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THE CAUCHY PROBLEM FOR THE DEGENERATE CONVECTIVE CAHN-HILLIARD EQUATION

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ABSTRACT. In this paper, we study the degenerate convective Cahn-Hilliard equation, which is a special case of the general convective Cahn-Hilliard equation with $M(u,\nabla u) = \operatorname{diag}(0,1,\dots,1)$. We obtain the uniform a priori decay estimates of a solution by use of the long-short wave method and the frequency decomposition method. We prove the existence of the unique global classical solution with small initial data by establishing the uniform estimates of the solution. Decay estimates are also discussed.

1. Introduction. In this paper, we study the following Cauchy problem of the degenerate convective Cahn-Hilliard equation

$$(1.1)\qquad \begin{cases} u_t + \Delta_{x'}^2 u - \Delta_{x'} A(u) - \vec{r}\cdot\nabla B(u) = 0 & x' \in \mathbb{R}^{n-1},\ t>0,\\ u(x,0) = u_0(x), \end{cases}$$

where $A(u)$ and $B(u)$ are given sufficiently smooth functions, and $\vec{r} = (r_1, r_2, \dots, r_n)$ is a constant vector. Here, $A(u) = O(|u|^{\theta+1})$, $B(u) = O(|u|^{\theta+1})$ with the same growth property and $\theta \geq 1$ is an integer. $\nabla = (\partial_{x_1}, \dots, \partial_{x_n})$ is the gradient operator, and the notation $\Delta_{x'} := \sum_{i=2}^{n} \partial_{x_i}^2$ denotes the x' direction Laplacian operator with respect to $x' = (x_2, x_3, \dots, x_n)$.

The equation in (1.1) is a special case of the general convective Cahn-Hilliard equation **[7]** with $M(u,\nabla u) = \operatorname{diag}(0,1,\dots,1)$,

$$(1.2)\qquad u_t + \operatorname{div}[M(u,\nabla u)\nabla(\Delta u - A(u))] - \vec{r}\cdot\nabla B(u) = 0.$$

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Heida [**7**] studied equation (1.2) with $M(u,\nabla u)$ such that $C^{-1}|\xi|^2 \leq (M(c,d)\xi)\cdot\xi \leq C|\xi|^2$, $C>0$, for all $(c,d)\in\mathbb{R}\times\mathbb{R}^n$ and all $\xi\in\mathbb{R}^n$. He proved the existence of solutions for equation (1.2) with dynamic boundary conditions.

During the past several years, many authors have paid much attention to the convective Cahn-Hilliard equation. It was Kwek [**8**] who first studied equation (1.2) for a special case with constant mobility and a special convection, namely, $M(u)=1$, $B(u)=u$. Based on the discontinuous Galerkin finite element method, he proved the existence of classical solutions. Liu [**10**] proved the existence and analyzed asymptotic behavior of classical solutions when $M(u)$ is a constant. Based on uniform Schauder-type estimates for local in-time solutions via the framework of Campanato spaces, Liu [**11**] studied the existence of weak solutions for the convective Cahn-Hilliard equation with degenerate mobility. The relevant equations have also been studied in [**4, 5, 6, 13, 14**].

Chen, Li and Wang [**2, 3**] considered the following conservation law with degenerate diffusion:

$$u_t - \Delta_{x'}u = \mathrm{div} f(u).$$

They proved the existence of the unique global classical solution for the initial-boundary value problem and the Cauchy problem.

In this paper, we investigate the existence of solutions. To prove the existence of classical solutions, the main difficulties are caused by the equation which is degenerate in the x_1 direction and the nonlinearity of $\Delta_{x'}A(u)$. The method for the convective Cahn-Hilliard equation with degenerate mobility, used in [**11**], seems not applicable to the present situation. Our method is based on the long-short wave and frequency decomposition methods. To estimate the low frequency part, we use Green's function methods, and to deal with the high frequency part, we employ energy estimates and a Poincaré-like inequality. For the standard continuity argument, we obtain the local solution first and then extend it to a global in-time solution by establishing the uniform estimates of the solution. For convenience, we suppose that $A(u)$ and $B(u)$ have the same growth property. If they have a different growth property, this creates a nonessential complexity.

Here, we introduce some notation that will be used throughout the paper. For any non-negative integer k and $1 \le p \le +\infty$, we define the following anisotropic Sobolev spaces

$$\|\omega\|_{\mathcal{A}^{k,p}(\mathbb{R}^n)} = \Bigg(\sum_{0\le s\le k} \|\partial_{x_1}^s \omega\|^2_{L^p(\mathbb{R}_{x_1};L^1(\mathbb{R}^{n-1}))} \Bigg)^{1/2}.$$

Now, we state the main results of this paper.

Theorem 1.1. *Assume that* $l \ge [n/2]+4$, $(n-1)(\theta-1/2) > 4$, *and* u_0 *satisfies*

$$\|u_0\|_{\mathcal{A}^{l,2}(\mathbb{R}^n)} + \|u_0\|_{\mathcal{A}^{l,\infty}(\mathbb{R}^n)} + \|u_0\|_{H^{l+1}(\mathbb{R}^n)} \le \delta_0$$

for some small constant $0 < \delta_0 \ll 1$. *Then, the problem* (1.1) *admits a unique global classical solution* $u(x,t)$ *satisfying*

$$u(x,t) \in L^\infty([0,\infty); H^l(\mathbb{R})).$$

Moreover, for any given α *and* β *satisfying* $|\alpha| \le l$ *and* $|\beta| \le l-[n/2]-1$, *there exists a constant* $C>0$, *such that*

$$\|\partial_x^\alpha u(x,t)\|_{L^2} \le C(1+t)^{-[(n-1)/8]-|\alpha'|/4}, \tag{1.3}$$

$$\|\partial_x^\beta u(x,t)\|_{L^\infty} \le C(1+t)^{-[(n-1)/4]-|\beta'|/4}, \tag{1.4}$$

where $\alpha=(\alpha_1,\alpha')$, $\alpha'=(\alpha_2,\ldots,\alpha_n)$, $\beta=(\beta_1,\beta')$ *and* $\beta'=(\beta_2,\ldots,\beta_n)$.

This paper is organized as follows. In Section 2, we state some important lemmas and notation. We prove the existence of local unique solutions of problem (1.1) for large initial data in Section 3, and then we give the existence and decay estimates for the global solutions in Section 4.

2. Some lemmas. In this section, we give some lemmas which will be used later.

Lemma 2.1 ([**9**]). *If the norms appearing on the right-hand side of the following inequalities exist, then, for any multi-index* $\alpha = (\alpha_1, \alpha_2,$

$\ldots, \alpha_n), |\alpha| \geq 0$, we have

$$\|\partial_x^\alpha(fg)\|_{L^r(\mathbb{R}^n)} \leq C(\|f\|_{L^p(\mathbb{R}^n)}\|\partial_x^\alpha g\|_{L^q(\mathbb{R}^n)} + \|\partial_x^\alpha f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}), \tag{2.1}$$

and, for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \geq 1$, we have

$$\begin{aligned} &\|\partial_x^\alpha(fg) - f\partial_x^\alpha g\|_{L^r(\mathbb{R}^n)} \\ &\quad \leq C(\|\partial_{x_i} f\|_{L^p(\mathbb{R}^n)}\|\partial_{x_1}^{\alpha_1}\cdots\partial_{x_i}^{\alpha_i-1}\cdots\partial_{x_n}^{\alpha_n} g\|_{L^q(\mathbb{R}^n)} \\ &\qquad + \|\partial_x^\alpha f\|_{L^q(\mathbb{R}^n)}\|g\|_{L^p(\mathbb{R}^n)}), \end{aligned} \tag{2.2}$$

where $1/r = 1/p + 1/q$, $1 \leq p,\ q, r \leq +\infty$.

Lemma 2.2 ([**9**]). *For scalar function w in $\mathbb{R}^n$, let $F = F(w)$ be a smooth function of ω satisfying $F(w) = O(|w|^{1+\theta})$, for $|w| \leq \nu_0$, where $\theta \geq 1$ is an integer. Then, for any integer $s \geq 0$, if $w_1, w_2 \in W^{s,q}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $\|w_1\|_{L^\infty} \leq \nu_0, \|w_2\|_{L^\infty} \leq \nu_0$, we have $F(w_1) - F(w_2) \in W^{s,r}(\mathbb{R}^n)$. Furthermore, the following inequalities hold*:

$$\|F(w_1)\|_{W^{s,r}} \leq C\|w_1\|_{W^{s,q}}\|w_1\|_{L^p}\|w_1\|_{L^\infty}^{\theta-1}, \tag{2.3}$$

$$\begin{aligned} &\|F(w_1) - F(w_2)\|_{W^{s,r}} \\ &\quad \leq C(\|w_1\|_{L^\infty} + \|w_2\|_{L^\infty})^{\theta-1}\big[\|w_1 - w_2\|_{W^{s,q}}(\|w_1\|_{L^p} + \|w_2\|_{L^p}) \\ &\qquad + \|w_1 - w_2\|_{L^p}(\|w_1\|_{W^{s,q}} + \|w_2\|_{W^{s,q}})\big], \end{aligned} \tag{2.4}$$

where $1/r = 1/p + 1/q$, $1 \leq p,\ q, r \leq +\infty$. In addition, the following holds:

$$\|\partial_x^\alpha F(w_1)\|_{L^r} \leq C\|\partial_x^\alpha w_1\|_{L^q}\|w_1\|_{L^p}\|w_1\|_{L^\infty}^{\theta-1} \quad \text{for } |\alpha| \leq s. \tag{2.5}$$

Lemma 2.3 ([**1**]). *Let $0 < p_1 \leq p_2 \leq +\infty$. Suppose that w is measurable on $\mathbb{R}^m \times \mathbb{R}^n$, that $w(\cdot, y) \in L^{p_2}(\mathbb{R}^m)$ for almost all $y \in \mathbb{R}^n$, and that the function $y \to \|w(\cdot, y)\|_{L^{p_2}(\mathbb{R}^m)}$ belongs to $L^{p_1}(\mathbb{R}^n)$. Then,* *the function* $x \to \int_{\mathbb{R}^n} w(x,y)\,dy$ *belongs to* $L^{p_2}(\mathbb{R}^m)$ *and*

$$\left(\int_{\mathbb{R}^m}\left(\int_{\mathbb{R}^n}|w(x,y)|^{p_1}dy\right)^{p_2/p_1}dx\right)^{1/p_2} \le \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m}|w(x,y)|^{p_2}dx\right)^{p_1/p_2}dy\right)^{1/p_1}.$$

Lemma 2.4 (**[2]**). *For scalar function* ω *in* $\mathbb{R}^n$, *let* $F = F(w)$ *be a smooth function of* ω *satisfying* $F(\omega) = O(|\omega|^{1+\theta})$, *for* $|w| \le \nu_0$, *where* $\theta \ge 1$ *is an integer. If the norms appearing on the right-hand side of the following inequalities exist, then, for any integer* $k \ge 0$, *we have*

$$\|\partial_{x_1}^k F(w)\|_{L^2(\mathbb{R}_{x_1};L^1(\mathbb{R}^{n-1}))} \le C\|\partial_{x_1}^k w\|_{L^2}\|w\|_{H^1}\|w\|_{L^\infty}^{\theta-1}$$

and

$$\|\partial_{x_1}^k F(w)\|_{L^\infty(\mathbb{R}_{x_1};L^1(\mathbb{R}^{n-1}))} \le C\|w\|_{H^{k+1}}\|w\|_{H^1}\|w\|_{L^\infty}^{\theta-1}.$$

Lemma 2.5 (**[12]**). *Let* α, β *and* γ *be positive constants,* $0 \le \tau < 1$, $t \ge 2\tau$. *Then*

(i) $\int_\tau^t (1+t-s)^{-\alpha}(1+s)^{-\beta}ds \le O(1)(1+t)^{-\min\{\alpha,\beta\}}$, *if* $\max\{\alpha,\beta\} > 1$;

(ii) $\int_\tau^t (1+t-s)^{-\alpha}(1+s)^{-\beta}ds \le O(1)(1+t)^{1-\alpha-\beta}$, *if* $\max\{\alpha,\beta\} < 1$, $\alpha+\beta > 1$;

(iii) $\int_\tau^{t/2} (1+t-s)^{-\beta}(1+s)^{-\gamma}ds \le O(1)(1+t)^{-\alpha}$, *if* $\alpha \le \beta, \alpha \le \gamma+\beta-1$, $\gamma \ne 1$, *or if* $\alpha < \beta$, $\alpha \le \gamma+\beta-1$, $\gamma = 1$;

(iv) $\int_{t/2}^{t} (1+t-s)^{-\beta}(1+s)^{-\gamma}ds \le O(1)(1+t)^{-\alpha}$, *if* $\alpha \le \gamma$, $\alpha \le \gamma+\beta-1$, $\beta \ne 1$, *or if* $\alpha < \gamma$, $\alpha \le \gamma+\beta-1$, $\beta = 1$.

3. Local existence. In this section, we shall prove the local existence of a solution to problem (1.1). For this purpose, we first consider the following linearized iteration scheme

$$
(3.1)\qquad \begin{cases} u_t^m + \Delta_{x'}^2 u^m - \nabla_{x'} \cdot (A'(u^{m-1})\nabla_{x'} u^m) \\ \quad -B'(u^{m-1})\vec{r} \cdot \nabla u^m = 0 & t > 0, \\ u^m(x,0) = u_0(x), \end{cases}
$$

with $m \geq 1$ and $u^0(x,t) = 0$. For convenience of calculations, we do not expand the term of $\nabla_{x'} \cdot (A'(u^{m-1})\nabla_{x'} u^m)$. For a given integer $s \geq [n/2] + 5$, we introduce

$$
X_T^{s+1} = \{u(x,t) \mid \|u\|_{X^{s+1}} < \infty\}
$$

as the suitable space for solutions, where

$$
\|u\|_{X^{s+1}} = \sup_{0 \leq t \leq T} \|u(\cdot,t)\|_{H^{s+1}(\mathbb{R}^n)}.
$$

It is easy to show that X_T^{s+1}, equipped with norm $\|\cdot\|_{X^{s+1}}$, is a non-empty Banach space. To obtain the local solution, we first claim that the sequence $\{u^m\}$ is bounded in X_T^{s+1}, i.e., that $\|u^m\|_{X^{s+1}} \leq D$ for some constant $D > 0$. To prove $\|u^m\|_{X^{s+1}} \leq D$, we adopt the inductive method. Clearly, by taking $D = 2\|u_0\|_{H^{s+1}}$, we have $\|u^1(x,t)\|_{X^{s+1}} \leq D$. Now, assuming that $\|u^j(x,t)\|_{X^{s+1}} \leq D$ for all $j \leq m-1$, we need to prove that it holds for $j = m$.

Lemma 3.1. *Assume that T is sufficiently small. Then, there exists some positive constant D such that $\|u^m\|_{X^{s+1}} \leq D$.*

Proof. Multiplying the equation in (3.1) by u_m, integrating by parts with respect to x_1, and employing the Cauchy inequality, we see that

$$
\begin{aligned}
&\frac{d}{dt}\|u^m\|_{L^2}^2 + 2\|\Delta_{x'} u^m\|_{L^2}^2 \\
&\quad = -2\int_{R^n} A'(u^{m-1})|\nabla_{x'} u^m|^2 dx + 2\int_{R^n} B'(u^{m-1})\vec{r'} \cdot \nabla_{x'} u^m u^m dx \\
&\qquad + 2r_1 \int_{R^n} B'(u^{m-1})\partial_{x_1} u^m u^m dx \\
&\quad = -2\int_{R^n} A'(u^{m-1})|\nabla_{x'} u^m|^2 dx + 2\int_{R^n} B'(u^{m-1})\vec{r'} \cdot \nabla_{x'} u^m u^m dx \\
&\qquad + r_1 \int_{R^n} \partial_{x_1} B'(u^{m-1})(u^m)^2 dx \\
&\quad \leq 2\|A'(u^{m-1})\|_{L^\infty} \|\nabla_{x'} u^m\|_{L^2}^2
\end{aligned}
$$

$$+ 2|\vec{r'}|\|B'(u^{m-1})\|_{L^\infty}\left(\frac{1}{2}\|u^m\|_{L^2}^2 + \frac{1}{2}\|\nabla_{x'}u^m\|_{L^2}^2\right)$$
$$+ |r_1|\|\partial_{x_1}B'(u^{m-1})\|_{L^\infty}\|u^m\|_{L^2}^2.$$

By Lemma 2.2 and the induction hypothesis, we conclude that

$$\begin{aligned}\frac{d}{dt}\|u^m\|_{L^2}^2+2\|\Delta_{x'}u^m\|_{L^2}^2 &\le C\|u^{m-1}\|_{L^\infty}^{\theta}\|\nabla_{x'}u^m\|_{L^2}^2\\ &\quad +C|\vec{r'}|\|u^{m-1}\|_{L^\infty}^{\theta}\left(\frac{1}{2}\|u^m\|_{L^2}^2+\frac{1}{2}\|\nabla_{x'}u^m\|_{L^2}^2\right)\\ &\quad + C\|u^{m-1}\|_{L^\infty}^{(\theta-1)}\|\partial_{x_1}u^{m-1}\|_{L^\infty}\|u^m\|_{L^2}^2\\ &\le CD^\theta(\|u^m\|_{L^2}^2 + \|\nabla_{x'}u^m\|_{L^2}^2).\end{aligned} \tag{3.2}$$

On the other hand, by the Gagliardo-Nirenberg interpolation inequality and the Cauchy inequality, it is easy to obtain that

$$\|\nabla_{x'}u^m\|_{L^2}^2 \le C_\varepsilon\|u^m\|_{L^2}^2 + \varepsilon\|\Delta_{x'}u^m\|_{L^2}^2.$$

Combining (3.2) and the above estimation, we have

$$\frac{d}{dt}\|u^m\|_{L^2}^2 + 2\|\Delta_{x'}u^m\|_{L^2}^2 \le C_\varepsilon D^\theta\|u^m\|_{L^2}^2 + \varepsilon D^\theta\|\Delta_{x'}u^m\|_{L^2}^2.$$

Taking ε small enough such that $2 - \varepsilon D^\theta > 0$, we derive

$$\sup_{0\le t\le T}\|u^m\|_{L^2}^2 \le \|u_0\|_{L^2}^2 + CD^\theta T\sup_{0\le\tau\le T}\|u^m\|_{L^2}^2.$$

For sufficiently small T, we obtain

$$\sup_{0\le t\le T}\|u^m\|_{L^2}^2 \le 2\|u_0\|_{L^2}^2 \le D.$$

Now, we establish the higher order estimates. Differentiating the equation with respect to the x_1 direction, we know that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial_{x_1}^h u^m\|_{L^2}^2 + \|\Delta_{x'}\partial_{x_1}^h u^m\|_{L^2}^2\\ &\quad = -\int_{R^n}\partial_{x_1}^h(A'(u^{m-1})\nabla_{x'}u^m)\partial_{x_1}^h\nabla_{x'}u^m dx\\ &\qquad + \int_{R^n}\partial_{x_1}^h(B'(u^{m-1})\vec{r'}\cdot\nabla_{x'}u^m)\partial_{x_1}^h u^m dx\\ &\qquad + \int_{R^n}\partial_{x_1}^h(B'(u^{m-1})r_1\partial_{x_1}u^m)\partial_{x_1}^h u^m dx = I_1 + I_2 + I_3.\end{aligned}$$

We now estimate the three terms on the right-hand side of the above formula. Recalling Lemma 2.2, Lemma 2.1 and induction hypothesis, we obtain

$$\begin{aligned} I_1 &\le \|\partial_{x_1}^h (A'(u^{m-1})\nabla_{x'} u^m)\|_{L^2} \|\partial_{x_1}^h \nabla_{x'} u^m\|_{L^2} \\ &\le C\|\partial_{x_1}^h A'(u^{m-1})\|_{L^2} \|\nabla_{x'} u^m\|_{L^\infty} \|\partial_{x_1}^h \nabla_{x'} u^m\|_{L^2} \\ &\quad + C\|A'(u^{m-1})\|_{L^\infty} \|\partial_{x_1}^h \nabla_{x'} u^m\|_{L^2} \|\partial_{x_1}^h \nabla_{x'} u^m\|_{L^2} \\ &\le C(1+D^\theta)\|\partial_{x_1}^h \nabla_{x'} u^m\|_{L^2}^2 + CD^{2\theta}\|\nabla_{x'} u^m\|_{L^\infty}^2. \end{aligned}$$

Similarly, for I_2, we obtain

$$\begin{aligned} I_2 &\le \|\partial_{x_1}^h (B'(u^{m-1})\vec{r'} \cdot \nabla_{x'} u^m)\|_{L^2} \|\partial_{x_1}^h u^m\|_{L^2} \\ &\le C\|\partial_{x_1}^h (B'(u^{m-1})\|_{L^2} \|\nabla_{x'} u^m\|_{L^\infty} \|\partial_{x_1}^h u^m\|_{L^2} \\ &\quad + C\|\partial_{x_1}^h \nabla_{x'} u^m\|_{L^2} \|B'(u^{m-1})\|_{L^\infty} \|\partial_{x_1}^h u^m\|_{L^2} \\ &\le CD^{2\theta}(\|\partial_{x_1}^h \nabla_{x'} u^m\|_{L^2}^2 + \|\nabla_{x'} u^m)\|_{L^\infty}^2) + C\|\partial_{x_1}^h u^m\|_{L^2}^2. \end{aligned}$$

I_3 can be split into two parts to avoid the difficulty of degenerate diffusion in the x_1 direction as follows:

$$\begin{aligned} I_3 &= \int_{R^n} \partial_{x_1}^h (B'(u^{m-1}) r_1 \partial_{x_1} u^m) \partial_{x_1}^h u^m dx \\ &= \int_{R^n} \left[\partial_{x_1}^h (B'(u^{m-1}) u_{x_1}^m) - B'(u^{m-1}) \partial_{x_1}^h u_{x_1}^m\right] \partial_{x_1}^h u^m dx \\ &\quad + \int_{R^n} B'(u^{m-1}) \partial_{x_1}^h u_{x_1}^m \partial_{x_1}^h u^m dx \\ &= \int_{R^n} \left[\partial_{x_1}^h (B'(u^{m-1}) u_{x_1}^m) - B'(u^{m-1}) \partial_{x_1}^h u_{x_1}^m\right] \partial_{x_1}^h u^m dx \\ &\quad + \frac{1}{2} \int_{R^n} \partial_{x_1} B'(u^{m-1}) (\partial_{x_1}^h u^m)^2 dx. \end{aligned}$$

Lemma 2.2, Lemma 2.1, Cauchy's inequality and the induction hypothesis imply that

$$\begin{aligned} I_3 &\le C(\|\partial_{x_1} B'(u^{m-1})\|_{L^\infty} \|\partial_{x_1}^h u^m\|_{L^2} \\ &\quad + \|\partial_{x_1}^h B'(u^{m-1})\|_{L^2} \|\partial_{x_1} u^m\|_{L^\infty}) \|\partial_{x_1}^h u^m\|_{L^2} \\ &\quad + \frac{1}{2} \|\partial_{x_1} B'(u^{m-1})\|_{L^\infty} \|\partial_{x_1}^h u^m\|_{L^2}^2 \\ &\le (1 + CD^\theta + CD^{2\theta}) \|\partial_{x_1}^h u^m\|_{L^2}^2 + CD^{2\theta} \|\partial_{x'} u^m\|_{L^\infty}^2. \end{aligned}$$

On the other hand, the Gagliardo-Nirenberg inequality yields

$$\|\partial_{x_1}^h \nabla_{x'} u^m\|_{L^2}^2 \le C_\varepsilon \|\partial_{x_1}^h u^m\|_{L^2}^2 + \varepsilon \|\partial_{x_1}^h \Delta_{x'} u^m\|_{L^2}^2.$$

Combining these inequalities and integrating the result over t yields

$$\sup_{0\le t\le T} \|\partial_{x_1}^h u^m\|_{L^2}^2 \le C D^{2\theta} T \sup_{0\le t\le T} \|\nabla u^m\|_{L^\infty}^2 + \left(1 + \frac{1}{2(h+1)}\right) \|\partial_{x_1}^h u_0\|_{L^2}^2.$$

For those terms involving first order derivatives on the x' direction, such as terms $\|\partial_{x_1}^{h-1}\partial_{x_i} u^m\|_{L^2}^2$, $i \ne 1$, analogously, we can obtain the same conclusion. Furthermore, we see that

$$\sup_{0\le t\le T} \|u^m\|_{H^{s+1}}^2 \le C D^{2\theta} T \sup_{0\le t\le T} \|\nabla u^m\|_{L^\infty}^2 + \frac{3}{2} \|\partial_{x_1}^h u_0\|_{H^{s+1}}^2$$

holds for sufficiently small T. By the Sobolev embedding theorem for $s \ge [n/2] + 5$, we arrive at

$$\|u^m\|_{H^{s+1}}^2 \le 2\|u_0\|_{H^{s+1}}^2.$$

The proof is complete. □

Next, we show that $\{u^m\}$ is a Cauchy sequence, which means that the Cauchy problem (1.1) admits a local solution.

Lemma 3.2. *Assume that T is small enough. Then, $\{u^m\}$ is a Cauchy sequence in X_T^s.*

Proof. Let $v^m = u^m - u^{m-1}$. It follows from (3.1) that

$$\begin{aligned} v_t^m + \Delta_{x'}^2 v^m &= \nabla_{x'}\big(A'(u^{m-1})\nabla_{x'} u^m - A'(u^{m-2})\nabla_{x'} u^{m-1}\big) \\ &\quad + \big(B'(u^{m-1})\vec{r}\cdot\nabla u^m - B'(u^{m-2})\vec{r}\cdot\nabla u^{m-1}\big) \end{aligned} \tag{3.3}$$

with $m \ge 2$ and $v^1(x,t) = u^1(x,t)$. Multiplying both sides of equation (3.3) by v^m, integrating the resulting relation and integrating by parts, we conclude that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|v^m\|_{L^2}^2 + \|\Delta_{x'} v^m\|_{L^2}^2 = &-\int_{R^n} \big(A'(u^{m-1})\nabla_{x'} v^m + (A'(u^{m-1}) \\ &- A'(u^{m-2}))\nabla_{x'} u^{m-1}\big)\nabla_{x'} v^m dx \\ &+ \int_{R^n} \big(B'(u^{m-1})\vec{r}\cdot\nabla v^m + (B'(u^{m-1}) \\ &- B'(u^{m-2}))\vec{r}\cdot\nabla u^{m-1}\big) v^m dx. \end{aligned}$$

By Lemma 2.2 and Cauchy's inequality, we know that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v^m\|_{L^2}^2+\|\Delta_{x'}v^m\|_{L^2}^2\\
&\quad\le \|A'(u^{m-1})\|_{L^\infty}\|\nabla_{x'}v^m\|_{L^2}^2+C\|u^{m-1}\|_{L^\infty}^{\theta-1}\|\nabla u^{m-1}\|_{L^\infty}\|v^m\|_{L^2}^2\\
&\qquad+\|\nabla_{x'}u^{m-1}\|_{L^\infty}\|(A'(u^{m-1})-A'(u^{m-2}))\|_{L^2}\|\nabla_{x'}v^m\|_{L^2}\\
&\qquad+\|\nabla u^{m-1}\|_{L^\infty}\|(B'(u^{m-1})-B'(u^{m-2}))\|_{L^2}\|v^m\|_{L^2}\\
&\quad\le C\|u^{m-1}\|_{L^\infty}^{\theta}\|\nabla_{x'}v^m\|_{L^2}^2+C\|u^{m-1}\|_{L^\infty}^{\theta-1}\|\nabla u^{m-1}\|_{L^\infty}\|v^m\|_{L^2}^2\\
&\qquad+\frac{1}{2}(\|\nabla u^{m-1}\|_{L^\infty}+\|\nabla_{x'}u^{m-1}\|_{L^\infty}^2)(\|u^{m-1}\|_{L^\infty}+\|u^{m-2}\|_{L^\infty})^{2(\theta-2)}\\
&\quad\cdot\Big[\|v^{m-1}\|_{L^\infty}^2(\|u^{m-1}\|_{L^2}+\|u^{m-1}\|_{L^2})^2\\
&\qquad\qquad\qquad+\|v^{m-1}\|_{L^2}^2(\|u^{m-1}\|_{L^\infty}+\|u^{m-2}\|_{L^\infty})^2\Big]\\
&\qquad+\frac{1}{2}(\|\nabla u^{m-1}\|_{L^\infty}\|v^m\|_{L^2}^2+\|\nabla_{x'}v^m\|_{L^2}^2).
\end{aligned}
$$

On the other hand, the Gagliardo-Nirenberg interpolation inequality implies

$$\|\nabla_{x'}v^m\|_{L^2}^2\le C_\varepsilon\|v^m\|_{L^2}^2+\varepsilon\|\Delta_{x'}v^m\|_{L^2}^2.$$

Hence, we obtain

$$\frac{d}{dt}\|v^m\|_{L^2}^2\le C(D^\theta+1)\|v^m\|_{L^2}^2+CD^{2\theta}(\|v^{m-1}\|_{L^\infty}^2+\|v^{m-1}\|_{L^2}^2).$$

Integrating the above inequality in time over $[0,T]$, and for T small enough, we get

$$\sup_{0\le t\le T}\|v^m\|_{L^2}^2\le CD^{2\theta}T\sup_{0\le t\le T}(\|v^{m-1}\|_{L^\infty}^2+\|v^{m-1}\|_{L^2}^2).$$

Similarly, we have the higher order derivative estimate as follows

$$\sup_{0\le t\le T}\|\partial_{x_1}^h v^m\|_{L^2}^2\le CD^{2\theta}T\sup_{0\le t\le T}(\|\nabla v^m\|_{L^\infty}^2+\|\partial_{x_1}^h v^{m-1}\|_{L^2}^2+\|v^{m-1}\|_{L^\infty}^2),$$

and

$$
\begin{aligned}
\sup_{0\le t\le T}\|\partial_{x_1}^{h-1}\nabla_{x'}v^m\|_{L^2}^2\le CD^{2\theta}T\sup_{0\le t\le T}&(\|\nabla v^m\|_{L^\infty}^2+\|\partial_{x_1}^{h-1}v^m\|_{L^2}^2\\
&+\|\partial_{x_1}^{h-1}v^{m-1}\|_{L^2}^2+\|v^{m-1}\|_{L^\infty}^2).
\end{aligned}
$$

It follows from summing up all of estimates that

$$\sup_{0\le t\le T}\|\upsilon^m\|_{H^s}^2 \le CD^{2\theta}T\sup_{0\le t\le T}(\|\nabla\upsilon^m\|_{L^\infty}^2+\|\upsilon^{m-1}\|_{L^\infty}^2+\|\upsilon^{m-1}\|_{H^s}^2),$$

for small enough T.

The Sobolev embedding theorem implies

$$\sup_{0\le t\le T}\|\upsilon^m\|_{H^s}^2 \le CD^{2\theta}T\sup_{0\le t\le T}(\|\upsilon^m\|_{H^s}^2+\|\upsilon^{m-1}\|_{H^s}^2)$$

for $s\ge[n/2]+2$. By taking T sufficiently small, there exists a constant $0<\eta<1$ such that

$$\|\upsilon^m\|_{X_T^s}\le\eta\|\upsilon^{m-1}\|_{X_T^s}.$$

We can choose η as follows (for T sufficiently small)

$$0<\eta=\frac{CD^{2\theta}T}{1-CD^{2\theta}T}<1,$$

which means that $\{u^m\}$ is convergent in X_T^s. Therefore, the proof of Lemma 3.2 is finished. □

Theorem 3.3. *Assume that* $s\ge[n/2]+4$ *and* $u_0\in H^{s+1}(\mathbb{R}^n)$. *Then, there exists a time* $T>0$, *such that problem* (1.1) *admits a unique classical solution* u *in* $[0,T)$ *satisfying*

$$u\in L^\infty([0,T);H^{s+1}(\mathbb{R}^n)).$$

Proof. By Lemma 3.2, we know that u^m is a Cauchy sequence in Banach space X_T^s with $s\ge[n/2]+5$. Therefore, the limit function u of u^m is a local solution of problem (1.1). In addition, using Lemma 3.1, we see that u^m is bounded in X_T^s. Hence u^m is also a Cauchy sequence in $X_T^{s'+1}$ for all $s'<s$, and the limit function u is in X_T^{s+1}. This means, when $s\ge[n/2]+4$, then u is a local solution in X_T^{s+1}. The proof is complete. □

4. Global existence. In this section, we are going to prove the global existence of solutions for problem (1.1).

First, assume that

$$\|\partial_x^\alpha u(\cdot,t)\|_{L^2}\le E(1+t)^{-[(n-1)/8]-|\alpha'|/4}, \tag{4.1}$$

$$\|\partial_x^\beta u(\cdot,t)\|_{L^\infty}\le E(1+t)^{-[(n-1)/4]-|\beta'|/4}, \tag{4.2}$$

for $l \geq [n/2] + 4$, $0 \leq |\alpha| \leq l$, $0 \leq |\beta| \leq l - [n/2] - 1$ and $0 < E \ll 1$. Here, $\alpha = (\alpha_1, \alpha')$, $\alpha' = (\alpha_2, \ldots, \alpha_n)$, $\beta = (\beta_1, \beta')$ and $\beta' = (\beta_2, \ldots, \beta_n)$.

Now, based on the a priori assumptions (4.1) and (4.2), for any integer $k \geq 0$, we establish H^k bounded estimates of the solutions for problem (1.1).

Lemma 4.1. *Assume that $u_0 \in H^k(\mathbb{R}^n)$, $(n-1)\theta > 4$ and $u(x,t)$ is a classical solution to problem* (1.1). *Assume also that* (4.2) *holds. Then, there exists a constant $C > 0$ such that, for all $k \geq 0$,*

$$\sup_{0 \leq t < \infty} \|u\|_{H^k}^2 \leq C\|u_0\|_{H^k}^2.$$

Proof. Multiplying (1.1) by u, integrating over $\mathbb{R}^n$ and integrating by parts, we deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\Delta_{x'} u\|_{L^2}^2 \\
&\quad = -\int_{\mathbb{R}^n} \nabla_{x'} A(u) \nabla_{x'} u \, dx + \int_{\mathbb{R}^n} \vec{r} \cdot \nabla B(u) u \, dx \\
&\quad = -\int_{\mathbb{R}^n} A'(u) |\nabla_{x'} u|^2 dx + \int_{\mathbb{R}^n} B'(u) \vec{r} \cdot \nabla u u \, dx.
\end{aligned}$$

Combining the Gagliardo-Nirenberg inequality, the Cauchy inequality and Lemma 2.2, we see that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \|\Delta_{x'} u\|_{L^2}^2 &\leq \|A(u)\|_{L^\infty} \|\nabla_{x'} u\|_{L^2}^2 \\
&\quad + |\vec{r'}| \|B'(u)\|_{L^\infty} \|\nabla_{x'} u\|_{L^2} \|u\|_{L^2} \\
&\quad + r_1 \|B'(u)\|_{L^2} \|u_{x_1}\|_{L^\infty} \|u\|_{L^2} \\
&\leq (C_\varepsilon \|u\|_{L^\infty}^{2\theta} + C\|u\|_{L^\infty}^{\theta-1} \|u_{x_1}\|_{L^\infty}) \|u\|_{L^2}^2 + \varepsilon \|\Delta_{x'} u\|_{L^2}.
\end{aligned}$$

Assumption (4.2) yields

$$\begin{aligned}
\frac{d}{dt}\|u\|_{L^2}^2 &\leq C(\|u\|_{L^\infty}^{2\theta} + \|u\|_{L^\infty}^{\theta-1} \|u_{x_1}\|_{L^\infty}) \|u\|_{L^2}^2 \\
&\leq CE^\theta (1+t)^{-[(n-1)\theta/4]} \|u\|_{L^2}^2.
\end{aligned}$$

Integrating the above inequality over time t, we know that

$$\|u\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + CE^\theta \sup_{0\leq t<\infty} \|u\|_{L^2}^2 \int_0^t (1+\tau)^{-[(n-1)\theta/4]} d\tau.$$

By $(n-1)\theta > 4$, we obtain

$$\|u\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 + CE^\theta \sup_{0\leq t<\infty} \|u\|_{L^2}^2.$$

Then, it follows from $0 < E \ll 1$ that

$$\sup_{0\leq t<\infty} \|u\|_{L^2}^2 \leq C\|u_0\|_{L^2}^2.$$

Similarly, we can deduce that, for $0 < E \ll 1$,

$$\sup_{0\leq t<\infty} \|\partial_{x_1}^k u\|_{L^2}^2 \leq C\|\partial_{x_1}^k u_0\|_{L^2}^2. \tag{4.3}$$

Next, we will estimate the terms involving derivatives in the x' direction. Applying $\partial_{x_1}^{k-1}\partial_{x_2}$ onto (1.1), multiplying by $\partial_{x_1}^{k-1}\partial_{x_2}u$, $k \geq 1$, and integrating over $\mathbb{R}^n$, we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\partial_{x_1}^{k-1}\partial_{x_2}u\|_{L^2}^2 + \|\Delta_{x'}\partial_{x_1}^{k-1}\partial_{x_2}u\|_{L^2}^2 \\
&\quad = -\int_{R^n} \partial_{x_1}^{k-1}\partial_{x_2}(A'(u)\nabla_{x'}u)\partial_{x_1}^{k-1}\partial_{x_2}\nabla_{x'}u\,dx \\
&\qquad + \int_{R^n} \partial_{x_1}^{k-1}\partial_{x_2}(B'(u)\vec{r'}\cdot\nabla_{x'}u)\partial_{x_1}^{k-1}\partial_{x_2}u\,dx \\
&\qquad + \int_{R^n} \partial_{x_1}^{k-1}\partial_{x_2}(B'(u)r_1\partial_{x_1}u)\partial_{x_1}^{k-1}\partial_{x_2}u\,dx.
\end{aligned}$$

Similarly to the estimate of term $\|\partial_{x_1}^h u^m\|_{L^2}^2$ in Section 3, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|\partial_{x_1}^{k-1}\partial_{x_2}u\|_{L^2}^2 \\
&\quad \leq C(\|\partial_{x_1}^k u\|_{L^2}^2 + \|\partial_{x_1}^{k-1}\partial_{x_2}u\|_{L^2}^2)\|u\|_{L^\infty}^{2\theta} \\
&\quad \leq CE^{2\theta}(1+t)^{-[(n-1)\theta/2]}(\|\partial_{x_1}^k u\|_{L^2}^2 + \|\partial_{x_1}^{k-1}\partial_{x_2}u\|_{L^2}^2).
\end{aligned}$$

Integrating this over t gives

$$\|\partial_{x_1}^{k-1}\partial_{x_2}u\|_{L^2}^2 \leq \|\partial_{x_1}^{k-1}\partial_{x_2}u_0\|_{L^2}^2 + CE^{2\theta}(\|\partial_{x_1}^k u\|_{L^2}^2 + \|\partial_{x_1}^{k-1}\partial_{x_2}u\|_{L^2}^2).$$

If $(n-1)\theta > 2$, then, for $0 < E \ll 1$ and (4.3), we see that

$$\sup_{0\le t<\infty} \|\partial_{x_1}^{k-1}\partial_{x_2} u\|_{L^2}^2 \le (\|\partial_{x_1}^{k} u_0\|_{L^2}^2 + \|\partial_{x_1}^{k-1}\partial_{x_2} u_0\|_{L^2}^2).$$

Similarly, for all $|\alpha| = k \ge 1$, we have

$$\sup_{0\le t<\infty} \|\partial^{\alpha} u\|_{L^2}^2 \le C \sum_{|\gamma|=k} \|\partial^{\gamma} u_0\|_{L^2}^2.$$

The proof is complete. □

Now, we discuss decay estimates of the solutions. Towards this purpose, we adopt the frequency-decomposition method. Let

$$\chi(\xi) = \begin{cases} 1 & |\xi'| \le 1, \\ 0 & |\xi'| > 1, \end{cases}$$

be a cut-off function. We define a Fourier multiplier operator $\chi(D)$ with the symbol $\chi(\xi)$. We decompose the solution to problem (1.1) into two parts: low frequency part u_L and high frequency u_H, which are defined as

$$u_L = \chi(D)u, \qquad u_H = (1-\chi(D))u.$$

First, we estimate the u_L by Green's function method. Green's function of problem (1.1) is given by

$$\begin{cases} \partial_t G(x,t) + \Delta_{x'}^2 G(x,t) = 0, \\ G(x,0) = \delta(x), \end{cases} \tag{4.4}$$

where $\delta(x)$ is the Dirac function. Taking the Fourier transform to (4.4), we know that

$$\begin{cases} \partial_t \widehat{G}(\xi,t) + |\xi'|^4 \widehat{G}(\xi,t) = 0, \\ \widehat{G}(\xi,0) = 1. \end{cases} \tag{4.5}$$

Hence, $\widehat{G}(\xi,t) = e^{-|\xi'|^4 t}$. Using the inverse Fourier transform, we conclude that

$$G(x,t) = \delta(x_1) G'(x',t),$$

where $G'(x',t)$ is obtained by taking the inverse Fourier transform to $e^{-|\xi'|^4 t}$ with respect to $|\xi'|$ in $\mathbb{R}^{n-1}$. For the low frequency part

$G'_L(x',t) = \chi(D)G'(x',t)$ of $G'(x',t)$, by direct calculation, we have the following lemma.

Lemma 4.2. *For any multi-index $\alpha' = (\alpha_2, \dots, \alpha_n)$ with $|\alpha'| \ge 0$,*

$$\|\partial_{x'}^{\alpha'} G'(\cdot,t)\|_{L^2(\mathbb{R}^{n-1})} \le C(1+t)^{-[(n-1)/8]-|\alpha'|/4}, \tag{4.6}$$

$$\|\partial_{x'}^{\alpha'} G'(\cdot,t)\|_{L^\infty(\mathbb{R}^{n-1})} \le C(1+t)^{-[(n-1)/4]-|\alpha'|/4}. \tag{4.7}$$

Lemma 4.3. *Let $\|u_0\|_{\mathcal{A}^{l,2}(\mathbb{R}^n)} + \|u_0\|_{H^{l+1}(\mathbb{R}^n)} \le \delta_0$ and $(n-1)(\theta - 1/2) > 4$. Assume that* (4.1) *and* (4.2) *hold. Then, for any $\alpha = (\alpha_1, \alpha')$, $|\alpha| \le l$, there exists a constant $C > 0$ such that*

$$\|\partial_x^\alpha u_L\|_{L^2} \le C(\delta_0 + E^{\theta+1})(1+t)^{-[(n-1)/8]-|\alpha'|/4}.$$

Proof. By the Duhamel principle, we see that the solution u of (1.1) satisfies

$$u(x,t) = G(\cdot,t) * u_0(\cdot) + \int_0^t G(\cdot, t-\tau) * (\Delta_{x'} A(u) + \vec{r} \cdot \nabla B(u))(\cdot,\tau)\, d\tau. \tag{4.8}$$

For any $\alpha = (\alpha_1, \alpha')$, $0 \le |\alpha| \le l$, applying the operator $\chi(D)\partial_x^\alpha$ to both sides of (4.8), we derive

$$\partial_x^\alpha u_L(x,t) = \partial_{x'}^{\alpha'} G_L(\cdot,t) * \partial_{x_1}^{\alpha_1} u_0(\cdot) + \int_0^t \partial_{x'}^{\alpha'} G_L(\cdot,t-\tau) * \partial_{x_1}^{\alpha_1}(\Delta_{x'} A(u) + \vec{r} \cdot \nabla B(u))(\cdot,\tau)\, d\tau, \tag{4.9}$$

where $G_L(x,t) = \chi(D)G(x,t)$. If $|\alpha'| = 0$ in (4.9), we see that

$$\begin{aligned} \partial_{x_1}^{\alpha_1} u_L(x,t) &= G_L(\cdot,t) * \partial_{x_1}^{\alpha_1} u_0(\cdot) + \int_0^t G_L(\cdot,t-\tau) \\ &\quad * \partial_{x_1}^{\alpha_1}(\Delta_{x'} A(u) + \vec{r} \cdot \nabla B(u))(\cdot,\tau)\, d\tau \\ &= \delta(x_1) G'_L(\cdot,t) \\ &\quad * \partial_{x_1}^{\alpha_1} u_0(\cdot) + \int_0^t \delta(x_1) G'_L(\cdot,t-\tau) \\ &\quad * \partial_{x_1}^{\alpha_1}(\Delta_{x'} A(u) + \vec{r} \cdot \nabla B(u))(\cdot,\tau)\, d\tau. \end{aligned} \tag{4.10}$$

Therefore,

$$\begin{aligned}\|\partial_{x_1}^{\alpha_1}u_L\|_{L^2} &= \|\delta(x_1)G_L'(\cdot,t)*\partial_{x_1}^{\alpha_1}u_0(\cdot)\|_{L^2}\\ &\quad+\int_0^t \|\delta(x_1)G_L'(\cdot,t-\tau)*\partial_{x_1}^{\alpha_1}(\Delta_{x'}A(u)+\vec{r}\cdot\nabla B(u))(\cdot,\tau)\|_{L^2}\,d\tau\\ &\le \left\{\int_{\mathbb{R}}\|G_L'(\cdot,t)*\partial_{x_1}^{\alpha_1}u_0(x_1,\cdot)\|_{L^2_{x'}}^2\,dx_1\right\}^{1/2}+\int_0^t H^{1/2}(\tau)\,d\tau,\end{aligned}$$

where

$$H(\tau)=\int_{\mathbb{R}}\|G_L'(\cdot,t-\tau)*\partial_{x_1}^{\alpha_1}(\Delta_{x'}A(u)+\vec{r}\cdot\nabla B(u))(x_1,\cdot,\tau)\|_{L^2_{x'}}^2\,dx_1.$$

Applying Young's inequality and Lemma 2.5, we get

$$\begin{aligned}&\|\partial_{x_1}^{\alpha_1}u_L\|_{L^2}\\ &\le \|G_L'\|_{L^2}\left\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1}u_0(x_1,\cdot)\|_{L^1}^2\,dx_1\right\}^{1/2}\\ &\quad+\int_0^t\|\Delta_{x'}G_L'(\cdot,t-\tau)\|_{L^2}\left\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1}A(u)(x_1,\cdot,\tau)\|_{L^1}^2\,dx_1\right\}^{1/2}d\tau\\ &\quad+|r_1|\int_0^t\|G_L'(\cdot,t-\tau)\|_{L^2}\left\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1+1}B(u)(x_1,\cdot,\tau)\|_{L^1}^2\,dx_1\right\}^{1/2}d\tau\\ &\quad+|\vec{r'}|\int_0^t\|\nabla_{x'}G_L'(\cdot,t-\tau)\|_{L^2}\left\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1}B(u)(x_1,\cdot,\tau)\|_{L^1}^2\,dx_1\right\}^{1/2}d\tau\\ &\le \|G_L'\|_{L^2}\|\partial_{x_1}^{\alpha_1}u_0\|_{L^2(\mathbb{R}_{x_1};L^1(\mathbb{R}^{n-1}))}\\ &\quad+C\int_0^t\|\Delta_{x'}G_L'(\cdot,t-\tau)\|_{L^2}\|\partial_{x_1}^{\alpha_1}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{H^1}\|u(\cdot,\tau)\|_{L^\infty}^{\theta-1}\,d\tau\\ &\quad+C\int_0^t\|G_L'(\cdot,t-\tau)\|_{L^2}\|\partial_{x_1}^{\alpha_1+1}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{H^1}\|u(\cdot,\tau)\|_{L^\infty}^{\theta-1}\,d\tau\\ &\quad+C\int_0^t\|\nabla_{x'}G_L'(\cdot,t-\tau)\|_{L^2}\|\partial_{x_1}^{\alpha_1}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{H^1}\|u(\cdot,\tau)\|_{L^\infty}^{\theta-1}\,d\tau.\end{aligned}$$

Recalling Lemma 2.5, Lemma 4.1 and Lemma 4.2, we deduce that

$$\begin{aligned}&\|\partial_{x_1}^{\alpha_1}u_L\|_{L^2}\le C\delta_0(1+t)^{-(n-1)/8}\\ &\quad+CE^{\theta+1}\int_0^t(1+t-\tau)^{-[(n-1)/8]-1/2}(1+\tau)^{-[((n-1)\theta)/4]+1/2}d\tau\end{aligned}$$

$$+ C\delta_0 E^{\theta}\int_0^t (1+t-\tau)^{-[(n-1)/8]}(1+\tau)^{-[((n-1)(\theta-1))/4-[(n-1)/8]}\,d\tau$$
$$+ CE^{\theta+1}\int_0^t (1+t-\tau)^{-[(n-1)/8]-1/4}(1+\tau)^{-[((n-1)\theta)/4]}\,d\tau.$$

Noting that $(n-1)(\theta-1/2)>4$, for $\alpha_1\le l$, we know that

$$\|\partial_{x_1}^{\alpha_1}u_L\|_{L^2}\le C\delta_0(1+E^{\theta})(1+t)^{-(n-1)/8}. \tag{4.11}$$

We now consider the case of $1\le|\alpha'|\le l$. Without loss of generality, we suppose that $\alpha_2\ge 1$, i.e., $\alpha'=(1,0,\dots,0)+\alpha''$ with $\alpha''=(\alpha_2-1,\alpha_3,\dots,\alpha_n)$. Then,

$$\begin{aligned}\|\partial_{x_1}^{\alpha_1}\partial_{x'}^{\alpha'}u_L\|_{L^2} &= \|\delta(x_1)\partial_{x'}^{\alpha'}G_L'(\cdot,t)*\partial_{x_1}^{\alpha_1}u_0(\cdot)\|_{L^2}\\ &\quad+\int_0^t \|\delta(x_1)\partial_{x'}^{\alpha'}G_L'(\cdot,t-\tau)\\ &\quad*\partial_{x_1}^{\alpha_1}(\Delta_{x'}A(u)+\vec r\cdot\nabla B(u))(\cdot,\tau)\|_{L^2}\,d\tau\\ &\le\left\{\int_{\mathbb R}\|\partial_{x'}^{\alpha'}G_L'(\cdot,t)*\partial_{x_1}^{\alpha_1}u_0(x_1,\cdot)\|_{L^2_{x'}}^2\,dx_1\right\}^{1/2}\\ &\quad+\int_0^{t/2}\Gamma^{1/2}d\tau+\int_{t/2}^t\Psi^{1/2}d\tau,\end{aligned}$$

where

$$\Gamma=\int_{\mathbb R}\|\partial_{x'}^{\alpha'}G_L'(\cdot,t-\tau)*\partial_{x_1}^{\alpha_1}(\Delta_{x'}A(u)+\vec r\cdot\nabla B(u))(x_1,\cdot,\tau)\|_{L^2_{x'}}^2\,dx_1,$$

and

$$\Psi=\int_{\mathbb R}\|\partial_{x_2}G_L'(\cdot,t-\tau)*\partial_{x_1}^{\alpha_1}\partial_{x'}^{\alpha''}(\Delta_{x'}A(u)+\vec r\cdot\nabla B(u))(x_1,\cdot,\tau)\|_{L^2_{x'}}^2\,dx_1.$$

By $(n-1)(\theta-1/2)>4$, we obtain

$$\begin{aligned}\|\partial_{x_1}^{\alpha_1}\partial_{x'}^{\alpha'}u_L\|_{L^2}&\le C\delta_0(1+t)^{-[(n-1)/8]-|\alpha'|/4}+CE^{\theta+1}(1+t)^{-[(n-1)/8]-|\alpha'|/4}\\ &\le C(\delta_0+E^{\theta+1})(1+t)^{-[(n-1)/8]-|\alpha'|/4}.\end{aligned}$$

In summary, this lemma is proven. □

Lemma 4.4. *Let* $\|u_0\|_{\mathcal{A}^{l,2}(\mathbb R^n)}+\|u_0\|_{H^{l+1}(\mathbb R^n)}\le\delta_0$ *and* $(n-1)(\theta-1/2)>4$. *Assume that* (4.1) *and* (4.2) *hold. Then, there exists a* *constant $C > 0$ such that*

$$\|\partial_{x_1}^{\alpha_1+1}\partial_{x'}^{\alpha'}u_L\|_{L^2} \le C(\delta_0 + E^\theta)(1+t)^{-[(n-1)/8]-|\alpha'|/4},$$

for $\alpha_1 + |\alpha'| \le l$ and $|\alpha'| \ge 1$.

Proof. For $|\alpha'| = 1$, without loss of generality, we suppose that $\alpha_2 = 1$. Then, we see that

$$\begin{aligned}
\|\partial_{x_1}^{\alpha_1+1}\partial_{x_2}u_L\|_{L^2} &\le \|\partial_{x_2}G_L'\|_{L^2}\bigg\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1+1}u_0(x_1,\cdot)\|_{L^1}^2\,dx_1\bigg\}^{1/2}\\
&\quad + \int_0^t\bigg\{\int_{\mathbb{R}}\|\partial_{x_2}G_L'(\cdot,t-\tau)\\
&\qquad\qquad * \partial_{x_1}^{\alpha_1+2}r_1B(u)(x_1,\cdot,\tau)\|_{L^2_{x'}}^2\,dx_1\bigg\}^{1/2}d\tau\\
&\quad + \int_0^t\bigg\{\int_{\mathbb{R}}\|\Delta_{x'}\partial_{x_2}G_L'(\cdot,t-\tau)\\
&\qquad\qquad * \partial_{x_1}^{\alpha_1+1}A(u)\|_{L^2_{x'}}^2\,dx_1\bigg\}^{1/2}d\tau\\
&\quad + \int_0^t\bigg\{\int_{\mathbb{R}}\|\partial_{x_2}G_L'(\cdot,t-\tau)\\
&\qquad\qquad * \partial_{x_1}^{\alpha_1+1}(r'\cdot\nabla_{x'}B(u))(x_1,\cdot,\tau)\|_{L^2_{x'}}^2\,dx_1\bigg\}^{1/2}d\tau\\
&= I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

First, Lemma 4.2 yields

$$I_1 \le C\delta_0(1+t)^{-[(n-1)/8]-1/4}.$$

Combining the Young inequality, Lemma 2.4 and Lemma 2.5 with $(n-1)(\theta-1/2) > 4$, we conclude that

$$\begin{aligned}
I_2 &\le \int_0^{t/2}\|\Pi\|_{L^2_{x'}}\bigg\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1+2}r_1B(u)(x_1,\cdot,\tau)\|_{L^1_{x'}}^2\,dx_1\bigg\}^{1/2}d\tau\\
&\quad + \int_{t/2}^t\|\Pi\|_{L^1_{x'}}\bigg\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1+2}r_1B(u)(x_1,\cdot,\tau)\|_{L^2_{x'}}^2\,dx_1\bigg\}^{1/2}d\tau
\end{aligned}$$

$$\leq C\int_0^{t/2}\|\Pi\|_{L^2_{x'}}\|\partial_{x_1}^{\alpha_1+2}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{H^1}\|u(\cdot,\tau)\|_{L^\infty}^{\theta-1}d\tau$$
$$+ C\int_{t/2}^{t}\|\Pi\|_{L^1_{x'}}\|\partial_{x_1}^{\alpha_1+2}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{L^\infty}^{\theta}\,d\tau,$$

where

$$\Pi = \partial_{x_2}G'_L(\cdot,\ t-\tau).$$

Noting that $\gamma+\beta-1=(1/4)(n-1)\theta+1/4-1\geq(n-1)/8+1/4=\alpha$, by direct calculations, we derive that

$$\|\partial_{x_2}G'_L(\cdot,t)\|_{L^1_{x'}}\leq C(1+t)^{-1/4},$$

which implies

$$I_2\leq C\delta_0E^\theta\int_0^{t/2}(1+t-\tau)^{-[(n-1)/8]-1/4}(1+\tau)^{-(1/4)(n-1)(\theta-1/2)}d\tau$$
$$+ C\delta_0E^\theta\int_{t/2}^{t}(1+t-\tau)^{-1/4}(1+\tau)^{-(1/4)(n-1)\theta}d\tau$$
$$\leq C\delta_0E^\theta(1+t)^{-[(n-1)/8]-1/4}.$$

Similarly, we have

$$I_3\leq\int_0^{t/2}\|\Delta_{x'}\Pi\|_{L^2_{x'}}\left\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1+1}A(u)(x_1,\cdot,\tau)\|_{L^1_{x'}}^2dx_1\right\}^{1/2}d\tau$$
$$+\int_{t/2}^{t}\|\Delta_{x'}\Pi\|_{L^1_{x'}}\left\{\int_{\mathbb{R}}\|\partial_{x_1}^{\alpha_1+1}A(u)(x_1,\cdot,\tau)\|_{L^2_{x'}}^2dx_1\right\}^{1/2}d\tau$$
$$\leq C\int_0^{t/2}\|\Delta_{x'}\Pi\|_{L^2_{x'}}\|\partial_{x_1}^{\alpha_1+1}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{H^1}\|u(\cdot,\tau)\|_{L^\infty}^{\theta-1}d\tau$$
$$+ C\int_{t/2}^{t}\|\Delta_{x'}\Pi\|_{L^1_{x'}}\|\partial_{x_1}^{\alpha_1+1}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{L^\infty}^{\theta}d\tau.$$

On the other hand, by $\gamma+\beta-1=(1/4)(n-1)\theta+3/4-1\geq(n-1)/8+3/4=\alpha$, we know that

$$\|\Delta_{x'}\Pi\|_{L^1_{x'}}=\|\Delta_{x'}\partial_{x_2}G'_L(\cdot,t)\|_{L^1_{x'}}\leq C(1+t)^{-3/4}.$$

Hence,

$$\begin{aligned}I_3 &\le C\delta_0 E^\theta \int_0^{t/2} (1+t-\tau)^{-[(n-1)/8]-3/4}(1+\tau)^{-(1/4)(n-1)(\theta-1/2)} d\tau \\ &\quad + C\delta_0 E^\theta \int_{t/2}^{t} (1+t-\tau)^{-3/4}(1+\tau)^{-(1/4)(n-1)\theta} d\tau \\ &\le C\delta_0 E^\theta (1+t)^{-[(n-1)/8]-3/4} \le C\delta_0 E^\theta (1+t)^{-[(n-1)/8]-1/4}.\end{aligned}$$

Similarly, we obtain

$$I_4 \le C\delta_0 E^\theta (1+t)^{-[(n-1)/8]-1/4}.$$

If $|\alpha'| \ge 2$, let $\alpha' = \gamma' + \gamma''$ with $|\gamma'| = 2$. Then, we derive

$$\begin{aligned}\|\partial_{x_1}^{\alpha_1+1}\partial_{x'}^{\alpha'} u_L\|_{L^2} &\le \|\partial_{x'}^{\alpha'} G_L'\|_{L^2} \bigg\{ \int_{\mathbb{R}} \|\partial_{x_1}^{\alpha_1+1} u_0(x_1,\cdot)\|_{L^1}^2 \, dx_1 \bigg\}^{1/2} \\ &\quad + \int_0^t \bigg\{ \int_{\mathbb{R}} \|\partial_{x'}^{\alpha'} G_L'(\cdot, t-\tau) \\ &\qquad\qquad * \partial_{x_1}^{\alpha_1+2} r_1 B(u)(x_1,\cdot,\tau)\|_{L^2_{x'}}^2 dx_1 \bigg\}^{1/2} d\tau \\ &\quad + \int_0^t \bigg\{ \int_{\mathbb{R}} \|\Delta_{x'}\partial_{x'}^{\alpha'} G_L'(\cdot, t-\tau) \\ &\qquad\qquad * \partial_{x_1}^{\alpha_1+1} A(u)\|_{L^2_{x'}}^2 dx_1 \bigg\}^{1/2} d\tau \\ &\quad + \int_0^t \bigg\{ \int_{\mathbb{R}} \|\partial_{x'}^{\alpha'} G_L'(\cdot, t-\tau) \\ &\qquad\qquad * \partial_{x_1}^{\alpha_1+1} (r'\cdot\nabla_{x'} B(u))(x_1,\cdot,\tau)\|_{L^2_{x'}}^2 dx_1 \bigg\}^{1/2} d\tau \\ &= J_1 + J_2 + J_3 + J_4.\end{aligned} \tag{4.12}$$

First, Lemma 4.2 shows that

$$J_1 \le C\delta_0 (1+t)^{-[(n-1)/8]-|\alpha'|/4}.$$

Now, we only estimate J_2, and the others are similarly estimated. By Lemma 2.4 and Lemma 2.5, we get

$$J_2 = \int_0^t \bigg\{ \int_{\mathbb{R}} \|\partial_{x'}^{\alpha'} G_L'(\cdot, t-\tau) * \partial_{x_1}^{\alpha_1+2} r_1 B(u)(x_1,\cdot,\tau)\|_{L^2_{x'}}^2 dx_1 \bigg\}^{1/2} d\tau$$

$$\leq C\int_0^{t/2}\|\partial_{x'}^{\alpha'}G_L'(\cdot,t-\tau)\|_{L^2_{x'}}\|\partial_{x_1}^{\alpha_1+2}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{H^1}\|u(\cdot,\tau)\|_{L^\infty}^{\theta-1}d\tau$$
$$+C\int_{t/2}^{t}\|\partial_{x'}^{\gamma'}G_L'(\cdot,t-\tau)\|_{L^1_{x'}}\|\partial_{x'}^{\gamma''}\partial_{x_1}^{\alpha_1+2}u(\cdot,\tau)\|_{L^2}\|u(\cdot,\tau)\|_{L^\infty}^{\theta}\,d\tau.$$

Noting that $\gamma+\beta-1=(n-1)/8+|\alpha'|/4+(1/4)(n-1)\theta-1\geq(n-1)/8+|\alpha'|/4=\alpha$, simple calculation shows that

$$\|\partial_{x'}^{\gamma'}G_L'(\cdot,t)\|_{L^1_{x'}}\leq C(1+t)^{-1/2}.$$

On the other hand, by $(1/4)(n-1)(\theta-1/2)>1$, we see that $(1/4)(n-1)\theta-1>0$. Hence, by $|\gamma''|+\alpha_1+2\leq l$, we have

$$J_2\leq C\delta_0E^\theta\int_0^{t/2}(1+t-\tau)^{-[(n-1)/8]-|\alpha'|/4}(1+\tau)^{-(1/4)(n-1)(\theta-1/2)}d\tau$$
$$+C\delta_0E^\theta\int_{t/2}^{t}(1+t-\tau)^{-1/2}(1+\tau)^{-[(n-1)/8]-|\alpha'|/4+(1/2)-(1/4)(n-1)\theta}d\tau$$
$$\leq C\delta_0E^\theta(1+t)^{-[(n-1)/8]-|\alpha'|/4}.$$

It is not difficult to see that J_3 and J_4 hold with the same estimate of J_2. Combining the above estimates, when $(n-1)(\theta-1/2)>4$, we deduce that

$$\|\partial_{x_1}^{\alpha_1+1}\partial_{x'}^{\alpha'}u_L\|_{L^2}\leq C(\delta_0+E^\theta)(1+t)^{-[(n-1)/8]-|\alpha'|/4}.$$

The proof is complete. □

Next, we estimate the L^2 decay for the high frequency part. First, we have the following Poincaré-like inequality.

Lemma 4.5. *For any multi-index α, $|\alpha|\geq 0$, there exist constants C_0 and C_1 such that*

$$\|\partial_x^\alpha u_H\|_{L^2}^2\leq C_0\|\Delta_{x'}\partial_x^\alpha u_H\|_{L^2}^2,\tag{4.13}$$

$$\|\nabla_{x'}\partial_x^\alpha u_H\|_{L^2}^2\leq C_1\|\Delta_{x'}\partial_x^\alpha u_H\|_{L^2}^2.\tag{4.14}$$

Proof. Using the Plancherel theorem and $u_H = (1-\chi(D))u$, we easily see that

$$\begin{aligned}\|\partial_x^\alpha u_H\|_{L^2}^2 = \|\widehat{\partial_x^\alpha u_H}\|_{L^2}^2 &= \left\|\frac{1}{|\xi'|^2}|\xi'|^2\widehat{\partial_x^\alpha u_H}\right\|_{L^2}^2 \\ &\le C_0\||\xi'|^2\widehat{\partial_x^\alpha u_H}\|_{L^2}^2 = C_0\|\Delta_{x'}\partial_x^\alpha u_H\|_{L^2}^2\end{aligned}$$

and

$$\begin{aligned}\|\nabla_{x'}\partial_x^\alpha u_H\|_{L^2}^2 = \||\xi'|\widehat{\partial_x^\alpha u_H}\|_{L^2}^2 &= \left\|\frac{1}{|\xi'|}|\xi'|^2\widehat{\partial_x^\alpha u_H}\right\|_{L^2}^2 \\ &\le C_1\||\xi'|^2\widehat{\partial_x^\alpha u_H}\|_{L^2}^2 = C_1\|\Delta_{x'}\partial_x^\alpha u_H\|_{L^2}^2.\end{aligned}$$

The proof is complete. □

Next, we shall prove the decay estimates for the high frequency part u_H of the solution.

Lemma 4.6. *Assume that* $\|u_0\|_{\mathcal{A}^{l,2}(\mathbb{R}^n)}+\|u_0\|_{H^{l+1}(\mathbb{R}^n)} \le \delta_0$, $(n-1)(\theta-1/2) > 4$ *and* (4.1), (4.2) *hold. Then, for any* α *with* $|\alpha| \le l$, *there exists a constant* $C > 0$ *such that*

$$\|\partial_x^\alpha u_H\|_{L^2} \le C(\delta_0 + E^{\theta+1})(1+t)^{-[(n-1)/4]-|\alpha'|/4}.$$

Proof. Applying the operator $(1-\chi(D))$ to problem (1.1), we easily derive

$$\begin{cases}\partial_t u_H + \Delta_{x'}^2 u_H = (1-\chi(D))\Delta_{x'}A(u) + \vec{r}\cdot(1-\chi(D))\nabla B(u), \\ u_H(x,0) = u_{0H}(x).\end{cases} \tag{4.15}$$

Multiplying the above equation by u_H and integrating over R^n, using integration by parts, we know that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_H\|_{L^2}^2 + \|\Delta_{x'}u_H\|_{L^2}^2 \\ &\quad= \int_{R^n}\big((1-\chi(D))\Delta_{x'}A(u)u_H + \vec{r}\cdot(1-\chi(D))\nabla B(u)u_H\big)\,dx.\end{aligned}$$

Using Cauchy's inequality, Lemma 2.2 and Lemma 4.5, we have

$$\begin{aligned}
&\frac{d}{dt}\|u_H\|_{L^2}^2 + \|\Delta_{x'} u_H\|_{L^2}^2 + \frac{1}{C_0}\|u_H\|_{L^2}^2 \\
&\quad \le 2\int_{R^n} \big((1-\chi(D))\Delta_{x'} A(u)u_H + \vec{r}\cdot(1-\chi(D))\nabla B(u)u_H\big)dx \\
&\quad \le C_\varepsilon(\|(1-\chi(D))\Delta_{x'} A(u)\|_{L^2}^2 + \|\vec{r}\cdot(1-\chi(D))\nabla B(u)\|_{L^2}^2) + \varepsilon\|u_H\|_{L^2}^2 \\
&\quad \le C_{\varepsilon,\vec{r}}\|u\|_{L^\infty}^{2\theta}(\|\Delta u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2) + \varepsilon\|u_H\|_{L^2}^2
\end{aligned}$$

for small $\varepsilon > 0$. Taking $\varepsilon \le 1/(2C_0)$, (4.1) and (4.2) imply that

$$\begin{aligned}
&\frac{d}{dt}\|u_H\|_{L^2}^2 + \|\Delta_{x'} u_H\|_{L^2}^2 + \frac{1}{2C_0}\|u_H\|_{L^2}^2 \\
&\quad \le CE^{2(1+\theta)}\big((1+t)^{-[(n-1)/4]-[\theta(n-1)/2]-1/2} \\
&\qquad\qquad + (1+t)^{-[(n-1)/4]-[\theta(n-1)/2]-1/4}\big) \\
&\quad \le CE^{2(1+\theta)}(1+t)^{-[(n-1)/4]-[\theta(n-1)/2]}.
\end{aligned}$$

Multiplying both sides of the above inequality by $e^{t/(2C_0)}$, we obtain

$$\begin{aligned}
\frac{d}{dt}(\|u_H\|_{L^2}^2 e^{t/(2C_0)}) + \|\Delta_{x'} u_H\|_{L^2}^2 e^{t/(2C_0)} \\
\le CE^{2(1+\theta)}(1+t)^{-[(n-1)/4]-[\theta(n-1)/2]}e^{t/(2C_0)}.
\end{aligned} \tag{4.16}$$

If $\theta > 1/2$, integrating the above inequality over t, we see that

$$\begin{aligned}
&\|u_H\|_{L^2}^2 + \int_0^t e^{-(t-\tau)/(2C_0)}\|\Delta_{x'} u_H\|_{L^2}^2\, d\tau \\
&\quad \le e^{-t/(2C_0)}\|u_0\|_{L^2}^2 \\
&\qquad + CE^{2(1+\theta)}\int_0^t e^{-(t-\tau)/(2C_0)}(1+\tau)^{-[(n-1)/4]-[\theta(n-1)/2]}\, d\tau \\
&\quad \le e^{-t/(2C_0)}\delta_0^2 + CE^{2(1+\theta)}(1+t)^{-[(n-1)/4]-[\theta(n-1)/2]}\, d\tau \\
&\quad \le C(\delta_0^2 + E^{2(1+\theta)})(1+t)^{-(n-1)/2}.
\end{aligned}$$

Next, we establish estimates for $\|\partial_{x_1}^{\alpha_1} u_H\|_{L^2}$, $0 < \alpha_1 \le l$. Applying the operator $\partial_{x_1}^{\alpha_1}$ to (4.15) and integrating by parts, for small $\varepsilon > 0$, we conclude that

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\|\Delta_{x'}\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\frac{1}{C_0}\|\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2\\
&\quad\le 2\int_{R^n}\big((1-\chi(D))\Delta_{x'}\partial_{x_1}^{\alpha_1}A(u)\partial_{x_1}^{\alpha_1}u_H\,dx\\
&\qquad+2\int_{R^n}\vec{r'}\cdot(1-\chi(D))\nabla_{x'}\partial_{x_1}^{\alpha_1}B(u)\partial_{x_1}^{\alpha_1}u_H\,dx\\
&\qquad+2\int_{R^n}r_1(1-\chi(D))\partial_{x_1}^{\alpha_1+1}B(u)\partial_{x_1}^{\alpha_1}u_H\,dx\\
&\quad\le C_\varepsilon(\|\partial_{x_1}^{\alpha_1}A(u)\|_{L^2}^2+\|\partial_{x_1}^{\alpha_1}B(u)\|_{L^2}^2+\|\partial_{x_1}^{\alpha_1+1}B(u)\|_{L^2}^2))\\
&\qquad+\varepsilon(\|\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\|\nabla_{x'}\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\|\Delta_{x'}\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2)\\
&\quad\le C_\varepsilon\|u\|_{L^\infty}^{2\theta}(\|\partial_{x_1}^{\alpha_1}u\|_{L^2}^2+\|\partial_{x_1}^{\alpha_1+1}u\|_{L^2}^2)\\
&\qquad+\varepsilon(\|\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\|\nabla_{x'}\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\|\Delta_{x'}\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2).
\end{aligned}
$$

Choosing $\varepsilon \le \max\{1/(2C_1+2), 1/(2C_0)\}$, by Lemma 4.1 and the a priori assumption, we know that

$$
\begin{aligned}
&\frac{d}{dt}\|\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\frac{1}{2}\|\Delta_{x'}\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\frac{1}{2C_0}\|\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2\\
&\quad\le C\|u\|_{L^\infty}^{2\theta}(\|\partial_{x_1}^{\alpha_1}u\|_{L^2}^2+\|\partial_{x_1}^{\alpha_1+1}u\|_{L^2}^2)\\
&\quad\le CE^{2\theta}\|\partial_{x_1}^{\alpha_1}u_0\|_{L^2}^2(1+t)^{-[(n-1)\theta]/2}.
\end{aligned}
$$

Similarly to (4.16), when $\theta \ge 1$, we obtain

$$
\begin{aligned}
&\|\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2+\frac{1}{2}\int_0^t e^{-(t-\tau)/(2C_0)}\|\Delta_{x'}\partial_{x_1}^{\alpha_1}u_H\|_{L^2}^2d\tau\\
&\quad\le C\delta_0^2(1+E^{2(1+\theta)})(1+t)^{-(n-1)/2}.
\end{aligned}
$$

In the same way, we can estimate $\|\partial_{x_1}^{\alpha_1}\partial_{x'}^{\alpha'}u_H\|_{L^2}$ for $|\alpha'| \ge 1$ as follows:

$$
\begin{aligned}
&\|\partial_{x_1}^{\alpha_1}\partial_{x'}^{\alpha'}u_H\|_{L^2}^2+\frac{1}{2}\int_0^t e^{-(t-\tau)/(2C_0)}\|\Delta_{x'}\partial_{x_1}^{\alpha_1}\partial_{x'}^{\alpha'}u_H\|_{L^2}^2\,d\tau\\
&\quad\le C(\delta_0^2+E^{2(1+\theta)})(1+t)^{-[(n-1)/2]-|\alpha'|/2},
\end{aligned}
$$

if $\theta \ge 1/2$. The proof is complete. □

By Lemma 4.3 and Lemma 4.6, we have the following estimate.

Lemma 4.7. *Assume that* $\|u_0\|_{\mathcal{A}^{l,2}(\mathbb{R}^n)}+\|u_0\|_{H^{l+1}(\mathbb{R}^n)} \le \delta_0$, $(n-1)(\theta-1/2) > 4$, *and the a priori assumptions hold. Then, for any* α *with* $|\alpha| \le l$, *there exists a constant* $C > 0$ *such that*

$$\|\partial_x^\alpha u\|_{L^2} \le C(\delta_0 + E^{\theta+1})(1+t)^{-[(n-1)/8]-|\alpha'|/4}. \tag{4.17}$$

Now, we also use the frequency decomposition method to derive L^∞ decay estimates of solutions.

Lemma 4.8. *Assume that* $\|u_0\|_{\mathcal{A}^{l,\infty}(\mathbb{R}^n)}+\|u_0\|_{H^{l}(\mathbb{R}^n)} \le \delta_0$, $(n-1)\theta > 4$ *and* (4.1), (4.2) *hold. Then, for any* β *with* $|\beta| \le l-[n/2]-1$, *there exists a constant* $C > 0$, *such that*

$$\|\partial_x^\beta u_L\|_{L^\infty} \le C(\delta_0 + E^{\theta+1})(1+t)^{-[(n-1)/4]-|\beta'|/4}.$$

Proof. For any multi-index $\beta = (\beta_1, \beta')$ with $0 \le |\beta| \le l-[n/2]-1$, we know that

$$\|\partial_x^\beta u_L(x,t)\|_{L^\infty} = \partial_{x'}^{\beta'} G_L(\cdot,t) * \partial_{x_1}^{\beta_1} u_0(\cdot) + \int_0^t \partial_{x'}^{\beta'} G(\cdot, t-\tau) * \partial_{x_1}^{\beta_1}(\Delta_{x'} A(u) + \vec{r}\cdot\nabla B(u))(\cdot,\tau)\, d\tau.$$

If $|\beta'| = 0$ in the above equality, Young's inequality yields

$$\begin{aligned}
\|\partial_{x_1}^{\beta_1} u_L\|_{L^\infty} &\le \|\delta(x_1) G_L'(\cdot,t) * \partial_{x_1}^{\beta_1} u_0(\cdot)\|_{L^\infty} \\
&\quad + \int_0^t \|\delta(x_1) G'(\cdot,t-\tau) * \partial_{x_1}^{\beta_1}(\Delta_{x'} A(u) \\
&\qquad\qquad + \vec{r}\cdot\nabla B(u))(\cdot,\tau)\|_{L^\infty}\, d\tau \\
&\le \sup_{x\in R} \|G_L'(\cdot,t) * \partial_{x_1}^{\beta_1} u_0(x_1,\cdot)\|_{L^\infty_{x'}} \\
&\quad + \int_0^t \sup_{x\in R} \|G'(\cdot,t-\tau) * \partial_{x_1}^{\beta_1}(\Delta_{x'} A(u) \\
&\qquad\qquad + \vec{r}\cdot\nabla B(u))(x_1,\cdot,\tau)\|_{L^\infty_{x'}}\, d\tau \\
&\le \|G_L'\|_{L^\infty_{x'}} \|\partial_{x_1}^{\beta_1} u_0\|_{L^\infty(R_{x_1}; L^1(R^{n-1}))} \\
&\quad + \int_0^t \|\Delta_{x'} G'(\cdot,t-\tau)\|_{L^\infty_{x'}} \sup_{x\in R} \|\partial_{x_1}^{\beta_1} A(u)(x_1,\cdot,\tau)\|_{L^1_{x'}}\, d\tau \\
&\quad + \int_0^t r_1 \|G'(\cdot,t-\tau)\|_{L^\infty_{x'}} \sup_{x\in R} \|\partial_{x_1}^{\beta_1+1} B(u)(x_1,\cdot,\tau)\|_{L^1_{x'}}\, d\tau
\end{aligned}$$

$$+\int_0^t |\vec{r'}| \|\nabla_{x'} G'(\cdot, t-\tau)\|_{L^\infty_{x'}} \sup_{x\in R} \|\partial_{x_1}^{\beta_1} B(u)(x_1,\cdot,\tau)\|_{L^1_{x'}} d\tau.$$

By Lemma 2.4, we deduce that

$$\begin{aligned}
&\|\partial_{x_1}^{\beta_1} u_L\|_{L^\infty} \\
&\quad \le \|G'_L\|_{L^\infty_{x'}} \|\partial_{x_1}^{\beta_1} u_0\|_{L^\infty(R_{x_1};L^1(R^{n-1}))} \\
&\qquad + C\int_0^t \|\Delta_{x'} G'(\cdot, t-\tau)\|_{L^\infty_{x'}} \|u(\cdot,\tau)\|_{H^{\beta_1+1}} \|u(\cdot,\tau)\|_{H^1} \|u(\cdot,\tau)\|_{L^\infty}^{\theta-1} d\tau \\
&\qquad + C\int_0^t \|\nabla_{x'} G'(\cdot, t-\tau)\|_{L^\infty_{x'}} \|u(\cdot,\tau)\|_{H^{\beta_1+1}} \|u(\cdot,\tau)\|_{H^1} \|u(\cdot,\tau)\|_{L^\infty}^{\theta-1} d\tau \\
&\qquad + C\int_0^t \|G'(\cdot, t-\tau)\|_{L^\infty_{x'}} \|u(\cdot,\tau)\|_{H^{\beta_1+2}} \|u(\cdot,\tau)\|_{H^1} \|u(\cdot,\tau)\|_{L^\infty}^{\theta-1} d\tau.
\end{aligned}$$

When $(n-1)\theta > 4$, by Lemma 4.2, Lemma 2.5 and the fact $\beta_1 + 2 \le l$ $(|\beta| \le l - [n]/2 - 1, |\beta| = |\beta'| + \beta_1, |\beta'| = 0,\ n \ge 2)$, we conclude that

$$\begin{aligned}
&\|\partial_{x_1}^{\beta_1} u_L\|_{L^\infty} \\
&\quad \le C\delta_0 (1+t)^{-[(n-1)/4]} \\
&\qquad + CE^{\theta+1} \int_0^t (1+t-\tau)^{-[(n-1)/4]-(1/2)} (1+\tau)^{-[(n-1)\theta/4]-(1/4)}\, d\tau \\
&\qquad + CE^{\theta+1} \int_0^t (1+t-\tau)^{-[(n-1)/4]-(1/4)} (1+\tau)^{-[(n-1)\theta/4]-(1/4)}\, d\tau \\
&\qquad + CE^{\theta+1} \int_0^t (1+t-\tau)^{-[(n-1)/4]} (1+\tau)^{-[(n-1)\theta/4]-(1/4)}\, d\tau \\
&\quad \le CE^{\theta+1} \int_0^t (1+t-\tau)^{-[(n-1)/4]} (1+\tau)^{-[(n-1)\theta/4]}\, d\tau \\
&\quad \le C(\delta_0 + E^{\theta+1})(1+t)^{-(n-1)/4}.
\end{aligned}$$

Here, we have used the fact

$$(1+\tau)^{-[(n-1)\theta]/4-(1/4)} \le (1+\tau)^{-[(n-1)\theta]/4} \quad \text{for any } \tau > 0.$$

Next, we consider the case that $|\beta'| \ge 1$. Without loss of generality, we assume that $|\beta_2| \ge 1$. By Young's inequality, we see that

$$
\begin{aligned}
\|\partial_{x_1}^{\beta_1}\partial_{x'}^{\beta'}u_L\|_{L^\infty} &\le \|\partial_{x'}^{\beta'}G_L'\|_{L^\infty_{x'}}\|\partial_{x_1}^{\beta_1}u_0\|_{L^\infty(R_{x_1};L^1(R^{n-1}))} \\
&\quad + \int_0^{t/2}\|\Delta_{x'}\partial_{x'}^{\beta'}G'(\cdot,t-\tau)\|_{L^\infty_{x'}}\sup_{x\in R}\|\partial_{x_1}^{\beta_1}A(u)(x_1,\cdot,\tau)\|_{L^1_{x'}}d\tau \\
&\quad + |\vec r|\int_0^{t/2}\|\nabla_{x'}\partial_{x'}^{\beta'}G'(\cdot,t-\tau)\|_{L^\infty_{x'}}\sup_{x\in R}\|\partial_{x_1}^{\beta_1}B(u)(x_1,\cdot,\tau)\|_{L^1_{x'}}d\tau \\
&\quad + |\vec r|\int_0^{t/2}\|\partial_{x'}^{\beta'}G'(\cdot,t-\tau)\|_{L^\infty_{x'}}\sup_{x\in R}\|\partial_{x_1}^{\beta_1+1}B(u)(x_1,\cdot,\tau)\|_{L^1_{x'}}d\tau \\
&\quad + \int_{t/2}^{t}\|\Delta_{x'}\partial_{x_2}G'(\cdot,t-\tau)\|_{L^1_{x'}}\sup_{x\in R}\|\partial_{x_1}^{\beta_1}\partial_{x'}^{\beta''}A(u)(x_1,\cdot,\tau)\|_{L^\infty_{x'}}d\tau \\
&\quad + \int_{t/2}^{t}\|\partial_{x_2}G'(\cdot,t-\tau)\|_{L^1_{x'}}\sup_{x\in R}\|\vec r\cdot\partial_{x_1}^{\beta_1}\partial_{x'}^{\beta''}\nabla B(u)(x_1,\cdot,\tau)\|_{L^\infty_{x'}}d\tau \\
&= \sum_{i=1}^{6}L_i,
\end{aligned}
$$

where $\beta''=(\beta_2-1,\beta_3,\dots,\beta_n)$. Similarly to the estimates of the terms I_i of Lemma 4.4, we can estimate L_i term-by-term. Therefore, we have

$$
\|\partial_{x_1}^{\beta_1}\partial_{x'}^{\beta'}u_L\|_{L^\infty}\le C(\delta_0+E^{\theta+1})(1+t)^{-[(n-1)/4]-|\beta'|/4},
$$

when $(n-1)\theta>4$. The proof is complete. □

Lemma 4.9. *Assume that* $\|u_0\|_{\mathcal{A}^{l,2}(\mathbb{R}^n)}+\|u_0\|_{H^{l+1}(\mathbb{R}^n)}\le\delta_0$, $(n-1)(\theta-1/2)>4$ *and* (4.1), (4.2) *hold. Then, for any* β *with* $|\beta|\le l-[n/2]-1$, *there exists a constant* $C>0$ *such that*

$$
\|\partial_x^{\beta}u_H\|_{L^\infty}\le C(\delta_0+E^{\theta+1})(1+t)^{-[(n-1)/4]-|\beta'|/4}.
$$

Proof. Lemma 4.6 and the Sobolev embedding theorem yield the desired conclusion, and the proof is complete. □

By Lemma 4.8 and Lemma 4.9, we immediately obtain the following lemma.

Lemma 4.10. *Let* $\|u_0\|_{\mathcal{A}^{l,2}(\mathbb{R}^n)}+\|u_0\|_{\mathcal{A}^{l,\infty}(\mathbb{R}^n)}+\|u_0\|_{H^{l+1}(\mathbb{R}^n)}\le\delta_0$, $(n-1)(\theta-1/2)>4$. *Assume also that* (4.1), (4.2) *hold. Then, for any*

β *with* $|\beta| \le l - [n/2] - 1$, *there exists a constant* $C > 0$ *such that*

$$\|\partial_x^\beta u\|_{L^\infty} \le C(\delta_0 + E^{\theta+1})(1+t)^{-[(n-1)/4]-|\beta'|/4}.$$

Combining Lemma 4.7, Lemma 4.10 and taking E sufficiently small, we obtain the a priori assumptions. Therefore, the proof of Theorem 1.1 is complete.

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