of exponential trichotomies for linear time-varying ordinary differential equations. In this context, different forms of trichotomy were established and discussed, such as, stronger exponential trichotomy [6, 7], generalized ℓ -exponential trichotomy [13],

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nonuniform exponential trichotomy [2], ρ -nonuniform exponential trichotomy [3], h -trichotomy [17], (h, k) -trichotomy [10, 12, 15], (μ, ν) -trichotomy [9] and (a, b, c) -trichotomy [11]. In view of the above notions of different trichotomies, an interesting problem is whether there is a more general trichotomy to unify the existing notions of trichotomies and characterize more trichotomous behaviors of linear nonautonomous differential equations.

Our work is concerned with a new trichotomous behavior of the following linear nonautonomous differential equation

$$(1.1) \quad \dot{x} = A(t)x$$

in a Banach space \mathcal{X} . Here, we propose a more general notion of trichotomy for (1.1), called the nonuniform (h, k, μ, ν) -trichotomy. This new notion, mainly motivated by the nonuniform (h, k, μ, ν) -dichotomy established in [24], not only unifies and extends the existing notions of trichotomies in the literature, but also characterizes more reasonable and general trichotomous behaviors of (1.1).

An interesting topic of the present paper is to establish the existence of nonuniform (h, k, μ, ν) -trichotomy for (1.1). In the development process of trichotomies, the existence of trichotomies is always a fundamental problem and has been extensively studied in different ways [3, 4, 6, 8, 14, 19, 20, 21, 22, 23, 25]. The approach adopted here is based on the construction of suitable Lyapunov exponents for (1.1). This study reveals that the nonuniform (h, k, μ, ν) -trichotomy widely exists and arises naturally in linear nonautonomous differential equations. In addition, roughness, one of the most important properties of trichotomies, describes the persistence of trichotomies under linear perturbations and plays an important role in the study of trichotomy theory and dynamical systems [1, 2, 3, 5, 6, 9, 16]. The roughness of the nonuniform (h, k, μ, ν) -trichotomy is also well explored.

This paper is organized as follows. In Section 2, a new notion called nonuniform (h, k, μ, ν) -trichotomy is formulated for (1.1), and the main findings on the existence and roughness of nonuniform (h, k, μ, ν) -trichotomy are presented. In Section 3, detailed proofs are provided for the main conclusions.

2. Nonuniform (h, k, μ, ν) -trichotomy and main results. Let $\mathcal{B}(\mathcal{X})$ be the space of bounded linear operators on a Banach space \mathcal{X}

and $A : \mathbb{R}^+ \rightarrow \mathcal{B}(\mathcal{X})$ in (1.1). Denote by $T(t, s)$ the evolution operator associated with (1.1) satisfying $T(t, s)x(s) = x(t)$ and $T(t, \tau)T(\tau, s) = T(t, s)$ for any $t, s, \tau \in \mathbb{R}^+$.

2.1. Definition of nonuniform (h, k, μ, ν) -trichotomy. An increasing function $u : \mathbb{R}^+ \rightarrow [1, +\infty)$ is said to be a growth rate if u satisfies $u(0) = 1$ and $\lim_{t \rightarrow +\infty} u(t) = +\infty$. Let Δ be the set of all growth rate functions and $h_i(t), k_i(t), \mu_i(t), \nu_i(t) \in \Delta$, $i = 1, 2$, throughout this paper.

Definition 2.1. Equation (1.1) is said to admit a nonuniform (h, k, μ, ν) -trichotomy in \mathbb{R}^+ if

(i) there exist projections $P(t), Q(t), R(t) : \mathcal{X} \rightarrow \mathcal{X}$ such that

$$T(t, s)P(s) = P(t)T(t, s),$$

$$T(t, s)Q(s) = Q(t)T(t, s),$$

$$T(t, s)R(s) = R(t)T(t, s)$$

for any $t, s \in \mathbb{R}^+$, where $P(t) + Q(t) + R(t) = Id$;

(ii) there exist constants $a \leq 0 < b$, $c \leq 0 < d$, $\epsilon \geq 0$, and $K_1 \geq 0$ such that, for any $t \geq s \geq 0$,

$$(2.1) \quad \begin{aligned} |T(t, s)P(s)| &\leq K_1 \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s), \\ |T(t, s)R(s)| &\leq K_1 \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s), \end{aligned}$$

and, for any $0 \leq t \leq s$,

$$(2.2) \quad \begin{aligned} |T(t, s)Q(s)| &\leq K_1 \left(\frac{k_1(s)}{k_1(t)} \right)^c \nu_1^\epsilon(s), \\ |T(t, s)R(s)| &\leq K_1 \left(\frac{k_2(s)}{k_2(t)} \right)^d \nu_2^\epsilon(s). \end{aligned}$$

By Definition 2.1, it can be verified that the nonuniform (h, k, μ, ν) -trichotomy is much more general and includes the existing uniform or nonuniform trichotomy as special cases: uniform exponential trichotomy, stronger exponential trichotomy [6, 7], generalized ℓ -expo-

nential trichotomy [13], nonuniform exponential trichotomy [2], h -trichotomy [17], (h, k) -trichotomy [10, 12, 15], (μ, ν) -trichotomy [9] and (a, b, c) -trichotomy [11].

The following example shows the generality of the nonuniform (h, k, μ, ν) -trichotomy.

Example 2.2. Let $\epsilon_1 > 0, \alpha > 0, \beta > 0$ and $\widehat{h}_i, \widehat{k}_i, \widehat{\mu}_i, \widehat{\nu}_i, i = 1, 2$, be differentiable growth rates. Consider the following differentiable equations in \mathbb{R}^3

$$\begin{aligned}
 \dot{x}(t) &= \left(-\frac{\alpha \widehat{h}'_1(t)}{\widehat{h}_1(t)} + \frac{\epsilon_1 \widehat{\mu}'_1(t)}{2\widehat{\mu}_1(t)}(\cos t - 1) - \frac{\epsilon_1}{2} \log \widehat{\mu}_1(t) \sin t \right) x, \\
 (2.3) \quad \dot{y}(t) &= 0, \\
 \dot{z}(t) &= \left(\frac{\beta \widehat{k}'_1(t)}{\widehat{k}_1(t)} - \frac{\epsilon_1 \widehat{\nu}'_1(t)}{2\widehat{\nu}_1(t)}(\cos t - 1) + \frac{\epsilon_1}{2} \log \widehat{\nu}_1(t) \sin t \right) z.
 \end{aligned}$$

Let

$$\begin{aligned}
 P(t)(x, y, z)^T &= (x, 0, 0)^T, \\
 Q(t)(x, y, z)^T &= (0, 0, z)^T, \\
 R(t)(x, y, z)^T &= (0, y, 0)^T
 \end{aligned}$$

for $t \geq 0$. Direct calculation shows that

$$\begin{aligned}
 T(t, s)P(s) &= (\widehat{X}(t, s), 0, 0)^T, \\
 T(t, s)Q(s) &= (0, 0, \widehat{Z}(t, s))^T, \\
 T(t, s)R(s) &= (0, \widehat{Y}(t, s), 0)^T,
 \end{aligned}$$

where

$$\begin{aligned}
 \widehat{X}(t, s) &= \left(\frac{\widehat{h}_1(t)}{\widehat{h}_1(s)} \right)^{-\alpha} \exp \left(\frac{\epsilon_1}{2} \log \widehat{\mu}_1(t)(\cos t - 1) - \frac{\epsilon_1}{2} \log \widehat{\mu}_1(s)(\cos s - 1) \right), \\
 \widehat{Y}(t, s) &= 1, \\
 \widehat{Z}(t, s) &= \left(\frac{\widehat{k}_1(t)}{\widehat{k}_1(s)} \right)^\beta \exp \left(-\frac{\epsilon_1}{2} \log \widehat{\nu}_1(t)(\cos t - 1) + \frac{\epsilon_1}{2} \log \widehat{\nu}_1(s)(\cos s - 1) \right).
 \end{aligned}$$

Then,

$$\begin{aligned}
 |T(t, s)P(s)| &= |\widehat{X}(t, s)| \leq \left(\frac{\widehat{h}_1(t)}{\widehat{h}_1(s)}\right)^{-\alpha} \widehat{\mu}_1^{\epsilon_1}(s), & t \geq s \geq 0, \\
 |T(t, s)R(s)| &= |\widehat{Y}(t, s)| = 1 < \left(\frac{\widehat{h}_2(t)}{\widehat{h}_2(s)}\right)^\alpha \widehat{\mu}_2^{\epsilon_2}(s), & t \geq s \geq 0, \\
 |T(t, s)Q(s)| &= |\widehat{Z}(t, s)| \leq \left(\frac{\widehat{k}_1(s)}{\widehat{k}_1(t)}\right)^{-\beta} \widehat{\nu}_1^{\epsilon_1}(s), & s \geq t \geq 0, \\
 |T(t, s)R(s)| &= |\widehat{Y}(t, s)| = 1 < \left(\frac{\widehat{k}_2(s)}{\widehat{k}_2(t)}\right)^\beta \widehat{\nu}_2^{\epsilon_2}(s), & s \geq t \geq 0.
 \end{aligned}$$

This shows that (2.3) admits a nonuniform (h, k, μ, ν) -trichotomy in \mathbb{R}^+ .

2.2. Existence of nonuniform (h, k, μ, ν) -trichotomy. In this subsection, for the linear nonautonomous differential equation (1.1) in block form, we formulate a simple criterion to characterize the existence of nonuniform (h, k, μ, ν) -trichotomy in a finite-dimensional space.

In (1.1), assume that $A(t)$ has the block form

$$(2.4) \quad A(t) = \begin{pmatrix} A_1(t) & 0 & 0 \\ 0 & A_2(t) & 0 \\ 0 & 0 & A_3(t) \end{pmatrix}$$

for a decomposition $\mathbb{R}^n = E \oplus F \oplus G$, where $\dim E = l$, $\dim F = m$, $\dim G = n - l - m$. For $t \geq 0$, equation (1.1) transforms into

$$(2.5) \quad x'_1 = A_1(t)x_1, \quad y'_1 = A_2(t)y_1, \quad z'_1 = A_3(t)z_1,$$

and the corresponding adjoint equations are

$$(2.6) \quad x'_2 = -A_1(t)^*x_2, \quad y'_2 = -A_2(t)^*y_2, \quad z'_2 = -A_3(t)^*z_2,$$

where $A_1(t)^*$, $A_2(t)^*$ and $A_3(t)^*$ are the transpose of $A_1(t)$, $A_2(t)$ and $A_3(t)$, respectively.

Assume that $\log 0 = -\infty$ and $H, K, L, \overline{H}, \overline{K}, \overline{L} \in \Delta$. Define $\varphi, \overline{\varphi} : E \rightarrow \mathbb{R} \cup \{-\infty\}$, $\psi, \overline{\psi} : F \rightarrow \mathbb{R} \cup \{-\infty\}$, $\phi, \overline{\phi} : G \rightarrow \mathbb{R} \cup \{-\infty\}$ by

$$\begin{aligned}
 (2.7) \quad & \varphi(x_1^0) = \limsup_{t \rightarrow +\infty} \frac{\log |x_1(t)|}{\log H(t)}, & \bar{\varphi}(x_2^0) &= \limsup_{t \rightarrow +\infty} \frac{\log |x_2(t)|}{\log \bar{H}(t)}, \\
 & \psi(y_1^0) = \limsup_{t \rightarrow +\infty} \frac{\log |y_1(t)|}{\log K(t)}, & \bar{\psi}(y_2^0) &= \limsup_{t \rightarrow +\infty} \frac{\log |y_2(t)|}{\log \bar{K}(t)}, \\
 & \phi(z_1^0) = \limsup_{t \rightarrow +\infty} \frac{\log |z_1(t)|}{\log L(t)}, & \bar{\phi}(z_2^0) &= \limsup_{t \rightarrow +\infty} \frac{\log |z_2(t)|}{\log \bar{L}(t)},
 \end{aligned}$$

where $(x_1(t), y_1(t), z_1(t))$ is a solution of (2.5) with $(x_1(0), y_1(0), z_1(0)) = (x_1^0, y_1^0, z_1^0)$, and $(x_2(t), y_2(t), z_2(t))$ is a solution of (2.6) with $(x_2(0), y_2(0), z_2(0)) = (x_2^0, y_2^0, z_2^0)$. Then, it is not difficult to conclude that

- (i) $\varphi(0) = \bar{\varphi}(0) = \psi(0) = \bar{\psi}(0) = \phi(0) = \bar{\phi}(0) = -\infty$;
- (ii) $\varphi(\tilde{c}x) = \varphi(x)$, $\bar{\varphi}(\tilde{c}x) = \bar{\varphi}(x)$, $\psi(\tilde{c}y) = \psi(y)$, $\bar{\psi}(\tilde{c}y) = \bar{\psi}(y)$, $\phi(\tilde{c}z) = \phi(z)$, $\bar{\phi}(\tilde{c}z) = \bar{\phi}(z)$ for $x \in E$, $y \in F$, $z \in G$ and $\tilde{c} \in \mathbb{R} \setminus \{0\}$;
- (iii) for $x', x'' \in E$, $y', y'' \in F$, $z', z'' \in G$, we have

$$\begin{aligned}
 & \varphi(x' + x'') \leq \max\{\varphi(x'), \varphi(x'')\}, \\
 & \bar{\varphi}(x' + x'') \leq \max\{\bar{\varphi}(x'), \bar{\varphi}(x'')\}, \\
 & \psi(y' + y'') \leq \max\{\psi(y'), \psi(y'')\}, \\
 & \bar{\psi}(y' + y'') \leq \max\{\bar{\psi}(y'), \bar{\psi}(y'')\}, \\
 & \phi(z' + z'') \leq \max\{\phi(z'), \phi(z'')\}, \\
 & \bar{\phi}(z' + z'') \leq \max\{\bar{\phi}(z'), \bar{\phi}(z'')\};
 \end{aligned}$$

(iv) if $\varphi(x^1), \dots, \varphi(x^{l_1})$ or $\bar{\varphi}(x^1), \dots, \bar{\varphi}(x^{l_1})$ are distinct for $x^1, \dots, x^{l_1} \in E \setminus \{0\}$, then x^1, \dots, x^{l_1} are linearly independent; if $\psi(y^1), \dots, \psi(y^{m_1})$ or $\bar{\psi}(y^1), \dots, \bar{\psi}(y^{m_1})$ are distinct for $y^1, \dots, y^{m_1} \in F \setminus \{0\}$, then y^1, \dots, y^{m_1} are linearly independent; if $\phi(z^1), \dots, \phi(z^{\kappa_1})$ or $\bar{\phi}(z^1), \dots, \bar{\phi}(z^{\kappa_1})$ are distinct for $z^1, \dots, z^{\kappa_1} \in G \setminus \{0\}$, then z^1, \dots, z^{κ_1} are linearly independent;

(v) $\varphi(\bar{\varphi})$ has at most $r_1 \leq l$ ($\bar{r}_1 \leq l$) distinct values in $E \setminus \{0\}$, say, $-\infty \leq \lambda_1 < \dots < \lambda_{r_1} \leq +\infty$ ($-\infty \leq \bar{\lambda}_{r_1} < \dots < \bar{\lambda}_1 \leq +\infty$); $\psi(\bar{\psi})$ has at most $r_2 \leq m$ ($\bar{r}_2 \leq m$) distinct values in $F \setminus \{0\}$, say, $-\infty \leq \chi_1 < \dots < \chi_{r_2} \leq +\infty$ ($-\infty \leq \bar{\chi}_{r_2} < \dots < \bar{\chi}_1 \leq +\infty$); $\phi(\bar{\phi})$ has at most $r_3 \leq n - l - m$ ($\bar{r}_3 \leq n - l - m$) distinct values in $G \setminus \{0\}$, say, $-\infty \leq \iota_1 < \dots < \iota_{r_3} \leq +\infty$ ($-\infty \leq \bar{\iota}_{r_3} < \dots < \bar{\iota}_1 \leq +\infty$).

If (i)–(iii) hold, then (φ, ψ, ϕ) (or $(\bar{\varphi}, \bar{\psi}, \bar{\phi})$) is a Lyapunov exponent with respect to (2.5) (or (2.6)). For convenience, this Lyapunov exponent is called the (H, K, L) (or $(\bar{H}, \bar{K}, \bar{L})$) Lyapunov exponent.

Let $\varrho_1, \dots, \varrho_n, \zeta_1, \dots, \zeta_n$ and ξ_1, \dots, ξ_n be three bases of \mathbb{R}^n and (\cdot, \cdot) the standard inner product in \mathbb{R}^n . The above three bases are said to be dual if $(\varrho_i, \zeta_j) = \omega_{ij}, (\varrho_i, \xi_j) = \omega_{ij}, (\zeta_i, \xi_j) = \omega_{ij}$ for every i, j , where ω_{ij} is the Kronecker symbol. The regularity coefficient of φ and $\bar{\varphi}$ is defined by

$$(2.8) \quad \gamma_1(\varphi, \bar{\varphi}) = \min \max \{ \varphi(\delta_i^{(1)}) + \bar{\varphi}(\bar{\delta}_i^{(1)}) : 1 \leq i \leq l \},$$

where the minimum is taken over all dual bases $\delta_1^{(1)}, \dots, \delta_l^{(1)}$ and $\bar{\delta}_1^{(1)}, \dots, \bar{\delta}_l^{(1)}$ of E . The regularity coefficient of ψ and $\bar{\psi}$ is defined by

$$(2.9) \quad \gamma_2(\psi, \bar{\psi}) = \min \max \{ \psi(\delta_i^{(2)}) + \bar{\psi}(\bar{\delta}_i^{(2)}) : 1 \leq i \leq m \},$$

where the minimum is taken over all dual bases $\delta_1^{(2)}, \dots, \delta_m^{(2)}$ and $\bar{\delta}_1^{(2)}, \dots, \bar{\delta}_m^{(2)}$ of F . The regularity coefficient of ϕ and $\bar{\phi}$ is defined by

$$(2.10) \quad \gamma_3(\phi, \bar{\phi}) = \min \max \{ \phi(\delta_i^{(3)}) + \bar{\phi}(\bar{\delta}_i^{(3)}) : 1 \leq i \leq n - l - m \},$$

where the minimum is taken over all dual bases $\delta_1^{(3)}, \dots, \delta_{n-l-m}^{(3)}$ and $\bar{\delta}_1^{(3)}, \dots, \bar{\delta}_{n-l-m}^{(3)}$ of G . Note that $\varphi, \bar{\varphi}, \psi, \bar{\psi}, \phi, \bar{\phi}$ takes only a finite number of values, which means that (2.8), (2.9) and (2.10) are well defined, and $\gamma_1(\varphi, \bar{\varphi}) \geq 0, \gamma_2(\psi, \bar{\psi}) \geq 0, \gamma_3(\phi, \bar{\phi}) \geq 0$.

Theorem 2.3. *Assume that $A(t)$ has the block form (2.4) for $t \geq 0$. If*

$$\lambda_{r_1} < 0, \quad \chi_1 < 0, \quad \chi_{r_2} > 0, \quad \iota_1 > 0,$$

then, for any sufficiently small $\tilde{\epsilon} > 0$, (1.1) admits a nonuniform (h, k, μ, ν) -trichotomy on \mathbb{R}^+ with

$$a = \lambda_{r_1} + \tilde{\epsilon}, \quad b = \chi_{r_2} + \tilde{\epsilon}, \quad c = -(\iota_1 + \tilde{\epsilon}), \quad d = -(\chi_1 + \tilde{\epsilon}),$$

$$\epsilon = \max \{ \gamma_1(\varphi, \bar{\varphi}), \gamma_2(\psi, \bar{\psi}), \gamma_3(\phi, \bar{\phi}) \} + \tilde{\epsilon},$$

$$h_1(t) = H(t), \quad h_2(t) = K(t), \quad k_1(t) = L(t), \quad k_2(t) = K(t),$$

$$\mu_1(t) = H(t)\bar{H}(t), \quad \mu_2(t) = K(t)\bar{K}(t),$$

$$\nu_1(t) = L(t)\bar{L}(t), \quad \nu_2(t) = K(t)\bar{K}(t).$$

Theorem 2.3 suggests that any linear nonautonomous differential equation admits a nonuniform (h, k, μ, ν) -trichotomy if it has at least two negative (H, K, L) Lyapunov exponents. This fact reveals that the nonuniform (h, k, μ, ν) -trichotomy widely exists and arises naturally in linear nonautonomous differential equations.

2.3. Roughness of nonuniform (h, k, μ, ν) -trichotomy. Consider the linear perturbed equation

$$(2.11) \quad \dot{x} = (A(t) + B(t))x.$$

The principle aim of this section is to investigate the roughness of nonuniform (h, k, μ, ν) -trichotomy and to prove that, if (1.1) admits a nonuniform (h, k, μ, ν) -trichotomy, then (2.11) also has a similar nonuniform (h, k, μ, ν) -trichotomy.

Let $\widehat{T}(t, s)$ be the evolution operator associated to (2.11), and assume that

(H_1) (1.1) admits a nonuniform (h, k, μ, ν) -trichotomy on \mathbb{R}^+ with $\mu_i, \nu_i \in \Delta (i = 1, 2)$ and $\mu'_i > 0, \nu'_i > 0$;

(H_2) there exist constants $c_1 > 0$ and $\omega_1 > 0$ such that

$$\sup_{t \in \mathbb{R}^+} \left\{ \frac{|B(t)|h_2^b(t)k_1^{-c}(t)k_2^d(t)\mu_i^{\omega_1+\epsilon+1}(t)}{\mu'_i(t)}, \frac{|B(t)|h_2^b(t)k_1^{-c}(t)k_2^d(t)\nu_i^{\omega_1+\epsilon+1}(t)}{\nu'_i(t)} \right\} \leq c_1$$

for $i = 1, 2$;

(H_3) $\lim_{t \rightarrow +\infty} h_1^a(t)\mu_1^\epsilon(t) = 0, \lim_{t \rightarrow +\infty} k_1^c(t)\nu_1^\epsilon(t) = 0$;

(H_4) $h_1^a(t)k_1^c(t)\nu_1^\epsilon(t), h_1^a(t)k_2^d(t)\nu_2^\epsilon(t), k_1^c(t)k_2^{-d}(t)\nu_2^\epsilon(t)$ are decreasing;

(H_5) there exist positive constants N_1 and N_2 such that

$$\int_0^\infty \nu_i^\epsilon(\tau)\mu_j^{-\omega_1-1}(\tau)\mu'_j(\tau) d\tau \leq N_1/2,$$

$$\int_0^\infty \mu_i^\epsilon(\tau)\nu_j^{-\omega_1-1}(\tau)\nu'_j(\tau) d\tau \leq N_2/2;$$

(H_6) $c_1 < [3K_1(N_1 + N_2) + \omega_1^{-1}]^{-1}$.

Theorem 2.4. *If (1.1) admits a nonuniform (h, k, μ, ν) -trichotomy on \mathbb{R}^+ ,*

$$(2.12) \quad 3K_1c_1 < \omega_1$$

and (H_1) – (H_6) hold, then (2.11) also admits a nonuniform (h, k, μ, ν) -trichotomy on \mathbb{R}^+ , i.e., there exist projections $\widehat{P}(t)$, $\widehat{Q}(t)$ and $\widehat{R}(t)$ for $t \in \mathbb{R}^+$ such that

$$(2.13) \quad \begin{aligned} \widehat{P}(t)\widehat{T}(t, s) &= \widehat{T}(t, s)\widehat{P}(s), \\ \widehat{Q}(t)\widehat{T}(t, s) &= \widehat{T}(t, s)\widehat{Q}(s), \\ \widehat{R}(t)\widehat{T}(t, s) &= \widehat{T}(t, s)\widehat{R}(s) \end{aligned}$$

and the following conclusions hold:

$$(2.14) \quad \begin{aligned} |\widehat{T}(t, s)\widehat{P}(s)| &\leq \frac{K_1\widehat{K}}{1 - 3K_1\widehat{K}c_1(N_1 + N_2)} \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) \\ &\quad \cdot \sum_{i=1}^2 (\mu_i^\epsilon(s) + \nu_i^\epsilon(s)), \\ |\widehat{T}(t, s)\widehat{R}(s)| &\leq \frac{K_1\widehat{K}}{1 - 3K_1\widehat{K}c_1(N_1 + N_2)} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) \\ &\quad \cdot \sum_{i=1}^2 (\mu_i^\epsilon(s) + \nu_i^\epsilon(s)) \end{aligned}$$

for $t \geq s \geq 0$, and

$$(2.15) \quad \begin{aligned} |\widehat{T}(t, s)\widehat{Q}(s)| &\leq \frac{K_1\widehat{K}}{1 - 3K_1\widehat{K}c_1(N_1 + N_2)} \left(\frac{k_1(s)}{k_1(t)} \right)^c \nu_1^\epsilon(s) \\ &\quad \cdot \sum_{i=1}^2 (\mu_i^\epsilon(s) + \nu_i^\epsilon(s)), \\ |\widehat{T}(t, s)\widehat{R}(s)| &\leq \frac{K_1\widehat{K}}{1 - 3K_1\widehat{K}c_1(N_1 + N_2)} \left(\frac{k_2(s)}{k_2(t)} \right)^d \nu_2^\epsilon(s) \\ &\quad \cdot \sum_{i=1}^2 (\mu_i^\epsilon(s) + \nu_i^\epsilon(s)) \end{aligned}$$

for $0 \leq t \leq s$, where $\widehat{P}(t) + \widehat{Q}(t) + \widehat{R}(t) = Id$ and

$$(2.16) \quad \widehat{K} = \frac{K_1}{1 - 3K_1c_1/\omega_1}.$$

Theorem 2.4 indicates that the nonuniform (h, k, μ, ν) -trichotomy persists under sufficiently small linear perturbations. Moreover, it also generalizes and extends some previous work in a certain range for differential equations, such as [1, 2, 3, 5, 6, 9, 16].

3. Proofs of main results. This section constitutes the detailed proofs of Theorem 2.3 and Theorem 2.4.

3.1. Proof of Theorem 2.3. Define $U(t, s) := X(t)X^{-1}(s)$ for $t \geq s$, where $X(t)$ is a fundamental solution matrix of the first equation of (2.5) with the columns $x_1(t), \dots, x_l(t)$. It is easy to show that $\overline{X}(t) = (X(t)^*)^{-1}$ with the columns $\overline{x}_1(t), \dots, \overline{x}_l(t)$ is a fundamental solution matrix of the first equation of (2.6). For any sufficiently small $\tilde{\epsilon} > 0$, set $m_j^{(1)} = \varphi(x_j(0))$ and $n_j^{(1)} = \overline{\varphi}(\overline{x}_j(0))$ for $j = 1, \dots, l$, and then, by (2.7), we conclude that there exists a positive constant \overline{M}_1 such that

$$(3.1) \quad |x_j(t)| \leq \overline{M}_1 H^{m_j^{(1)} + \tilde{\epsilon}}(t), \quad |\overline{x}_j(t)| \leq \overline{M}_1 \overline{H}^{n_j^{(1)} + \tilde{\epsilon}}(t)$$

for $t \geq 0$ and $j = 1, \dots, l$. If one appropriately chooses the matrix $X(t)$, then

$$\gamma_1(\varphi, \overline{\varphi}) = \max\{m_j^{(1)} + n_j^{(1)} : j = 1, \dots, l\}.$$

Note that $\overline{X}(t)^*X(t) = Id$, $(x_i(t), x_j(t)) = \omega_{ij}$ for each i and j , $U(t, s) = X(t)X^{-1}(s) = X(t)\overline{X}(s)^*$, and any entry of $U(t, s)$ is $u_{ik}(t, s) = \sum_{j=1}^l x_{ij}(t)\overline{x}_{kj}(s)$. By (3.1), we have

$$\begin{aligned} |u_{ik}(t, s)| &= \left| \sum_{j=1}^l x_{ij}(t)\overline{x}_{kj}(s) \right| \leq \sum_{j=1}^l |x_j(t)||\overline{x}_j(s)| \\ &\leq \sum_{j=1}^l \overline{M}_1^2 H^{m_j^{(1)} + \tilde{\epsilon}}(t) \overline{H}^{n_j^{(1)} + \tilde{\epsilon}}(s) \end{aligned}$$

$$\begin{aligned} &\leq \sum_{j=1}^l \overline{M}_1^2 \left(\frac{H(t)}{H(s)} \right)^{m_j^{(1)} + \tilde{\epsilon}} H^{m_j^{(1)} + \tilde{\epsilon}}(s) \overline{H}^{n_j^{(1)} + \tilde{\epsilon}}(s) \\ &\leq \overline{M}_1^2 l \left(\frac{H(t)}{H(s)} \right)^{\lambda_{r_1} + \tilde{\epsilon}} (H(s) \overline{H}(s))^{\gamma_1(\varphi, \overline{\varphi}) + \tilde{\epsilon}}. \end{aligned}$$

Let $e_1^{(1)}, \dots, e_l^{(1)}$ be the standard orthogonal basis of E and $\eta^{(1)} = \sum_{j=1}^l l_j^{(1)} e_j^{(1)}$ with $|\eta^{(1)}|^2 = \sum_{j=1}^l (l_j^{(1)})^2 = 1$. Then,

$$\begin{aligned} |U(t, s)\eta^{(1)}|^2 &= \left| \sum_{i=1}^l \sum_{k=1}^l l_k^{(1)} u_{ik}(t, s) e_i^{(1)} \right|^2 \\ &\leq \sum_{i=1}^l \left(\sum_{k=1}^l (l_k^{(1)})^2 \sum_{k=1}^l u_{ik}^2(t, s) \right) \\ &\leq \sum_{i=1}^l \sum_{k=1}^l u_{ik}^2(t, s), \end{aligned}$$

so we have

$$\begin{aligned} |U(t, s)| &= \left(\sum_{i=1}^l \sum_{k=1}^l u_{ik}^2(t, s) \right)^{1/2} \\ &\leq \overline{M}_1^2 l^2 \left(\frac{H(t)}{H(s)} \right)^{\lambda_{r_1} + \tilde{\epsilon}} (H(s) \overline{H}(s))^{\gamma_1(\varphi, \overline{\varphi}) + \tilde{\epsilon}} \\ &\leq \overline{M}_1^2 l^2 \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s), \end{aligned}$$

where $a = \lambda_{r_1} + \tilde{\epsilon} < 0$, $h_1(t) = H(t)$, $\mu_1(t) = H(t) \overline{H}(t) = h_1(t) \overline{h}_1(t)$.

Let $Y(t)$ be a fundamental solution matrix of the second equation of (2.5) and $m_j^{(2)} = \psi(y_j(0))$ for $j = 1, \dots, m$, where $y_1(t), \dots, y_m(t)$ are the columns of $Y(t)$. Then, $\overline{Y}(t) = (Y(t)^*)^{-1}$ is a fundamental solution matrix of the second equation of (2.6). Let $n_j^{(2)} = \overline{\psi}(\overline{y}_j(0))$ for $j = 1, \dots, m$, where $\overline{y}_1(t), \dots, \overline{y}_m(t)$ are the columns of $\overline{Y}(t)$. By (2.7), there exists a positive constant \overline{M}_2 such that

$$(3.2) \quad |y_j(t)| \leq \overline{M}_2 K^{m_j^{(2)} + \tilde{\epsilon}}(t), \quad |\overline{y}_j(t)| \leq \overline{M}_2 \overline{K}^{n_j^{(2)} + \tilde{\epsilon}}(t)$$

for $t \geq 0$ and $j = 1, \dots, m$. On the other hand, since $\bar{Y}(t)^*Y(t) = Id$, we have $(y_i(t), y_j(t)) = \omega_{ij}$ for any i and j . Choosing the appropriate matrix $Y(t)$, we have

$$\gamma_2(\psi, \bar{\psi}) = \max\{m_j^{(2)} + n_j^{(2)} : j = 1, \dots, m\}.$$

Let $V(t, s) = Y(t)Y^{-1}(s) = Y(t)\bar{Y}(s)^*$ for $t \geq s$, and the entries of $V(t, s)$ are $v_{ik}(t, s) = \sum_{j=1}^m y_{ij}(t)\bar{y}_{kj}(s)$. From (3.2), we know

$$\begin{aligned} |v_{ik}(t, s)| &\leq \sum_{j=1}^m |y_j(t)| |\bar{y}_j(s)| \\ &\leq \sum_{j=1}^m \bar{M}_2^2 K(t)^{m_j^{(2)} + \tilde{\epsilon}} \bar{K}(s)^{n_j^{(2)} + \tilde{\epsilon}} \\ &\leq \sum_{j=1}^m \bar{M}_2^2 \left(\frac{K(t)}{K(s)}\right)^{m_j^{(2)} + \tilde{\epsilon}} K(s)^{m_j^{(2)} + \tilde{\epsilon}} \bar{K}(s)^{n_j^{(2)} + \tilde{\epsilon}} \\ &\leq \bar{M}_2^2 m \left(\frac{K(t)}{K(s)}\right)^{\chi_{r_2} + \tilde{\epsilon}} (K(s)\bar{K}(s))^{\gamma_2(\psi, \bar{\psi}) + \tilde{\epsilon}}. \end{aligned}$$

Let $e_1^{(2)}, \dots, e_m^{(2)}$ be the standard orthogonal basis of F and $\eta^{(2)} = \sum_{j=1}^m l_j^{(2)} e_j^{(2)}$ with $|\eta^{(2)}|^2 = \sum_{j=1}^m (l_j^{(2)})^2 = 1$. Then,

$$\begin{aligned} |V(t, s)\eta^{(2)}|^2 &= \left| \sum_{i=1}^m \sum_{k=1}^m l_k^{(2)} v_{ik}(t, s) e_i^{(2)} \right|^2 \\ &\leq \sum_{i=1}^m \left(\sum_{k=1}^m (l_k^{(2)})^2 \sum_{k=1}^m v_{ik}^2(t, s) \right) \\ &\leq \sum_{i=1}^m \sum_{k=1}^m v_{ik}^2(t, s), \end{aligned}$$

whence

$$\begin{aligned} |V(t, s)| &= \left(\sum_{i=1}^m \sum_{k=1}^m v_{ik}^2(t, s) \right)^{1/2} \\ &\leq \bar{M}_2^2 m^2 \left(\frac{K(t)}{K(s)}\right)^{\chi_{r_2} + \tilde{\epsilon}} (K(s)\bar{K}(s))^{\gamma_2(\psi, \bar{\psi}) + \tilde{\epsilon}} \\ &\leq \bar{M}_2^2 m^2 \left(\frac{h_2(t)}{h_2(s)}\right)^b \mu_2^\epsilon(s), \end{aligned}$$

where $b = \chi_{r_2} + \tilde{\epsilon} > 0$, $h_2(t) = K(t)$, $\mu_2(t) = K(t)\bar{K}(t) = h_2(t)\bar{h}_2(t)$.

Let $\widehat{V}(t, s) = Y(t)Y^{-1}(s) = Y(t)\overline{Y}(s)^*$ for $0 \leq t \leq s$, and the entries of $\widehat{V}(t, s)$ are $\widehat{v}_{ik}(t, s) = \sum_{j=1}^m y_{ij}(t)\overline{y}_{kj}(s)$. From (3.2), we know that

$$\begin{aligned} |v_{ik}(t, s)| &\leq \sum_{j=1}^m |y_j(t)||\overline{y}_j(s)| \\ &\leq \sum_{j=1}^m \overline{M}_2^2 K(t)^{m_j^{(2)} + \tilde{\epsilon}} \overline{K}(s)^{n_j^{(2)} + \tilde{\epsilon}} \\ &\leq \sum_{j=1}^m \overline{M}_2^2 \left(\frac{K(s)}{K(t)}\right)^{-m_j^{(2)} - \tilde{\epsilon}} K(s)^{m_j^{(2)} + \tilde{\epsilon}} \overline{K}(s)^{n_j^{(2)} + \tilde{\epsilon}} \\ &\leq \overline{M}_2^2 m \left(\frac{K(s)}{K(t)}\right)^{-(\chi_1 + \tilde{\epsilon})} (K(s)\overline{K}(s))^{\gamma_2(\psi, \overline{\psi}) + \tilde{\epsilon}}. \end{aligned}$$

Let $e_1^{(2)}, \dots, e_m^{(2)}$ be the standard orthogonal basis of F and $\eta^{(2)} = \sum_{j=1}^m l_j^{(2)} e_j^{(2)}$ with $|\eta^{(2)}|^2 = \sum_{j=1}^m (l_j^{(2)})^2 = 1$. Then,

$$\begin{aligned} |\widehat{V}(t, s)\eta^{(2)}|^2 &= \left| \sum_{i=1}^m \sum_{k=1}^m l_k^{(2)} \widehat{v}_{ik}(t, s) e_i^{(2)} \right|^2 \\ &\leq \sum_{i=1}^m \left(\sum_{k=1}^m (l_k^{(2)})^2 \sum_{k=1}^m \widehat{v}_{ik}^2(t, s) \right) \leq \sum_{i=1}^m \sum_{k=1}^m \widehat{v}_{ik}^2(t, s), \end{aligned}$$

and

$$\begin{aligned} |\widehat{V}(t, s)| &= \left(\sum_{i=1}^m \sum_{k=1}^m \widehat{v}_{ik}(t, s)^2 \right)^{1/2} \\ &\leq \overline{M}_2^2 m^2 \left(\frac{K(s)}{K(t)}\right)^{-(\chi_1 + \tilde{\epsilon})} (K(s)\overline{K}(s))^{\gamma_2(\psi, \overline{\psi}) + \tilde{\epsilon}} \\ &\leq \overline{M}_2^2 m^2 \left(\frac{k_2(s)}{k_2(t)}\right)^d \nu_2^\epsilon(s), \end{aligned}$$

where $d = -(\chi_1 + \tilde{\epsilon}) > 0$, $k_2(t) = K(t)$, $\nu_2(t) = K(t)\overline{K}(t) = k_2(t)\overline{k}_2(t)$.

Let $Z(t)$ be a fundamental solution matrix of the third equation of (2.5) and $m_j^{(3)} = \phi(z_j(0))$ for $j = 1, \dots, n - l - m$, where $z_1(t), \dots, z_{n-l-m}(t)$ are the columns of $Z(t)$. Then, $\overline{Z}(t) = (Z(t)^*)^{-1}$ is a fundamental solution matrix of the third equation of (2.6). Let

$n_j^{(3)} = \bar{\phi}(\bar{z}_j(0))$ for $j = 1, \dots, n - l - m$, where $\bar{z}_1(t), \dots, \bar{z}_{n-l-m}(t)$ are the columns of $\bar{Z}(t)$. By (2.7), we know that there exists a positive constant \bar{M}_3 such that

$$(3.3) \quad |z_j(t)| \leq \bar{M}_3 L^{m_j^{(3)} + \tilde{\epsilon}}(t) \quad \text{and} \quad |\bar{z}_j(t)| \leq \bar{M}_3 \bar{L}^{n_j^{(3)} + \tilde{\epsilon}}(t)$$

for $t \geq 0$ and $j = 1, \dots, n - l - m$. On the other hand, since $\bar{Z}(t)^* Z(t) = Id$, we have $(z_i(t), z_j(t)) = \omega_{ij}$ for any i and j . Choosing the appropriate matrix $Z(t)$, we have

$$\gamma_3(\phi, \bar{\phi}) = \max\{m_j^{(3)} + n_j^{(3)} : j = 1, \dots, n - l - m\}.$$

Let $W(t, s) = Z(t)Z^{-1}(s) = Z(t)\bar{Z}(s)^*$ for $0 \leq t \leq s$, and the entries of $W(t, s)$ are $w_{ik}(t, s) = \sum_{j=1}^{n-l-m} z_{ij}(t)\bar{z}_{kj}(s)$. From (3.3), we know that

$$\begin{aligned} |w_{ik}(t, s)| &\leq \sum_{j=1}^{n-l-m} |z_j(t)||\bar{z}_j(s)| \\ &\leq \sum_{j=1}^{n-l-m} \bar{M}_3^2 L^{m_j^{(3)} + \tilde{\epsilon}} \bar{L}^{n_j^{(3)} + \tilde{\epsilon}} \\ &\leq \sum_{j=1}^{n-l-m} \bar{M}_3^2 \left(\frac{L(s)}{L(t)}\right)^{-(m_j^{(3)} + \tilde{\epsilon})} L^{m_j^{(3)} + \tilde{\epsilon}} \bar{L}^{n_j^{(3)} + \tilde{\epsilon}} \\ &\leq \bar{M}_3^2 (n - l - m) \left(\frac{L(s)}{L(t)}\right)^{-(\iota_1 + \tilde{\epsilon})} (L(s)\bar{L}(s))^{\gamma_3(\phi, \bar{\phi}) + \tilde{\epsilon}}. \end{aligned}$$

Letting $e_1^{(3)}, \dots, e_{n-l-m}^{(3)}$ be the standard orthogonal basis of G and $\eta^{(3)} = \sum_{j=1}^{n-l-m} l_j^{(3)} e_j^{(3)}$ with $|\eta^{(3)}|^2 = \sum_{j=1}^{n-l-m} (l_j^{(3)})^2 = 1$, then we have

$$\begin{aligned} |W(t, s)\eta^{(3)}|^2 &= \left| \sum_{i=1}^{n-l-m} \sum_{k=1}^{n-l-m} l_k^{(3)} w_{ik}(t, s) e_i^{(3)} \right|^2 \\ &\leq \sum_{i=1}^{n-l-m} \left(\sum_{k=1}^{n-l-m} (l_k^{(3)})^2 \sum_{k=1}^{n-l-m} w_{ik}^2(t, s) \right) \\ &\leq \sum_{i=1}^{n-l-m} \sum_{k=1}^{n-l-m} w_{ik}^2(t, s) \end{aligned}$$

and

$$\begin{aligned} |W(t, s)| &= \left(\sum_{i=1}^{n-l-m} \sum_{k=1}^{n-l-m} w_{ik}^2(t, s) \right)^{1/2} \\ &\leq \bar{M}_3^2 (n-l-m)^2 \left(\frac{L(s)}{L(t)} \right)^{-(\iota_1 + \tilde{\epsilon})} (L(s)\bar{L}(s))^{\gamma_3(\phi, \bar{\phi}) + \tilde{\epsilon}} \\ &\leq \bar{M}_3^2 (n-l-m)^2 \left(\frac{k_1(s)}{k_1(t)} \right)^c \nu_1^\epsilon(s), \end{aligned}$$

where

$$\begin{aligned} c &= -(\iota_1 + \tilde{\epsilon}) < 0, \quad k_1(t) = L(t), \quad \nu_1(t) = L(t)\bar{L}(t) = k_1(t)\bar{k}_1(t), \\ \epsilon &= \max\{\gamma_1(\varphi, \bar{\varphi}), \gamma_2(\psi, \bar{\psi}), \gamma_3(\phi, \bar{\phi})\} + \tilde{\epsilon}. \end{aligned}$$

3.2. Proof of Theorem 2.4. In order to establish the roughness of nonuniform (h, k, μ, ν) -trichotomies, we establish a series of lemmas and then present the proof of Theorem 2.4.

First, we define

$$\begin{aligned} \Omega := \{ \Phi(t, s) \in \mathcal{B}(\mathcal{X}) : \Phi(t, s) \text{ is continuous} \\ \text{and } |\Phi(t, s)|_i < \infty, (t, s) \in \mathbb{R}^+ \times \mathbb{R}^+ \}, \end{aligned}$$

where $i = 1, 2, 3, 4$, and

$$\begin{aligned} |\Phi|_1 &= \sup \left\{ |\Phi(t, s)| \left(\frac{h_1(t)}{h_1(s)} \right)^{-a} \mu_1^{-\epsilon}(s) : t \geq s \right\}, \\ |\Phi|_2 &= \sup \left\{ |\Phi(t, s)| \left(\frac{h_2(t)}{h_2(s)} \right)^{-b} \mu_2^{-\epsilon}(s) : t \geq s \right\}, \\ |\Phi|_3 &= \sup \left\{ |\Phi(t, s)| \left(\frac{k_1(s)}{k_1(t)} \right)^{-c} \nu_1^{-\epsilon}(s) : t \leq s \right\}, \\ |\Phi|_4 &= \sup \left\{ |\Phi(t, s)| \left(\frac{k_2(s)}{k_2(t)} \right)^{-d} \nu_2^{-\epsilon}(s) : t \leq s \right\}. \end{aligned}$$

It is not difficult to show that $(\Omega, |\cdot|_1)$, $(\Omega, |\cdot|_2)$, $(\Omega, |\cdot|_3)$, and $(\Omega, |\cdot|_4)$ are all Banach spaces. Set $\Omega_i := (\Omega, |\cdot|_i)$, $i = 1, 2, 3, 4$.

Lemma 3.1. *Assume that (H_1) and (H_2) hold. If*

$$(3.4) \quad 3K_1c_1 < \omega_1,$$

then, for any $s \in \mathbb{R}^+, t \geq s$, there exists a unique solution $U \in \Omega_1$ of (2.11) satisfying

$$(3.5) \quad \begin{aligned} U(t, s) = & T(t, s)P(s) + \int_s^t T(t, \tau)P(\tau)B(\tau)U(\tau, s) d\tau \\ & - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)U(\tau, s) d\tau. \end{aligned}$$

Proof. It is not difficult to show that $U(t, s)_{t \geq s}$ satisfying (3.5) is a solution of (2.11). Define an operator J_1 on Ω_1 by

$$\begin{aligned} (J_1U)(t, s) = & T(t, s)P(s) + \int_s^t T(t, \tau)P(\tau)B(\tau)U(\tau, s) d\tau \\ & - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)U(\tau, s) d\tau. \end{aligned}$$

For $t \geq s$, by (H_1) and (H_2) , we have

$$\begin{aligned} A^1 := & \int_s^t |T(t, \tau)P(\tau)||B(\tau)||U(\tau, s)| d\tau \\ & + \int_t^\infty |T(t, \tau)(Q(\tau) + R(\tau))||B(\tau)||U(\tau, s)| d\tau \\ \leq & K_1c_1 \left(\frac{h_1(t)}{h_1(s)}\right)^a \mu_1^\epsilon(s)|U|_1 \\ & \cdot \left(\int_s^t h_2^{-b}(\tau)k_1^c(\tau)k_2^{-d}(\tau)\mu_1^{-\omega_1-1}(\tau)\mu_1'(\tau) d\tau \right. \\ & + \int_t^\infty \left(\frac{k_1(\tau)}{k_1(t)}\right)^c \nu_1'(\tau)\nu_1^{-\omega_1-1}(\tau)h_2^{-b}(\tau)k_1^c(\tau)k_2^{-d}(\tau) \left(\frac{h_1(\tau)}{h_1(t)}\right)^a d\tau \\ & \left. + \int_t^\infty \left(\frac{k_2(\tau)}{k_2(t)}\right)^d \nu_2'(\tau)\nu_2^{-\omega_1-1}(\tau)h_2^{-b}(\tau)k_1^c(\tau)k_2^{-d}(\tau) \left(\frac{h_1(\tau)}{h_1(t)}\right)^a d\tau \right) \\ \leq & \frac{3K_1c_1}{\omega_1} \left(\frac{h_1(t)}{h_1(s)}\right)^a \mu_1^\epsilon(s)|U|_1. \end{aligned}$$

Then,

$$\begin{aligned} |(J_1 U)(t, s)| &\leq |T(t, s)P(s)| + A^1 \\ &\leq K_1 \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) + \frac{3K_1 c_1}{\omega_1} \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) |U|_1, \end{aligned}$$

which implies

$$(3.6) \quad |J_1 U|_1 \leq K_1 + \frac{3K_1 c_1}{\omega_1} |U|_1 < \infty.$$

Therefore, $J_1 U$ is well defined and $J_1 : \Omega_1 \rightarrow \Omega_1$. Moreover, for $U_1, U_2 \in \Omega_1$ and $t \geq s$, we have

$$\begin{aligned} A^2 &:= \int_s^t |T(t, \tau)P(\tau)||B(\tau)||U_1(\tau, s) - U_2(\tau, s)| d\tau \\ &\quad + \int_t^\infty |T(t, \tau)(Q(\tau) + R(\tau))||B(\tau)||U_1(\tau, s) - U_2(\tau, s)| d\tau \\ &\leq \frac{3K_1 c_1}{\omega_1} \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) |U_1 - U_2|_1, \end{aligned}$$

whence,

$$|(J_1 U_1)(t, s) - (J_1 U_2)(t, s)| \leq \frac{3K_1 c_1}{\omega_1} \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) |U_1 - U_2|_1$$

and

$$|J_1 U_1 - J_1 U_2|_1 \leq \frac{3K_1 c_1}{\omega_1} |U_1 - U_2|_1.$$

Hence, the operator J_1 is a contraction. By the principle of contraction mapping, there exists a unique $U \in \Omega_1$ such that $J_1 U = U$. The proof is complete. \square

Lemma 3.2. *If $U(t, s)$ is the unique solution of (2.11) in Ω_1 satisfying (3.5), then $U(t, \sigma)U(\sigma, s) = U(t, s)$ for $t \geq \sigma \geq s$.*

Proof. From (3.5), it follows that

$$\begin{aligned} U(t, \sigma)U(\sigma, s) &= T(t, s)P(s) + \int_s^\sigma T(t, \tau)P(\tau)B(\tau)U(\tau, s) d\tau \\ &\quad + \int_\sigma^t T(t, \tau)P(\tau)B(\tau)U(\tau, \sigma) d\tau U(\sigma, s) \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)U(\tau, \sigma) d\tau U(\sigma, s). \end{aligned}$$

Define the operator H_1 by

$$(H_1l)(t, \sigma) = \int_{\sigma}^t T(t, \tau)P(\tau)B(\tau)l(\tau, \sigma) d\tau - \int_t^{\infty} T(t, \tau)(Q(\tau) + R(\tau))B(\tau)l(\tau, \sigma) d\tau$$

for $l \in \Omega_1^{\sigma}$ and $t \geq \sigma$, where Ω_1^{σ} is obtained from Ω_1 by replacing s with σ . By carrying out similar arguments to the above, we have

$$|H_1l|_1 \leq \frac{3K_1c_1}{\omega_1}|l|_1, \quad |H_1l_1 - H_1l_2|_1 \leq \frac{3K_1c_1}{\omega_1}|l_1 - l_2|_1$$

for any $l, l_1, l_2 \in \Omega_1^{\sigma}$. Hence, there exists a unique $l \in \Omega_1^{\sigma}$ such that $H_1l = l$. On the other hand, we have $H_1(U(t, \sigma)U(\sigma, s) - U(t, s)) = U(t, \sigma)U(\sigma, s) - U(t, s)$, $H_10 = 0$ for $t \geq \sigma \geq s$ and $U(t, \sigma)U(\sigma, s) - U(t, s), 0 \in \Omega_1^{\sigma}$. Therefore, $U(t, \sigma)U(\sigma, s) - U(t, s) = l = 0$ for $t \geq \sigma \geq s$. □

Lemma 3.3. *Assume that (H_1) , (H_2) and (H_4) hold. If (3.4) holds, then, for any $s \in \mathbb{R}^+, t \geq s$, there exists a unique solution $V \in \Omega_2$ of (2.11) satisfying*

$$(3.7) \quad V(t, s) = T(t, s)R(s) + \int_s^t T(t, \tau)P(\tau)B(\tau)V(\tau, s) d\tau - \int_t^{\infty} T(t, \tau)(Q(\tau) + R(\tau))B(\tau)V(\tau, s) d\tau.$$

Proof. It is not difficult to show that $V(t, s)_{t \geq s}$ satisfying (3.7) is a solution of (2.11). Define an operator J_2 on Ω_2 by

$$(J_2V)(t, s) = T(t, s)R(s) + \int_s^t T(t, \tau)P(\tau)B(\tau)V(\tau, s) d\tau - \int_t^{\infty} T(t, \tau)(Q(\tau) + R(\tau))B(\tau)V(\tau, s) d\tau.$$

For $t \geq s$, by (H_1) , (H_2) and (H_4) , we have

$$B^1 := \int_s^t |T(t, \tau)P(\tau)||B(\tau)||V(\tau, s)| d\tau + \int_t^{\infty} |T(t, \tau)(Q(\tau) + R(\tau))||B(\tau)||V(\tau, s)| d\tau$$

$$\begin{aligned}
&\leq K_1 c_1 \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |V|_2 \\
&\quad \cdot \left(\int_s^t \left(\frac{h_1(t)}{h_1(\tau)} \right)^a \mu_1'(\tau) \mu_1^{-\omega_1-1}(\tau) h_2^{-b}(t) k_1^c(\tau) k_2^{-d}(\tau) d\tau \right. \\
&\quad + \int_t^\infty \left(\frac{k_1(\tau)}{k_1(t)} \right)^c \nu_1'(\tau) \nu_1^{-\omega_1-1}(\tau) h_2^{-b}(t) k_1^c(\tau) k_2^{-d}(\tau) d\tau \\
&\quad \left. + \int_t^\infty \left(\frac{k_2(\tau)}{k_2(t)} \right)^d \nu_2'(\tau) \nu_2^{-\omega_1-1}(\tau) h_2^{-b}(t) k_1^c(\tau) k_2^{-d}(\tau) d\tau \right) \\
&\leq \frac{3K_1 c_1}{\omega_1} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |V|_2.
\end{aligned}$$

Then

$$\begin{aligned}
|(J_2 V)(t, s)| &\leq |T(t, s)R(s)| + B^1 \\
&\leq K_1 \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) + \frac{3K_1 c_1}{\omega_1} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |V|_2
\end{aligned}$$

and

$$(3.8) \quad |J_2 V|_2 \leq K_1 + \frac{3K_1 c_1}{\omega_1} |V|_2 < \infty.$$

Hence, $J_2 V$ is well defined and $J_2 : \Omega_2 \rightarrow \Omega_2$. Moreover, for any $V_1, V_2 \in \Omega_2$ and $t \geq s$, we have

$$\begin{aligned}
B^2 &:= \int_s^t |T(t, \tau)P(\tau)| |B(\tau)| |V_1(\tau, s) - V_2(\tau, s)| d\tau \\
&\quad + \int_t^\infty |T(t, \tau)(Q(\tau) + R(\tau))| |B(\tau)| |V_1(\tau, s) - V_2(\tau, s)| d\tau \\
&\leq \frac{3K_1 c_1}{\omega_1} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |V_1 - V_2|_2.
\end{aligned}$$

Hence,

$$|(J_2 V_1)(t, s) - (J_2 V_2)(t, s)| \leq \frac{3K_1 c_1}{\omega_1} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |V_1 - V_2|_2$$

and

$$|J_2 V_1 - J_2 V_2|_2 \leq \frac{3K_1 c_1}{\omega_1} |V_1 - V_2|_2.$$

Then, J_2 is a contraction, and there exists a unique $V \in \Omega_2$ such that $J_2V = V$ since (3.4) holds. □

Lemma 3.4. *If $V(t, s)$ is the unique solution of (2.11) in Ω_2 satisfying (3.7), then $V(t, \sigma)V(\sigma, s) = V(t, s)$ for $t \geq \sigma \geq s$.*

Proof. From (3.7), we have

$$\begin{aligned} V(t, \sigma)V(\sigma, s) &= T(t, s)R(s) + \int_s^\sigma T(t, \tau)P(\tau)B(\tau)V(\tau, s) d\tau \\ &\quad + \int_\sigma^t T(t, \tau)P(\tau)B(\tau)V(\tau, \sigma) d\tau V(\sigma, s) \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)V(\tau, \sigma) d\tau V(\sigma, s). \end{aligned}$$

Define the operator H_2 by

$$\begin{aligned} (H_2\tilde{l})(t, \sigma) &= \int_\sigma^t T(t, \tau)P(\tau)B(\tau)\tilde{l}(\tau, \sigma) d\tau \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\tilde{l}(\tau, \sigma) d\tau \end{aligned}$$

for $\tilde{l} \in \Omega_2^g$ and $t \geq \sigma$, where Ω_2^g is obtained from Ω_2 by replacing s with σ . By arguments similar to the above, we have

$$|H_2\tilde{l}|_2 \leq \frac{3K_1c_1}{\omega_1}|\tilde{l}|_2, \quad |H_2\tilde{l}_1 - H_2\tilde{l}_2|_2 \leq \frac{3K_1c_1}{\omega_1}|\tilde{l}_1 - \tilde{l}_2|_2$$

for any $\tilde{l}, \tilde{l}_1, \tilde{l}_2 \in \Omega_2^g$. Hence, there exists a unique $\tilde{l} \in \Omega_2^g$ such that $H_2\tilde{l} = \tilde{l}$. On the other hand, $H_2(V(t, \sigma)V(\sigma, s) - V(t, s)) = V(t, \sigma)V(\sigma, s) - V(t, s)$, $H_20 = 0$ for $t \geq \sigma \geq s$ and $V(t, \sigma)V(\sigma, s) - V(t, s), 0 \in \Omega_2^g$. Then, $V(t, \sigma)V(\sigma, s) - V(t, s) = \tilde{l} = 0$ for $t \geq \sigma \geq s$; hence, $V(t, \sigma)V(\sigma, s) = V(t, s)$. □

Similarly, we have

Lemma 3.5. *If $(H_1), (H_2), (H_4)$ and (3.4) hold, then, for any $s \in \mathbb{R}^+$, there exists a unique solution $W \in \Omega_3$ and another unique solution $\hat{V} \in \Omega_4$ of (2.11) satisfying*

$$\begin{aligned}
 (3.9) \quad W(t, s) &= T(t, s)Q(s) + \int_0^t T(t, \tau)P(\tau)B(\tau)W(\tau, s) d\tau \\
 &\quad - \int_t^s T(t, \tau)(Q(\tau) + R(\tau))B(\tau)W(\tau, s) d\tau, \\
 \widehat{V}(t, s) &= T(t, s)R(s) + \int_0^t T(t, \tau)P(\tau)B(\tau)\widehat{V}(\tau, s) d\tau \\
 &\quad - \int_t^s T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\widehat{V}(\tau, s) d\tau
 \end{aligned}$$

for $t \leq s$ and $W(t, \sigma)W(\sigma, s) = W(t, s)$, $\widehat{V}(t, \sigma)\widehat{V}(\sigma, s) = \widehat{V}(t, s)$ for $t \leq \sigma \leq s$.

Let $U(t, s)_{t \geq s}$, $V(t, s)_{t \geq s}$, $W(t, s)_{t \leq s}$, $\widehat{V}(t, s)_{t \leq s}$ be the unique solutions characterized by Lemmas 3.1, 3.3 and 3.5, respectively. Given $\gamma \in \mathbb{R}^+$, note that $U_1(t, \gamma) = U(t, \gamma)P(\gamma)$ and $V_1(t, \gamma) = V(t, \gamma)R(\gamma)$ satisfy (3.5), (3.7) with $s = \gamma$ and $W_1(t, \gamma) = W(t, \gamma)Q(\gamma)$, $\widehat{V}_1(t, \gamma) = \widehat{V}(t, \gamma)R(\gamma)$ satisfy (3.9) with $s = \gamma$, by the lemmas proved above, we have

$$\begin{aligned}
 U(t, \gamma)P(\gamma) &= U(t, \gamma), & V(t, \gamma)R(\gamma) &= V(t, \gamma), \\
 W(t, \gamma)Q(\gamma) &= W(t, \gamma), & \widehat{V}(t, \gamma)R(\gamma) &= \widehat{V}(t, \gamma).
 \end{aligned}$$

Define the linear operators

$$\begin{aligned}
 \widetilde{P}(t) &:= \widehat{T}(t, \gamma)U(\gamma, \gamma)\widehat{T}(\gamma, t), & \widetilde{Q}(t) &:= \widehat{T}(t, \gamma)W(\gamma, \gamma)\widehat{T}(\gamma, t), \\
 \widetilde{R}_1(t) &:= \widehat{T}(t, \gamma)V(\gamma, \gamma)\widehat{T}(\gamma, t), & \widetilde{R}_2(t) &:= \widehat{T}(t, \gamma)\widehat{V}(\gamma, \gamma)\widehat{T}(\gamma, t)
 \end{aligned}$$

and

$$\widetilde{R}(t) = \begin{cases} \widetilde{R}_1(t) & t \geq s, \\ \widetilde{R}_2(t) & s \geq t \end{cases}$$

for any $t \in \mathbb{R}^+$. From Lemmas 3.2, 3.4, 3.5 and

$$\begin{aligned}
 (3.10) \quad \widetilde{P}(\gamma) &= U(\gamma, \gamma) = P(\gamma) - \int_\gamma^\infty T(\gamma, \tau)(Q(\tau) + R(\tau))B(\tau)U(\tau, \gamma) d\tau, \\
 \widetilde{R}_1(\gamma) &= V(\gamma, \gamma) = R(\gamma) - \int_\gamma^\infty T(\gamma, \tau)(Q(\tau) + R(\tau))B(\tau)V(\tau, \gamma) d\tau,
 \end{aligned}$$

$$\begin{aligned} \tilde{Q}(\gamma) &= W(\gamma, \gamma) = Q(\gamma) + \int_0^\gamma T(\gamma, \tau)P(\tau)B(\tau)W(\tau, \gamma) d\tau, \\ \tilde{R}_2(\gamma) &= \hat{V}(\gamma, \gamma) = R(\gamma) + \int_0^\gamma T(\gamma, \tau)P(\tau)B(\tau)\hat{V}(\tau, \gamma) d\tau, \end{aligned}$$

it follows that

- (b₁) $\tilde{P}(t), \tilde{Q}(t), \tilde{R}(t)$ are projections for each $t \in \mathbb{R}^+$;
- (b₂) $\tilde{P}(t)\hat{T}(t, s) = \hat{T}(t, s)\tilde{P}(s), \tilde{Q}(t)\hat{T}(t, s) = \hat{T}(t, s)\tilde{Q}(s), \tilde{R}(t)\hat{T}(t, s) = \hat{T}(t, s)\tilde{R}(s)$ for each $t, s \in \mathbb{R}^+$;
- (b₃) $P(\gamma)\tilde{P}(\gamma) = P(\gamma), Q(\gamma)\tilde{Q}(\gamma) = Q(\gamma), R(\gamma)\tilde{R}(\gamma) = R(\gamma)$;
- (b₄) $\tilde{P}(\gamma)P(\gamma) = \tilde{P}(\gamma), \tilde{Q}(\gamma)Q(\gamma) = \tilde{Q}(\gamma), \tilde{R}(\gamma)R(\gamma) = \tilde{R}(\gamma)$.

Lemma 3.6. *If (H₁)–(H₄) and (3.4) hold, then*

$$\begin{aligned} (3.11) \quad |\hat{T}(t, s)|\text{Im}\tilde{P}(s) &\leq \frac{K_1}{1 - 3K_1c_1/\omega_1} \left(\frac{h_1(t)}{h_1(s)}\right)^a \mu_1^\epsilon(s), \quad t \geq s, \\ |\hat{T}(t, s)|\text{Im}\tilde{R}(s) &\leq \frac{K_1}{1 - 3K_1c_1/\omega_1} \left(\frac{h_2(t)}{h_2(s)}\right)^b \mu_2^\epsilon(s), \quad t \geq s \end{aligned}$$

and

$$\begin{aligned} (3.12) \quad |\hat{T}(t, s)|\text{Im}\tilde{Q}(s) &\leq \frac{K_1}{1 - 3K_1c_1/\omega_1} \left(\frac{k_1(s)}{k_1(t)}\right)^c \nu_1^\epsilon(s), \quad t \leq s, \\ |\hat{T}(t, s)|\text{Im}\tilde{R}(s) &\leq \frac{K_1}{1 - 3K_1c_1/\omega_1} \left(\frac{k_2(s)}{k_2(t)}\right)^d \nu_2^\epsilon(s), \quad t \leq s. \end{aligned}$$

Proof. First, it is trivial to verify that $z_1(t)_{t \geq s}$ and $z_2(t)_{t \geq s}$ are bounded solutions of (2.11), where

$$\begin{aligned} (3.13) \quad z_1(t) &= T(t, s)P(s)z_1(s) + \int_s^t T(t, \tau)P(\tau)B(\tau)z_1(\tau) d\tau \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)z_1(\tau) d\tau, \\ z_2(t) &= T(t, s)R(s)z_2(s) + \int_s^t T(t, \tau)P(\tau)B(\tau)z_2(\tau) d\tau \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)z_2(\tau) d\tau. \end{aligned}$$

Let $z_1(t) = \widehat{T}(t, s)\widetilde{P}(s)\xi$ be a solution of (2.11) for $t \geq s$ and each given $\xi \in X$. Note that $\widehat{T}(t, \gamma)U(\gamma, \gamma)$ and $U(t, \gamma)$ are solutions of (2.11), which coincide for $t = \gamma$. Then,

$$\begin{aligned} z_1(t) &= \widehat{T}(t, s)\widetilde{P}(s)\xi = \widehat{T}(t, \gamma)\widetilde{P}(\gamma)\widehat{T}(\gamma, s)\xi \\ &= \widehat{T}(t, \gamma)U(\gamma, \gamma)\widehat{T}(\gamma, s)\xi = U(t, \gamma)\widehat{T}(\gamma, s)\xi \end{aligned}$$

is a bounded solution of (2.11) with the initial value $z_1(s) = \widetilde{P}(s)\xi$ since $U(t, \gamma)$ is bounded for $t \in \mathbb{R}^+$. Similarly, we also have

$$\begin{aligned} z_2(t) &= \widehat{T}(t, s)\widetilde{R}(s)\xi = \widehat{T}(t, \gamma)\widetilde{R}(\gamma)\widehat{T}(\gamma, s)\xi \\ &= \widehat{T}(t, \gamma)V(\gamma, \gamma)\widehat{T}(\gamma, s)\xi = V(t, \gamma)\widehat{T}(\gamma, s)\xi, \end{aligned}$$

which is a bounded solution of (2.11) with the initial value $z_2(s) = \widetilde{R}(s)\xi$ for $t \geq s$, $\xi \in X$. It follows from (3.13) that

$$\begin{aligned} \widehat{T}(t, s)\widetilde{P}(s)\xi &= z_1(t) = T(t, s)P(s)\widetilde{P}(s)\xi \\ &\quad + \int_s^t T(t, \tau)P(\tau)B(\tau)\widetilde{P}(\tau)\widehat{T}(\tau, s)\xi \, d\tau \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\widetilde{P}(\tau)\widehat{T}(\tau, s)\xi \, d\tau \end{aligned}$$

and

$$\begin{aligned} \widehat{T}(t, s)\widetilde{R}(s)\xi &= z_2(t) = T(t, s)R(s)\widetilde{R}(s)\xi \\ &\quad + \int_s^t T(t, \tau)P(\tau)B(\tau)\widetilde{R}(\tau)\widehat{T}(\tau, s)\xi \, d\tau \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\widetilde{R}(\tau)\widehat{T}(\tau, s)\xi \, d\tau \end{aligned}$$

for $t \geq s$. Combining (H_1) , (H_2) and (H_3) leads to

$$\begin{aligned} A^3 &:= \int_s^t |T(t, \tau)P(\tau)||B(\tau)||\widetilde{P}(\tau)\widehat{T}(\tau, s)\xi| \, d\tau \\ &\quad + \int_t^\infty |T(t, \tau)(Q(\tau) + R(\tau))||B(\tau)||\widetilde{P}(\tau)\widehat{T}(\tau, s)\xi| \, d\tau \\ &\leq K_1 c_1 \int_s^t \left(\frac{h_1(t)}{h_1(\tau)} \right)^a h_2^{-b}(\tau) k_1^c(\tau) k_2^{-d}(\tau) \end{aligned}$$

$$\begin{aligned}
 & \cdot \mu_1^{-\omega_1-1}(\tau)\mu'_1(\tau)|\tilde{P}(\tau)\hat{T}(\tau, s)||\tilde{P}(s)\xi| d\tau \\
 & + K_1c_1 \int_t^\infty \left(\frac{k_1(\tau)}{k_1(t)}\right)^c h_2^{-b}(\tau)k_1^c(\tau)k_2^{-d}(\tau) \\
 & \cdot \nu_1^{-\omega_1-1}(\tau)\nu'_1(\tau)|\tilde{P}(\tau)\hat{T}(\tau, s)||\tilde{P}(s)\xi| d\tau \\
 & + K_1c_1 \int_t^\infty \left(\frac{k_2(\tau)}{k_2(t)}\right)^d h_2^{-b}(\tau)k_1^c(\tau)k_2^{-d}(\tau) \\
 & \cdot \nu_2^{-\omega_1-1}(\tau)\nu'_2(\tau)|\tilde{P}(\tau)\hat{T}(\tau, s)||\tilde{P}(s)\xi| d\tau \\
 \leq & K_1c_1 \left(\frac{h_1(t)}{h_1(s)}\right)^a \mu_1^\epsilon(s)|\tilde{P}\hat{T}|_1|\tilde{P}(s)\xi| \int_s^t \mu_1^{-\omega_1-1}(\tau)\mu'_1(\tau) d\tau \\
 & + K_1c_1 \left(\frac{h_1(t)}{h_1(s)}\right)^a \mu_1^\epsilon(s)|\tilde{P}\hat{T}|_1|\tilde{P}(s)\xi| \\
 & \cdot \int_t^\infty \left(\frac{k_1(\tau)}{k_1(t)}\right)^c \nu_1^{-\omega_1-1}(\tau)\nu'_1(\tau) \left(\frac{h_1(\tau)}{h_1(t)}\right)^a d\tau \\
 & + K_1c_1 \left(\frac{h_1(t)}{h_1(s)}\right)^a \mu_1^\epsilon(s)|\tilde{P}\hat{T}|_1|\tilde{P}(s)\xi| \\
 & \cdot \int_t^\infty \left(\frac{k_2(\tau)}{k_2(t)}\right)^d \nu_2^{-\omega_1-1}(\tau)\nu'_2(\tau) \left(\frac{h_1(\tau)}{h_1(t)}\right)^a d\tau \\
 \leq & \frac{3K_1c_1}{\omega_1} \left(\frac{h_1(t)}{h_1(s)}\right)^a \mu_1^\epsilon(s)|\tilde{P}\hat{T}|_1|\tilde{P}(s)\xi|
 \end{aligned}$$

and

$$\begin{aligned}
 B^3 := & \int_s^t |T(t, \tau)P(\tau)||B(\tau)||\tilde{R}(\tau)\hat{T}(\tau, s)\xi| d\tau \\
 & + \int_t^\infty |T(t, \tau)(Q(\tau) + R(\tau))||B(\tau)||\tilde{R}(\tau)\hat{T}(\tau, s)\xi| d\tau \\
 \leq & K_1c_1 \int_s^t \left(\frac{h_1(t)}{h_1(\tau)}\right)^a h_2^{-b}(\tau)k_1^c(\tau)k_2^{-d}(\tau) \\
 & \cdot \mu_1^{-\omega_1-1}(\tau)\mu'_1(\tau)|\tilde{R}(\tau)\hat{T}(\tau, s)||\tilde{R}(s)\xi| d\tau \\
 & + K_1c_1 \int_t^\infty \left(\frac{k_1(\tau)}{k_1(t)}\right)^c h_2^{-b}(\tau)k_1^c(\tau)k_2^{-d}(\tau) \\
 & \cdot \nu_1^{-\omega_1-1}(\tau)\nu'_1(\tau)|\tilde{R}(\tau)\hat{T}(\tau, s)||\tilde{R}(s)\xi| d\tau
 \end{aligned}$$

$$\begin{aligned}
& + K_1 c_1 \int_t^\infty \left(\frac{k_2(\tau)}{k_2(t)} \right)^d h_2^{-b}(\tau) k_1^c(\tau) k_2^{-d}(\tau) \\
& \cdot \nu_2^{-\omega_1-1}(\tau) \nu_2'(\tau) |\tilde{R}(\tau) \hat{T}(\tau, s)| |\tilde{R}(s) \xi| d\tau \\
\leq & K_1 c_1 |\tilde{R} \hat{T}|_2 \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\tilde{R}(s) \xi| \\
& \cdot \int_s^t \left(\frac{h_1(t)}{h_1(\tau)} \right)^a h_2^{-b}(t) \mu_1^{-\omega_1-1}(\tau) \mu_1'(\tau) d\tau \\
& + K_1 c_1 |\tilde{R} \hat{T}|_2 \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\tilde{R}(s) \xi| \\
& \cdot \int_t^\infty \left(\frac{k_1(\tau)}{k_1(t)} \right)^c h_2^{-b}(t) \nu_1^{-\omega_1-1}(\tau) \nu_1'(\tau) d\tau \\
& + K_1 c_1 |\tilde{R} \hat{T}|_2 \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\tilde{R}(s) \xi| \\
& \cdot \int_t^\infty k_2^{-d}(t) h_2^{-b}(t) \nu_2^{-\omega_1-1}(\tau) \nu_2'(\tau) d\tau \\
\leq & \frac{3K_1 c_1}{\omega_1} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\tilde{R} \hat{T}|_2 |\tilde{R}(s) \xi|.
\end{aligned}$$

Then,

$$\begin{aligned}
|\tilde{P}(t) \hat{T}(t, s) \tilde{P}(s) \xi| & \leq K_1 \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) |\tilde{P}(s) \xi| + A^3 \\
& \leq K_1 \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) |\tilde{P}(s) \xi| \\
& \quad + \frac{3K_1 c_1}{\omega_1} \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) |\tilde{P} \hat{T}|_1 |\tilde{P}(s) \xi| \\
|\tilde{R}(t) \hat{T}(t, s) \tilde{R}(s) \xi| & \leq K_1 \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\tilde{R}(s) \xi| + B^3 \\
& \leq K_1 \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\tilde{R}(s) \xi| \\
& \quad + \frac{3K_1 c_1}{\omega_1} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\tilde{R} \hat{T}|_2 |\tilde{R}(s) \xi|,
\end{aligned}$$

i.e.,

$$|\widetilde{P\widehat{T}}|_1 \leq K_1 + \frac{3K_1c_1}{\omega_1}|\widetilde{P\widehat{T}}|_1, \quad |\widetilde{R\widehat{T}}|_2 \leq K_1 + \frac{3K_1c_1}{\omega_1}|\widetilde{R\widehat{T}}|_2,$$

which implies that the inequality (3.11) holds. Similarly, the inequality (3.12) holds for $t \leq s$. □

Lemma 3.7. *Let $S(\gamma) = \widetilde{P}(\gamma) + \widetilde{Q}(\gamma) + \widetilde{R}(\gamma)$ and (H_1) – (H_6) hold. Then, $S(\gamma)$ is invertible for any fixed $\gamma \in \mathbb{R}^+$.*

Proof. By (b_3) and (b_4) , we have

$$(3.14) \quad \begin{aligned} \widetilde{P}(\gamma) + \widetilde{Q}(\gamma) + \widetilde{R}(\gamma) - Id \\ = Q(\gamma)\widetilde{P}(\gamma) + R(\gamma)\widetilde{P}(\gamma) + P(\gamma)\widetilde{Q}(\gamma) + P(\gamma)\widetilde{R}(\gamma). \end{aligned}$$

By (3.10), we have

$$\begin{aligned} P(\gamma)(\widetilde{Q}(\gamma) + \widetilde{R}(\gamma)) &= P(\gamma)(W(\gamma, \gamma) + \widehat{V}(\gamma, \gamma)) \\ &= \int_0^\gamma T(\gamma, \tau)P(\tau)B(\tau)(W(\tau, \gamma) + \widehat{V}(\tau, \gamma)) d\tau, \end{aligned}$$

and

$$\begin{aligned} (Q(\gamma) + R(\gamma))\widetilde{P}(\gamma) &= (Q(\gamma) + R(\gamma))U(\gamma, \gamma) \\ &= - \int_\gamma^\infty T(\gamma, \tau)(Q(\tau) + R(\tau))B(\tau)U(\tau, \gamma) d\tau. \end{aligned}$$

By (2.16) and Lemmas 3.1, 3.3 and 3.5, we obtain

$$(3.15) \quad \begin{aligned} |U(t, s)| &\leq \widehat{K} (h_1(t)/h_1(s))^a \mu_1^\epsilon(s), \\ |V(t, s)| &\leq \widehat{K} (h_2(t)/h_2(s))^b \mu_2^\epsilon(s), \quad t \geq s, \\ |W(t, s)| &\leq \widehat{K} (k_1(s)/k_1(t))^c \nu_1^\epsilon(s), \\ |\widehat{V}(t, s)| &\leq \widehat{K} (k_2(s)/k_2(t))^d \nu_2^\epsilon(s), \quad t \leq s. \end{aligned}$$

From (3.15) and (H_3) – (H_5) , it follows that

$$(3.16) \quad \begin{aligned} A^4 &:= \int_0^\gamma |T(\gamma, \tau)P(\tau)||B(\tau)||W(\tau, \gamma) + \widehat{V}(\tau, \gamma)| d\tau \\ &\leq \widehat{K} K_1 c_1 \int_0^\gamma \left(\frac{h_1(\gamma)}{h_1(\tau)} \right)^a \mu_1^{-\omega_1-1}(\tau) \mu_1'(\tau) h_2^{-b}(\tau) \end{aligned}$$

$$\begin{aligned}
 & \cdot k_1^\epsilon(\tau)k_2^{-d}(\tau)\left(\frac{k_1(\gamma)}{k_1(\tau)}\right)^c \nu_1^\epsilon(\gamma) d\tau \\
 & + \widehat{K}K_1c_1 \int_0^\gamma \left(\frac{h_1(\gamma)}{h_1(\tau)}\right)^a \mu_1^{-\omega_1-1}(\tau)\mu_1'(\tau)h_2^{-b}(\tau) \\
 & \cdot k_1^c(\tau)k_2^{-d}(\tau)\left(\frac{k_2(\gamma)}{k_2(\tau)}\right)^d \nu_2^\epsilon(\gamma) d\tau \\
 \leq & \widehat{K}K_1c_1 \left(\int_0^\gamma \nu_1^\epsilon(\tau)\mu_1^{-\omega_1-1}(\tau)\mu_1'(\tau) d\tau \right. \\
 & \left. + \int_0^\gamma \nu_2^\epsilon(\tau)\mu_1^{-\omega_1-1}(\tau)\mu_1'(\tau) d\tau \right) \\
 \leq & \widehat{K}K_1N_1c_1
 \end{aligned}$$

and

$$\begin{aligned}
 (3.17) \quad B^4 & := \int_\gamma^\infty |T(\gamma, \tau)(Q(\tau) + R(\tau))||B(\tau)||U(\tau, \gamma)| d\tau \\
 & \leq \widehat{K}K_1c_1 \int_\gamma^\infty \left(\frac{k_1(\tau)}{k_1(\gamma)}\right)^c \nu_1^{-\omega_1-1}(\tau)\nu_1'(\tau)h_2^{-b}(\tau) \\
 & \cdot k_1^c(\tau)k_2^{-d}(\tau)\left(\frac{h_1(\tau)}{h_1(\gamma)}\right)^a \mu_1^\epsilon(\gamma) d\tau \\
 & + \widehat{K}K_1c_1 \int_\gamma^\infty \left(\frac{k_2(\tau)}{k_2(\gamma)}\right)^d \nu_2^{-\omega_1-1}(\tau)\nu_2'(\tau)h_2^{-b}(\tau) \\
 & k_1^c(\tau)k_2^{-d}(\tau)\left(\frac{h_1(\tau)}{h_1(\gamma)}\right)^a \mu_1^\epsilon(\gamma) d\tau \\
 \leq & \widehat{K}K_1c_1 \left(\int_\gamma^\infty \mu_1^\epsilon(\tau)\nu_1^{-\omega_1-1}(\tau)\nu_1'(\tau) d\tau \right. \\
 & \left. + \int_\gamma^\infty \mu_1^\epsilon(\tau)\nu_2^{-\omega_1-1}(\tau)\nu_2'(\tau) d\tau \right) \\
 \leq & \widehat{K}K_1N_2c_1.
 \end{aligned}$$

Then, we have

$$|\widetilde{P}(\gamma) + \widetilde{Q}(\gamma) + \widetilde{R}(\gamma) - Id| \leq A^4 + B^4 \leq \widehat{K}K_1(N_1 + N_2)c_1.$$

This means that the operator $S(\gamma)$ is invertible if (H_6) holds. □

We are now in the right position to prove Theorem 2.4.

Proof of Theorem 2.4. Let

$$\begin{aligned}
 \widehat{P}(t) &= \widehat{T}(t, \gamma)S(\gamma)P(\gamma)S(\gamma)^{-1}\widehat{T}(\gamma, t), \\
 \widehat{Q}(t) &= \widehat{T}(t, \gamma)S(\gamma)Q(\gamma)S(\gamma)^{-1}\widehat{T}(\gamma, t), \\
 \widehat{R}(t) &= \widehat{T}(t, \gamma)S(\gamma)R(\gamma)S(\gamma)^{-1}\widehat{T}(\gamma, t)
 \end{aligned}
 \tag{3.18}$$

for any $t \in \mathbb{R}^+$. Obviously, $\widehat{P}(t) + \widehat{Q}(t) + \widehat{R}(t) = Id$, and (2.13) is valid.

In order to establish (2.14) and (2.15), we first show

$$\begin{aligned}
 |\widehat{T}(t, s)\widehat{P}(s)| &\leq \widehat{K} \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) |\widehat{P}(s)|, \\
 |\widehat{T}(t, s)\widehat{R}(s)| &\leq \widehat{K} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\widehat{R}(s)|
 \end{aligned}
 \tag{3.19}$$

for $t \geq s$, and

$$\begin{aligned}
 |\widehat{T}(t, s)\widehat{Q}(s)| &\leq \widehat{K} \left(\frac{k_1(s)}{k_1(t)} \right)^c \nu_1^\epsilon(s) |\widehat{Q}(s)|, \\
 |\widehat{T}(t, s)\widehat{R}(s)| &\leq \widehat{K} \left(\frac{k_2(s)}{k_2(t)} \right)^d \nu_2^\epsilon(s) |\widehat{R}(s)|
 \end{aligned}
 \tag{3.20}$$

for $t \leq s$. It follows from (b_4) that

$$\begin{aligned}
 S(\gamma)P(\gamma) &= (\widetilde{P}(\gamma) + \widetilde{Q}(\gamma) + \widetilde{R}(\gamma))P(\gamma) = \widetilde{P}(\gamma), \\
 S(\gamma)Q(\gamma) &= (\widetilde{P}(\gamma) + \widetilde{Q}(\gamma) + \widetilde{R}(\gamma))Q(\gamma) = \widetilde{Q}(\gamma), \\
 S(\gamma)R(\gamma) &= (\widetilde{P}(\gamma) + \widetilde{Q}(\gamma) + \widetilde{R}(\gamma))R(\gamma) = \widetilde{R}(\gamma).
 \end{aligned}$$

Note that $S(t) = \widehat{T}(t, \gamma)S(\gamma)\widehat{T}(\gamma, t)$ for $t \in \mathbb{R}^+$. Then,

$$\begin{aligned}
 \widehat{P}(t)S(t) &= \widehat{T}(t, \gamma)S(\gamma)P(\gamma)S(\gamma)^{-1}\widehat{T}(\gamma, t)\widehat{T}(t, \gamma)S(\gamma)\widehat{T}(\gamma, t) \\
 &= \widehat{T}(t, \gamma)S(\gamma)P(\gamma)\widehat{T}(\gamma, t) \\
 &= \widehat{T}(t, \gamma)\widetilde{P}(\gamma)\widehat{T}(\gamma, t) \\
 &= \widetilde{P}(t).
 \end{aligned}$$

On the other hand, we have

$$\widehat{Q}(t)S(t) = \widehat{T}(t, \gamma)S(\gamma)Q(\gamma)\widehat{T}(\gamma, t) = \widehat{T}(t, \gamma)\widetilde{Q}(\gamma)\widehat{T}(\gamma, t) = \widetilde{Q}(t),$$

$$\widehat{R}(t)S(t) = \widehat{T}(t, \gamma)S(\gamma)R(\gamma)\widehat{T}(\gamma, t) = \widehat{T}(t, \gamma)\widetilde{R}(\gamma)\widehat{T}(\gamma, t) = \widetilde{R}(t).$$

Then, $S(t)$ is invertible and

$$\text{Im } \widehat{P}(t) = \text{Im } \widetilde{P}(t), \quad \text{Im } \widehat{Q}(t) = \text{Im } \widetilde{Q}(t), \quad \text{Im } \widehat{R}(t) = \text{Im } \widetilde{R}(t).$$

By Lemma 3.6, we have

$$\begin{aligned} |\widehat{T}(t, s)\widehat{P}(s)| &\leq |\widehat{T}(t, s)|\widehat{P}(s)|\widehat{P}(s)| \leq |\widehat{T}(t, s)|\widetilde{P}(s)|\widehat{P}(s)| \\ &\leq \widehat{K} \left(\frac{h_1(t)}{h_1(s)} \right)^a \mu_1^\epsilon(s) |\widehat{P}(s)|, \\ |\widehat{T}(t, s)\widehat{R}(s)| &\leq |\widehat{T}(t, s)|\widehat{R}(s)|\widehat{R}(s)| \leq |\widehat{T}(t, s)|\widetilde{R}(s)|\widehat{R}(s)| \\ &\leq \widehat{K} \left(\frac{h_2(t)}{h_2(s)} \right)^b \mu_2^\epsilon(s) |\widehat{R}(s)| \end{aligned}$$

for $t \geq s$, and

$$\begin{aligned} |\widehat{T}(t, s)\widehat{Q}(s)| &\leq |\widehat{T}(t, s)|\widehat{Q}(s)|\widehat{Q}(s)| \leq |\widehat{T}(t, s)|\widetilde{Q}(s)|\widehat{Q}(s)| \\ &\leq \widehat{K} \left(\frac{k_1(s)}{k_1(t)} \right)^c \nu_1^\epsilon(s) |\widehat{Q}(s)|, \\ |\widehat{T}(t, s)\widehat{R}(s)| &\leq |\widehat{T}(t, s)|\widehat{R}(s)|\widehat{R}(s)| \leq |\widehat{T}(t, s)|\widetilde{R}(s)|\widehat{R}(s)| \\ &\leq \widehat{K} \left(\frac{k_2(s)}{k_2(t)} \right)^d \nu_2^\epsilon(s) |\widehat{R}(s)| \end{aligned}$$

for $t \leq s$.

Next, we establish norm bounds for the projects $\widehat{P}(t)$, $\widehat{Q}(t)$, $\widehat{R}(t)$. For any $\xi \in X$, set

$$\begin{aligned} z_1(t) &= \widehat{T}(t, s)\widehat{P}(s)\xi, \quad z_2(t) = \widehat{T}(t, s)\widehat{R}(s)\xi, \quad t \geq s, \\ z_3(t) &= \widehat{T}(t, s)\widehat{Q}(s)\xi, \quad z_4(t) = \widehat{T}(t, s)\widehat{R}(s)\xi, \quad t \leq s. \end{aligned}$$

By (3.13), (3.19) and (3.20), it follows that $(z_1(t))_{t \geq s}$, $(z_2(t))_{t \geq s}$, $(z_3(t))_{t \leq s}$ and $(z_4(t))_{t \leq s}$ are bounded solutions of (2.11),

$$\begin{aligned} \widehat{P}(t)\widehat{T}(t, s)\xi &= T(t, s)P(s)\widehat{P}(s)\xi \\ &\quad + \int_s^t T(t, \tau)P(\tau)B(\tau)\widehat{P}(\tau)\widehat{T}(\tau, s)\xi \, d\tau \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\widehat{P}(\tau)\widehat{T}(\tau, s)\xi \, d\tau, \end{aligned}$$

$$\begin{aligned} \widehat{R}(t)\widehat{T}(t, s)\xi &= T(t, s)R(s)\widehat{R}(s)\xi + \int_s^t T(t, \tau)P(\tau)B(\tau)\widehat{R}(\tau)\widehat{T}(\tau, s)\xi d\tau \\ &\quad - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\widehat{R}(\tau)\widehat{T}(\tau, s)\xi d\tau, \end{aligned}$$

and

$$\begin{aligned} \widehat{Q}(t)\widehat{T}(t, s)\xi &= T(t, s)Q(s)\widehat{Q}(s)\xi \\ &\quad + \int_0^t T(t, \tau)P(\tau)B(\tau)\widehat{Q}(\tau)\widehat{T}(\tau, s)\xi d\tau \\ &\quad - \int_t^s T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\widehat{Q}(\tau)\widehat{T}(\tau, s)\xi d\tau, \\ \widehat{R}(t)\widehat{T}(t, s)\xi &= T(t, s)R(s)\widehat{R}(s)\xi \\ &\quad + \int_0^t T(t, \tau)P(\tau)B(\tau)\widehat{R}(\tau)\widehat{T}(\tau, s)\xi d\tau \\ &\quad - \int_t^s T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\widehat{R}(\tau)\widehat{T}(\tau, s)\xi d\tau. \end{aligned}$$

Taking $t = s$ leads to

$$\begin{aligned} (Q(t) + R(t))\widehat{P}(t)\xi &= - \int_t^\infty T(t, \tau)(Q(\tau) + R(\tau))B(\tau)\widehat{P}(\tau)\widehat{T}(\tau, t)\xi d\tau, \\ P(t)(\widehat{Q}(t) + \widehat{R}(t))\xi &= \int_0^t T(t, \tau)P(\tau)B(\tau)(\widehat{Q}(\tau) + \widehat{R}(\tau))\widehat{T}(\tau, t)\xi d\tau. \end{aligned}$$

Similarly to (3.16) and (3.17), by applying (3.15), (3.19), (3.20) and (H_3) – (H_5) , we obtain

$$\begin{aligned} |(Q(t) + R(t))\widehat{P}(t)| &\leq \widehat{K}K_1N_2c_1|\widehat{P}(t)|, \\ |P(t)(\widehat{Q}(t) + \widehat{R}(t))| &\leq \widehat{K}K_1N_1c_1|\widehat{Q}(t) + \widehat{R}(t)|, \\ |(Q(t) + P(t))\widehat{R}(t)| &\leq \widehat{K}K_1N_2c_1|\widehat{R}(t)|, \\ |R(t)(\widehat{Q}(t) + \widehat{P}(t))| &\leq \widehat{K}K_1N_1c_1|\widehat{Q}(t) + \widehat{P}(t)|, \\ |(R(t) + P(t))\widehat{Q}(t)| &\leq \widehat{K}K_1N_1c_1|\widehat{Q}(t)|, \\ |Q(t)(\widehat{R}(t) + \widehat{P}(t))| &\leq \widehat{K}K_1N_2c_1|\widehat{R}(t) + \widehat{P}(t)|. \end{aligned}$$

Since

$$|P(t)| \leq K_1\mu_1^\epsilon(t), \quad |R(t)| \leq K_1\mu_2^\epsilon(t),$$

$$|Q(t)| \leq K_1 \nu_1^\epsilon(t), \quad |R(t)| \leq K_1 \nu_2^\epsilon(t),$$

we have

$$\begin{aligned} |\widehat{P}(t)| &\leq |(Id - P(t))\widehat{P}(t) - P(t)(Id - \widehat{P}(t))| + |P(t)| \\ &\leq \widehat{K} K_1 N_2 c_1 |\widehat{P}(t)| + \widehat{K} K_1 N_1 c_1 (|\widehat{Q}(t)| + |\widehat{R}(t)|) + |P(t)| \\ &\leq \widehat{K} K_1 (N_2 + N_1) c_1 (|\widehat{P}(t)| + |\widehat{Q}(t)| + |\widehat{R}(t)|) + K_1 \mu_1^\epsilon(t), \\ |\widehat{R}(t)| &\leq |(Id - R(t))\widehat{R}(t) - R(t)(Id - \widehat{R}(t))| + |R(t)| \\ &\leq \widehat{K} K_1 N_2 c_1 |\widehat{R}(t)| + \widehat{K} K_1 N_1 c_1 (|\widehat{Q}(t)| + |\widehat{P}(t)|) + |R(t)| \\ &\leq \widehat{K} K_1 (N_2 + N_1) c_1 (|\widehat{P}(t)| + |\widehat{Q}(t)| + |\widehat{R}(t)|) + K_1 \mu_2^\epsilon(t), \end{aligned}$$

and

$$\begin{aligned} |\widehat{Q}(t)| &\leq |(Id - Q(t))\widehat{Q}(t) - Q(t)(Id - \widehat{Q}(t))| + |Q(t)| \\ &\leq \widehat{K} K_1 N_1 c_1 |\widehat{Q}(t)| + \widehat{K} K_1 N_2 c_1 (|\widehat{P}(t)| + |\widehat{R}(t)|) + |Q(t)| \\ &\leq \widehat{K} K_1 (N_1 + N_2) c_1 (|\widehat{P}(t)| + |\widehat{Q}(t)| + |\widehat{R}(t)|) + K_1 \nu_1^\epsilon(t), \\ |\widehat{R}(t)| &\leq |(Id - R(t))\widehat{R}(t) - R(t)(Id - \widehat{R}(t))| + |R(t)| \\ &\leq \widehat{K} K_1 N_2 c_1 |\widehat{R}(t)| + \widehat{K} K_1 N_1 c_1 (|\widehat{Q}(t)| + |\widehat{P}(t)|) + |R(t)| \\ &\leq \widehat{K} K_1 (N_2 + N_1) c_1 (|\widehat{P}(t)| + |\widehat{Q}(t)| + |\widehat{R}(t)|) + K_1 \nu_2^\epsilon(t). \end{aligned}$$

Therefore,

$$\begin{aligned} |\widehat{P}(t)| + |\widehat{Q}(t)| + |\widehat{R}(t)| &\leq 3\widehat{K} K_1 c_1 (|\widehat{P}(t)| + |\widehat{Q}(t)| + |\widehat{R}(t)|) (N_1 + N_2) \\ &\quad + K_1 (\mu_1^\epsilon(t) + \mu_2^\epsilon(t) + \nu_1^\epsilon(t) + \nu_2^\epsilon(t)). \end{aligned}$$

The proof is complete. \square

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