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K3 SURFACES WITH $\mathbb{Z}_2^2$ SYMPLECTIC ACTION

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ABSTRACT. Let G be a finite abelian group which acts symplectically on a K3 surface. The Néron-Severi lattice of the projective K3 surfaces admitting G symplectic action and with minimal Picard number was computed by Garbagnati and Sarti **[8]**. We consider a four-dimensional family of projective K3 surfaces with $\mathbb{Z}_2^2$ symplectic action which do not fall into the above cases. If X is one of these K3 surfaces, then it arises as the minimal resolution of a specific $\mathbb{Z}_2^3$-cover of $\mathbb{P}^2$ branched along six general lines. We show that the Néron-Severi lattice of X with minimal Picard number is generated by 24 smooth rational curves and that X specializes to the Kummer surface $\mathrm{Km}(E_i \times E_i)$. We relate X to the K3 surfaces given by the minimal resolution of the $\mathbb{Z}_2$-cover of $\mathbb{P}^2$, branched along six general lines, and the corresponding Hirzebruch-Kummer covering of exponent 2 of $\mathbb{P}^2$.

1. Introduction. Let X be a K3 surface over $\mathbb{C}$. A subgroup of the automorphism group of X acts symplectically on X if the induced action on $H^0(X, \omega_X)$ is the identity. Finite abelian groups of automorphisms acting symplectically on K3 surfaces were classified by Nikulin **[19]**, and by Mukai in the non-commutative case, see **[17, 28]**.

Let G be a finite abelian group that acts symplectically on a K3 surface X. If Ω_G denotes the orthogonal complement in $H^2(X;\mathbb{Z})$ of the fixed sublattice $H^2(X;\mathbb{Z})^G$, then Ω_G is a negative definite primitive sublattice of the Néron-Severi lattice $\mathrm{NS}(X)$. All of the possible lattices Ω_G and their orthogonal complements in $H^2(X;\mathbb{Z})$

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are computed in [**7, 8**]. If X is projective, then the Picard number $\rho(X)$ satisfies $\rho(X) \geq \mathrm{rk}(\Omega_G) + 1$.

The projective K3 surfaces X admitting a G symplectic action form a family of dimension $19 - \mathrm{rk}(\Omega_G)$ [**8,** Remark 6.2]. If $\rho(X) = \mathrm{rk}(\Omega_G) + 1$, then the lattices $\mathrm{NS}(X)$ are computed in [**8,** Proposition 6.2]. The case $G = \mathbb{Z}_2^4$ received special attention: in [**9**], among other results, the authors compute $\mathrm{NS}(\widetilde{X/G})$, where $\widetilde{X/G}$ is the minimal resolution of X/G, if $\rho(\widetilde{X/G}) = 16$.

There is a seven-dimensional family of projective K3 surfaces with $\mathbb{Z}_2^2$ symplectic action and, in this paper, we analyze the four-dimensional subfamily, which arises as follows. Consider six general lines in $\mathbb{P}^2$, and divide them into three pairs. Consider the chain of double covers

$$X_3 \xrightarrow{\mathbb{Z}_2} X_2 \xrightarrow{\mathbb{Z}_2} X_1 \xrightarrow{\mathbb{Z}_2} \mathbb{P}^2,$$

where the first cover is branched along the first pair of lines, the second cover is branched along the preimage of the second pair of lines, and so on. The minimal resolution of X_3 is a projective K3 surface X, which we call a *triple-double* K3 surface, see subsection 3.1. Equivalently, X can be viewed as an appropriate $\mathbb{Z}_2^3$-cover of $\mathrm{Bl}_3\,\mathbb{P}^2$, which denotes the blow up of $\mathbb{P}^2$ at three general points. The surface X admits a $\mathbb{Z}_2^2$ symplectic action and an Enriques involution, see Proposition 3.5. The minimal Picard number for X is 16, as shown in Section 4 (this occurs for a very general X). Moduli compactifications of the corresponding four-dimensional family of Enriques surfaces were studied in [**22, 23**]. The main goal of this paper is to compute the lattice $\mathrm{NS}(X)$ for X with minimal Picard number and relate it to the geometry of X.

Theorem 1.1. *Let X be a triple-double* K3 *surface with minimal Picard number. Then, the following hold*:

(i) *The Néron-Severi lattice* $\mathrm{NS}(X)$ *has rank* 16 *and is generated by the irreducible components of the preimage of the* (-1)*-curves on* $\mathrm{Bl}_3\mathbb{P}^2$. *The dual graph of this configuration of* 24 *smooth rational curves is shown in Figure* 3 (*see Theorem* 4.6);

(ii) $\mathrm{NS}(X)$ *has discriminant group* $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$ *and is isometric to* $U \oplus E_8 \oplus Q$, *where Q is the lattice in Lemma* 4.1 (iii). *An explicit* $\mathbb{Z}$*-basis of* $\mathrm{NS}(X)$, *which realizes it as a direct sum of* U, E_8 *and* Q *can be found in Remark* 4.7;

(iii) *The transcendental lattice* T_X *is isometric to* $U \oplus U(2) \oplus \langle -4 \rangle^{\oplus 2}$ (*see Proposition* 4.9);

(iv) *The Kummer surface* $\mathrm{Km}(E_i \times E_i)$ (*see* **[11]**) *appears as a specialization of the four-dimensional family of triple-double* K3 *surfaces* (*see Theorem* 4.11). *The line arrangement in* $\mathbb{P}^2$ *that gives rise to this Kummer surface is shown in Figure* 5;

(v) *Let* ι *be an involution on* X *coming from the* $\mathbb{Z}_2^2$ *symplectic action, and denote by* X' *the minimal resolution of* X/ι. *For* X' *with minimal Picard number, the Néron-Severi lattice of* X' *has rank* 16 *and discriminant group* $\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2$. *An explicit* $\mathbb{Z}$*-basis for* $\mathrm{NS}(X')$ *is given in Theorem* 6.2. *Finally, the transcendental lattice* $T_{X'}$ *is isometric to* $U(2)^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$.

The Néron-Severi lattice of a triple-double K3 surface X with $\rho(X) = 16$ is not included in **[7, 8, 9]** for the following reasons:

- If G is a finite abelian group acting symplectically on a projective K3 surface, then $16 = \mathrm{rk}(\Omega_G) + 1$ if and only if $G \cong \mathbb{Z}_2^4$, see **[8,** Proposition 5.1];
- X does not admit $\mathbb{Z}_2^4$ symplectic action, see Proposition 4.12 (i);
- X is not isomorphic to the minimal resolution of the quotient of a K3 surface by a symplectic action of the group $\mathbb{Z}_2^4$, see Proposition 4.12 (ii).

Triple-double K3 surfaces are closely related to two classical examples of K3 surfaces. The first is the Hirzebruch-Kummer covering Y of exponent 2 of $\mathbb{P}^2$ branched along six general lines, see **[3,** subsection 4.3]. The covering map $Y \to \mathbb{P}^2$ is a $\mathbb{Z}_2^5$-cover, and there is a subgroup of $\mathbb{Z}_2^5$ isomorphic to $\mathbb{Z}_2^4$, which acts symplectically on Y. The second example is the minimal resolution of the double cover of $\mathbb{P}^2$ branched along six general lines, which we denote by Z. Now, let X be a triple-double K3 surface. Given the corresponding six lines in $\mathbb{P}^2$, we may consider Y and Z as above. Then, in Proposition 5.5, we show that X is isomorphic to $\widetilde{Y/\mathbb{Z}_2^2}$ for an appropriate subgroup $\mathbb{Z}_2^2 < \mathbb{Z}_2^4$, and the minimal resolution of the quotient of X by the leftover $\mathbb{Z}_2^4/\mathbb{Z}_2^2 \cong \mathbb{Z}_2^2$ symplectic action which gives rise to Z. Summarizing, we have the following diagram:

$$
\begin{tikzcd}
Y \arrow[r] \arrow[drr] & Y/\mathbb{Z}_2^2 & X \arrow[l] \arrow[r] \arrow[d] & X/\mathbb{Z}_2^2 & Z \arrow[l] \arrow[dll] \\
 & & \mathbb{P}^2. & &
\end{tikzcd}
$$

The Néron-Severi lattices $\mathrm{NS}(Y), \mathrm{NS}(Z)$ for $\rho(Y) = \rho(Z) = 16$ are well known, and the interaction between them is described in **[9]** (see **[12,** Section 5] for $\mathrm{NS}(Z)$).

We remark that similar problems are considered in **[2]**, although different families of K3 surfaces are studied there. We also mention **[21]**, where Nikulin classified the main part of the Néron-Severi lattice of an arbitrary Kählerian K3 surface **[21,** Section 3] with a large enough group of symplectic automorphisms (see **[21,** Theorem 2] for a precise statement). There are more examples of K3 surfaces that are the minimal resolution of a $\mathbb{Z}_2^n$-cover of $\mathbb{P}^2$ branched along six general lines, and that differ from X, X', Y, Z (see the notation introduced above). These other K3 surfaces will be studied later.

In Section 2, we recall some preliminary facts regarding K3 surfaces, symplectic automorphisms, and even lattices. In Section 3, we define triple-double K3 surfaces and characterize them in several different ways. We classify the involutions coming from the $\mathbb{Z}_2^3$-action of the cover. We also analyze the configuration of 24 smooth rational curves on a triple-double K3 surface X coming from the preimage of the (-1)-curves under the covering map $X \to \mathrm{Bl}_3 \mathbb{P}^2$. In Section 4, we show that the sublattice of $\mathrm{NS}(X)$ generated by these curves equals $\mathrm{NS}(X)$ for X with minimal Picard number. We also compute T_X and show that $\mathrm{Km}(E_i \times E_i)$ is a specialization of this four-dimensional family. In Section 5, we relate triple-double K3 surfaces with the Hirzebruch-Kummer covering of exponent 2 of $\mathbb{P}^2$ branched along six general lines. Finally, in Section 6, we compute $\mathrm{NS}(X')$ and $T_{X'}$, where X' is the minimal resolution of the quotient of a triple-double K3 surface by an involution coming from the $\mathbb{Z}_2^2$ symplectic action and $\rho(X') = 16$. We work over $\mathbb{C}$.

2. Preliminaries: Lattice theory and symplectic automorphisms of K3 surfaces.

2.1. Even lattices and the discriminant quadratic form. A *lattice* $L = (L, b_L)$ is a finitely generated free abelian group L, together with a symmetric bilinear form $b_L\colon L \times L \to \mathbb{Z}$. In what follows, we consider *non-degenerate* lattices, which means that b_L is a non-degenerate symmetric bilinear form, which we always assume. If $e_1, \dots, e_n$ is a $\mathbb{Z}$-basis for L, then the *Gram* matrix of L associated to the chosen $\mathbb{Z}$-basis is the matrix $(b_L(e_i, e_j))_{1\leq i,j\leq n}$. The determinant of a Gram matrix of L does not depend upon the choice of $\mathbb{Z}$-basis and is called the *discriminant* of L. We say that L is *even* if $b_L(x,x) \in 2\mathbb{Z}$ for all $x \in L$, and *odd*, otherwise. Lattices of discriminant ± 1 are called *unimodular*. For the classification of unimodular lattices, we refer to [**24,** Chapter 5]. Denote by U (respectively, E_8) the unique even unimodular lattice of signature $(1,1)$ (respectively, $(0,8)$).

Let L be a lattice. The dual $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ comes with an induced $\mathbb{Q}$-valued symmetric bilinear form b_{L^*}. L embeds into $L^* = \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ since b_L is non-degenerate, and the quotient $A_L = L^*/L$ is called the *discriminant group* of L. We denote by $\ell(A_L)$ the minimum number of generators of A_L. The lattice L is 2*-elementary* if A_L is isomorphic to $\mathbb{Z}_2^\alpha$ for some α. The discriminant group A_L comes equipped with the quadratic form

$$\begin{aligned} q_L\colon A_L &\longrightarrow \mathbb{Q}/2\mathbb{Z} \\ x + L &\longmapsto b_{L^*}(x,x) \bmod 2\mathbb{Z}, \end{aligned}$$

which is called the *discriminant* quadratic form of L.

A sublattice $S \subseteq L$ is called *primitive* if the quotient L/S is torsion free, and L is called an *overlattice* of S if L/S is finite. We recall the following standard results for the reader's convenience.

Theorem 2.1 ([**20,** Corollary 1.13.3])**.** *Let L be an even indefinite lattice of signature (t_+, t_-) such that $t_+ + t_- \geq 2 + \ell(A_L)$. Then, L is unique up to isometry.*

Theorem 2.2 ([**20,** Corollary 1.13.5])**.** *Let L be an even lattice of signature (t_+, t_-).*

(i) *If* $t_+ \geq 1$, $t_- \geq 8$ *and* $t_+ + t_- \geq 9 + \ell(A_L)$, *then* $L \cong E_8 \oplus P$ *for some lattice* P;
(ii) *if* $t_+ \geq 1$, $t_- \geq 1$ *and* $t_+ + t_- \geq 3 + \ell(A_L)$, *then* $L \cong U \oplus P$ *for some lattice* P.

Theorem 2.3 ([**20,** Proposition 1.4.1(a)])**.** *Let* L *be an even lattice. Then, there is a one-to-one correspondence between overlattices of* L *and subgroups of* A_L, *which are isotropic with respect to the discriminant quadratic form* q_L.

Theorem 2.4 ([**20,** Section 1, 5°])**.** *Let* L *be an even unimodular lattice and* $S \subseteq L$ *a primitive sublattice. Thus,* $S^\perp \subseteq L$ *is also primitive, and the two discriminant groups* A_S, $A_{S^\perp}$ *are isomorphic. Moreover,* $q_S = -q_{S^\perp}$.

Theorem 2.5 ([**20,** Theorem 3.6.2])**.** *An indefinite* 2*-elementary even lattice* L *is determined, up to isometry, by its signature,* $\ell(A_L)$ *and* $\delta(L)$ (*for the definition of* $\delta(L)$, *the reader is referred to* [**20,** Section 3, 6°]).

2.2. K3 surfaces. A K3 *surface* X is a smooth irreducible projective two-dimensional variety with $K_X \sim 0$ and $h^1(X, \mathcal{O}_X) = 0$. On a K3 surface X, numerical algebraic and linear equivalence between divisors coincide [**10,** Chapter 1, Proposition 2.4], and therefore,

$$\mathrm{Pic}(X) = \mathrm{NS}(X) = \mathrm{Num}(X).$$

In the case of a K3 surface X, we have that $\mathrm{NS}(X)$ is a primitive sublattice of $H^2(X;\mathbb{Z})$ (the fact that $h^1(X, \mathcal{O}_X) = 0$ implies that $\mathrm{NS}(X)$ embeds into $H^2(X;\mathbb{Z})$ and $H^2(X;\mathbb{Z})/\mathrm{NS}(X)$ embeds into $H^2(X, \mathcal{O}_X) \cong \mathbb{C}$). Recall that $H^2(X;\mathbb{Z})$ is an even unimodular lattice of signature $(3,19)$ and, therefore, isometric to $U^{\oplus 3} \oplus E_8^{\oplus 2}$. It follows that the *transcendental* lattice $T_X = \mathrm{NS}(X)^\perp$ has the same discriminant group as $\mathrm{NS}(X)$, see Theorem 2.4. The *Picard number* of a K3 surface X is the rank of $\mathrm{NS}(X)$ and is denoted by $\rho(X)$.

2.3. Symplectic automorphisms of K3 surfaces. Let X be a K3 surface. An automorphism f of X is called *symplectic* if the induced map $f^*\colon H^0(X, \omega_X) \to H^0(X, \omega_X)$ is the identity. The effective (left) action of a group G on the K3 surface X is called *symplectic* if, for all

$g \in G$, the automorphism of X given by $x \mapsto g \cdot x$ is symplectic. It is well known that a finite group G acts symplectically on X if and only if the minimal resolution $\widetilde{X/G}$ of X/G is also a K3 surface (see [**17,** (8.10) Proposition (1)], [**5,** Theorem 0.4.2]).

Let G be a finite abelian group. Nikulin [**19**] showed that G acts symplectically on a K3 surface X if and only if G is one of the following groups:

$$\begin{aligned} &\mathbb{Z}_n, \quad 2 \le n \le 8, \ \mathbb{Z}_m^2, \quad m = 2,3,4, \\ &\mathbb{Z}_2 \oplus \mathbb{Z}_4, \qquad \mathbb{Z}_2 \oplus \mathbb{Z}_6, \ \mathbb{Z}_2^\ell, \quad \ell = 3,4. \end{aligned}$$

Moreover, the action of G on $H^2(X;\mathbb{Z})$ is unique up to isometry; hence, it depends only upon G and not the K3 surface X. Denote the orthogonal complement of the fixed sublattice $H^2(X;\mathbb{Z})^G$ by Ω_G.

The next theorem briefly summarizes many results of Garbagnati and Sarti [**8**], which are fundamental for this paper.

Theorem 2.6 ([**8**]). *Let G be a finite abelian group that acts symplectically on a K3 surface X. Then, the orthogonal complement of Ω_G in $H^2(X;\mathbb{Z})$ is computed in* [**8,** Proposition 5.1]. *Moreover, if $\rho(X) = \mathrm{rk}(\Omega_G) + 1$, then $\mathrm{NS}(X)$ is given in* [**8,** Proposition 6.2].

Remark 2.7. The results for $G = \mathbb{Z}_2$ in Theorem 2.6 are from [**16, 26**], and the cases $G = \mathbb{Z}_p$ for $p = 3,5,7$ appeared in [**7**].

Definition 2.8. A symplectic automorphism of order 2 on a K3 surface X is called a *symplectic involution* or *Nikulin involution.* A Nikulin involution on X has exactly eight fixed points [**17,** subsection 0.1], and in Proposition 2.9, (which is well known), we show that an involution on X with exactly eight fixed points is necessarily symplectic.

Proposition 2.9. *Let X be a* K3 *surface, and let ι be an involution on X with exactly eight fixed points. Then, ι is a Nikulin involution.*

Proof. If $Y = X/\iota$, then we show that $\widetilde{Y}$ is a K3 surface. Let $X' \to X$ be the blow up at the eight fixed points of ι, and denote the corresponding exceptional divisors by $E_1, \ldots, E_8$. Then, we have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & \widetilde{Y} \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y, \end{array}$$

where π is the $\mathbb{Z}_2$-cover ramified at $E_1 + \cdots + E_8$. Therefore,

$$E_1 + \cdots + E_8 = K_{X'} \sim \pi^*(K_{\widetilde{Y}}) + E_1 + \cdots + E_8 \Longrightarrow \pi^*(K_{\widetilde{Y}}) \sim 0.$$

We can conclude that $2K_{\widetilde{Y}} \sim \pi_*\pi^*(K_{\widetilde{Y}}) \sim 0$.

Since $\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_X) = 2$, Noether's formula gives that the topological Euler characteristic $\chi_{\mathrm{top}}(X')$ of X' equals 32. From this, we may argue that $\chi_{\mathrm{top}}(\widetilde{Y}) = 24$. Noether's formula again applied to $\widetilde{Y}$ implies that $\chi(\mathcal{O}_{\widetilde{Y}}) = 2$. If B is the branch locus of π, then $\pi_*\mathcal{O}_{X'} = \mathcal{O}_{\widetilde{Y}} \oplus \mathcal{O}_{\widetilde{Y}}(-B/2)$. This guarantees that $q(\widetilde{Y}) = 0$, and hence, $p_g(\widetilde{Y}) = 1$ since $\chi(\mathcal{O}_{\widetilde{Y}}) = 2$. Summing up, $2K_{\widetilde{Y}} \sim 0, q(\widetilde{Y}) = 0$, and $p_g(\widetilde{Y}) = 1$, which is sufficient to show that $\widetilde{Y}$ is a K3 surface. $\square$

2.4. Even eights on a K3 surface.

Definition 2.10. Let $R_1, \ldots, R_m$ be smooth, disjoint, rational curves on a K3 surface X. Then, $\{R_1, \ldots, R_m\}$ is called an *even set* if $R_1 + \cdots + R_m$ is divisible by 2 in $\mathrm{NS}(X)$.

The following result of Nikulin is a combination of [**18,** Lemma 3] and [**18,** Corollary 1].

Theorem 2.11 ([**18**]). *Let $\{R_1, \ldots, R_m\}$, with $m \geq 1$, be an even set on a* K3 *surface. Then, $m = 8$ or* 16.

Observation 2.12. In Theorem 2.11, it is simple to show that an even set on a K3 surface X cannot have cardinality 4 (which is what we need in this paper). By contradiction, let $\{C_1, C_2, C_3, C_4\}$ be an even set on X, and let $C = (C_1+C_2+C_3+C_4)/2$. Using the Riemann-Roch theorem, we know that C or $-C$ is effective since $C^2 = -2$. Therefore, C is effective since the ample class on X positively intersects the curves C_i. However, $C_i \subset \mathrm{Supp}(C)$ for all $i \in \{1, 2, 3, 4\}$ since $C_i \cdot C < 0$. This implies that $C - C_1 - C_2 - C_3 - C_4 = -C$ is also effective. However, then $C = 0$, which cannot be.

Definition 2.13. An *even eight* on a K3 surface is an even set of cardinality 8.

Example 2.14. In the proof of Proposition 2.9, the irreducible curves in the branch locus of the double cover $\pi\colon X' \to \widetilde{Y}$ form an even eight since they are eight disjoint smooth rational curves and their sum is divisible by 2 in $\mathrm{NS}(\widetilde{Y})$.

3. The four-dimensional subfamily of K3 surfaces with $\mathbb{Z}_2^2$ symplectic action.

3.1. Triple-double K3 surfaces. Consider six lines $\ell_0, \dots, \ell_5$ in $\mathbb{P}^2$ without triple intersection points. Divide the six lines into three pairs (ℓ_0, ℓ_1), (ℓ_2, ℓ_3) and (ℓ_4, ℓ_5). The double cover $X_1 \to \mathbb{P}^2$ exists, branched along $\ell_0 + \ell_1$, since $\ell_0 + \ell_1 \in 2\,\mathrm{Pic}(\mathbb{P}^2)$. The pullback to X_1 of $\ell_2 + \ell_3$ is also divisible by 2 in $\mathrm{Pic}(X_1)$; thus, we have another double cover $X_2 \to X_1$. Repeat this construction on X_2 with respect to the last pair of lines to obtain a triple-double cover $X_3 \to \mathbb{P}^2$ branched along $\sum_{i=0}^{5} \ell_i$. The preimage of $\ell_0 \cap \ell_1$ under $X_3 \to \mathbb{P}^2$ consists of four A_1 singularities, and the same applies to $\ell_2 \cap \ell_3$ and $\ell_4 \cap \ell_5$. X_3 is smooth away from these 12 singular points. Let $\sigma\colon X \to X_3$ be the minimal resolution of X_3, obtained by blowing up the 12 A_1 singularities.

Proposition 3.1. *With the notation above, X is a* K3 *surface.*

Proof. A projective realization of X_3 can be constructed as follows. Let $[W_0 : W_1 : W_2]$ be coordinates in $\mathbb{P}^2$ and $L_i = L_i(W_0, W_1, W_2) = 0$ the equation of the line ℓ_i. Then, $X_3 \subset \mathbb{P}^5$ is given by the vanishing of the following three quadrics:

$$\begin{cases} W_3^2 = L_0 L_1, \\ W_4^2 = L_2 L_3, \\ W_5^2 = L_4 L_5. \end{cases}$$

From this, it immediately follows that $K_{X_3} \sim 0$ and $h^1(X_3, \mathcal{O}_{X_3}) = 0$. We conclude that X is a K3 surface since $\sigma^* K_{X_3} \sim K_X$ and $\sigma_* \mathcal{O}_X \cong \mathcal{O}_{X_3}$. $\square$

Definition 3.2. As in Proposition 3.1, we call a K3 surface X a *triple-double* K3 surface.

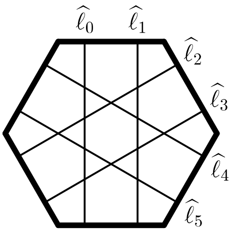

FIGURE 1. Toric picture of $\mathrm{Bl}_3\,\mathbb{P}^2$ together with the divisor $\widehat{\ell}_0 + \cdots + \widehat{\ell}_5$.

Observation 3.3. Let X be the triple-double K3 surface as defined above. Let $\mathrm{Bl}_3\,\mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $\ell_0 \cap \ell_1,\ \ell_2 \cap \ell_3,\ \ell_4 \cap \ell_5$. Then, X can be viewed as a $\mathbb{Z}_2^3$-cover of $\mathrm{Bl}_3\,\mathbb{P}^2$ branched along the strict preimages $\widehat{\ell}_0, \ldots, \widehat{\ell}_5$ of the six lines in $\mathbb{P}^2$, see Figure 1. More precisely, we first take the double cover of $\mathrm{Bl}_3\,\mathbb{P}^2$ branched along $\widehat{\ell}_0 + \widehat{\ell}_1$, etc. In this manner, the $\mathbb{Z}_2^3$-cover is automatically smooth. We can also realize X as a hypersurface in $(\mathbb{P}^1)^3$, as follows. The blow up $\mathrm{Bl}_3\,\mathbb{P}^2$ can be embedded in $(\mathbb{P}^1)^3$ as a hypersurface of an equation, given by

$$\sum_{i,j,k=0,1} c_{ijk} X_i Y_j Z_k = 0,$$

where the coefficients c_{ijk} are general, nonzero and the lines $\widehat{\ell}_0, \ldots, \widehat{\ell}_5$ are given by the restriction of the toric boundary of $(\mathbb{P}^1)^3$ **[23]**. Then, X is the following hypersurface in $(\mathbb{P}^1)^3$:

$$\sum_{i,j,k=0,1} c_{ijk} X_i^2 Y_j^2 Z_k^2 = 0, \tag{3.1}$$

where the $\mathbb{Z}_2^3$-covering map is given by the restriction to X of

$$([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]) \longmapsto ([X_0^2 : X_1^2], [Y_0^2 : Y_1^2], [Z_0^2 : Z_1^2]).$$

At this point, we have several equivalent descriptions of triple-double K3 surfaces. In what follows, we switch from one perspective to another, dependent upon whichever is more convenient for our purposes.

Definition 3.4. Consider the realization of a triple-double K3 surface X as a hypersurface in $(\mathbb{P}^1)^3$, given by equation (3.1) in Observation 3.3. Up to rescaling the coefficients, and up to the torus action on the variables, we can assume that $c_{000}, c_{100}, c_{010}, c_{001} = 1$. Let $\mathcal{U} \subset \mathbb{G}_m^4$ be the dense open subset consisting of quadruples $(c_{110}, c_{101}, c_{011}, c_{111})$, which give a triple-double K3 surface. Considering $c_{110}, c_{101}, c_{011}, c_{111}$ as variables, let $\mathfrak{X}$ be the subvariety of $\mathcal{U} \times (\mathbb{P}^1)^3$, given by

$$X_0^2Y_0^2Z_0^2 + X_1^2Y_0^2Z_0^2 + X_0^2Y_1^2Z_0^2 + X_0^2Y_0^2Z_1^2 + c_{110}X_1^2Y_1^2Z_0^2 \\ + c_{101}X_1^2Y_0^2Z_1^2 + c_{011}X_0^2Y_1^2Z_1^2 + c_{111}X_1^2Y_1^2Z_1^2 = 0.$$

We refer to the *family* of triple-double K3 surfaces as $\mathfrak{X} \to \mathcal{U}$.

3.2. Involutions of a triple-double K3 surface.

Proposition 3.5. *Let X be a triple-double* K3 *surface, and view it as a hypersurface in $(\mathbb{P}^1)^3$ of equation*

$$\sum_{i,j,k=0,1} c_{ijk}X_i^2Y_j^2Z_k^2 = 0,$$

as explained in Observation 3.3. *Denote by ι_{ijk} the restriction to X of*

$$(\mathbb{P}^1)^3 \longrightarrow (\mathbb{P}^1)^3,$$
$$([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]) \longmapsto ([X_0 : (-1)^i X_1], \\ [Y_0 : (-1)^j Y_1], [Z_0 : (-1)^k Z_1]).$$

Then, the following hold:

(i) ι_{111} *is an Enriques involution, i.e., X/ι_{111} is an Enriques surface*;
(ii) *if $\iota \in \{\iota_{100}, \iota_{010}, \iota_{001}\}$, then ι is a non-symplectic involution and X/ι is a smooth rational surface. The fixed points locus of ι consists of two disjoint smooth genus* 1 *curves* (*and hence, ι is of parabolic type according to* [**1,** Section 2.8]);
(iii) $\iota_{110}, \iota_{101}, \iota_{011}$ *are Nikulin involutions with pairwise disjoint sets of fixed points.*

In particular, the group of automorphisms $\{\mathrm{id}_X, \iota_{110}, \iota_{101}, \iota_{011}\} \cong \mathbb{Z}_2^2$ *acts symplectically on* X.

Proof. ι_{111} is an Enriques involution since it has no fixed points (recall that the coefficients c_{ijk} are nonzero). $\iota_{110}, \iota_{101}, \iota_{011}$ are symplectic involutions since each has exactly eight fixed points, see Proposition 2.9. It is easy to observe that $\iota_{110}, \iota_{101}, \iota_{011}$ have pairwise disjoint sets of fixed points.

We prove (ii) for ι_{100} (similar arguments hold for ι_{010}, ι_{001}). The fixed points locus of ι_{100} is given by

$$\left\{X_0 = 0, \sum_{j,k=0,1} c_{1jk} Y_j^2 Z_k^2 = 0\right\} \amalg \left\{X_1 = 0, \sum_{j,k=0,1} c_{0jk} Y_j^2 Z_k^2 = 0\right\},$$

which are two disjoint smooth genus 1 curves. This is sufficient to show that X/ι_{100} is a smooth rational surface [**1,** Chapter 2]. $\square$

Observation 3.6. Let $\mathfrak{X} \to \mathcal{U}$ be the family of triple-double K3 surfaces in Definition 3.4, which has a four-dimensional parameter space $\mathcal{U}$. Denote the fiber over a point $u \in \mathcal{U}$ by X_u. Now, we show that $\mathfrak{X} \to \mathcal{U}$ is a *four-dimensional family* of K3 surfaces, and, by this, we mean that no positive-dimensional subvariety $C \subset \mathcal{U}$ exists such that $X_u \cong X_v$ for all $u, v \in C$. Assume, by contradiction, that there exists such $C \subset \mathcal{U}$. Let G be the group generated by the involutions $\{\iota_{110}, \iota_{101}, \iota_{011}\}$ in Proposition 3.5. Then, G acts on the entire space $\mathfrak{X}$ inducing a symplectic action on each fiber of $\mathfrak{X} \to \mathcal{U}$. If $\mathcal{Z}$ denotes the minimal resolution of the quotient of $\mathfrak{X}$ by G, then $\mathcal{Z} \to \mathcal{U}$ is the family of K3 surfaces, given by the minimal resolutions of the double covers of $\mathbb{P}^2$ branched along six lines without triple intersection points (more about this aspect may be found in Section 5). We have that $\mathcal{Z} \to \mathcal{U}$ is a four-dimensional family of K3 surfaces, see [**12,** Remark 5.9]. However, the fibers of $\mathcal{Z} \to \mathcal{U}$ over the positive-dimensional subvariety $C \subset \mathcal{U}$ would be isomorphic to each other, which cannot be.

Observation 3.7. A triple-double K3 surface X has infinite automorphism group since it covers an Enriques surface by Proposition 3.5 (i), see [**14,** page 194]. Moreover, X contains smooth rational curves (see Proposition 3.8 for a more detailed discussion). Therefore, X contains infinitely many smooth rational curves by [**10,** Corollary 4.7].

3.3. A configuration of 24 smooth rational curves on a triple-double K3 surface.

Proposition 3.8. *Let X be a triple-double* K3 *surface, and let $X \to \mathrm{Bl}_3\,\mathbb{P}^2$ be the corresponding $\mathbb{Z}_2^3$-cover as described in Observation* 3.3*. Then, the preimage of the six (-1)-curves on $\mathrm{Bl}_3\,\mathbb{P}^2$ under the covering map gives a configuration of* 24 *smooth rational curves on X whose dual graph is shown in Figure* 3.

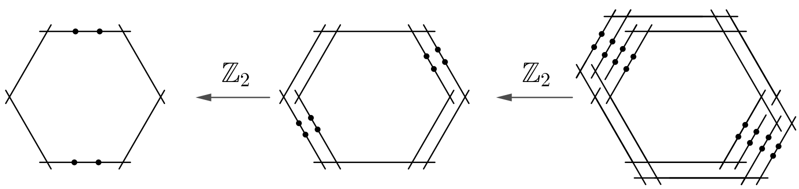

FIGURE 2. First two double covers of the six (-1)-curves of $\mathrm{Bl}_3\,\mathbb{P}^2$. The marked points represent the branch locus.

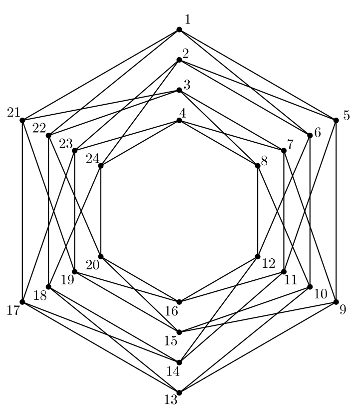

FIGURE 3. Dual graph of the smooth rational curves on a triple-double K3 surface arising as the $\mathbb{Z}_2^3$-cover of the six (-1)-curves of $\mathrm{Bl}_3\,\mathbb{P}^2$.

Proof. Split the $\mathbb{Z}_2^3$-cover $X \to \mathrm{Bl}_3 \mathbb{P}^2$ into three double covers, each branched along a pair of curves. Then, the configuration of 24 smooth rational curves on X can be computed by taking appropriate branched double covers beginning with the six (-1)-curves on $\mathrm{Bl}_3 \mathbb{P}^2$, as shown in Figure 2. The result of the last double cover is shown in Figure 3. □

Definition 3.9. Let X be a triple-double K3 surface. Denote by $R_1, \ldots, R_{24}$ the 24 smooth rational curves on X described in Proposition 3.8, and label them as shown in Figure 3.

4. The Néron-Severi lattice of a triple-double K3 surface with minimal Picard number.

Let X be a triple-double K3 surface with minimal Picard number. In this section, we show that $\mathrm{NS}(X)$ is generated by the 24 smooth rational curves $R_1, \ldots, R_{24}$. Moreover, we give an explicit $\mathbb{Z}$-basis for $\mathrm{NS}(X)$, which decomposes $\mathrm{NS}(X)$ as $U \oplus E_8 \oplus Q$, where Q is explicitly described in Lemma 4.1 (iii).

4.1. The sublattices $S, Q \subset \mathrm{NS}(X)$.

Lemma 4.1. *Let X be a triple-double* K3 *surface. Let S be the sublattice of* $\mathrm{NS}(X)$*, generated by*

$$\mathcal{S} = \{R_1,\ R_5,\ R_9,\ R_{13},\ R_{17},\ R_{23},\ R_4,\ R_{15},\ R_8,\ R_3\}.$$

Let Q be the sublattice of $\mathrm{NS}(X)$*, generated by*

$$\begin{aligned}\mathcal{Q} = \{&R_{16},\ R_{14} - R_{21} + R_{22},\ R_{11} - R_2 + R_{19} - R_{20},\\ &R_{17} + 2R_{14} - R_{18} - R_{19} + R_{20},\ R_{12} - R_{10} + R_{18} + R_{20},\\ &\qquad\qquad\qquad\qquad R_3 + 2R_{22} - 2R_6 - R_{12}\}.\end{aligned}$$

Then, the following hold:

(i) $\mathcal{S}$ *is a $\mathbb{Z}$-basis for S and $S \cong U \oplus E_8$;*
(ii) $Q \subset S^{\perp}$ *in* $\mathrm{NS}(X)$;
(iii) $\mathcal{Q}$ *is a $\mathbb{Z}$-basis for Q and its corresponding Gram matrix is given by*

$$\begin{pmatrix} -2 & 0 & 1 & 0 & 2 & -1 \\ 0 & -6 & -1 & -4 & 4 & -5 \\ 1 & -1 & -8 & 6 & 2 & 0 \\ 0 & -4 & 6 & -16 & 4 & -2 \\ 2 & 4 & 2 & 4 & -8 & 6 \\ -1 & -5 & 0 & -2 & 6 & -12 \end{pmatrix}.$$

Proof. All of the above statements can be explicitly verified. We only remark that S is isometric to $U \oplus E_8$ since S is an even unimodular lattice of signature $(1, 9)$, see Figure 4. □

Corollary 4.2. *Let X be a triple-double* K3 *surface with minimal Picard number. Then, $\rho(X) = 16$, and* $\mathrm{NS}(X)$ *is generated over $\mathbb{Q}$ by $R_1, \ldots, R_{24}$. In particular, given $D_1, D_2 \in \mathrm{NS}(X)$, we have that $D_1 = D_2$ if and only if $D_1 \cdot R_i = D_2 \cdot R_i$ for all $i \in \{1, \ldots, 24\}$.*

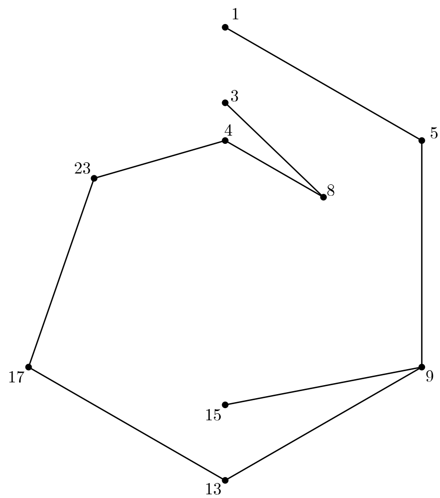

FIGURE 4. Subgraph of the dual graph of the curves $R_1, \ldots, R_{24}$ with vertices the $\mathbb{Z}$-basis of the lattice S in Lemma 4.1. These constitute a type II^* singular fiber of an elliptic pencil (see **[13,** Figure 1]) together with the section R_3.

Proof. If X is any triple-double K3 surface, then the sublattice $S \oplus Q \subseteq \mathrm{NS}(X)$ has rank 16; hence, $\rho(X) \geq \mathrm{rk}(S \oplus Q) = 16$. In order to show that the minimum for $\rho(X)$ equals 16, consider the four-dimensional coarse moduli space $\mathcal{M}$ of $(S \oplus Q)$-polarized K3 surfaces [**4**]. Let $\mathfrak{X} \to \mathcal{U}$ be the family of triple-double K3 surfaces constructed in Definition 3.4. Then, the induced morphism $p\colon \mathcal{U} \to \mathcal{M}$ is dominant since $\mathfrak{X} \to \mathcal{U}$ is a four-dimensional family by Observation 3.6. This implies that the minimal Picard number for X is 16.

Let $L \subseteq \mathrm{NS}(X)$ be the sublattice of $\mathrm{NS}(X)$ generated by $R_1, \ldots, R_{24}$. Then, $S \oplus Q \subseteq L \subseteq \mathrm{NS}(X)$, which implies that L generates $\mathrm{NS}(X)$ over $\mathbb{Q}$ if $\rho(X) = 16$.

The last statement about D_1, D_2 follows from this and the fact that, on a K3 surface, numerical equivalence of divisors coincides with linear equivalence, see subsection 2.2. $\square$

Remark 4.3. Consider the family of triple-double K3 surfaces $\mathfrak{X} \to \mathcal{U}$ in Definition 3.4. Denote the fiber over a point $u \in \mathcal{U}$ by X_u. Then, by [**10,** Chapter 6, subsection 2.5] the K3 surface X_u has minimal Picard number for a very general $u \in \mathcal{U}$ (i.e., for all $u \in \mathcal{U} \setminus Z$, where Z is the union of countably many Zariski closed subsets of $\mathcal{U}$).

4.2. The discriminant group of Q and proof that $S \oplus Q = \mathrm{NS}(X)$.

Lemma 4.4. *The discriminant group of Q (and hence, the discriminant group of $S \oplus Q$) is isomorphic to $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$, and it is generated by the classes* modulo Q *of the following elements in Q^*:*

$$v_1 = \left(\frac{1}{2}, -\frac{1}{2}, 0, 0, \frac{1}{2}, 0\right), \qquad v_2 = \left(-\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right),$$
$$w_1 = \left(\frac{1}{2}, 0, 0, -\frac{1}{4}, 0, 0\right), \qquad w_2 = \left(0, 0, \frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}\right),$$

where the coordinates lie with respect to the $\mathbb{Z}$-basis $\mathcal{Q}$, see Lemma 4.1.

Proof. Let B be the Gram matrix of Q associated to the $\mathbb{Z}$-basis $\mathcal{Q}$ in Lemma 4.1 (iii). Then the lattice Q^* is generated over $\mathbb{Z}$ by the rows of B^{-1}. In order to understand the discriminant group of Q, we compute the Smith normal form of B^{-1}. This can be done using `Sage` [**25,** function smith_form()], which yields $M_1, M_2 \in \mathrm{SL}_6(\mathbb{Z})$ such that $M_1B^{-1}M_2$ is the diagonal matrix $\mathrm{diag}(1,1,1/2,1/2,1/4,1/4)$. This implies that $A_Q \cong \mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$. Moreover, the rows of M_1B^{-1} yield an alternative $\mathbb{Z}$-basis of Q^* with which to work. We have

$$M_1B^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}.$$

The first two rows represent elements in Q, and the last four are generators of the discriminant group A_Q, which we denote by v_1, v_2, w_1 and w_2, respectively. □

Lemma 4.5. *Let $v_1, v_2, w_1, w_2 \in Q^*$ as in Lemma 4.4. Then the values of the symmetric bilinear form b_{Q^*} evaluated at pairs of these vectors are shown in Table 1. In particular, the only isotropic elements in the discriminant group A_Q with respect to the discriminant quadratic form $q_Q\colon A_Q \to \mathbb{Q}/2\mathbb{Z}$ are the classes of*

$$2w_1,\ v_2,\ v_2+2w_1,\ v_1+2w_2,\ v_1+2w_1+2w_2,\ v_1+v_2,\ v_1+v_2+2w_1.$$

Moreover, these are not contained in $\mathrm{NS}(X)$.

TABLE 1.

b_{Q^*}	v_1	v_2	w_1	w_2
v_1	-5	$5/2$	-1	$-1/2$
v_2	$5/2$	-2	1	$1/2$
w_1	-1	1	$-3/2$	$-5/4$
w_2	$-1/2$	$1/2$	$-5/4$	$-11/4$

Proof. Direct calculation yields the values in Table 1. Recall that the coordinates of the vectors v_1, v_2, w_1, w_2 are with respect to the

$\mathbb{Z}$-basis $\mathcal{Q}$ in Lemma 4.4. Thus, for instance, we have that

$$v_1 = \tfrac{1}{2}R_{16} - \tfrac{1}{2}(R_{14} - R_{21} + R_{22}) + \tfrac{1}{2}(R_{12} - R_{10} + R_{18} + R_{20}).$$

The listed isotropic vectors are easily obtained after computing all possible $(av_1 + bv_2 + cw_1 + dw_2)^2$ with $((a,b),(c,d)) \in \{0,1\}^2 \times \{0,1,2,3\}^2$.

Given two vectors $u_1,\ u_2 \in Q^*$, we write $u_1 \approx u_2$ if $u_1 - u_2 \in \mathrm{NS}(X)$. Then, we have that

$$\begin{aligned} 2w_1 &\approx \frac{R_{17}+R_{18}+R_{19}+R_{20}}{2}, \\ v_2 &\approx \frac{R_{14}+R_{16}+R_{21}+R_{22}}{2}, \\ v_1+v_2 &\approx \frac{R_{10}+R_{12}+R_{18}+R_{20}}{2}, \\ v_1+v_2+2w_1 &\approx \frac{R_{10}+R_{12}+R_{17}+R_{19}}{2}. \end{aligned}$$

This implies that $2w_1, v_2, v_1+v_2, v_1+v_2+2w_1 \notin \mathrm{NS}(X)$ by Observation 2.12.

It remains to show that $v_2+2w_1, v_1+2w_2, v_1+2w_1+2w_2 \notin \mathrm{NS}(X)$. We analyze them separately.

• Assume, by contradiction, that $v_2+2w_1 \in \mathrm{NS}(X)$. Then, we have that

$$v_2 + 2w_1 \approx \frac{R_{14}+R_{16}+R_{17}+R_{18}+R_{19}+R_{20}+R_{21}+R_{22}}{2} = \alpha,$$

and $\alpha \in \mathrm{NS}(X)$. Using Corollary 4.2, it is easy to observe that $R_{13} + R_{14} + R_{17} + R_{18} = R_{15} + R_{16} + R_{19} + R_{20}$ due to the fact that $(R_{13}+R_{14}+R_{17}+R_{18}) \cdot R_i = (R_{15}+R_{16}+R_{19}+R_{20}) \cdot R_i$ for all $i = 1, \ldots, 24$. In particular, we have that

$$\frac{R_{13}+R_{14}+R_{17}+R_{18}+R_{15}+R_{16}+R_{19}+R_{20}}{2} = \alpha_1 \in \mathrm{NS}(X).$$

However, then $\alpha + \alpha_1 \in \mathrm{NS}(X)$, which contradicts Observation 2.12 since

$$\alpha + \alpha_1 \approx \frac{R_{13}+R_{15}+R_{21}+R_{22}}{2}.$$

• Similarly, assume that $v_1 + 2w_2 \in \mathrm{NS}(X)$. Then, we have that

$$v_1+2w_2 \approx \frac{R_2+R_3+R_{11}+R_{12}+R_{14}+R_{16}+R_{17}+R_{18}+R_{21}+R_{22}}{2} = \beta,$$

and $\beta \in \mathrm{NS}(X)$. Using Corollary 4.2, we can verify that $R_{11} + R_{12} + R_{14} + R_{16} = R_1 + R_3 + R_{21} + R_{22}$ so that

$$\frac{R_{11} + R_{12} + R_{14} + R_{16} + R_1 + R_3 + R_{21} + R_{22}}{2} = \beta_1 \in \mathrm{NS}(X).$$

However, then $\beta + \beta_1 \in \mathrm{NS}(X)$, which contradicts Observation 2.12 since

$$\beta + \beta_1 \approx \frac{R_1 + R_2 + R_{17} + R_{18}}{2}.$$

• If $v_1 + 2w_1 + 2w_2 \in \mathrm{NS}(X)$, then

$$v_1+2w_1+2w_2 \approx \frac{R_2+R_3+R_{11}+R_{12}+R_{14}+R_{16}+R_{19}+R_{20}+R_{21}+R_{22}}{2} = \gamma,$$

and $\gamma \in \mathrm{NS}(X)$. Let β_1 be as in the previous point. Then, $\gamma + \beta_1 \in \mathrm{NS}(X)$, which is not allowed by Observation 2.12 since

$$\gamma + \beta_1 \approx \frac{R_1 + R_2 + R_{19} + R_{20}}{2}. \qquad \square$$

Theorem 4.6. *Let X be a triple-double* K3 *surface with $\rho(X) = 16$. Then, the following hold*:

(i) $\mathrm{NS}(X) = S \oplus Q \cong U \oplus E_8 \oplus Q$;
(ii) $\mathrm{NS}(X)$ *is generated by the smooth rational curves $R_1, \ldots, R_{24}$*;
(iii) *the discriminant group of* $\mathrm{NS}(X)$ *is* $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$.

Proof. If $\rho(X) = 16$, then $\mathrm{NS}(X)$ is an even overlattice of $S \oplus Q$. Therefore, $\mathrm{NS}(X)/(S\oplus Q)$ corresponds to a subgroup of $A_{(S\oplus Q)} \cong A_Q$, which is isotropic with respect to the discriminant quadratic form $q_Q : A_Q \to \mathbb{Q}/2\mathbb{Z}$, see Theorem 2.3. However, Proposition 4.5 shows that no isotropic vector of A_Q can be contained in $\mathrm{NS}(X)$. This implies that $\mathrm{NS}(X)$ must be equal to $S\oplus Q$, which is isometric to $U\oplus E_8\oplus Q$ by Lemma 4.1 (i). It also follows that $\mathrm{NS}(X)$ is generated by $R_1, \ldots, R_{24}$ since the sublattice of $\mathrm{NS}(X)$ generated by these 24 curves contains

$S \oplus Q$. Finally, $A_{\mathrm{NS}(X)} = A_{(S\oplus Q)} \cong A_Q$, which is isomorphic to $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$ by Lemma 4.4. □

Remark 4.7. If X is a triple-double K3 surface with $\rho(X) = 16$, then the following $\mathbb{Z}$-basis of $\mathrm{NS}(X)$ realizes it as a direct sum of U, E_8 and Q, see Lemma 4.1:

$$\begin{aligned}\mathrm{NS}(X) &= S \oplus Q \\ &= \langle 2R_1 + 2R_4 + 4R_5 + R_8 + 6R_9 \\ &\qquad + 5R_{13} + 3R_{15} + 4R_{17} + 3R_{23},\ R_3\rangle_{\mathbb{Z}} \\ &\oplus \langle R_1,\ R_5,\ R_9,\ R_{13},\ R_{17},\ R_{23},\ R_4,\ R_{15}\rangle_{\mathbb{Z}} \\ &\oplus \langle R_{16},\ R_{14} - R_{21} + R_{22},\ R_{11} - R_2 + R_{19} - R_{20}, R_{17} \\ &\quad + 2R_{14} - R_{18} - R_{19} + R_{20}, R_{12} - R_{10} \\ &\qquad + R_{18} + R_{20}, R_3 + 2R_{22} - 2R_6 - R_{12}\rangle_{\mathbb{Z}} \\ &\cong U \oplus E_8 \oplus Q.\end{aligned}$$

Remark 4.8. For a triple-double K3 surface X with $\rho(X) = 16$, a splitting $\mathrm{NS}(X) \cong U \oplus E_8 \oplus P$ for some lattice P is abstractly predicted by Theorem 2.2, as follows. We know that $\mathrm{NS}(X)$ is even, of signature $(1, 15)$, and discriminant group $\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^4$. It follows from Theorem 2.2 (i) that $\mathrm{NS}(X)$ is isometric to $E_8 \oplus P'$ for some lattice P'. However, then P' is even, of signature $(1, 7)$, and its discriminant group is also $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$. In particular, we can apply Theorem 2.2 (ii) to P' to argue that $P' = U \oplus P$ for some lattice P. In conclusion, we have that $\mathrm{NS}(X) \cong U \oplus E_8 \oplus P$, but all we know about P is that it is even, of signature $(0, 6)$ and discriminant group $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$. However, in our case, we were able to provide an explicit lattice Q (see Lemma 4.1 (iii)) which realizes the splitting $\mathrm{NS}(X) \cong U \oplus E_8 \oplus Q$.

4.3. The transcendental lattice of a triple-double K3 surface with minimal Picard number.

Proposition 4.9. *Let X be a triple-double* K3 *surface with $\rho(X) = 16$. Then, the transcendental lattice T_X is isometric to $U \oplus U(2) \oplus \langle -4\rangle^{\oplus 2}$.*

Proof. The transcendental lattice T_X is an even lattice of signature $(2,4)$. Here, we study its discriminant quadratic form. Table 1 gives the discriminant quadratic form of the lattice $\mathrm{NS}(X) = S \oplus Q$ with respect to $v_1,\ v_2,\ w_1,\ w_2$, whose classes modulo Q generate the discriminant group. Changing basis to $v_2,\ v_1 + v_2 + 2w_1,\ v_1 + 2w_1 - w_2,\ v_1 + w_1 - w_2$, the discriminant quadratic form becomes

$$\begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix},$$

where the entries on the main diagonal (respectively, off the main diagonal) are considered modulo $2\mathbb{Z}$ (respectively, modulo $\mathbb{Z}$). It follows from Theorem 2.4 that the discriminant group of T_X is $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4^2$ ($\mathrm{NS}(X)$ is a primitive sublattice of the even unimodular lattice $H^2(X;\mathbb{Z})$), and its discriminant quadratic form is the opposite of the above matrix. Observe that the lattice $U \oplus U(2) \oplus \langle -4\rangle^{\oplus 2}$ is even, of signature $(2,4)$, and its discriminant quadratic form equals the discriminant quadratic form of T_X. We can conclude from Theorem 2.1 that T_X is isometric to $U \oplus U(2) \oplus \langle -4\rangle^{\oplus 2}$. $\square$

Recall that projective Kummer surfaces are special types of projective K3 surfaces, obtained as the minimal resolution of the quotient of an abelian surface A by the inversion morphism $a \mapsto -a$. We denote such a Kummer surface by $\mathrm{Km}(A)$. A renowned example of Kummer surface is $\mathrm{Km}(E_i \times E_i)$, where E_i is the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus i\mathbb{Z})$. The automorphism group of $\mathrm{Km}(E_i \times E_i)$ was studied in **[11]**, and in **[8]**, this Kummer surface was used to compute Ω_G and $\Omega_G^\perp$ for $G = \mathbb{Z}_2 \oplus \mathbb{Z}_4, \mathbb{Z}_2^2, \mathbb{Z}_4$.

Observation 4.10. The transcendental lattice of $\mathrm{Km}(E_i \times E_i)$ is isometric to $\langle 4\rangle^{\oplus 2}$, see **[11,** Section 1], and $\langle 4\rangle^{\oplus 2}$ admits a primitive embedding into the transcendental lattice of a triple-double K3 surface with minimal Picard number, which is $U \oplus U(2) \oplus \langle -4\rangle^{\oplus 2}$ by Proposition 4.9. In order to show this, consider the sublattice $L \subset U \oplus U(2) \oplus \langle -4\rangle^{\oplus 2}$ generated by $\alpha = (1,2,0,0,0,0)$, $\beta = (0,0,1,1,0,0)$. We have that L is isometric to $\langle 4\rangle^{\oplus 2}$ since $\alpha^2 = \beta^2 = 4$

and $\alpha \cdot \beta = 0$. In order to prove that L is a primitive sublattice, assume that we have $m(x, y, z, w, u, v) \in L$ for some integer $m > 1$ and $(x, y, z, w, u, v) \in U \oplus U(2) \oplus \langle -4 \rangle^{\oplus 2}$. Then, $y = 2x$, $z = w$ and $u = v = 0$, which implies that $(x, y, z, w, u, v) \in L$.

Therefore, we ask whether the Kummer surface $\mathrm{Km}(E_i \times E_i)$ is a specialization of the family of triple-double K3 surfaces. If this is true, then there should be a configuration of three pairs of lines in $\mathbb{P}^2$ such that the minimal resolution of the appropriate $\mathbb{Z}_2^3$-cover of $\mathbb{P}^2$ gives $\mathrm{Km}(E_i \times E_i)$ (by appropriate, we mean the usual chain of three double covers). The next theorem explicitly describes this line arrangement.

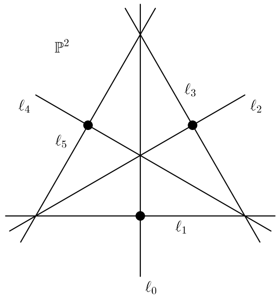

FIGURE 5. The three pairs of lines $(\ell_0, \ell_1), (\ell_2, \ell_3), (\ell_4, \ell_5)$ in $\mathbb{P}^2$ such that the minimal resolution of the appropriate $\mathbb{Z}_2^3$-cover of $\mathbb{P}^2$ branched along these three pairs gives $\mathrm{Km}(E_i \times E_i)$.

Theorem 4.11. *Consider three pairs of lines (ℓ_0, ℓ_1), (ℓ_2, ℓ_3) and (ℓ_4, ℓ_5) in $\mathbb{P}^2$ such that the resulting line arrangement has exactly four triple intersection points, as shown in Figure* 5 (*this is unique up to an automorphism of* $\mathbb{P}^2$)*. Then, the minimal resolution of the appropriate $\mathbb{Z}_2^3$-cover of $\mathbb{P}^2$ branched along these three pairs is isomorphic to the singular Kummer surface* $\mathrm{Km}(E_i \times E_i)$.

Proof. Let $\mathrm{Bl}_3 \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at the three marked points in Figure 5. Let $X \to \mathrm{Bl}_3 \mathbb{P}^2$ be the appropriate $\mathbb{Z}_2^3$-cover branched along the three pairs of lines $(\widehat{\ell_0}, \widehat{\ell_1})$, $(\widehat{\ell_2}, \widehat{\ell_3})$, $(\widehat{\ell_4}, \widehat{\ell_5})$. In particular, X has exactly four A_1 singularities, one over each triple intersection point of the line arrangement. Following Observation 3.3, the surface X can be viewed as the following hypersurface in $(\mathbb{P}^1)^3$:

$$X_1^2Y_0^2Z_0^2 + X_0^2Y_1^2Z_0^2 + X_0^2Y_0^2Z_1^2 + X_1^2Y_1^2Z_1^2 = 0.$$

The blow up of X at the four A_1 singularities is a K3 surface.

Consider the genus 1 fibration on X given by the restriction to X of the projection $\pi_3\colon ([X_0 : X_1], [Y_0 : Y_1], [Z_0 : Z_1]) \mapsto [Z_0 : Z_1]$. The general fiber of this fibration is a genus 1 curve in $\mathbb{P}^1 \times \mathbb{P}^1$, given by

$$C\colon \lambda^2X_1^2Y_0^2 + \lambda^2X_0^2Y_1^2 + \mu^2X_0^2Y_0^2 + \mu^2X_1^2Y_1^2 = 0,$$

for a general $[\lambda : \mu] \in \mathbb{P}^1$. The restriction to C of the projection

$$([X_0 : X_1], [Y_0 : Y_1]) \longmapsto [Y_0 : Y_1]$$

realizes C as a double cover of $\mathbb{P}^1$ branched along

$$[i\lambda : \mu],\ [-i\lambda : \mu],\ [i\mu : \lambda],\ [-i\mu : \lambda].$$

If we set $\sigma = i(\lambda/\mu)$ in the affine patch of $\mathbb{P}^1$, where $Y_1 \neq 0$, then the four branch points above become, respectively,

$$\sigma,\ -\sigma,\ -\frac{1}{\sigma},\ \frac{1}{\sigma}.$$

Using the automorphism of $\mathbb{P}^1$, given by

$$z \longmapsto \left(\frac{\sigma + \sigma^3}{2}\right) \cdot \frac{z - \sigma}{\sigma z - 1},$$

we can move these branch points to

$$0,\ \sigma^2,\ \frac{(1+\sigma^2)^2}{4},\ \infty,$$

respectively. However, then, the elliptic fibration $\pi_3|_X\colon X \to \mathbb{P}^1$ is isomorphic to the elliptic fibration (6) in [**8**, Section 4] (set $\tau = 1$), which gives $\mathrm{Km}(E_i \times E_i)$. $\square$

4.4. Further properties of triple-double K3 surfaces. The next proposition, among other things, shows that triple-double K3 surfaces with minimal Picard number are disjoint from the family of K3 surfaces with $\mathbb{Z}_2^3$ symplectic action.

Proposition 4.12. *Let X be a triple-double* K3 *surface with $\rho(X) = 16$. Then, the following hold*:

(i) *X does not admit $\mathbb{Z}_2^3$ symplectic action. In particular, X does not admit $\mathbb{Z}_2^4$ symplectic action*; *hence,* [**8,** Proposition 6.2] *cannot be used to compute* $\mathrm{NS}(X)$;

(ii) *X is not isomorphic to the minimal resolution of the quotient of a* K3 *surface by a symplectic action of the group $\mathbb{Z}_2^3$. In particular, X is not isomorphic to the minimal resolution of the quotient of a* K3 *surface by a symplectic action of the group $\mathbb{Z}_2^4$*; *hence,* $\mathrm{NS}(X)$ *cannot be computed using* [**9,** Theorem 8.3].

Proof. In order to prove (i), X admits $\mathbb{Z}_2^3$ symplectic action if and only if the transcendental lattice T_X primitively embeds into $\Omega_{\mathbb{Z}_2^3}^{\perp}$, see [**19**]. However, $T_X \cong U \oplus U(2) \oplus \langle -4\rangle^{\oplus 2}$ from Proposition 4.9, and $\Omega_{\mathbb{Z}_2^3}^{\perp} \cong U(2)^{\oplus 3} \oplus \langle -4\rangle^{\oplus 2}$ from [**8,** Proposition 5.1]. Moreover, $(1,1,0,0,0,0) \in U \oplus U(2) \oplus \langle -4\rangle^{\oplus 2}$ squares, giving 2 and $4 \mid x^2$ for all $x \in U(2)^{\oplus 3} \oplus \langle -4\rangle^{\oplus 2}$. Therefore, T_X cannot primitively embed into $\Omega_{\mathbb{Z}_2^3}^{\perp}$.

For (ii), we first need some preliminaries. Let $M_{\mathbb{Z}_2^3}$ be the abstract lattice generated by the following vectors:

$$m_1, \ldots, m_{14}, \frac{\sum_{i=1}^{8} m_i}{2}, \frac{\sum_{i=5}^{12} m_i}{2}, \frac{m_1+m_2+m_5+m_6+m_9+m_{10}+m_{13}+m_{14}}{2},$$

where $m_i^2 = -2$ and $m_1 \cdot m_j = 0$ for all $i, j \in \{1, \ldots, 14\}$, $i \neq j$. The lattice $M_{\mathbb{Z}_2^3}$ is a negative definite of rank 14 and has discriminant group $\mathbb{Z}_2^8$. Using similar calculations to those used to determine the discriminant quadratic form of T_X given generators for $\mathrm{NS}(X)$, we can show that the discriminant quadratic form of $M_{\mathbb{Z}_2^3}^{\perp}$, the orthogonal complement of $M_{\mathbb{Z}_2^3}$ in $H^2(X;\mathbb{Z})$, is isomorphic to the discriminant quadratic form of $\langle 2\rangle^{\oplus 2} \oplus U(2) \oplus \langle -2\rangle^{\oplus 4}$. From Theorem 2.5, this implies that $M_{\mathbb{Z}_2^3}^{\perp}$ is isometric to the lattice $\langle 2\rangle^{\oplus 2} \oplus U(2) \oplus \langle -2\rangle^{\oplus 4}$. Now, X is isomorphic to the minimal resolution of the quotient of a K3 surface by a symplectic action of the group $\mathbb{Z}_2^3$ if and only if the lattice $M_{\mathbb{Z}_2^3}$ primitively embeds into $\mathrm{NS}(X)$ (see [**6,** Definition 2.5], [**9,** Corollary 8.9]), or equivalently, if and only if T_X primitively embeds into $M_{\mathbb{Z}_2^3}^{\perp}$. However, $T_X \cong U \oplus U(2) \oplus \langle -4\rangle^{\oplus 2}$, and, in particular, there are two vectors $x, y \in T_X$ such that $x \cdot y = 1$. However, for any two vectors $x, y \in M_{\mathbb{Z}_2^3}^{\perp} \cong \langle 2\rangle^{\oplus 2} \oplus U(2) \oplus \langle -2\rangle^{\oplus 4}$, the product $x \cdot y$ is even. $\square$

Corollary 4.13. *Let X be a triple-double* K3 *surface with $\rho(X) = 16$. Then, X does not contain* 14 *disjoint smooth rational curves.*

Proof. If X contains 14 disjoint smooth rational curves, then the lattice $M_{\mathbb{Z}_2^3}$ (see the proof of Proposition 4.12 (ii)) would be primitively embedded in $\mathrm{NS}(X)$. However, this would contradict Proposition 4.12 (ii) [**9,** Corollary 8.9]. $\square$

5. $\mathbb{Z}_2^n$-covers of $\mathbb{P}^2$ branched along six lines and K3 surfaces.

5.1. Hirzebruch-Kummer coverings.

Definition 5.1. Let n be a positive integer, and let $D \subset \mathbb{P}^2$ be a line arrangement such that no point in $\mathbb{P}^2$ belongs to all of the lines. Let $Y' \to \mathbb{P}^2$ be the normal finite covering such that its restriction to $\mathbb{P}^2 \setminus D$ is the Galois unramified covering associated to the monodromy homomorphism

$$H_1(\mathbb{P}^2 \setminus D; \mathbb{Z}) \longrightarrow H_1(\mathbb{P}^2 \setminus D; \mathbb{Z}) \otimes \mathbb{Z}_n.$$

Then, the minimal resolution Y of Y' is called the *Hirzebruch-Kummer covering of exponent n* of $\mathbb{P}^2$ branched along the configuration of lines. An explicit construction of the Hirzebruch-Kummer covering can be found in [**3,** Definition 63]. It turns out that Y' is a $\mathbb{Z}_n^{r-1}$-cover of $\mathbb{P}^2$ branched along D, where r equals the number of lines in D.

Example 5.2. We are interested in the Hirzebruch-Kummer covering of exponent 2 of $\mathbb{P}^2$ branched along six general lines (which is a $\mathbb{Z}_2^5$-cover of $\mathbb{P}^2$ branched along the six lines). This is constructed as follows. Let $[W_0 : \cdots : W_5]$ be coordinates in $\mathbb{P}^5$. Then, for general coefficients a_i, b_i, c_i, $i = 0, \ldots, 5$, let $Y \subset \mathbb{P}^5$ be the following intersection:

$$\begin{cases} a_0 W_0^2 + \cdots + a_5 W_5^2 = 0, \\ b_0 W_0^2 + \cdots + b_5 W_5^2 = 0, \\ c_0 W_0^2 + \cdots + c_5 W_5^2 = 0. \end{cases}$$

Y is smooth for a general choice of the coefficients; hence, Y is a K3 surface. Consider the restriction to Y of the morphism $\mathbb{P}^5 \to \mathbb{P}^5$, given by

$$[W_0 : \cdots : W_5] \longmapsto [W_0^2 : \cdots : W_5^2].$$

Then, this realizes Y as a $\mathbb{Z}_2^5$-cover of the linear subspace $H \subset \mathbb{P}^5$, given by

$$\begin{cases} a_0 W_0 + \cdots + a_5 W_5 = 0, \\ b_0 W_0 + \cdots + b_5 W_5 = 0, \\ c_0 W_0 + \cdots + c_5 W_5 = 0, \end{cases}$$

which is isomorphic to $\mathbb{P}^2$. The branch locus of this cover is given by the restrictions to H of the six coordinate hyperplanes of $\mathbb{P}^5$, which correspond to six general lines in $\mathbb{P}^2$. Label the six lines $\ell_0, \ldots, \ell_5$, and denote by π the covering map $Y \to \mathbb{P}^2$. Then, π is the Hirzebruch-Kummer covering of exponent 2 of $\mathbb{P}^2$ branched along $\ell_0, \ldots, \ell_5$. Since the coefficients a_i, b_j and c_k vary, Y describes a four-dimensional family of K3 surfaces, which was also considered in [**9,** subsection 10.2].

Remark 5.3. There is no $\mathbb{Z}_2^n$-cover of $\mathbb{P}^2$ branched along $\ell_0, \ldots, \ell_5$, for $n \geq 6$, and any $\mathbb{Z}_2^n$ cover of $\mathbb{P}^2$ branched along $\ell_0, \ldots, \ell_5$, for $n \leq 4$ is an appropriate quotient of the Hirzebruch-Kummer covering Y constructed above, see [**3,** Section 4].

5.2. Relation with triple-double K3 surfaces.

Observation 5.4. Let Y be the Hirzebruch-Kummer covering described in Example 5.2. Let ι_{01} be the restriction to Y of the involution of $\mathbb{P}^5$, defined by

$$[W_0 : W_1 : W_2 : W_3 : W_4 : W_5] \longmapsto [-W_0 : -W_1 : W_2 : W_3 : W_4 : W_5]. \tag{5.1}$$

Define ι_{ij} analogously for all $i, j \in \{0, \ldots, 5\}$, $i \neq j$. The involutions ι_{ij} form a subgroup of $\mathrm{Aut}(Y)$ isomorphic to $\mathbb{Z}_2^4$ acting symplectically on Y since each involution has exactly eight fixed points. The eight fixed points of ι_{ij} are mapped to $\ell_i \cap \ell_j$ under π. We have that $\widetilde{Y/\mathbb{Z}_2^4}$ is the K3 surface given by the minimal resolution of the double cover of $\mathbb{P}^2$ branched along $\ell_0, \ldots, \ell_5$. This observation may also be found in [**3,** Section 4.3], [**9,** Section 10.2].

Proposition 5.5. *Let X be a triple-double* K3 *surface, and let $X_3 \to \mathbb{P}^2$ be the corresponding $\mathbb{Z}_2^3$-cover branched along the three pairs of lines (ℓ_0, ℓ_1), (ℓ_2, ℓ_3) and (ℓ_4, ℓ_5), so that $X = \widetilde{X}_3$. Let Y be the corresponding Hirzebruch-Kummer covering in Example* 5.2. *Consider the symplectic action on Y of the group $\{\mathrm{id}_Y, \iota_{01}, \iota_{23}, \iota_{45}\} \cong \mathbb{Z}_2^2$, see Observation* 5.4. *Then, $X \cong \widetilde{Y/\mathbb{Z}_2^2}$.*

Proof. By Remark 5.3, we have that X is isomorphic to the minimal resolution of the quotient of Y by a subgroup $G \subset \mathbb{Z}_2^4$ of symplectic automorphisms. It follows that G is isomorphic to $\mathbb{Z}_2^2$, and the only possibility for G is to equal $\{\mathrm{id}_Y, \iota_{01}, \iota_{23}, \iota_{45}\}$. The reason for the latter statement is due to the fact that this is the only choice so that Y/G has exactly four singularities of type A_1 over each point $\ell_0 \cap \ell_1$, $\ell_2 \cap \ell_3$, $\ell_4 \cap \ell_5$. $\square$

5.3. Other K3 surfaces which are $\mathbb{Z}_2^n$-covers of $\mathbb{P}^2$ branched along six lines. Consider six general lines in $\mathbb{P}^2$, and denote by D the divisor on $\mathbb{P}^2$ given by their sum. Let Y be the Hirzebruch-Kummer covering of exponent 2 of $\mathbb{P}^2$ branched along D in Example 5.2. Let X be a triple-double K3 surface obtained as a $\mathbb{Z}_2^3$-cover of $\mathbb{P}^2$ branched along D. Lastly, let Z be the minimal resolution of the double cover of $\mathbb{P}^2$ branched along D. As stated in the introduction, the lattices $\mathrm{NS}(Y)$ and $\mathrm{NS}(Z)$ are well known for $\rho(Y) = \rho(Z) = 16$, and the lattice $\mathrm{NS}(X)$ is computed for $\rho(X) = 16$. Let X' be the minimal resolution of the quotient of X by one of the three symplectic involutions $\iota_{110}, \iota_{101}, \iota_{011}$ in Proposition 3.5. In Section 6, we compute $\mathrm{NS}(X')$ for X' with minimal Picard number. The surface X' can be viewed as an appropriate $\mathbb{Z}_2^2$ cover of $\mathbb{P}^2$ branched along D. There are other K3 surfaces which differ from X, X', Y, Z that are $\mathbb{Z}_2^n$-covers of $\mathbb{P}^2$ branched along six general lines with $n = 3, 4$. We shall investigate their Néron-Severi and transcendental lattices in future work.

Remark 5.6. If X' is as above, after computing $\mathrm{NS}(X')$ and $T_{X'}$ in Section 6, we show that X' admits $\mathbb{Z}_2^4$ symplectic action, see

Proposition 6.4. Since $\rho(X') = 16$, this implies that $\mathrm{NS}(X')$ can be computed using [**9**]. However, in Section 6, we find an explicit $\mathbb{Z}$-basis for $\mathrm{NS}(X')$ and relate it to the geometry of X'.

6. The Néron-Severi lattice of X' with minimal Picard number.

6.1. Construction of the family. Let $\mathfrak{X} \to \mathcal{U}$ be the family of triple-double K3 surfaces in Definition 3.4. The involution ι_{011} in Proposition 3.5 acts on $\mathfrak{X}$ inducing a symplectic action on each fiber of $\mathfrak{X} \to \mathcal{U}$ (ι_{110}, ι_{101} are treated similarly). Let $\mathfrak{X}'$ be the minimal resolution of the quotient of $\mathfrak{X}$ by ι_{011}. Then, $\mathfrak{X}' \to \mathcal{U}$ is a four-dimensional family of K3 surfaces, and we denote one of its fibers by X'. We compute the lattices $\mathrm{NS}(X')$ and $T_{X'}$ for X' with minimal Picard number in several steps.

6.2. A configuration of 20 smooth rational curves on X'. First, we want to understand the quotient by ι_{011} of the configuration of 24 smooth rational curves $R_1, \ldots, R_{24}$ on X in Figure 3 (observe that the eight fixed points of ι_{011} are in the complement of $R_1 \cup \cdots \cup R_{24}$). Therefore, we need to understand how the involutions ι_{001} and ι_{010} act on these curves. In Figure 2, we can see how ι_{001} acts by a base change on the configuration fixing the eight curves with two branches on each one. Therefore, we have that

$$\begin{aligned}\iota_{001} \cdot (1, \ldots, 24) = (&3, 4, 1, 2, 7, 8, 5, 6, 9, 10, 11, 12, \\ &15, 16, 13, 14, 19, 20, 17, 18, 21, 22, 23, 24),\end{aligned}$$

where we only carry the indices of the curves $R_1, \ldots, R_{24}$, for simplicity. Similarly, we have that

$$\begin{aligned}\iota_{010} \cdot (1, \ldots, 24) = (&2, 1, 4, 3, 5, 6, 7, 8, 11, 12, 9, 10, 14, 13, \\ &16, 15, 17, 18, 19, 20, 23, 24, 21, 22),\end{aligned}$$

and hence,

$$\begin{aligned}\iota_{011} \cdot (1, \ldots, 24) = (&4, 3, 2, 1, 7, 8, 5, 6, 11, 12, 9, 10, \\ &16, 15, 14, 13, 19, 20, 17, 18, 23, 24, 21, 22).\end{aligned}$$

In particular, we have the following classes of curves modulo ι_{011}:

$$\{R_1, R_4\}, \{R_2, R_3\}, \{R_5, R_7\}, \{R_6, R_8\}, \{R_9, R_{11}\}, \{R_{10}, R_{12}\},$$
$$\{R_{13}, R_{16}\}, \{R_{14}, R_{15}\}, \{R_{17}, R_{19}\}, \{R_{18}, R_{20}\}, \{R_{21}, R_{23}\}, \{R_{22}, R_{24}\}.$$

Consider the commutative diagram (which we discussed in the proof of Proposition 2.9)

$$\begin{tikzcd} \mathrm{Bl}_8 X \arrow[r, "\pi"] \arrow[d] & X' \arrow[d] \\ X \arrow[r] & X/\iota_{011}, \end{tikzcd}$$

where $\mathrm{Bl}_8 X$ is the blow up of X at the eight fixed points of ι_{011}. Thus, $\pi\colon \mathrm{Bl}_8 X \to X'$ is the $\mathbb{Z}_2$-cover of X' branched along the eight exceptional divisors of the resolution $X' \to X/\iota_{011}$, which we denote by $N_1, \dots, N_8$. If we set $C_1 = \pi(R_1) = \pi(R_4)$, $C_2 = \pi(R_2) = \pi(R_3)$, etc., then the corresponding curve arrangement on X' is shown in Figure 6. In conclusion, we have two disjoint sets of smooth rational curves on X' given by $\{C_1, \dots, C_{12}\}$, $\{N_1, \dots, N_8\}$.

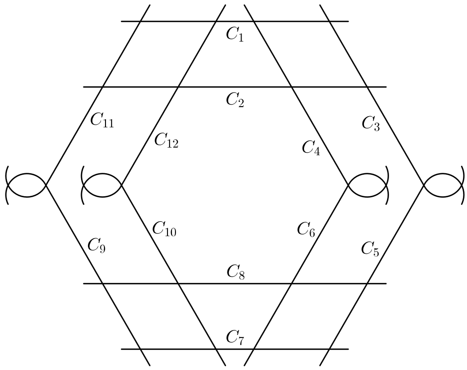

FIGURE 6. Configuration of smooth rational curves on X'.

6.3. Even eights on X'. With the notation introduced above, we know that

$$N=\frac{N_1+\cdots+N_8}{2}\in \mathrm{NS}(X'),$$

and the lattice generated by

$$N_1,\ldots,N_8,\frac{N_1+\cdots+N_8}{2}$$

is called the *Nikulin lattice*. Next, we find more sets of curves on X' whose sum is 2-divisible.

Recall the following result on Nikulin [**6,** subsection 2.3]. We state it in a convenient form.

Proposition 6.1. *Let $L_1,\ldots,L_k$, be smooth disjoint rational curves on a* K3 *surface W.*

- *If $k=13$, then, up to reordering the indices,*

$$\frac{\sum_{i=1}^{8}L_i}{2},\quad \frac{\sum_{i=5}^{12}L_i}{2}\in \mathrm{NS}(W).$$

- *If $k=14$, then, up to reordering the indices, the following vectors are in* $\mathrm{NS}(W)$:

$$\frac{\sum_{i=1}^{8}L_i}{2},\quad \frac{\sum_{i=5}^{12}L_i}{2},\quad \frac{L_1+L_2+L_5+L_6+L_9+L_{10}+L_{13}+L_{14}}{2}.$$

Now, we apply Proposition 6.1 for $k=13$ to the curves

$$C_1,\ C_5,\ C_6,\ C_9,\ C_{10},\ N_1,\ldots,\ N_8.$$

It is clear that $N\in \mathrm{NS}(X')$. This implies that, up to reordering the indices of the curves N_i, the sum of $N_1+N_2+N_3+N_4$ with the other four curves among $C_1,\ C_5,\ C_6,\ C_9,\ C_{10}$ is 2-divisible. If we use C_1 and $\{C_i,C_j,C_k\}\subset\{C_5,C_6,C_9,C_{10}\}$, then the intersection number

$$\frac{C_1+C_i+C_j+C_k+N_1+N_2+N_3+N_4}{2}\cdot C_3,$$

is not an integer. Therefore, we have that

$$\Lambda_1 = \frac{C_5 + C_6 + C_9 + C_{10} + N_1 + N_2 + N_3 + N_4}{2} \in \mathrm{NS}(X').$$

Now, we apply Proposition 6.1 for $k = 14$ to the curves

$$C_1,\ C_2,\ C_5,\ C_6,\ C_9,\ C_{10},\ N_1, \dots,\ N_8.$$

Then, we have that

$$\frac{C_1 + C_2 + C_i + C_j + N_1 + N_2 + N_5 + N_6}{2} \in \mathrm{NS}(X'),$$

where $C_i,\ C_j$ is a choice of two curves among C_5, C_6, C_9, C_{10}. This is true up to permuting $N_1, \dots, N_4$, and up to permuting $N_5, \dots, N_8$. Up to permuting C_5, C_6, C_9, C_{10}, we can assume that $\{i, j\} = \{5, 6\}$; hence,

$$\Lambda_2 = \frac{C_1 + C_2 + C_5 + C_6 + N_1 + N_2 + N_5 + N_6}{2} \in \mathrm{NS}(X').$$

6.4. Computation of $\mathrm{NS}(X')$ and $T_{X'}$.

Theorem 6.2. *Let X' be a fiber of the family of* K3 *surfaces $\mathfrak{X}' \to \mathcal{U}$, defined in Section* 6.1. *Assume that X' has minimal Picard number. Then, the lattice* $\mathrm{NS}(X')$ *has rank* 16 *and discriminant group $\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2$. A $\mathbb{Z}$-basis for* $\mathrm{NS}(X')$ *is given by*

$$\{C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_7,\ C_8,\ C_9,\ N_1,\ N_2,\ N_3,\ N_5,\ N_7,\ N,\ \Lambda_1,\ \Lambda_2\}.$$

The discriminant quadratic form of $\mathrm{NS}(X')$ *is given by*

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

Hence, the transcendental lattice $T_{X'}$ is isometric to $U(2)^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$.

Proof. Some of the computations that follow are computer-assisted (we used **[27]**). Define

$$M = \langle C_1, \ldots, C_{12}, N_1, \ldots, N_8, N, \Lambda_1, \Lambda_2\rangle_{\mathbb{Z}} \subseteq \mathrm{NS}(X').$$

The lattice M has rank 16, implying that $\rho(X') \geq 16$. However, since the K3 surfaces X' vary in a four-dimensional family, we have that the minimum Picard number for X' is 16. Therefore, M is a sublattice of $\mathrm{NS}(X')$ of finite index. We show that $M = \mathrm{NS}(X')$.

A $\mathbb{Z}$-basis for M is given by

$$\{C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_7,\ C_8,\ C_9,\ N_1,\ N_2,\ N_3,\ N_5,\ N_7,\ N,\ \Lambda_1,\ \Lambda_2\}.$$

This is true since

$$\begin{aligned}
C_6 &= C_3 - C_4 + C_5,\ C_{10} = C_1 + C_2 + C_3 + C_4 - C_7 - C_8 - C_9,\\
C_{11} &= C_3 + C_5 - C_9,\ C_{12} = -C_1 - C_2 - C_4 + C_5 + C_7 + C_8 + C_9,\\
N_4 &= -C_1 - C_2 - 2C_3 - 2C_5 + C_7 + C_8 - N_1 - N_2 - N_3 + 2\Lambda_1,\\
N_6 &= -C_1 - C_2 - C_3 + C_4 - 2C_5 - N_1 - N_2 - N_5 + 2\Lambda_2,\\
N_8 &= 2C_1 + 2C_2 + 3C_3 - C_4 + 4C_5 - C_7 - C_8 + N_1 + N_2 - N_7\\
&\qquad + 2N - 2\Lambda_1 - 2\Lambda_2.
\end{aligned}$$

In order to show that $M = \mathrm{NS}(X')$, we use the same strategy used in the proof of Theorem 4.6. We prove that any isotropic element of the discriminant group A_M cannot be an element in $\mathrm{NS}(X')$. If B is the intersection matrix of the curves in the $\mathbb{Z}$-basis of M above, then the dual M^* is generated over $\mathbb{Z}$ by the rows of B^{-1}. Using `Sage`, we can find matrices $M_1, M_2 \in \mathrm{SL}_{16}(\mathbb{Z})$ such that $M_1 B^{-1} M_2$ is the diagonal matrix

$$\mathrm{diag}\left(1, \ldots, 1,\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{4},\ \tfrac{1}{4}\right).$$

This tells us that the discriminant group of M is isomorphic to $\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^2$. In particular, the rows of $M_1 B^{-1}$ give us an alternative $\mathbb{Z}$-basis of M^* to work with. The matrix $M_1 B^{-1}$ is explicitly given by

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&0&0&0\\
0&0&0&0&0&0&0&0&\frac{1}{2}&0&-\frac{1}{2}&-\frac{1}{2}&-\frac{1}{2}&0&0&0\\
0&0&0&0&\frac{1}{2}&0&0&-\frac{1}{2}&0&0&0&-\frac{1}{2}&-\frac{1}{2}&0&0&0\\
0&0&\frac{1}{2}&-\frac{1}{2}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&\frac{1}{4}&-\frac{1}{4}&-\frac{1}{2}&0&0&-\frac{1}{2}&0&-\frac{1}{2}&0&0&0\\
\frac{1}{4}&-\frac{1}{4}&0&-\frac{1}{2}&0&0&0&0&0&0&-\frac{1}{2}&0&-\frac{1}{2}&0&0&0
\end{pmatrix}.$$

Denote by v_1, v_2, v_3, v_4, w_1, w_2, respectively the last six rows of the above matrix, which generate the discriminant group A_M. We can then enumerate the isotropic elements of A_M, given by the classes modulo M of the following vectors:

$$(C_1+C_2+C_7+C_8)/2,\ (C_1+C_2+C_3+C_4)/2,\ (C_3+C_4+C_7+C_8)/2,$$
$$(C_5+C_9+N_5+N_7)/2,\ (N_1+N_3+N_5+N_7)/2,\ (C_5+C_9+N_1+N_3)/2,$$
$$(N_2+N_3+N_5+N_7)/2,\ (C_5+C_9+N_2+N_3)/2,\ (C_1+C_2+N_1+N_2)/2,$$
$$(C_7+C_8+N_1+N_2)/2,\ (C_3+C_4+N_1+N_2)/2,$$
$$(C_3+C_4+C_5+C_9+N_5+N_7)/2,$$
$$(C_3+C_4+C_5+C_9+N_1+N_3)/2,$$
$$(C_3+C_4+C_5+C_9+N_2+N_3)/2,$$
$$(C_1+C_2+C_5+C_7+C_8+C_9+N_5+N_7)/2,$$
$$(C_1+C_2+C_7+C_8+N_1+N_3+N_5+N_7)/2,$$
$$(C_1+C_2+C_3+C_4+N_1+N_3+N_5+N_7)/2,$$
$$(C_3+C_4+C_7+C_8+N_1+N_3+N_5+N_7)/2,$$
$$(C_1+C_2+C_5+C_7+C_8+C_9+N_1+N_3)/2,$$
$$(C_1+C_2+C_7+C_8+N_2+N_3+N_5+N_7)/2,$$
$$(C_1+C_2+C_3+C_4+N_2+N_3+N_5+N_7)/2,$$
$$(C_3+C_4+C_7+C_8+N_2+N_3+N_5+N_7)/2,$$

$$(C_1 + C_2 + C_5 + C_7 + C_8 + C_9 + N_2 + N_3)/2,$$
$$(C_1 + C_2 + C_3 + C_4 + C_7 + C_8 + N_1 + N_2)/2,$$
$$(C_1 + C_2 + C_5 + C_9 + N_1 + N_2 + N_5 + N_7)/2,$$
$$(C_5 + C_7 + C_8 + C_9 + N_1 + N_2 + N_5 + N_7)/2,$$
$$(C_1 + C_2 + C_3 + C_4 + C_5 + C_7 + C_8 + C_9 + N_5 + N_7)/2,$$
$$(C_1 + C_2 + C_3 + C_4 + C_5 + C_7 + C_8 + C_9 + N_1 + N_3)/2,$$
$$(C_1 + C_2 + C_3 + C_4 + C_5 + C_7 + C_8 + C_9 + N_2 + N_3)/2,$$
$$(C_1 + C_2 + C_3 + C_4 + C_5 + C_9 + N_1 + N_2 + N_5 + N_7)/2,$$
$$(C_3 + C_4 + C_5 + C_7 + C_8 + C_9 + N_1 + N_2 + N_5 + N_7)/2.$$

It is easy to show that all of the above elements are not vectors of the lattice $\mathrm{NS}(X')$. In order to show this, let v be one of these vectors, and assume, by contradiction, that $v \in \mathrm{NS}(X')$. Then, we can

- Add or subtract elements of M to v;
- Use the relations:

$$C_1 + C_2 + C_3 + C_4 = C_7 + C_8 + C_9 + C_{10},$$
$$C_1 + C_2 + C_{11} + C_{12} = C_5 + C_6 + C_7 + C_8,$$
$$C_3 + C_5 = C_4 + C_6 = C_9 + C_{11} = C_{10} + C_{12}.$$

Using these operations on v, we can produce a vector $v' \in \mathrm{NS}(X')$ equal to half the sum of four disjoint smooth rational curves (see Example 6.3 following this proof), which is impossible by Theorem 2.11 (observe that, in some cases, v is already half the sum of four disjoint smooth rational curves). This implies that $M = \mathrm{NS}(X')$.

If we choose $\{v_1,\ v_2,\ v_4 + 2w_2,\ v_3 + v_4,\ w_1,\ w_2\}$ as a basis for the discriminant group of $\mathrm{NS}(X')$, we obtain the discriminant quadratic form claimed in the statement of Theorem 6.2. From Theorem 2.4 and [**15,** Chapter 8, Corollary 7.8(3)], it follows that $T_{X'}$ is isometric to the lattice $U(2)^{\oplus 2} \oplus \langle -4 \rangle^{\oplus 2}$. □

Example 6.3. Say $v = (C_1 + C_2 + C_7 + C_8 + N_2 + N_3 + N_5 + N_7)/2$. Each time we use one of the above operations, we write "≈." Then

$$\frac{C_1 + C_2 + C_7 + C_8 + N_2 + N_3 + N_5 + N_7}{2}$$
$$\approx \frac{C_3 + C_4 + C_9 + C_{10} + N_2 + N_3 + N_5 + N_7}{2}$$

$$\approx \frac{C_5+C_6+C_9+C_{10}+N_2+N_3+N_5+N_7}{2} \approx \frac{N_1+N_4+N_5+N_7}{2}.$$

6.5. Additional properties of X'. The following proposition is the analogue of Proposition 4.12 for the K3 surfaces X'.

Proposition 6.4. *Let X' be a fiber of the family of* K3 *surfaces $\mathfrak{X}' \to \mathcal{U}$, defined in Section* 6.1 *with $\rho(X') = 16$. Then, the following hold*:

(i) *X' admits $\mathbb{Z}_2^4$ symplectic action*;
(ii) *X' is not isomorphic to the minimal resolution of the quotient of a* K3 *surface by a symplectic action of the group $\mathbb{Z}_2^4$*;
(iii) *X' is isomorphic to the minimal resolution of the quotient of a* K3 *surface by a symplectic action of the group $\mathbb{Z}_2^3$.*

Proof. We know from **[19]** that X' admits $\mathbb{Z}_2^4$ symplectic action if and only if the lattice $T_{X'}$ primitively embeds into $\Omega^{\perp}_{\mathbb{Z}_2^4}$. However, $T_{X'} \cong U(2)^{\oplus 2} \oplus \langle -4\rangle^{\oplus 2}$ by Theorem 6.2, and $\Omega^{\perp}_{\mathbb{Z}_2^4} \cong U(2)^{\oplus 3} \oplus \langle -8\rangle$ by **[8,** Proposition 5.1]. Therefore, it is sufficient to show that $\langle -4\rangle^{\oplus 2}$ primitively embeds into $U(2) \oplus \langle -8\rangle$. The vectors $(1,1,1)$, $(-1,1,0)$ in $U(2)\oplus\langle -8\rangle$ generate a sublattice L isometric to $\langle -4\rangle^{\oplus 2}$. In order to show that L is primitive, assume that $m(x,y,z) = (a-b, a+b, a) \in L$ for some integers a,b,m with $m>1$ and $(x,y,z) \in U(2)\oplus\langle -8\rangle$. Then, $x = 2z-y$, which implies $(x,y,z) = (z)(1,1,1)+(y-z)(-1,1,0) \in L$, proving (i). Part (ii) follows from the fact that $T_{X'}$ is not isometric to any of the transcendental lattices listed in **[9,** Theorem 8.3]. For part (iii), we know, from the discussion in Section 5, that X' is the minimal resolution of the quotient of the Hirzebruch-Kummer covering Y, see Example 5.2, by a symplectic action of $\mathbb{Z}_2^3$. □

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