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# EIGENVALUES OF SOME $p(x)$-BIHARMONIC PROBLEMS UNDER NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. In this paper, we study the following $p(x)$-biharmonic problem in Sobolev spaces with variable exponents

$$\begin{cases} \triangle^2_{p(x)} u = \lambda(\partial F(x,u)/\partial u) & x \in \Omega, \\ \partial u/\partial n = 0 & x \in \partial\Omega, \\ \partial(|\triangle u|^{p(x)-2}\triangle u)/\partial n = a(x)|u|^{p(x)-2}u & x \in \partial\Omega. \end{cases}$$

By means of the variational approach and Ekeland's principle, we establish that the above problem admits a nontrivial weak solution under appropriate conditions.

**1. Introduction.** Stimulated by the development of the study of elastic mechanics, see [**29**], electrorheological fluids, see [**26**], image processing, see [**5**], and mathematical description of the filtration processes of an ideal baroscopic gas through a porous medium, see [**1**], interest in variational problems and differential equations with variable exponents has grown in recent decades. Meanwhile, elliptic problems involving operators in divergence form can be found in [**4, 22**]. Some other results dealing with the $p(x)$-Laplace and the $p(x)$-biharmonic operators in Sobolev spaces with variable exponents can be found in [**12, 15, 16, 17, 18, 20, 21**].

The purpose of this paper is to study the existence of an eigenvalue for the following $p(x)$-biharmonic problem

$$\begin{cases} \triangle^2_{p(x)} u = \lambda(\partial F(x,u)/\partial u) & x \in \Omega, \\ \partial u/\partial n = 0 & x \in \partial\Omega, \\ \partial(|\triangle u|^{p(x)-2}\triangle u)/\partial n = a(x)|u|^{p(x)-2}u & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$ with sufficiently smooth boundary $\partial\Omega$, $\Delta^2_{p(x)} u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator of fourth order, $n$ is a unit outward normal to $\partial\Omega$, $a \in L^\infty(\partial\Omega)$ with $a^- := \inf_{x\in\partial\Omega} a(x) > 0$, $\lambda$ is a positive real number and the functions $p$ and $F$ satisfy the following assumptions:

$$p \in C(\overline{\Omega}) \text{ with } p^- := \inf_{x\in\overline{\Omega}} p(x) > 1 \quad \text{and} \quad F \in C^1(\overline{\Omega}\times\mathbb{R}, \mathbb{R}).$$

The $p(x)$-biharmonic problem under Neumann boundary conditions has been studied by many authors in recent years. Let us recall that Ben Haddouch, et al. **[3]**, studied the following problem:

$$\begin{cases} \triangle^2_{p(x)} u = \lambda |u|^{q(x)-2}u & x \in \Omega, \\ \partial u/\partial n = \partial(|\triangle u|^{p(x)-2}\triangle u)/\partial n = 0 & x \in \partial\Omega. \end{cases} \tag{1.2}$$

The authors established the existence of a continuous family of eigenvalues by using the Mountain pass lemma and Ekeland's variational principle. Moreover, Taarabti, et al. **[27]**, studied the following nonhomogeneous eigenvalue problem

$$\begin{cases} \triangle^2_{p(x)} u = \lambda V(x) |u|^{q(x)-2}u & x \in \Omega, \\ \partial u/\partial n = \partial(|\triangle u|^{p(x)-2}\triangle u)/\partial n = 0 & x \in \partial\Omega. \end{cases} \tag{1.3}$$

They used Ekeland's variational principle to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin. Moreover, Bin Ge, et al. **[13]**, proved the existence of a continuous family of eigenvalues by considering different situations concerning the growth rates involved in the above-quoted problem. Inspired by the above-mentioned papers, we study problem (1.1) under the following assumptions.

(**H1**) $F : \overline{\Omega}\times\mathbb{R} \to \mathbb{R}$ is a $C^1$ function such that

$$F(x, tu) = t^{q(x)} F(x, u), \ t > 0, \text{ for all } x \in \Omega, \ u \in \mathbb{R}.$$

(**H2**)
$$\left|\frac{\partial F}{\partial t}(x,t)\right| \leq c_1 V(x) |t|^{q(x)-1},$$

for all $t \in \mathbb{R}$, for all $x \in \overline{\Omega}$, where $c$ is a positive constant, $V \in L^{s(x)}(\Omega)$ and $s, q \in C(\overline{\Omega})$ are such that, for all $x \in \overline{\Omega}$, we have $1 < q(x) < p(x) < N/2 < s(x)$.

(**H3**) There exists an $\Omega_0 \subset\subset \Omega$ with $|\Omega_0| > 0$ such that $F(x,t) > 0$ in $\Omega_0$.

**Remark 1.1.** Due to assumption **(H1)**, $F$ leads to the so-called Euler identity

$$t\frac{\partial F}{\partial t}(x,t) = q(x)F(x,t), \quad \text{for all } x \in \Omega,\ t \in \mathbb{R}. \tag{1.4}$$

Our main results establish, for small perturbation, the existence of a continuous family of eigenvalues in a neighborhood of the origin. On the other hand, we show the existence of a global minimizer of the Euler Lagrange functional associated to problem (1.1).

**2. Terminology and abstract setting.** In order to study $p(x)$-biharmonic problems, we need some results on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$, see **[10, 14, 24, 25]** for details, complements and proofs.

Set

$$C_+(\overline{\Omega}) := \{h : h \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$, we denote $1 < p^- := \min_{x\in\overline{\Omega}} p(x) \le p^+ = \max_{x\in\overline{\Omega}} p(x) < \infty$ and

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)}dx < \infty\right\}.$$

The spaces $L^{p(x)}(\Omega)$ were introduced by Orlicz **[23]**.

The space $L^{p(x)}(\Omega)$ is endowed with the Luxemburg norm, defined by

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Clearly, when $p(x) \equiv p$, the space $L^{p(x)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, and the norm $|u|_{p(x)}$ reduces to the standard norm

$$\|u\|_{L^p} = \left(\int_\Omega |u|^p dx\right)^{1/p} \quad \text{in } L^p(\Omega).$$

For any positive integer $k$, let

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is a multi-index,

$$|\alpha| = \sum_{i=1}^{N} \alpha_i \quad \text{and} \quad D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_N} x_n}.$$

Then, $W^{k,p(x)}(\Omega)$ is a separable and reflexive Banach space, equipped with the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)}.$$

Let $L^{p'(x)}(\Omega)$ be the conjugate space of $L^{p(x)}(\Omega)$ with $1/p + 1/p' = 1$. Then, the following Hölder-type inequality

$$\left| \int_\Omega uv \, dx \right| \le \left( \frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)}, \quad u \in L^{p(x)}(\Omega),\ v \in L^{p'(x)}(\Omega), \tag{2.1}$$

holds. Moreover, if $h_1$, $h_2$ and $h_3 : \overline{\Omega} \to (1, \infty)$ are Lipschitz continuous functions such that $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$, then, for any $u \in L^{h_1(x)}(\Omega), v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$, the following inequality holds [**9,** Proposition 2.5]:

$$\left| \int_\Omega uvw \, dx \right| \le \left( \frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-} \right) |u|_{h_1(x)} |v|_{h_2(x)} |w|_{h_3(x)}. \tag{2.2}$$

Inequality (2.1) and its generalized version (2.2) are due to Orlicz [**23**].

The modular on the space $L^{p(x)}(\Omega)$ is the map $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx.$$

**Proposition 2.1** ([**19**])**.** *For all $u, v \in L^{p(x)}(\Omega)$, we have*

(i) $|u|_{p(x)} < 1$ (respectively, $= 1, > 1$) $\Leftrightarrow \rho_{p(x)}(u) < 1$ (respectively, $= 1, > 1$).

(ii) $\min(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}) \le \rho_{p(x)}(u) \le \max(|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+})$.

(iii) $\rho_{p(x)}(u - v) \to 0 \Leftrightarrow |u - v|_{p(x)} \to 0$.

Another interesting property of the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the following.

**Proposition 2.2** ([**6**]). *Let $p$ and $q$ be measurable functions such that $p \in L^{\infty}(\Omega)$ and $1 \le p(x)q(x) \le \infty$, for almost every $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$\min(|u|_{p(x)q(x)}^{p^+}, |u|_{p(x)q(x)}^{p^-}) \le ||u|^{p(x)}|_{q(x)} \le \max(|u|_{p(x)q(x)}^{p^-}, |u|_{p(x)q(x)}^{p^+}).$$

In order to prove the existence of a weak solution for problem (1.1), we introduce the space

$$X = \left\{ u \in W^{2,p(x)}(\Omega) : \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0 \right\}.$$

This space was first considered by El Amrouss, et al. [**7**], who proved that $X$ is a nonempty and well-defined closed subspace of $W^{2,p(x)}(\Omega)$.

Let

$$\|u\|_a := \inf \left\{ \mu > 0 : \int_{\Omega} \left| \frac{\Delta u}{\mu} \right|^{p(x)} dx + \int_{\partial\Omega} a(x) \left| \frac{u}{\mu} \right|^{p(x)} d\sigma \le 1 \right\}$$

for $u \in X$. Since $a \in L^{\infty}(\partial\Omega)$ and $\operatorname{essinf}_{x\in\Omega} a > 0$, we deduce that $\|u\|_a$ is an equivalent norm to $\|u\|_{2,p(x)}$ in $X$. Here, we will use the norm $\|u\|_a$, and the modular is defined as $\rho_{p(x)}^a : X \to \mathbb{R}$ by

$$\rho_{p(x)}^a(u) = \int_{\Omega} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} a(x)|u|^{p(x)} d\sigma,$$

which satisfies the same properties as Proposition 2.1. Accordingly, we have, similar to [**11,** Theorem 1.3], the following propositions.

**Proposition 2.3.** *For all $u \in L^{p(x)}(\Omega)$, we have*

(i) $\|u\|_a < 1$ (respectively, $= 1, > 1$) $\Leftrightarrow \rho_{p(x)}^a(u) < 1$ (respectively, $= 1, > 1$).

(ii) $\min(\|u\|_a^{p^-}, \|u\|_a^{p^+}) \le \rho_{p(x)}^a(u) \le \max(\|u\|_a^{p^-}, \|u\|_a^{p^+})$.

(iii) $\|u_n\|_a \to 0$ (respectively, $\to \infty$) $\Leftrightarrow \rho_{p(x)}^a(u_n) \to 0$ (respectively, $\to \infty$).

Arguments similar to those used in the proof of [**2,** Proposition 4.2] showed the following.

**Proposition 2.4.** *Let*

$$I_a(u) = \int_\Omega \frac{1}{p(x)} |\Delta u|^{p(x)} dx + \int_{\partial\Omega} \frac{1}{p(x)} a(x)|u|^{p(x)} d\sigma.$$

*Then*

(i) $I_a : X \to \mathbb{R}$ *is sequentially weakly lower semi-continuous,* $I_a \in C^1(X, \mathbb{R})$.

(ii) *The mapping* $I'_a : X \to X^*$ *is a strictly monotone, bounded homeomorphism, and is of type* $(S_+)$*, that is, if* $u_n \rightharpoonup u$ *and* $\limsup_{n\to+\infty} I'_a(u_n)(u_n - u) \leq 0$*, then* $u_n \to u$.

We recall that the critical Sobolev exponent is defined as follows:

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & p(x) < \dfrac{N}{2}, \\ +\infty & p(x) \geq \dfrac{N}{2}. \end{cases}$$

We point out that, if $q \in C^+(\overline{\Omega})$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then $X$ is continuously and compactly embedded in $L^{q(x)}(\Omega)$. The Lebesgue and Sobolev spaces with variable exponents coincide with the usual Lebesgue and Sobolev spaces, provided that $p$ is constant. According to [**25,** pages 8–9], these function spaces $L^{p(x)}$ and $W^{1,p(x)}$ have some unusual properties, such as:

(i) Assuming that $1 < p^- \leq p^+ < \infty$, and $p : \overline{\Omega} \to [1, \infty)$ is a smooth function, then the following co-area formula

$$\int_\Omega |u(x)|^p dx = p \int_0^\infty t^{p-1} \left|\{x \in \Omega;\ |u(x)| > t\}\right| dt$$

has no analog in the framework of variable exponents.

(ii) Spaces $L^{p(x)}$ do *not* satisfy the *mean continuity property.* More exactly, if $p$ is nonconstant and continuous in an open ball $B$, then there is some $u \in L^{p(x)}(B)$ such that $u(x + h) \notin L^{p(x)}(B)$ for every $h \in \mathbb{R}^N$ with arbitrary small norm.

(iii) Function spaces with variable exponents are *never* invariant with respect to translations. The convolution is also limited. For instance, the classical Young inequality

$$|f * g|_{p(x)} \leq c\,|f|_{p(x)}\,\|g\|_{L^1}$$

remains true if and only if $p$ is constant.

**3. Main results and auxiliary properties.** Throughout the paper, the letters $c, c_i$, $i = 1, 2, \ldots$, denote positive constants which may change from line to line. In the sequel, denote by $s'(x)$ the conjugate exponent of the function $s(x)$, and put $\alpha(x) := s(x)q(x)/(s(x) - q(x))$. Then, we have:

**Remark 3.1.** Under assumption $(\mathbf{H}_2)$, we have $s'(x)q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$; hence, the embeddings $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ and $X \hookrightarrow L^{\alpha(x)}(\Omega)$ are compact and continuous.

**Proposition 3.2** ([**8,** Theorem 2.4])**.** *Let $\Omega \in \mathbb{R}^N$ be an open bounded domain with Lipschitz boundary. Let $m$ be a positive integer. Suppose that $p \in C^0(\overline{\Omega})$ with $p^- > 1$ and $mp^+ < N$. If $q \in S(\partial\Omega)$, where $S(\partial\Omega)$ is the set of all measurable real functions defined on $\Omega$, and there exists a positive constant $\varepsilon$ such that*

$$1 \leq q(x) < q(x) + \varepsilon \leq \frac{(N-1)p(x)}{N - mp(x)} \quad \text{for } x \in \partial\Omega,$$

*then the boundary trace embedding $W^{m,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega)$ is compact.*

**Remark 3.3.** Since $p > 1/2$, then, by Proposition 3.2, we have that $W^{2,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$ is compact.

Note that an eigenvalue for problem (1.1) satisfies the following definition.

**Definition 3.4.** We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (1.1), if there exists a $u \in X \setminus \{0\}$ such that

$$\int_\Omega |\triangle u|^{p(x)-2}\triangle u \triangle v\, dx + \int_{\partial\Omega} a(x)|u|^{p(x)-2}uv\, d\sigma = \lambda \int_\Omega \frac{\partial F}{\partial u}(x,u)v\, dx,$$

for any $v \in X$, and we recall that, if $\lambda$ is an eigenvalue of problem (1.1), then, the corresponding $u \in X \setminus \{0\}$ is a weak solution of (1.1).

**Proposition 3.5.** *If $u \in X$ is a weak solution of* (1.1) *and $u \in C^4(\overline{\Omega})$, then, $u$ is a classical solution of* (1.1).

*Proof.* Let $u \in C^4(\overline{\Omega})$ be a weak solution of problem (1.1). Then, for every $v \in X$, we have

$$\int_\Omega |\triangle u|^{p(x)-2}\triangle u \triangle v\, dx + \int_{\partial\Omega} a(x)|u|^{p(x)-2}uv\, d\sigma = \lambda \int_\Omega \frac{\partial F}{\partial u}(x,u)v\, dx.$$

By applying Green's formula, we have:

$$\begin{aligned}\int_\Omega \triangle(|\triangle u|^{p(x)-2}\triangle u)v\, dx = &-\int_\Omega \nabla(|\triangle u|^{p(x)-2}\triangle u)\cdot \nabla v\, dx\\ &+\int_{\partial\Omega} v\frac{\partial}{\partial n}(|\triangle u|^{p(x)-2}\triangle u)\, d\sigma,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega |\triangle u|^{p(x)-2}\triangle u\triangle v\, dx = &-\int_\Omega \nabla(|\triangle u|^{p(x)-2}\triangle u)\cdot \nabla v\, dx\\ &+\int_{\partial\Omega} (|\triangle u|^{p(x)-2}\triangle u)\frac{\partial}{\partial n}(v)\, d\sigma.\end{aligned}$$

Since $v \in X$, then $\partial(v)/\partial n = 0$. For $v \in D(\Omega)$, we have

$$\triangle(|\triangle u|^{p(x)-2}\triangle u) = \lambda\frac{\partial F}{\partial u}(x,u) \text{ almost everywhere } x \in \Omega.$$

For each $v \in X$, we have

$$\int_{\partial\Omega} \frac{\partial}{\partial n}(|\triangle u|^{p(x)-2}\triangle u)v d\sigma = \int_{\partial\Omega} a(x)|u|^{p(x)-2}uv\, d\sigma.$$

Then, for all $v \in D(\Omega)$, we have

$$\int_{\partial\Omega} \frac{\partial}{\partial n}(|\triangle u|^{p(x)-2}\triangle u)v\, d\sigma = \int_{\partial\Omega} a(x)|u|^{p(x)-2}uv\, d\sigma,$$

which implies that

$$\frac{\partial}{\partial n}(|\triangle u|^{p(x)-2}\triangle u) - a(x)|u|^{p(x)-2}u = 0$$
$$\text{almost everywhere } x \in \Omega.$$
□

The first result in this paper is the following.

**Theorem 3.6.** *Assume that hypotheses* (**H1**), (**H2**) *and* (**H3**) *are fulfilled. Then, there exists a* $\lambda^* > 0$, *such that any* $\lambda \in (0, \lambda^*)$ *is an eigenvalue of problem* (1.1).

In the second, we establish that the Euler-Lagrange functional associated to problem (1.1) has a global minimizer.

**Theorem 3.7.** *Assume that hypotheses* (**H1**), (**H2**) *and* (**H3**) *hold. Then, any* $\lambda > 0$ *is an eigenvalue of problem* (1.1).

In order to formulate the variational problem (1.1), we introduce the functionals $\Phi$ and $J : X \to \mathbb{R}$, defined by:

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} |\triangle u|^{p(x)} dx + \int_{\partial\Omega} \frac{a(x)}{p(x)} |u|^{p(x)} d\sigma$$

and

$$J(u) = \int_\Omega F(x, u)\, dx.$$

The Euler Lagrange functional corresponding to problem (1.1) is defined by $\Psi_\lambda : X \to \mathbb{R}$, where

$$\Psi_\lambda(u) := \Phi(u) - \lambda J(u).$$

Standard arguments show that $\Psi_\lambda \in C^1(X, \mathbb{R})$ and

$$\langle d\Psi_\lambda(u), v\rangle = \int_\Omega |\triangle u|^{p(x)-2} \triangle u \triangle v\, dx + \int_{\partial\Omega} a(x)|u|^{p(x)-2} uv\, d\sigma - \lambda \int_\Omega \frac{\partial F}{\partial u}(x, u) v\, dx,$$

for any $v \in X$. Hence, a solution to problem (1.1) is a critical point of $\Psi_\lambda$.

We begin with the following auxiliary lemmas.

**Lemma 3.8.** *Suppose that we are under the hypotheses of Theorem* 3.6. *Then, for all* $\rho \in (0, 1)$, *there exist* $\lambda^* > 0$ *and* $b > 0$ *such that, for all*

*$u \in X$ with $\|u\|_a = \rho$,*

$$\Psi_\lambda(u) \geq b > 0 \quad \textit{for all } \lambda \in (0, \lambda^*).$$

*Proof.* Since the embedding $X \hookrightarrow L^{s'(x)q(x)}(\Omega)$ is continuous, then

$$|u|_{s'(x)q(x)} \leq c_2 \|u\|_a, \quad \text{for all } u \in X. \tag{3.1}$$

We assume that $\|u\|_a < \min(1, 1/c_2)$, where $c_2$ is the positive constant of inequality (3.1). Then, we have $|u|_{s'(x)q(x)} < 1$, using Hölder inequality (2.1), Proposition 2.3, Remark 1.1 and inequality (3.1), we deduce that, for any $u \in X$ with $\|u\|_a = \rho$, the following inequalities hold:

$$\begin{aligned}
\Psi_\lambda(u) &= \int_\Omega \frac{1}{p(x)} |\triangle u|^{p(x)} dx + \int_{\partial\Omega} \frac{a(x)}{p(x)} |u|^{p(x)} d\sigma - \lambda \int_\Omega F(x,u)\, dx \\
&\geq \frac{1}{p^+} \|u\|_a^{p^+} - \lambda c_1 |V|_{s(x)} ||u|^{q(x)}|_{s'(x)} \\
&\geq \frac{1}{p^+} \|u\|_a^{p^+} - \lambda c_1 |V|_{s(x)} |u|^{q^-}_{s'(x)q(x)} \\
&\geq \frac{1}{p^+} \|u\|_a^{p^+} - \lambda c_1 |V|_{s(x)} c_2^{q^-} \|u\|_a^{q^-} \\
&= \frac{1}{p^+} \rho^{p^+} - \lambda c_1 c_2^{q^-} |V|_{s(x)} \rho^{q^-} \\
&= \rho^{q^-} \left( \frac{1}{p^+} \rho^{p^+ - q^-} - \lambda c_1 c_2^{q^-} |V|_{s(x)} \right).
\end{aligned}$$

From the above inequality, we remark that, if we define

$$\lambda^* = \frac{\rho^{p^+ - q^-}}{2p^+} \; \frac{1}{c_1 \; c_2^{q^-} |V|_{s(x)}}, \tag{3.2}$$

then, for any $\lambda \in (0, \lambda^*)$ and $u \in X$ with $\|u\|_a = \rho$, there exists a $b > 0$ such that

$$\Psi_\lambda(u) \geq b > 0.$$

The proof of Lemma 3.8 is complete. □

The next result asserts the existence of a valley for $\Psi_\lambda$ near the origin.

**Lemma 3.9.** *There exists a* $\phi \in X$ *such that* $\phi \geq 0$, $\phi \neq 0$ *and* $\Psi_\lambda(t\phi) < 0$, *for* $t > 0$ *small enough.*

*Proof.* Assumption (**H2**) implies that $q(x) < p(x)$ for all $x \in \overline{\Omega}_0$. In the sequel, denote $q_0^- = \inf_{\Omega_0} q(x)$ and $p_0^- = \inf_{\Omega_0} p(x)$. Let $\epsilon_0$ be such that $q_0^- + \epsilon_0 < p_0^-$. On the other hand, since $q \in C(\overline{\Omega}_0)$, there exists an open set $\Omega_1 \subset \Omega_0$ such that $|q(x) - q_0^-| < \epsilon_0$ for all $x \in \Omega_1$. It follows that $q(x) \leq q_0^- + \epsilon_0 < p_0^-$, for all $x \in \Omega_1$.

Let $\phi \in C_0^\infty(\Omega)$ be such that $\text{supp}(\phi) \subset \Omega_1 \subset \Omega_0$, $\phi = 1$ in a subset $\Omega'_1 \subset \text{supp}(\phi)$, $0 \leq \phi \leq 1$ in $\Omega_1$. We obtain

$$\begin{aligned}
\Psi_\lambda(t\phi) &= \int_\Omega \frac{1}{p(x)} |\triangle(t\phi)|^{p(x)} dx + \int_{\partial\Omega} \frac{a(x)}{p(x)} |t\phi|^{p(x)} d\sigma - \lambda \int_\Omega F(x, t\phi)\, dx \\
&\leq \frac{1}{p_0^-}\left( \int_{\Omega_0} t^{p(x)} |\triangle\phi|^{p(x)} dx + \int_{\partial\Omega} t^{p(x)} a(x) |\phi|^{p(x)} d\sigma \right) \\
&\quad - \lambda \int_{\Omega_1} t^{q(x)} F(x, \phi)\, dx \\
&\leq \frac{t^{p_0^-}}{p_0^-} \rho^a_{p(x)}(\phi) - \lambda t^{q_0^- + \epsilon_0} \int_{\Omega_1} F(x, \phi)\, dx, \\
&\leq \frac{t^{p_0^-}}{p_0^-} \max(\|\phi\|_a^{p^-}, \|\phi\|_a^{p^+}) - \lambda t^{q_0^- + \epsilon_0} \int_{\Omega_1} F(x, \phi)\, dx.
\end{aligned}$$

Therefore,

$$\Psi_\lambda(t\phi) < 0$$

for $t < \delta^{1/(p_0^- - q_0^- - \epsilon_0)}$, with

$$0 < \delta < \min\left\{ 1, \frac{\lambda p_0^- \int_{\Omega_1} F(x, \phi) dx}{\max(\|\phi\|_a^{p^+}, \|\phi\|_a^{p^-})} \right\}.$$

Since $\phi = 1$ in $\Omega'_1$, then $\|\phi\|_a > 0$; thus, the proof of Lemma 3.9 is complete. $\square$

*Proof of Theorem* 3.6. Let $\lambda^* > 0$ be defined as in (3.2) and $\lambda \in (0, \lambda^*)$. By Lemma 3.8 it follows that, on the boundary of the ball centered at the origin and of radius $\rho$ in $X$, denoted by $B_\rho(0)$, we have

$$\inf_{\partial B_\rho(0)} \Psi_\lambda > 0. \tag{3.3}$$

On the other hand, by Lemma 3.9, there exists a $\phi \in X$ such that $\Psi_\lambda(t\phi) < 0$ for all $t > 0$ small enough. Moreover, using Hölder inequality (2.1), Proposition 2.3 and inequality (3.1), we deduce that, for any $u \in B_\rho(0)$, we have

$$\Psi_\lambda(u) \geq \frac{1}{p^+}\|u\|_a^{p^+} - \lambda c_1 c_2^{q^-}|V|_{s(x)}\|u\|_a^{q^-}.$$

It follows that

$$-\infty < \underline{c} := \inf_{\overline{B_\rho(0)}} \Psi_\lambda < 0.$$

Let $0 < \epsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda - \inf_{B_\rho(0)} \Psi_\lambda$. Using the above information, the functional $\Psi_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ is lower bounded on $\overline{B_\rho(0)}$ and $\Psi_\lambda \in C^1(\overline{B_\rho(0)}, \mathbb{R})$. Then, by Ekeland's variational principle, there exists a $u_\epsilon \in \overline{B_\rho(0)}$ such that

$$\begin{cases} \underline{c} \leq \Psi_\lambda(u_\epsilon) \leq \underline{c} + \epsilon, \\ 0 < \Psi_\lambda(u) - \Psi_\lambda(u_\epsilon) + \epsilon \cdot \|u - u_\epsilon\|_a \quad u \neq u_\epsilon. \end{cases}$$

Since

$$\Psi_\lambda(u_\epsilon) \leq \inf_{\overline{B_\rho(0)}} \Psi_\lambda + \epsilon \leq \inf_{B_\rho(0)} \Psi_\lambda + \epsilon < \inf_{\partial B_\rho(0)} \Psi_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$.

Now, we define $I_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $I_\lambda(u) = \Psi_\lambda(u) + \epsilon \cdot \|u - u_\epsilon\|_a$. It is clear that $u_\epsilon$ is a minimum point of $I_\lambda$, and thus,

$$\frac{I_\lambda(u_\epsilon + t \cdot v) - I_\lambda(u_\epsilon)}{t} \geq 0,$$

for small $t > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{\Psi_\lambda(u_\epsilon + t \cdot v) - \Psi_\lambda(u_\epsilon)}{t} + \epsilon \cdot \|v\|_a \geq 0.$$

Letting $t \to 0$, it follows that $\langle d\Psi_\lambda(u_\epsilon), v\rangle + \epsilon \cdot \|v\|_a \geq 0$, and we infer that $\|d\Psi_\lambda(u_\epsilon)\|_a \leq \epsilon$. We deduce that there exists a sequence $\{w_n\} \subset B_\rho(0)$ such that

$$\Psi_\lambda(w_n) \longrightarrow \underline{c} < 0 \quad \text{and} \quad d\Psi_\lambda(w_n) \longrightarrow 0_{X^*}. \tag{3.4}$$

It is clear that $\{w_n\}$ is bounded in $X$. Thus, there exists a $w$ in $X$ such that, up to a subsequence, $\{w_n\}$ weakly converges to $w$ in $X$. Since $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$, we deduce that there exists a compact embedding $E \hookrightarrow L^{\alpha(x)}(\Omega)$, and consequently, $\{w_n\}$ strongly converges in $L^{\alpha(x)}(\Omega)$. For the strong convergence of $\{w_n\}$ in $X$, we need the following proposition.

**Proposition 3.10.**

$$\lim_{n\to\infty}\int_\Omega \frac{\partial F}{\partial u}(x,w_n)(w_n-w)\,dx=0.$$

*Proof.* Using Hölder inequality (2.1), we have:

$$\begin{aligned}\int_\Omega\left|\frac{\partial F}{\partial u}(x,w_n)(w_n-w)\right|dx &\le c_1|V|_{s(x)}\big||w_n|^{q(x)-2}w_n(w_n-w)\big|_{s'(x)}\\ &\le c_1|V|_{s(x)}\big||w_n|^{q(x)-2}w_n\big|_{q(x)/(q(x)-1)}|w_n-w|_{\alpha(x)}.\end{aligned}$$

Now, if $||w_n|^{q(x)-2}w_n|_{q(x)/(q(x)-1)} > 1$, by Proposition 2.2, we get $||w_n|^{q(x)-2}w_n|_{q(x)/(q(x)-1)} \le |w_n|^{q^+}_{q(x)}$. The compact embedding $X \hookrightarrow L^{q(x)}(\Omega)$ concludes the proof. □

Since $d\Psi_\lambda(w_n)\to 0$, and $w_n$ is bounded in $X$, we have

$$\begin{aligned}|\langle d\Psi_\lambda(w_n),w_n-w\rangle| &\le |\langle d\Psi_\lambda(w_n),w_n\rangle|+|\langle d\Psi_\lambda(w_n),w\rangle|\\ &\le \|d\Psi_\lambda(w_n)\|_a\|w_n\|_a+\|d\Psi_\lambda(w_n)\|_a\|w\|_a.\end{aligned}$$

Moreover, using Proposition 3.10, we have

$$\lim_{n\to\infty}\langle d\Psi_\lambda(w_n),w_n-w\rangle=0.$$

Hence,

$$\lim_{n\to\infty}\int_\Omega|\triangle w_n|^{p(x)-2}\triangle w_n(\triangle w_n-\triangle w)\,dx + \int_{\partial\Omega}a(x)|w_n|^{p(x)-2}w_n(w_n-w)\,d\sigma=0.$$

Now, Proposition 2.4 ensures that $\{w_n\}$ strongly converges to $w$ in $X$. Since $\Psi_\lambda\in C^1(X,\mathbb{R})$, we conclude

$$d\Psi_\lambda(w_n)\longrightarrow d\Psi_\lambda(w)\quad\text{as } n\to\infty. \tag{3.5}$$

Relations (3.4) and (3.5) show that $d\Psi_\lambda(w)=0$, and thus, $w$ is a weak solution for problem (1.1). Moreover, by relation (3.4), it follows that $\Psi_\lambda(w) < 0$, and thus, $w$ is a nontrivial weak solution for (1.1). The proof of Theorem 3.6 is complete. □

*Proof of Theorem* 3.7. Using Hölder inequality (2.1) for $\|u\|_a > 1$, we have

$$\begin{aligned}\Psi_\lambda(u) &= \int_\Omega \frac{1}{p(x)}|\triangle u|^{p(x)}dx + \int_{\partial\Omega} \frac{1}{p(x)}|a(x)|u|^{p(x)}d\sigma - \lambda \int_\Omega F(x,u)\,dx \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \lambda c_1 |V|_{s(x)}||u|^{q(x)}|_{s'(x)} \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \lambda c_1 |V|_{s(x)}|u|^{q^+}_{s'(x)q(x)} \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \lambda c_1 |V|_{s(x)} c_2^{q^+}\|u\|_a^{q^+} \longrightarrow +\infty \quad \text{as } \|u\|_a \to +\infty.\end{aligned}$$

In conclusion, since $\Psi_\lambda$ is weakly lower semi-continuous, then it has a global minimizer which is a solution of problem (1.1); moreover, Lemma 3.9 ensures that this minimizer is nontrivial, which ends the proof. □

**Example 3.11.** Put $F(x,t) = V(x)t^{q(x)}$, where the function $V(\cdot)$ was as in the assumption **(H2)**, and consider the problem

$$\begin{cases} \triangle^2_{p(x)} u = \lambda(\partial F(x,u)/\partial u) & x \in \Omega, \\ \partial u/\partial n = 0 & x \in \partial\Omega, \\ \partial(|\triangle u|^{p(x)-2}\triangle u)/\partial n = a(x)|u|^{p(x)-2}u & x \in \partial\Omega, \end{cases} \tag{3.6}$$

where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$, with sufficiently smooth boundary $\partial\Omega$, $n$ is a unit outward normal to $\partial\Omega$, $a \in L^\infty(\partial\Omega)$ with $a^- := \inf_{x\in\partial\Omega} a(x) > 0$ and $\lambda$ is a positive real number.

First, observe that the function $F$ satisfies assumptions (**H1**), (**H2**) and (**H3**). Then, Theorem 3.6 asserts that there exists a $\lambda^* > 0$, under which problem (3.6) has a nontrivial weak solution. Moreover, due to Theorem 3.7, we have a solution for any $\lambda > 0$.

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