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WEIGHTED PERSISTENT HOMOLOGY

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ABSTRACT. In this paper, we develop the theory of weighted persistent homology. In 1990, Dawson **[9]** was the first to study in depth the homology of weighted simplicial complexes. We generalize the definitions of weighted simplicial complex and the homology of weighted simplicial complex to allow weights in an integral domain R. Then, we study the resulting weighted persistent homology. We show that weighted persistent homology can distinguish between filtrations that ordinary persistent homology does not distinguish. For example, if there is a point considered as special, weighted persistent homology can tell when a cycle containing the point is formed or has disappeared.

1. Introduction. In topological data analysis, point cloud data refers to a finite set X of points in the Euclidean space $\mathbb{R}^n$, and the computation of persistent homology usually begins with the point cloud data X. In the classical approach of the persistent homology of X, each point in X plays an equally important role, or, in other words, each point has the same weight, cf., **[25]**. Then, X is converted into a simplicial complex, for example, the Čech complex and the Vietoris-Rips complex, cf., **[11,** Chapter 3].

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In this paper, we consider the situation where different points in X may have varying importance. Our point cloud data X is weighted, that is, each point in X has a weight. Some practical examples where it is useful to consider weighted point cloud data are described in subsection 2.1.

Our approach is to weight the boundary map. This is different from existing techniques of introducing weights to persistent homology. For instance, in [**23**] by Petri, et al., weights are introduced via the *weight rank clique filtration* with a thresholding of weights, where, at each step t, the thresholded graph with links of weight are larger than a threshold ϵ_t. In [**13**], the weight of edges is also used to construct a filtration. The theory of weighted simplicial complexes we use is additionally significantly different from the theory of *weighted alpha shapes* [**10**], which are polytopes uniquely determined by points, their weights, and a parameter $\alpha \in \mathbb{R}$ that controls the level of detail.

In his thesis [**7**], Curry utilizes the barcode descriptor from persistent homology to interpret cellular cosheaf homology in terms of Borel-Moore homology of the barcode. In [**7,** page 244], it is briefly mentioned that, for applications, cosheaves should allow us to weight different models of the real world. In a subsequent work by Curry, Ghrist and Nanda [**8**], it was shown how sheaves and sheaf cohomology are powerful tools in computational topology, greatly generalizing persistent homology. An algorithm for simplifying the computation of cellular sheaf cohomology via (discrete) Morse-theoretic techniques is included in [**8**]. In the recent paper [**18**], Kashiwara and Schapira show that many results in persistent homology can be interpreted in the language of microlocal sheaf theory. We note that, in [**18,** page 8], a notion of weight is used, where the closed ball $B(s;t)$ is replaced by $B(s;\rho(s)t)$, where $\rho(s) \in \mathbb{R}_{\geq 0}$ is the weight. This notion of weight is more geometrical, which differs from our more algebraic approach of weighting the boundary operator.

In the seminal paper by Carlsson and Zomorodian [**3**], the theory of multidimensional persistence of multidimensional filtrations is developed. In a subsequent work by Carlsson, Singh and Zomorodian [**2**], a polynomial time algorithm for computing multidimensional persistence is presented. In [**5**], the authors showed that Betti numbers in multidimensional persistent homology are stable functions, in the sense that small changes of the vector-valued filtering functions imply only small changes of persistent Betti number functions. In [**24**], Xia and Wei introduced two families of multidimensional persistence, namely, pseudomultidimensional persistence and multiscale multidimensional persistence, and applied them to analyze biomolecular data. The utility and robustness of the proposed topological methods are effectively demonstrated via protein folding, protein flexibility analysis and various other applications. In [**24,** page 1509], a particle type-dependent weight function w_j is introduced in the definition of the atomic rigidity index μ_i. The atomic rigidity index μ_i can be generalized to a position ($\mathbf{r}$)-dependent rigidity density $\mu(\mathbf{r})$. Subsequently [**24,** page 1512], filtration is performed over the density $\mu(\mathbf{r})$.

The main aim of our paper is to construct weighted persistent homology to study the topology of weighted point cloud data. A weighted simplicial complex is a simplicial complex where each simplex is assigned with a weight. We convert a weighted point cloud data X into a weighted simplicial complex. In [**9**], Dawson was the first to study in depth the homology of weighted simplicial complexes. We use an adaptation of [**9**] to compute the homology of weighted simplicial complexes. In [**9**], the weights take values in the set of non-negative integers $\{0, 1, 2, \ldots\}$. We generalize [**9**] such that the weights can take values in any integral domain R with multiplicative identity 1_R. Finally, we study and analyze the weighted persistent homology of filtered weighted simplicial complexes.

2. Background.

In this section, we review some background knowledge and give some preliminary definitions. We provide some examples of weighted cloud data in subsection 2.1. We review the definitions of simplicial complexes in subsection 2.2 and some properties of rings and integral domains in subsection 2.3. We give the formal definitions of weighted point cloud data and weighted simplicial complex in subsection 2.4.

2.1. Examples of weighted cloud data.

As the motivation of this paper, we describe some practical problems with weight function on data. We look at typical applications of persistent homology and consider the situation where data points may not be equally important. When some data points may be more important than others, mathematically, a weight function is required to give the difference between points.

In the field of computer vision, Carlsson, et al., [**1**] developed a framework for using persistent homology to analyze natural images such as digital photographs. The natural image may be viewed as a vector in a very high-dimensional space $\mathcal{P}$. In the paper, the dimension of $\mathcal{P}$ is the number of pixels in the format used by the camera, and the image is associated to the vector whose coordinates are grey scale values of the pixels. In certain scenarios, such as color detection in computer vision [**6, 20, 22**], each pixel may play different roles depending on its color. In this case, each pixel can then be given a different weight depending on its color. More generally, pixels in images can be weighted depending on their wavelength in the electromagnetic spectrum, which includes infrared and ultraviolet light.

In [**4**], persistent homology was used to study collaboration networks, which measures how scientists collaborate on papers. In the collaboration network, there is a connection between two scientists if they are coauthors on at least one paper. Depending upon the purpose for the research, weights can be used to differentiate different groups of scientists, for example, Ph.D. students, postdoctoral researchers and professors, or researchers in different fields.

Lee, et al., [**19**] proposed a new framework for modeling brain connectivity using persistent homology. The connectivity of the human brain, also known as human connectome, is usually represented as a graph consisting of nodes and edges connecting the nodes. In this scenario, different weights could be assigned to different neurons in different parts of the brain, for example, left/right brain, frontal lobe or temporal lobe.

There are many different ways in which to define the theory of weighted persistent homology, cf., [**9, 16, 23**], and our definition is not unique. We will show that our definition satisfies some nice properties, including some properties related to category theory [**21**], which is an important part of modern mathematics.

In this section, we review the mathematical background necessary for our work. We assume all rings have the multiplicative identity 1. First, we define the concept of weighted point cloud data (WPCD). Then, similar to the unweighted case, we can convert the WPCD to a simplicial complex, using either the Čech complex or the Vietoris-Rips complex. Then, we define a weight function for the simplices so as to compute the weighted simplicial homology.

2.2. Simplicial complexes. The following definition of simplicial complexes can be found in [**15,** page 107]. An (*abstract*) *simplicial complex* is a collection K of nonempty finite sets, called (*abstract*) *simplices* such that, if $\sigma \in K$, then every nonempty subset of σ is in K. Let K be a simplicial complex, and let $\sigma \in K$. An element v of σ is called a *vertex*, and any nonempty subset of σ is called a *face*. For convenience, we do not distinguish between a vertex v and the corresponding face $\{v\}$.

The definition of orientations of simplices is given in [**15,** page 105]. Let $\sigma = \{v_0, v_1, \ldots, v_n\}$ be a simplex of a simplicial complex K. An orientation of σ is given by an ordering of its vertices $v_0, v_1, \ldots, v_n$, with the rule that two orderings define the same orientation if and only if they differ by an even permutation. An oriented simplex σ is written as $[v_0, v_1, \ldots, v_n]$.

Let $\{x_\alpha\}$ be a set of points in the Euclidean space $\mathbb{R}^n$. Let $\epsilon > 0$. The *Čech complex*, denoted as $\mathcal{C}_\epsilon$, is the abstract simplicial complex where $k+1$ vertices span a k-simplex if and only if the $k+1$ corresponding closed $\epsilon/2$-ball neighborhoods of the vertices have nonempty intersection, cf., [**11,** page 72]. The *Vietoris-Rips complex*, denoted as $\mathcal{R}_\epsilon$, is the abstract simplicial complex where $k+1$ vertices span a k-simplex if and only if the distance between any pair of the $k+1$ vertices is at most ϵ, cf., [**11,** page 74].

2.3. Rings. Throughout this section, we let R be a commutative ring with multiplicative identity. A nonzero element $a \in R$ is said to *divide* an element $b \in R$ (denoted $a \mid b$) if there exists an $x \in R$ such that $ax = b$. A nonzero element a in a ring R is called a *zero divisor* if there exists a nonzero $x \in R$ such that $ax = 0$. A commutative ring R

with $1_R \neq 0$ and no zero divisors is called an *integral domain*, cf., [**17,** page 116].

Let R be an integral domain. Let S be the set of all nonzero elements in R. Then, we can construct the *quotient field* $S^{-1}R$, cf., [**17,** page 142].

Proposition 2.1 ([**17,** page 144])**.** *The map $\varphi_s : R \to S^{-1}R$, given by $r \mapsto rs/s$ (for any $s \in S$) is a monomorphism. Hence, the integral domain R can be embedded in its quotient field.*

Remark 2.2. Due to Proposition 2.1, we may identify $rs/s \in S^{-1}R$ with $r \in R$. We denote this as $\varphi_s^{-1}(rs/s) = r$, or simply $rs/s = r$, if there is no danger of confusion.

2.4. Weighted simplicial complexes. In the following definitions, we generalize Dawson's work [**9**] and define *weighted point cloud data* and *weighted simplicial complexes*, with weights in rings.

Definition 2.3 (Weighted point cloud data)**.** Let n be a positive integer. The *point cloud data* X in $\mathbb{R}^n$ is a finite subset of $\mathbb{R}^n$. Given some point cloud data X, a *weight* on X is a function $w_0 : X \to R$, where R is a commutative ring. The pair (X, w_0) is called *weighted point cloud data*, or WPCD for short.

Next, in Definition 2.4, we generalize the definition of *weighted simplicial complex* in [**9,** page 229] to allow for weights in a commutative ring.

Definition 2.4 (cf., [**9,** page 229])**.** A *weighted simplicial complex* (or *WSC*) is a pair (K, w) consisting of a simplicial complex K and a function

$$w : K \longrightarrow R,$$

where R is a commutative ring such that, for any $\sigma_1, \sigma_2 \in K$ with $\sigma_1 \subseteq \sigma_2$, we have $w(\sigma_1) \mid w(\sigma_2)$.

Given any weighted point cloud data (X, w_0), we allow for flexible definitions of extending the weight function w_0 to all higher-dimensional simplices, where the only condition to be satisfied is the divisibility condition in Definition 2.4. One such definition is what we call the *product weighting*.

Definition 2.5 (Product weighting)**.** Let (X, w_0) be a weighted point cloud data, with weight function

$$w_0 : X \longrightarrow R$$

(where R is a commutative ring). Let K be a simplicial complex whose set of vertices is X. We define a weight function $w : K \to R$ by

$$w(\sigma) = \prod_{i=0}^{k} w_0(v_i), \tag{2.1}$$

where $\sigma = [v_0, v_1, \ldots, v_k]$ is a k-simplex of K. We call w, defined as such, the *product weighting*.

Proposition 2.6. *Let (X, w_0) be a weighted point cloud data. Let w be the product weighting defined in Definition* 2.5*. Then, the following hold*:

(i) *the restriction of w to the vertices of K is w_0;*
(ii) *for any $\sigma_1, \sigma_2 \in K$, if $\sigma_1 \subseteq \sigma_2$, then $w(\sigma_1) \mid w(\sigma_2)$.*

Proof. Firstly, if $\sigma = [v_0]$ is a vertex of K (0-simplex), then $w(\sigma) = w_0(v_0)$ by (2.1). For the second assertion, suppose that $\sigma_1 \subseteq \sigma_2$, where $\sigma_1 = [v_0, \ldots, v_k]$ and $\sigma_2 = [v_0, \ldots, v_k, \ldots, v_l]$. Then,

$$w(\sigma_2) = w(\sigma_1) \cdot \prod_{i=k+1}^{l} w_0(v_i). \qquad \square$$

For commutative rings such that every two elements have an LCM (for instance, UFDs), we can use the economical weighting in [**9,** page 231] instead, where the weight of any simplex is the LCM of the weights of its faces.

3. Properties of weighted simplicial complexes. In this section, we prove some properties of weighted simplicial complexes. We consider the case where R is a commutative ring with 1. We now consider subcomplexes given by the preimage of the weight function with values in ideals. This may have the meaning to take out partial data according to the values of the weight function.

Lemma 3.1. *Let I be an ideal of a commutative ring R. Let (K, w) be a* WSC*, where*

$$w : K \longrightarrow R$$

is a weight function. Let $w^{-1}(I)$ denote the preimage of I under w. If $\sigma \in w^{-1}(I)$, then, for all simplices τ containing σ, we have $\tau \in w^{-1}(I)$.

Proof. Let $\sigma \in w^{-1}(I)$, i.e., $w(\sigma) \in I$. By Definition 2.4, for $\sigma \subseteq \tau$, we have $w(\sigma) \mid w(\tau)$. Hence, $w(\tau) = w(\sigma)x$ for some $x \in R$. Since I is an ideal, thus, $w(\tau) \in I$. $\square$

Theorem 3.2. *Let I be an ideal of a commutative ring R. Let (K, w) be a* WSC*, where $w : K \to R$ is a weight function. Then, $K \setminus w^{-1}(I)$ is a simplicial subcomplex of K.*

Proof. If $K \setminus w^{-1}(I) = \emptyset$, then it is the empty subcomplex of K. Otherwise, let $\tau \in K \setminus w^{-1}(I)$. Let σ be a nonempty subset of τ. Suppose, to the contrary, that $\sigma \in w^{-1}(I)$. Then, by Lemma 3.1, we have $\tau \in w^{-1}(I)$, which is a contradiction. Hence, $\sigma \in K \setminus w^{-1}(I)$; thus, we have proved that $K \setminus w^{-1}(I)$ is a simplicial complex. $\square$

Proposition 3.3. *Let I and J be ideals of a commutative ring R. Let (K, w) be a* WSC*. Then*:

$$K \setminus w^{-1}(I \cap J) = (K \setminus w^{-1}(I)) \cup (K \setminus w^{-1}(J)) \tag{3.1}$$

is a simplicial subcomplex of K.

Proof. We have that

$$\sigma \in K \setminus w^{-1}(I \cap J) \iff w(\sigma) \notin I \cap J$$

$$\iff w(\sigma) \notin I \text{ or } w(\sigma) \notin J$$
$$\iff \sigma \in (K \setminus w^{-1}(I)) \cup (K \setminus w^{-1}(J)).$$

Hence, equation (3.1) holds. Since I, J are ideals, by Theorem 3.2, both $K \setminus w^{-1}(I)$ and $K \setminus w^{-1}(J)$ are simplicial subcomplexes of K and so is their union. Alternatively, we can apply Theorem 3.2 to the ideal $I \cap J$ to conclude that $K \setminus w^{-1}(I \cap J)$ is a simplicial subcomplex of K. □

3.1. Categorical properties of WSC. Let K and L be simplicial complexes. A map $f : K \to L$ is called a *simplicial map* if it sends each simplex of K to a simplex of L by a linear map taking vertices to vertices, that is, if the vertices $v_0, \dots, v_n$ of K span a simplex of K, the points $f(v_0), \dots, f(v_n)$ (not necessarily distinct) span a simplex of L.

Next, we will use some terminology from category theory. For an introduction to the subject, the book by Mac Lane [**21**] is recommended. The categorical properties of WSCs have been studied in [**9**]. Here, we mainly show that it easily generalizes to the case where weights lie in a ring, and we write it in greater detail.

Definition 3.4 ([**21,** page 13])**.** Let C and B be categories. A *functor* $T : C \to B$ with domain C and codomain B consists of two suitably related functions: the object function T, which assigns to each object c of C an object Tc of B, and the arrow function (also written as T) which assigns to each arrow $f : c \to c'$ of C an arrow

$$Tf : Tc \to Tc' \quad \text{of } B,$$

such that

$$T(1_c) = 1_{Tc}, \qquad T(g \circ f) = Tg \circ Tf,$$

where the latter holds whenever the composite $g \circ f$ is defined in C.

In [**9,** page 229], morphisms of weighted simplicial complexes with integral weights have been studied. In the following definition, we generalize [**9**] and define morphisms of weighted simplicial complexes with weights in general commutative rings.

Definition 3.5 (cf., [**9,** page 229])**.** Let (K, w_K) and (L, w_L) be WSCs. A *morphism of WSCs* is a simplicial map $f : K \to L$ such that

$$w_L(f(\sigma)) \mid w_K(\sigma) \quad \text{for all } \sigma \in K.$$

These form the morphisms of a category **WSC**. We may omit the subscripts in w_K and w_L, for instance, writing $w(f(\sigma)) \mid w(\sigma)$, if there is no danger of confusion.

The next example generalizes [**9,** page 229].

Example 3.6 (cf., [**9,** page 229])**.** For any simplicial complex K and every $a \in R$, there is a WSC (K, a) in which every simplex (in particular, every vertex) has weight a. We call this construction a *constant weighting.*

Let **SC** denote the category of simplicial complexes.

Proposition 3.7 (cf., [**9,** page 229])**.** *Constant weightings are functorial*: *let*

$$T : \mathbf{SC} \longrightarrow \mathbf{WSC}$$

be defined by $TK = (K, a)$ *for each simplicial complex* $K \in \mathbf{SC}$ *and* $Tf = f$ *for each simplicial map* $f \in \mathbf{SC}$. *Then,* T *is a functor.*

Proof. Straightforward verification. Note that the condition $w(f(\sigma)) \mid w(\sigma)$ in Definition 3.5 is trivially satisfied since $a \mid a$ for all $a \in R$. $\square$

Definition 3.8 ([**21,** page 80])**.** Let A and X be categories. An *adjunction* from X to A is a triple $\langle F, G, \varphi \rangle$, where

$$F : X \longrightarrow A \quad \text{and} \quad G : A \longrightarrow X$$

are functors, and φ is a function which assigns to each pair of objects $x \in X$, $a \in A$ a bijection of sets

$$\varphi = \varphi_{x,a} : A(Fx, a) \cong X(x, Ga),$$

which is natural in x and a. An adjunction may also be directly described in terms of arrows. It is a bijection which assigns to each arrow $f : Fx \to a$ an arrow

$$\varphi f = \operatorname{rad} f : x \longrightarrow Ga,$$

the *right adjunct* of f, such that

$$\varphi(k \circ f) = Gk \circ \varphi f, \qquad \varphi(f \circ Fh) = \varphi f \circ h$$

hold for all f and all arrows $h : x' \to x$ and $k : a \to a'$. Given such an adjunction, the functor F is said to be a *left adjoint* for G, while G is called a *right adjoint* for F.

One reason for generalizing the weights to take values in rings with 1 is to keep the following, nice proposition true.

Proposition 3.9 (cf., [**9,** page 229])**.** *The constant weighting functors* $T_1 := (-, 1_R)$ *and* $T_0 := (-, 0_R)$ *are, respectively, right and left adjoint to the forgetful functor* U *from* **WSC** *to the category* **SC** *of simplicial complexes.*

Proof. Let φ be a bijection that assigns to each arrow

$$f : U(K, w) \longrightarrow L$$

an arrow

$$\varphi f : (K, w) \longrightarrow (L, 1),$$

where $\varphi f(\sigma) = f(\sigma)$. The key point is that the condition for WSC morphism (Definition 3.5), namely, $1 \mid w(\sigma)$, always holds for all $\sigma \in (K, w)$. Then, for all arrows

$$h : (K, w) \longrightarrow (K', w') \quad \text{and} \quad k : L \longrightarrow L',$$

we have $\varphi(k \circ f) = k \circ f = Uk \circ \varphi f$ and $\varphi(f \circ Uh) = f \circ Uh = \varphi f \circ h$. Thus, T_1 is the right adjoint for U.

Let ψ be a bijection that assigns to each arrow

$$f' : (K, 0) \longrightarrow (L, w')$$

an arrow

$$\psi f' : K \longrightarrow U(L, w'),$$

where $\psi f'(\sigma) = f'(\sigma)$. The key point is that $w'(f'(\sigma)) \mid 0$ always holds for all $\sigma \in (K, 0)$. Similarly, we can conclude that T_0 is the left adjoint for U. $\square$

4. Homology of weighted simplicial complexes. In this section, we let R be an integral domain, in order to form the field of fractions (also known as quotient field) which is necessary for our purposes.

4.1. Chain complex. A *chain complex* $(C_\bullet, \partial_\bullet)$ is a sequence of abelian groups or modules

$$\dots, C_2, C_1, C_0, C_{-1}, C_{-2}, \dots$$

connected by homomorphisms (called boundary homomorphisms)

$$\partial_n : C_n \longrightarrow C_{n-1},$$

such that $\partial_n \circ \partial_{n+1} = 0$ for each n. A chain complex is usually written out as:

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$
$$\longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \longrightarrow \cdots .$$

A *chain map* f between two chain complexes $(A_\bullet, \partial_{A,\bullet})$ and $(B_\bullet, \partial_{B,\bullet})$ is a sequence $f_\bullet$ of module homomorphisms

$$f_n : A_n \longrightarrow B_n$$

for each n that commutes with the boundary homomorphisms on the two chain complexes:

$$\partial_{B,n} \circ f_n = f_{n-1} \circ \partial_{A,n}.$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial_{A,n+1}} & A_n & \xrightarrow{\partial_{A,n}} & A_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial_{B,n+1}} & B_n & \xrightarrow{\partial_{B,n}} & B_{n-1} & \longrightarrow & \cdots .
\end{array}$$

4.2. Homology groups. For a topological space X and a chain complex $C(X)$, the *nth homology group* of X is $H_n(X) := \ker(\partial_n)/\mathrm{Im}(\partial_{n+1})$. Elements of $B_n(X) := \mathrm{Im}(\partial_{n+1})$ are called *boundaries*, and elements of $Z_n(X) := \ker(\partial_n)$ are called *cycles*.

Proposition 4.1. *A chain map $f_\bullet$ between chain complexes $(A_\bullet, \partial_{A,\bullet})$ and $(B_\bullet, \partial_{B,\bullet})$ induces homomorphisms between the homology groups of the two complexes.*

Proof. The relation $\partial f = f\partial$ implies that f takes cycles to cycles since $\partial\alpha = 0$ implies $\partial(f\alpha) = f(\partial\alpha) = 0$. In addition, f takes boundaries to boundaries since $f(\partial\beta) = \partial(f\beta)$.

For $\beta \in \mathrm{Im}\,\partial_{A,n+1}$, we have $\pi_{B,n} f_n(\beta) = \mathrm{Im}\,\partial_{B,n+1}$. Therefore,

$$\mathrm{Im}\,\partial_{A,n+1} \subseteq \ker(\pi_{B,n} \circ f_n).$$

By the universal property of quotient groups, there exists a unique homomorphism $(f_n)_*$ such that the following diagram commutes.

$$\begin{array}{ccccc} \ker\partial_{A,n} & \xrightarrow{f_n} & \ker\partial_{B,n} & \xrightarrow{\pi_{B,n}} & H_n(B_\bullet) = \ker\partial_{B,n}/\mathrm{Im}\,\partial_{B,n+1} \\ & \searrow^{\pi_{A,n}} & & \nearrow_{(f_n)_*} & \\ & & H_n(A_\bullet) = \ker\partial_{A,n}/\mathrm{Im}\,\partial_{A,n+1} & & \end{array}$$

Hence, $f_\bullet$ induces a homomorphism

$$(f_\bullet)_* : H_\bullet(A_\bullet) \longrightarrow H_\bullet(B_\bullet). \qquad \square$$

Definition 4.2. Let $C_n(K, w)$ (or simply $C_n(K)$, where unambiguous) be the free R-module with basis the n-simplices of K with nonzero weight. Elements of $C_n(K)$, called *n-chains*, are finite formal sums $\sum_\alpha n_\alpha \sigma_\alpha$ with coefficients $n_\alpha \in R$ and $\sigma_\alpha \in K$.

Definition 4.3. Given a simplicial map $f : K \to L$, the induced homomorphism

$$f_\sharp : C_n(K) \longrightarrow C_n(L)$$

is defined on the generators of $C_n(K)$ (and extended linearly) as follows. For $\sigma = [v_0, v_1, \ldots, v_n] \in C_n(K)$, we define

$$f_\sharp(\sigma) = \begin{cases} \frac{w(\sigma)}{w(f(\sigma))} f(\sigma) & \text{if } f(v_0), \ldots, f(v_n) \text{ are distinct,} \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where $w(\sigma)/w(f(\sigma)) \in S^{-1}R$ is identified with the corresponding element in R, as described in Remark 2.2.

Note that this is well defined since, if $w(\sigma) \neq 0$, then $w(f(\sigma)) \mid w(\sigma)$ in Definition 3.5 implies $w(f(\sigma)) \neq 0$. Thus, $w(\sigma)/w(f(\sigma)) \in S^{-1}R$. Furthermore,

$$\frac{w(\sigma)}{w(f(\sigma))} = \frac{xw(f(\sigma))}{w(f(\sigma))}$$

for some $x \in R$, so that $w(\sigma)/w(f(\sigma)) = x \in R$.

Definition 4.4 (cf., [**9,** page 234])**.** The *weighted boundary map* $\partial_n : C_n(K) \to C_{n-1}(K)$ is the map:

$$\partial_n(\sigma) = \sum_{i=0}^{n} \frac{w(\sigma)}{w(d_i(\sigma))} (-1)^i d_i(\sigma),$$

where the *face maps* d_i are defined as:

$$d_i(\sigma) = [v_0, \ldots, \widehat{v_i}, \ldots, v_n] \quad \text{(deleting the vertex } v_i\text{)}$$

for any n-simplex $\sigma = [v_0, \ldots, v_n]$.

Again, if $w(\sigma) \neq 0$, then $w(d_i(\sigma)) \neq 0$ so ∂_n is well defined. Similarly, we identify $w(\sigma)/w(d_i(\sigma)) \in S^{-1}R$ with the corresponding element in R, as described in Remark 2.2.

Next, we show that, after generalization to weights in an integral domain, the relation $\partial^2 = 0$ [**9,** page 234] of the weighted boundary map remains true.

Proposition 4.5 (cf., [**9,** page 234])**.** $\partial^2 = 0$. *To be precise, the composition*

$$C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} C_{n-2}(K)$$

is the zero map.

Proof. Let $\sigma = [v_0, \dots, v_n]$ be an n-simplex. We have

$$\partial_n(\sigma) = \sum_{i=0}^{n} \frac{w(\sigma)}{w([v_0, \dots, \widehat{v_i}, \dots, v_n])} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n].$$

Hence,

$$\begin{aligned}
&\partial_{n-1}\partial_n(\sigma) \\
&= \sum_{j<i} \frac{w(\sigma)}{w(d_i(\sigma))} (-1)^i \frac{w(d_i(\sigma))}{w([v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n])} \\
&\quad \cdot (-1)^j [v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n] \\
&\quad + \sum_{j>i} \frac{w(\sigma)}{w(d_i(\sigma))} (-1)^i \frac{w(d_i(\sigma))}{w([v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n])} \\
&\quad \cdot (-1)^{j-1} [v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n] \\
&= \sum_{j<i} \frac{w(\sigma)}{w([v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n])} (-1)^{i+j} [v_0, \dots, \widehat{v_j}, \dots, \widehat{v_i}, \dots, v_n] \\
&\quad + \sum_{j>i} \frac{w(\sigma)}{w([v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n])} \\
&\quad \cdot (-1)^{i+j-1} [v_0, \dots, \widehat{v_i}, \dots, \widehat{v_j}, \dots, v_n] \\
&= 0.
\end{aligned}$$

The latter two summations cancel since, after switching i and j in the second sum, it becomes the additive inverse of the first. □

Lemma 4.6. *Let $f : K \to L$ be a simplicial map and d_i the ith face map. Then*

$$d_i(f(\sigma)) = f(d_i(\sigma)) \tag{4.2}$$

for all $\sigma = [v_0, v_1, \dots, v_n] \in K$ with $f(v_0), \dots, f(v_n)$ are distinct.

Proof. Let $\sigma = [v_0, \dots, v_n]$. Then, we have

$$\begin{aligned}
d_i(f(\sigma)) &= d_i[f(v_0), \dots, f(v_n)] \\
&= [f(v_0), \dots, \widehat{f(v_i)}, \dots, f(v_n)]
\end{aligned}$$

$$
\begin{aligned}
&= f([v_0, \ldots, \widehat{v_i}, \ldots, v_n]) \\
&= f(d_i(\sigma)). \qquad \square
\end{aligned}
$$

Proposition 4.7. *Let $f : K \to L$ be a simplicial map. Then, $f_\sharp \partial = \partial f_\sharp$.*

Proof. Let $\sigma = [v_0, \ldots, v_n] \in C_n(K)$. Let τ be the simplex of L spanned by $f(v_0), \ldots, f(v_n)$. We consider three cases.

Case 1. $\dim \tau = n$. In this case, the vertices $f(v_0), \ldots, f(v_n)$ are distinct. We have:

$$
\begin{aligned}
f_\sharp \partial(\sigma) &= f_\sharp\left(\sum_{i=0}^{n} \frac{w(\sigma)}{w(d_i(\sigma))}(-1)^i d_i(\sigma)\right) \\
&= \sum_{i=0}^{n} \frac{w(\sigma)}{w(d_i(\sigma))}(-1)^i f_\sharp(d_i(\sigma)) \\
&= \sum_{i=0}^{n} \frac{w(\sigma)}{w(d_i(\sigma))}(-1)^i \frac{w(d_i(\sigma))}{w(f(d_i(\sigma)))} f(d_i(\sigma)) \\
&= \sum_{i=0}^{n} \frac{w(\sigma)}{w(f(d_i(\sigma)))}(-1)^i f(d_i(\sigma)).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\partial f_\sharp(\sigma) &= \partial\left(\frac{w(\sigma)}{w(f(\sigma))} f(\sigma)\right) \\
&= \sum_{i=0}^{n} \frac{w(\sigma)}{w(f(\sigma))} \cdot \frac{w(f(\sigma))}{w(d_i(f(\sigma)))}(-1)^i d_i(f(\sigma)) \\
&= \sum_{i=0}^{n} \frac{w(\sigma)}{w(d_i(f(\sigma)))}(-1)^i d_i(f(\sigma)) \\
&= f_\sharp \partial(\sigma)
\end{aligned}
$$

since $d_i(f(\sigma)) = f(d_i(\sigma))$ by Lemma 4.6.

Case 2. $\dim \tau \leq n - 2$. In this case, $f_\sharp(d_i(\sigma)) = 0$ for all i, since at least two of the points $f(v_0), \ldots, f(v_{i-1}), f(v_{i+1}), \ldots, f(v_n)$ are the same. Thus, $f_\sharp\partial(\sigma)$ vanishes. Note that $\partial f_\sharp(\sigma)$ also vanishes since $f_\sharp(\sigma) = 0$, due to the fact that $f(v_0), \dots, f(v_n)$ are not distinct.

Case 3. $\dim\tau = n - 1$. Without loss of generality, we may assume that the vertices are ordered such that $f(v_0) = f(v_1)$, and $f(v_1), \dots, f(v_n)$ are distinct. Then, $\partial f_\sharp(\sigma)$ vanishes. Now,

$$f_\sharp\partial(\sigma) = \sum_{i=0}^{n} \frac{w(\sigma)}{w(d_i(\sigma))}(-1)^i f_\sharp(d_i(\sigma))$$

has only two nonzero terms, which sum up to

$$\begin{aligned} &\frac{w(\sigma)}{w(d_0(\sigma))} \cdot \frac{w(d_0(\sigma))}{w(f(d_0(\sigma)))} f(d_0(\sigma)) \\ &\qquad - \frac{w(\sigma)}{w(d_1(\sigma))} \cdot \frac{w(d_1(\sigma))}{w(f(d_1(\sigma)))} f(d_1(\sigma)) \\ &\quad = \frac{w(\sigma)}{w(f(d_0(\sigma)))} f(d_0(\sigma)) - \frac{w(\sigma)}{w(f(d_1(\sigma)))} f(d_1(\sigma)). \end{aligned}$$

Since $f(v_0) = f(v_1)$, we have $f(d_0(\sigma)) = f(d_1(\sigma))$, and hence, the two terms cancel each other, as desired. □

Definition 4.8. We define the weighted homology group

$$H_n(K, w) := \ker(\partial_n)/\operatorname{Im}(\partial_{n+1}), \tag{4.3}$$

where ∂_n is the weighted boundary map, defined in Definition 4.4.

Since the maps

$$f_\sharp : C_n(K, w_K) \longrightarrow C_n(L, w_L)$$

satisfy $f_\sharp\partial = \partial f_\sharp$, the $f_\sharp$'s define a chain map from the chain complex of (K, w_K) to that of (L, w_L). By Proposition 4.1, $f_\sharp$ induces a homomorphism

$$f_* : H_n(K, w_K) \longrightarrow H_n(L, w_L).$$

We may then view the map

$$(K, w_K) \longmapsto H_n(K, w_K)$$

as a functor

$$H_n : \mathbf{WSC} \longrightarrow \mathbf{R\text{-}Mod}$$

from the category of weighted simplicial complexes (**WSC**) to the category of R-modules (**R-Mod**).

4.3. Calculation of homology groups in WSC. The homology functor we define is different from the standard simplicial homology functor. For instance, it is possible for H_0 of a weighted simplicial complex to have torsion when the coefficient ring is $\mathbb{Z}$, as shown in [**9,** page 237]. We illustrate this more generally in the following example.

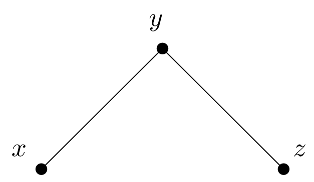


FIGURE 1. Simplicial complex with three vertices x, y and z.

Example 4.9 (cf., [**9,** page 237])**.** Let $R = \mathbb{Z}$. Consider (K, w), where w is the product weighting, to be the WSC shown in Figure 1, with

$$w(x) = 1, \qquad w(y) = n \quad \text{and} \quad w(z) = 1,$$

where $n \in \mathbb{Z}$, $n \geq 2$. Then,

$$\partial_1([x,y]) = \frac{w([x,y])}{w(y)} y - \frac{w([x,y])}{w(x)} x = \frac{n}{n} y - \frac{n}{1} x = y - nx.$$

Similarly, $\partial_1([y,z]) = nz - y$. Thus,

$$\begin{aligned} H_0(K,w) &= \ker \partial_0 / \operatorname{Im} \partial_1 \\ &\cong \langle x, y, z \mid nx = y, y = nz \rangle \\ &\cong \langle x, z \mid nx = nz \rangle \\ &\cong \mathbb{Z} \oplus \mathbb{Z}_n. \end{aligned}$$

Proposition 4.10 (cf., [**9,** page 239])**.** *For the constant weighting* (K, a), $a \in R \setminus \{0\}$, *the weighted homology functor is the same as the standard simplicial homology functor.*

Proof. If every simplex has weight $a \in R \setminus \{0\}$, note that the chain maps in Definition 4.3 and the weighted boundary maps in Definition 4.4 reduce to the usual ones in standard simplicial homology. Hence, the resulting weighted homology functor reduces to the standard one. □

5. Weighted persistent homology. After defining weighted homology, we proceed to define weighted persistent homology, following the example of the seminal paper by Zomorodian and Carlsson [**25**]. First, we provide a review of persistence [**14, 25**], with generalizations to the weighted case.

5.1. Persistence.

Definition 5.1. A *weighted filtered complex* is an increasing sequence of weighted simplicial complexes $(\mathcal{K}, w) = \{(K^i, w)\}_{i\geq 0}$, such that $K^i \subseteq K^{i+1}$ for all integers $i \geq 0$. (The weighting on K^i is a restriction of that on K^j for $i < j$.)

Given a weighted filtered complex, for the ith complex K^i, we define the associated weighted boundary maps ∂_k^i and groups C_k^i, Z_k^i, B_k^i and H_k^i for all integers $i, k \geq 0$, following our development in Section 4.

Definition 5.2. The *weighted boundary map* $\partial_k^i : C_k(K^i) \to C_{k-1}(K^i)$ is the map

$$\partial_k : C_k(K^i) \longrightarrow C_{k-1}(K^i)$$

as defined in Definition 4.4. The *weighted chain group* C_k^i is the group $C_k(K^i, w)$ in Definition 4.2. The *weighted cycle group* Z_k^i is the group $\ker(\partial_k^i)$, while the *weighted boundary group* B_k^i is the group $\mathrm{Im}(\partial_{k+1}^i)$. The *weighted homology group* H_k^i is the quotient group Z_k^i/B_k^i. (If the context is clear, we may omit the adjective "weighted".)

Definition 5.3 (cf., [**25,** page 6]). The weighted *p-persistent kth homology group* of $(\mathcal{K}, w) = \{(K^i, w)\}_{i\geq 0}$ is defined as

$$H_k^{i,p}(\mathcal{K}, w) := Z_k^i/(B_k^{i+p} \cap Z_k^i). \tag{5.1}$$

If the coefficient ring R is a PID and all the K^is are finite simplicial complexes, then $H_k^{i,p}$ is a finitely generated module over a PID. We can then define the *p-persistent kth Betti number* of (K^i, w), denoted by $\beta_k^{i,p}$, to be the rank of the free submodule of $H_k^{i,p}$. This is well defined by the structure theorem for finitely generated modules over a PID.

Consider the homomorphism

$$\eta_k^{i,p} : H_k^i \longrightarrow H_k^{i+p}$$

that maps a homology class into the one that contains it. To be precise,

$$n_k^{i,p}(\alpha + B_k^i) = \alpha + B_k^{i+p}. \tag{5.2}$$

The homomorphism $\eta_k^{i,p}$ is well defined since, if $\alpha_1 + B_k^i = \alpha_2 + B_k^i$, then $\alpha_1 - \alpha_2 \in B_k^i \subseteq B_k^{i+p}$. We prove that, similar to the unweighted case (cf., [**12, 25, 26**]), we have $\operatorname{Im} \eta_k^{i,p} \cong H_k^{i,p}$.

Proposition 5.4 (cf., [**25,** page 6]). $\operatorname{Im} \eta_k^{i,p} \cong H_k^{i,p}$.

Proof. By the first isomorphism theorem, we have

$$\operatorname{Im} \eta_k^{i,p} \cong H_k^i / \ker \eta_k^{i,p}.$$

Note that

$$\begin{aligned} &\alpha + B_k^i \in \ker \eta_k^{i,p} \\ &\quad \iff \alpha + B_k^{i+p} = B_k^{i+p} \quad \text{and} \quad \alpha \in Z_k^i \\ &\quad \iff \alpha \in B_k^{i+p} \cap Z_k^i \\ &\quad \iff \alpha + B_k^i \in (B_k^{i+p} \cap Z_k^i)/B_k^i. \end{aligned} \tag{5.3}$$

Hence,

$$\ker \eta_k^{i,p} = (B_k^{i+p} \cap Z_k^i)/B_k^i.$$

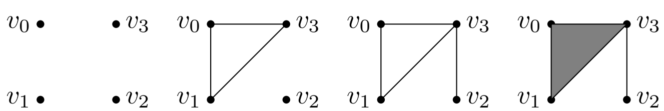
Hence, we have

$$\begin{aligned}\operatorname{Im}\eta_k^{i,p} &\cong H_k^i/\ker\eta_k^{i,p}\\ &= \frac{Z_k^i/B_k^i}{(B_k^{i+p}\cap Z_k^i)/B_k^i}\\ &\cong Z_k^i/(B_k^{i+p}\cap Z_k^i)\end{aligned}$$

by the third isomorphism theorem

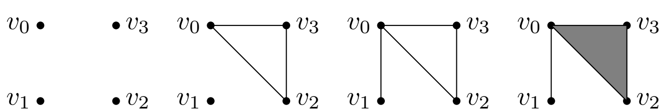
$$= H_k^{i,p}.$$ □

6. Applications. Weighted persistent homology can distinguish filtrations that ordinary persistent homology does not distinguish. For instance, if there is a special point, weighted persistent homology can tell when a cycle containing the point is formed or has disappeared. This is a generalization of the main feature of persistent homology, which is to detect the "birth" and "death" of cycles. This is illustrated in the following example.



(A) K^0. (B) K^1. (C) K^2. (D) K^3.

FIGURE 2. The filtration $\mathcal{K} = \{K^0, K^1, K^2, K^3\}$, where the shaded region denotes the 2-simplex $[v_0, v_1, v_3]$.



(A) L^0. (B) L^1. (C) L^2. (D) L^3.

FIGURE 3. The filtration $\mathcal{L} = \{L^0, L^1, L^2, L^3\}$, where the shaded region denotes the 2-simplex $[v_0, v_2, v_3]$.

Example 6.1. Consider the two filtrations as shown in Figures 2 and 3. By symmetry, it is clear that the (unweighted) persistent homology groups of the two filtrations will be the same.

Suppose that we consider v_2 as a special point and wish to tell through weighted persistent homology whether a 1-cycle containing v_2 is formed or has disappeared. We can achieve it by the following weight function (choosing $R = \mathbb{Z}$). Let w be the weight function such that all two-dimensional (and higher) simplices containing v_2 have weight 2, while all other simplices have weight 1. In our example, this means that $w([v_0, v_2, v_3]) = 2$, while $w(\sigma) = 1$ for all $\sigma \neq [v_0, v_2, v_3]$.

Then, for the filtration $\mathcal{K} = \{K^0, K^1, K^2, K^3\}$, we have

$$\begin{aligned} Z_1^1 &= \ker(\partial_1^1) = \langle [v_0, v_1] - [v_0, v_3] + [v_1, v_3] \rangle \\ \partial_2^3([v_0, v_1, v_3]) &= [v_1, v_3] - [v_0, v_3] + [v_0, v_1] \\ \partial_2^1 &= \partial_2^2 = 0. \end{aligned}$$

Hence, we have

$$H_1^{1,p}(\mathcal{K}, w) = \begin{cases} \mathbb{Z} & \text{for } p = 0, 1 \\ 0 & \text{for } p = 2. \end{cases}$$

However, for the filtration $\mathcal{L} = \{L^0, L^1, L^2, L^3\}$, we have

$$\begin{aligned} Z_1^1 &= \ker(\partial_1^1) = \langle [v_2, v_3] - [v_0, v_3] + [v_0, v_2] \rangle \\ \partial_2^3([v_0, v_2, v_3]) &= 2[v_2, v_3] - 2[v_0, v_3] + 2[v_0, v_2] \\ \partial_2^1 &= \partial_2^2 = 0 \end{aligned}$$

so that

$$H_1^{1,p}(\mathcal{L}, w) = \begin{cases} \mathbb{Z} & \text{for } p = 0, 1 \\ \mathbb{Z}_2 & \text{for } p = 2. \end{cases} \tag{6.1}$$

Referring to equation (6.1), we can interpret the presence of torsion in $H_1^{1,2}(\mathcal{L}, w)$ to mean that a 1-cycle containing v_2 is formed in L^1, persists in L^2, and disappears in L^3.

Remark 6.2. Let $R = \mathbb{Z}$. Generalizing Example 6.1, if there is a special point v, we can tell if a k-cycle containing v is formed or has disappeared by setting all $k+1$-dimensional and higher simplices containing v to have weight $m \geq 2$, and all other simplices to have weight 1.

6.1. Algorithm for PIDs. For coefficients in a PID, we show that the weighted persistent homology groups are computable. In the seminal paper [**25**] by Zomorodian and Carlsson, the authors gave an algorithm for persistent homology over a PID. We present an algorithm in this section, which is a weighted modification of the algorithm in [**25**] based on the reduction algorithm. We use Figure 3 as a running example to illustrate the algorithm.

Let R be a PID. We represent the weighted boundary operator $\partial_n : C_n(K, w) \to C_{n-1}(K, w)$ relative to the standard bases (the standard basis for $C_n(K, w)$ is the set of n-simplices of K with nonzero weight, see Definition 4.2) of the respective weighted chain groups as a matrix M_n with entries in R. The matrix M_n is called the *standard matrix representation* of ∂_n. It has m_n columns and m_{n-1} rows, where m_n and m_{n-1} are the number of n- and $(n-1)$-simplices with nonzero weights, respectively.

In general, due to the weights, the matrix M_n for the weighted boundary map is *different* from that of the unweighted case. For instance, for the unweighted case, the matrix representation is restricted to having entries in $\{-1_R, 0_R, 1_R\}$, while the weighted matrix representation can have entries taking arbitrary values in the ring$R$. In particular, when performing the reduction algorithm, we need to make the modification to allow the following *elementary row operations* on M_k:

(1) exchange row i and row j;
(2) multiply row i by a unit $u \in R \setminus \{0\}$;
(3) replace row i by (row i)+q(row j), where $q \in R\setminus\{0\}$ and $j \neq i$.

Note that, for the unweighted case [**25,** page 5], the second elementary row operation was "multiply row i by -1." A similar modification is also needed for the *elementary column operations*.

The subsequent steps are similar to those of the unweighted case, cf., [**25,** pages 5, 12]. We summarize the algorithm (Algorithm 1) and refer the reader to [**25,** page 5] for more information on the reduction algorithm and the Smith normal form.

Given a weighted filtered complex $\{(K^i, w)\}_{i\geq 0}$, we write M_k^i to denote the standard matrix representation of ∂_k^i. We perform Algorithm 1 to obtain the weighted homology groups.

Algorithm 1 Weighted persistent homology algorithm for PIDs, cf., [**25,** page 12].

Input: Weighted filtered complex $(\mathcal{K}, w) = \{(K^i, w)\}_{i\geq 0}$
Output: Weighted p-persistent kth homology group $H_k^{i,p}(\mathcal{K}, w)$

(1) Reduce the matrix M_k^i to its Smith normal form and obtain a basis $\{z^j\}$ for Z_k^i.
(2) Reduce the matrix M_{k+1}^{i+p} to its Smith normal form and obtain a basis $\{b^l\}$ for B_k^{i+p}.
(3) Let $A = [\{b^l\} \ \{z^j\}] = [B \ Z]$, i.e., the columns of matrix A consist of the basis elements computed in the previous steps, with respect to the standard basis of $C_k(K^{i+p}, w)$. We reduce A to its Smith normal form to find a basis $\{a^q\}$ for its nullspace.
(4) Each $a^q = [\alpha^q \ \beta^q]$, where α^q, β^q are column vectors of coefficients of $\{b^l\}$, $\{z^j\}$, respectively. Since $Au^q = B\alpha^q + Z\beta^q = 0$, the element $\beta\alpha^q = -Z\beta^q$ belongs to the span of both bases $\{z^j\}$ and $\{b^l\}$. Hence, both $\{B\alpha^q\}$ and $\{Z\beta^q\}$ are bases for $B_k^{i,p} = B_k^{i+p}\cap Z_k^i$. Using either, we form the matrix $M_{k+1}^{i,p}$ using the basis. The number of columns of $M_{k+1}^{i,p}$ is the cardinality of the basis for $B_k^{i,p}$, while the number of rows is the cardinality of the standard basis for $C_k(K^{i+p}, w)$.
(5) We reduce $M_{k+1}^{i,p}$ to Smith normal form to read off the torsion coefficients of $H_k^{i,p}(\mathcal{K}, w)$ and the rank of $B_k^{i,p}$.
(6) The rank of the free submodule of $H_k^{i,p}(\mathcal{K}, w)$ is the rank of Z_k^i minus the rank of $B_k^{i,p}$.

We illustrate the algorithm using Example 6.3.

Example 6.3. Consider the filtration $\mathcal{L} = \{L^0, L^1, L^2, L^3\}$ in Figure 3. We have

(6.2)
$$M_1^1 = \left[\begin{array}{c|ccc} & [v_0,v_3] & [v_0,v_2] & [v_2,v_3] \\ \hline v_0 & -1 & -1 & 0 \\ v_1 & 0 & 0 & 0 \\ v_2 & 0 & 1 & -1 \\ v_3 & 1 & 0 & 1 \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{c|ccc} & [v_0,v_3] & [v_0,v_2] & [v_2,v_3]-[v_0,v_3]+[v_0,v_2] \\ \hline v_3-v_0 & 1 & 0 & 0 \\ v_2-v_0 & 0 & 1 & 0 \\ v_1 & 0 & 0 & 0 \\ v_2 & 0 & 0 & 0 \end{array}\right].$$

Hence, a basis for Z_1^1 is $\{[v_2, v_3] - [v_0, v_3] + [v_0, v_2]\}$.

(6.3)
$$M_2^3 = \left[\begin{array}{c|c} & [v_0,v_2,v_3] \\ \hline [v_0,v_1] & 0 \\ [v_0,v_2] & 2 \\ [v_0,v_3] & -2 \\ [v_2,v_3] & 2 \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{c|c} & [v_0,v_2,v_3] \\ \hline [v_0,v_2]-[v_0,v_3]+[v_2,v_3] & 2 \\ [v_0,v_3] & 0 \\ [v_2,v_3] & 0 \\ [v_0,v_1] & 0 \end{array}\right]$$

Hence, a basis for B_1^3 is $\{2[v_0, v_2] - 2[v_0, v_3] + 2[v_2, v_3]\}$. Let $b = 2[v_0, v_2] - 2[v_0, v_3] + 2[v_2, v_3]$ and $z = [v_0, v_2] - [v_0, v_3] + [v_2, v_3]$.

(6.4)
$$A = [B\ Z] = \left[\begin{array}{c|cc} & b & z \\ \hline [v_0,v_1] & 0 & 0 \\ [v_0,v_2] & 2 & 1 \\ [v_0,v_3] & -2 & -1 \\ [v_2,v_3] & 2 & 1 \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{c|cc} & z & b-2z \\ \hline z & 1 & 0 \\ [v_0,v_3] & 0 & 0 \\ [v_2,v_3] & 0 & 0 \\ [v_0,v_1] & 0 & 0 \end{array}\right]$$

Hence, a basis for the nullspace of A is $\{b - 2z\}$. In this context, a basis for $B_1^{1,2}$ is $\{B\alpha^q\} = \{b\}$. Hence, we form a matrix

$$M_2^{1,2} = \left[\begin{array}{c|c} & b \\ \hline [v_0, v_1] & 0 \\ [v_0, v_2] & 2 \\ [v_0, v_3] & -2 \\ [v_2, v_3] & 2 \end{array}\right] \xrightarrow{\text{reduce}} \left[\begin{array}{c|c} & b \\ \hline z & 2 \\ [v_0, v_1] & 0 \\ [v_0, v_3] & 0 \\ [v_2, v_3] & 0 \end{array}\right]. \tag{6.5}$$

Since both Z_1^1 and $B_1^{1,2}$ have rank 1, the rank of the free part of $H_1^{1,2}(\mathcal{L}, w)$ is $1-1 = 0$. We read off (6.5), and conclude that $H_1^{1,2}(\mathcal{L}, w) = \mathbb{Z}_2$, which agrees with our previous computation in Example 6.1.

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