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DOMINATING SETS IN INTERSECTION GRAPHS OF FINITE GROUPS

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ABSTRACT. Let G be a group. The intersection graph $\Gamma(G)$ of G is an undirected graph without loops and multiple edges, defined as follows: the vertex set is the set of all proper non-trivial subgroups of G, and there is an edge between two distinct vertices H and K if and only if $H \cap K \neq 1$, where 1 denotes the trivial subgroup of G. In this paper, we study the dominating sets in intersection graphs of finite groups. We classify abelian groups by their domination number and find upper bounds for some specific classes of groups. Subgroup intersection is related to Burnside rings. We introduce the notion of an intersection graph of a G-set (somewhat generalizing the ordinary definition of an intersection graph of a group) and establish a general upper bound for the domination number of $\Gamma(G)$ in terms of subgroups satisfying a certain property in the Burnside ring. The intersection graph of G is the 1-skeleton of the simplicial complex. We name this simplicial complex intersection complex of G and show that it shares the same homotopy type with the order complex of proper non-trivial subgroups of G. We also prove that, if the domination number of $\Gamma(G)$ is 1, then the intersection complex of G is contractible.

1. Introduction. Let $\mathcal{F}$ be the set of proper subobjects of an object with an algebraic structure. In **[34]**, the *intersection graph* of $\mathcal{F}$ is defined in the following way: there is a vertex for each subobject in $\mathcal{F}$ other than the zero object, where the zero object is the object having a unique endomorphism, and there is an edge between two vertices whenever the intersection of the subobjects representing the vertices is not the zero object. In particular, if $\mathcal{F}$ is the set of proper subgroups of

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a group G, then the zero object is the trivial subgroup. The intersection graph of (the proper subgroups of) G will be denoted by $\Gamma(G)$.

Intersection graphs were first defined for semigroups by Bosák [**3**]. Let S be a semigroup. The intersection graph of the semigroup S is defined in the following way: the vertex set is the set of proper subsemigroups of S, and there is an edge between two distinct vertices A and B if and only if $A \cap B \neq \varnothing$. Interestingly, this definition is not in the scope of the abstract generalization given in the preceding paragraph. Later on, in [**9**], Csákány and Pollák adapted this definition for groups in the standard manner. However, there are analogous definitions to be used. For example, in [**7**], the authors studied the intersection graphs of ideals of a ring. In particular, they determined the values of n for which the intersection graph of the ideals of $\mathbb{Z}_n$ is connected, complete, bipartite, planar or has a cycle. For the corresponding literature, the reader may also refer to [**18, 19, 23, 30, 36**], and some of the references therein.

As is well known, subgroups of a group G form a lattice $L(G)$ ordered by set inclusion. Some of the structural properties of a group may be inferred by studying its subgroup lattice. Intersection graphs of groups are natural objects and are intimately related with subgroup lattices. In actuality, given the subgroup lattice, we can recover the intersection graph, but not vice versa, in general. Intuitively, by passing from $L(G)$ to $\Gamma(G)$, a certain amount of knowledge would be lost. In [**21**], the authors showed that finite abelian groups can sometimes be distinguished by their intersection graphs. The same result was previously proven for subgroup lattices in [**2**] (see, also, [**27,** Corollary 1.2.8]). Therefore, rather surprisingly, subgroup lattices and intersection graphs hold the same amount of information on the subgroup structure if the group is abelian.

By defining intersection graphs, we attach a graph to a group, as in the case of Cayley graphs. Thus, there are two natural directions we may follow. First, we may study the graph theoretical properties of intersection graphs by means of group theoretical arguments. This is straightforward. For example, we may ask for which groups their intersection graphs are planar [**1, 22**] or connected [**20, 24, 30**]. Second, we may study the algebraic properties of groups by means of combinatorial arguments applied to the intersection graphs, although this path seems to require more ingenuity.

In this paper, we study the dominating sets in intersection graphs. A *dominating set* D of a graph Γ is a subset of the vertex set V such that any vertex not in D is adjacent to some vertex in D. The *domination number* $\gamma(\Gamma)$ of Γ is the smallest cardinal of the dominating sets for Γ. Vizing's conjecture from 1968 asserts that, for any two graphs Γ and Γ', the product $\gamma(\Gamma)\gamma(\Gamma')$ is, at most, the domination number of the Cartesian product of Γ and Γ'. Despite the efforts of many mathematicians, this conjecture is still open, see [**4**]. Given a graph Γ and an integer n, the dominating set problem asks whether there is a dominating vertex set of size at most n. It is a classical instance of an NP-complete decision problem. There are many papers on domination theory covering algorithmic aspects, as well. More may be found on this subject, for example, in [**5, 11, 16**], and the references therein.

It is easy to observe that a subset $\mathcal{D}$ of the vertex set $V(G)$ of the intersection graph of the group G is a dominating set if and only if, for any minimal subgroup A of G, there exists an $H \in \mathcal{D}$ such that $A \leq H$. This, in turn, implies that $\mathcal{D}$ be a dominating set if and only if the union of the subgroups in $\mathcal{D}$ were to contain all of the minimal subgroups of G as a subset, or equivalently, if and only if the union of the subgroups in $\mathcal{D}$ were to contain all of the elements of G of prime order. In particular, the set of minimal subgroups and the set of maximal subgroups are dominating sets. We denote the domination number of $\Gamma(G)$ simply by $\gamma(G)$ and call this invariant of the group the *domination number* of G. Let G be a finite group. We shall note that a dominating set $\mathcal{D}$ of minimal size may be assumed to consist of maximal subgroups since any proper subgroup of a finite group is contained in a maximal subgroup. In particular, there exists a dominating set $\mathcal{D}$ such that each element of $\mathcal{D}$ is a maximal subgroup, and the cardinality of $\mathcal{D}$ is $\gamma(G)$.

In [**8**], Cohn defined a group as an n-sum group if it can be written as the union of n of its proper subgroups and is of no smaller number. Let G be an n-sum group. In light of the previous paragraphs, it is clear that $\gamma(G) \leq n$. Note that any non-trivial finite group can be written as the union of its proper subgroups unless it is cyclic; hence, it is reasonable to call the cyclic group C_m of order m an $\aleph_0$-*sum group*. On the other hand, a cyclic group of order p^s with p a prime contains a unique minimal subgroup; therefore, $\gamma(C_{p^s}) = 1$, provided $s > 1$. The intersection graph $\Gamma(G)$ is the empty graph whenever G is trivial or isomorphic to a cyclic group of prime order and, in such a case, we adopt the convention $\gamma(G) = \aleph_0$. The reason for this will be justified when we prove Lemma 2.5.

Let G be an n-sum group. It can easily be observed that $\gamma(G) = n$ if G is isomorphic to one of the following groups:

$$C_p, \qquad C_p \times C_p, \qquad C_p \times C_q, \qquad C_p \rtimes C_q,$$

where p and q are some distinct prime numbers. The reader may refer, for example, to **[8, 10, 13]** for literature on n-sum groups.

Classifying groups by their domination number is a difficult problem. Even the determination of groups with domination number 1 seems to be intractable. In Sections 3, 4, and 5, we determine upper bounds for the domination number of particular classes of groups. For example, abelian groups can be classified by their domination number (see Theorem 3.1), and the domination number of a supersolvable group is at most $p+1$ for some prime divisor p of its order (see Proposition 4.5). It turns out that symmetric groups form an interesting class in our context. In Section 5, we find some upper bounds for the symmetric groups by their degree (see Theorem 5.2) and show that those bounds are also applicable for the primitive subgroups containing an odd permutation (see Corollary 5.3).

In Section 6, we introduce *intersection graphs* of G-sets. This notion, in a sense, generalizes the ordinary definition of intersection graphs of groups (see Proposition 6.2). Subgroup intersection is related to the multiplication operator of Burnside rings, and the ultimate aim in this section is to incorporate the Burnside ring context into our discussion. We show that the domination number $\gamma(G)$ can be bounded by the sum of the indices of normalizers of some subgroups in G satisfying a certain property, such as a collection in the Burnside ring (see Proposition 6.3).

There is extensive literature on combinatorial objects associated with algebraic structures. An alternative path in this direction is to introduce order complexes of subgroups and thereby render the use of topological terms possible (see, for example, **[6, 15, 25, 31]**). Similar work using subgroup lattices, frames, coset posets and quandles has also appeared in the literature (see, **[12, 17, 28, 29]**).

A natural construction in which a simplicial complex $K(G)$ is associated with a group G is the following: the underlying set of $K(G)$ is the vertex set of $\Gamma(G)$ and, for each vertex H in $\Gamma(G)$, there is an associated simplex σ_H in $K(G)$, which is defined as the set of proper subgroups of G containing H. Clearly, the common face of σ_H and σ_K is $\sigma_{\langle H,K\rangle}$. Alternatively, $K(G)$ is the simplicial complex whose faces are the sets of proper subgroups of G which intersect non-trivially. Observe that $\Gamma(G)$ is the 1-skeleton of $K(G)$. By an argument due to Welker [**33**], $K(G)$ shares the same homotopy type with the order complex of proper non-trivial subgroups of G (see Proposition 7.3). In Section 7, we study the *intersection complex* $K(G)$ and prove that, if the domination number of $\Gamma(G)$ is 1, then the intersection complex of G is contractible (see Corollary 7.7).

2. Preliminaries.

First, we recall some of the basic facts from standard group theory.

Remark 2.1.

(a) (Product formula, see [**26,** Theorem 2.20]). $|XY||X \cap Y| = |X||Y|$ for any two subgroups X and Y of a finite group.

(b) (Sylow's theorem, see [**26,** Theorem 4.12]).

- (i) If P is a Sylow p-subgroup of a finite group G, then all Sylow p-subgroups of G are conjugate to P.
- (ii) If there are r Sylow p-subgroups, then r is a divisor of $|G|$ and $r \equiv 1 \pmod{p}$.

(c) (Hall's theorem, see [**14,** Theorem 4.1]). If G is a finite solvable group, then any π-subgroup is contained in a Hall π-subgroup. Moreover, any two Hall π-subgroups are conjugate.

(d) (Correspondence theorem, see [**26,** Theorem 2.28]). Let $N \trianglelefteq G$, and let $\nu\colon G \to G/N$ be the canonical morphism. Then, $S \mapsto \nu(S) = S/N$ is a bijection from the family of all of those subgroups S of G which contain N to the family of all of the subgroups of G/N.

(e) (Dedekind's lemma, see [**26,** Exercise 2.49]). Let H, K and L be subgroups of G with $H \leq L$. Then, $HK \cap L = H(K \cap L)$.

Let G be a finite group. We denote by N_G the subgroup of G generated by its minimal subgroups. Obviously, N_G is a characteristic subgroup. If $G \cong C_p$ with p a prime, then we take $N_G = G$. Adapting the module theoretical parlance, we might call a subgroup of a group *essential*, provided that it contains all minimal subgroups. Thus, N_G is the smallest essential subgroup. Note that, if G is abelian and $G \ncong C_p$, then N_G is the *socle* of G.

Lemma 2.2. *For a finite group G, the following statements are equivalent*:

(i) *The domination number of* $\Gamma(G)$ *is* 1.
(ii) N_G *is a proper* (*normal*) *subgroup of* G.
(iii) G *is a* (*non-split*) *extension of* N_G *by a non-trivial group* H.

Proof.

(i) $\Rightarrow$ (ii). There must be a proper subgroup H of G such that $H \cap K \neq 1$ for any non-trivial subgroup K of G. In particular, H non-trivially intersects, and hence, contains, any minimal subgroup in N_G, that is, $H \geq N_G$. However, H is a proper subgroup of G; thus, so is N_G.

(ii) $\Rightarrow$ (iii). Since N_G is a proper normal subgroup, G is an extension of N_G by a non-trivial group U. Note that this extension cannot split since, otherwise, there would be a subgroup of G isomorphic to H which intersects N_G trivially. However, this contradicts the definition of N_G.

(iii) $\Rightarrow$ (i). Clearly, N_G is a proper subgroup non-trivially intersecting any subgroup; hence, $\{N_G\}$ is a dominating set for $\Gamma(G)$. $\square$

Corollary 2.3. *If G is a finite simple group, then* $\gamma(G) > 1$.

In general, there is *no* relation between the domination number of a group and its subgroups. As a simple example, consider the dihedral group

$$D_8 = \langle a, b \mid a^4 = b^2 = 1,\ bab = a^3\rangle$$

of order 8. It has three maximal subgroups $\langle a^2, b\rangle$, $\langle a^2, ab\rangle$, $\langle a\rangle$, and the combination of the first two dominate $\Gamma(D_8)$. Moreover, as $D_8 = \langle ab, b\rangle$, we have $\gamma(D_8) = 2$ by Lemma 2.2. On the other hand, $\langle a^2, b\rangle \cong C_2 \times C_2$ and, since its intersection graph consists of isolated vertices, we have $\gamma(\langle a^2, b\rangle) = 3$; whereas, $\gamma(\langle a\rangle) = 1$ since it is isomorphic to the cyclic group of order four. However, by imposing some conditions on the subgroup H of G, it is easy to prove that $\gamma(H) \leq \gamma(G)$ holds.

Lemma 2.4. *Let H be a subgroup of G, and let $\mathcal{D}$ be a dominating set of $\Gamma(G)$. Then, $\gamma(H) \leq |\mathcal{D}|$, provided that none of the elements of $\mathcal{D}$ contains H. In particular, $\gamma(H) \leq \gamma(G)$, if there is such a dominating set with cardinality $\gamma(G)$.*

Proof. Observe that, if none of the elements of $\mathcal{D}$ contains H, then the set $\mathcal{D}_H := \{D \cap H : D \in \mathcal{D}\}$ is a dominating set for $\Gamma(H)$. $\square$

The next result will be very useful in our later arguments.

Lemma 2.5. *Let N be a normal subgroup of G. If $G/N \ncong C_p$, then $\gamma(G) \leq \gamma(G/N)$. Moreover, the condition $G/N \ncong C_p$ can be removed if G is a finite group.*

Proof. Let $\mathcal{D}$ be a dominating set for $\Gamma(G/N)$, and set $\overline{\mathcal{D}} := \{N < \overline{W} < G : \overline{W}/N \in \mathcal{D}\}$. By the Correspondence theorem, $|\overline{\mathcal{D}}| = |\mathcal{D}|$. We want to show that $\overline{\mathcal{D}}$ is a dominating set for $\Gamma(G)$. Let H be a proper non-trivial subgroup of G. If $H \cap N \neq 1$, then any element of $\overline{\mathcal{D}}$ non-trivially intersects H. Suppose that $H \cap N = 1$. Take $W \in \mathcal{D}$ such that $(NH/N) \cap W := Y \neq 1$. If $Y = NH/N$, then clearly, $\overline{W}$ contains H where $\overline{W} \in \overline{\mathcal{D}}$ is such that $\overline{W}/N = W$. Suppose that $Y \neq NH/N$, and let $\overline{Y}/N = Y$. Obviously, $\overline{Y} < \overline{W}$. Moreover, from Dedekind's lemma, $\overline{Y} = NK$, where $K := \overline{Y} \cap H$, that is, $\overline{W} \cap H \geq K \neq 1$. Since H is an arbitrary subgroup, $\overline{\mathcal{D}}$ dominates $\Gamma(G)$. This proves the first part. The second part follows from the convention $\gamma(C_p) = \aleph_0$. $\square$

Corollary 2.6. *Let $G \cong H \times K$ be a finite group. Then, $\gamma(G) \leq \min\{\gamma(H), \gamma(K)\}$.*

Let $\mathcal{S}$ be a subset of the vertex set $V(G)$ of $\Gamma(G)$. It is natural to define the intersection graph $\Gamma(\mathcal{S})$ of the vertex set $\mathcal{S}$ in the following manner. There is an edge between two vertices $H, K \in \mathcal{S}$ if and only if $H \cap K \in \mathcal{S}$. We denote the domination number of $\Gamma(\mathcal{S})$ simply by $\gamma(\mathcal{S})$. Let $V(G)_{>N}$ be the set of proper subgroups of G strictly containing the normal subgroup N of G. From the Correspondence theorem, $\Gamma(V(G)_{>N}) \cong \Gamma(G/N)$; hence, $\gamma(V(G)_{>N}) = \gamma(G/N)$. Moreover, from the proof of Lemma 2.5, a (non-empty) subset of $V(G)_{>N}$ dominates $\Gamma(G)$, provided that it dominates $\Gamma(V(G)_{>N})$. Let

$$\varphi_N \colon V(G) \cup \{1, G\} \longrightarrow V(G)_{>N} \cup \{N, G\}$$

be the map taking H to NH, and let $\mathcal{D}$ be a dominating set of $\Gamma(G)$ consisting of maximal subgroups. Note that, even if $\varphi_N(\mathcal{D}) \subseteq V(G)_{>N}$, the image set $\varphi_N(\mathcal{D})$ may not be a dominating set for $\Gamma(V(G)_{>N})$.

Let $\mathcal{S}_p(G)$ be the set of all proper non-trivial p-subgroups of G. Observe that, if G is a p-group, $\Gamma(G)$ and $\Gamma(\mathcal{S}_p(G))$ coincide. It is also true, in general, that there is an edge between two vertices of $\Gamma(\mathcal{S}_p(G))$ if and only if there is an edge between the corresponding vertices in $\Gamma(G)$.

Lemma 2.7. *Let G be a finite group, and let N be a normal p-subgroup of G such that the set $\mathcal{S}_p(G)_{>N}$ is non-empty. Then*:

$$\gamma(\mathcal{S}_p(G)) \leq \gamma(\mathcal{S}_p(G)_{>N}).$$

Proof. Let $\mathcal{D}$ be a dominating set of $\mathcal{S}_p(G)_{>N}$. Similarly to the proof of Lemma 2.5, we want to show that $\mathcal{D}$ dominates $\Gamma(\mathcal{S}_p(G))$. Let $H \in \mathcal{S}_p(G)$. If $H \cap N \neq 1$, then clearly, $H \cap W \neq 1$ for any $W \in \mathcal{D}$. Suppose that $H \cap N = 1$. Then, $NH \in \mathcal{S}_p(G)_{>N} \cup \{G\}$. Let Y be a minimal subgroup in $\mathcal{S}_p(G)_{>N}$ contained by NH, and let $W \in \mathcal{D}$ contain Y. If $Y = NH$, then $H < W$. Suppose that $Y \neq NH$. Since $N < Y < NH$ and $N \cap H = 1$, $Y \cap H \in \mathcal{S}_p(G)$ by the Product formula; hence, $W \cap H \in \mathcal{S}_p(G)$, as well. $\square$

3. Abelian groups. In this section, we classify finite abelian groups by their domination number. Recall that the exponent of a group G, denoted $\exp(G)$, is the least common multiple of the orders of elements of G. Let G be a finite group, and consider the function f from the set of non-empty subsets of G to the set of positive integers taking $X \subseteq G$ to the lowest common multiple of the orders of elements of X. Clearly, the image of the whole group G is $\exp(G)$. By a renowned theorem of Frobenius, if X is a maximal subset of G satisfying the condition

$x^k = 1$ for all $x \in X$ with k a fixed integer dividing $|G|$, then k divides $|X|$. Let g be the function taking the integer k to the maximal subset $X_k := \{x \in G \colon x^k = 1\}$. Then, f and g define a Galois connection between the poset of non-empty subsets of G and the poset of positive integers ordered by the divisibility relation. In general, such a maximal subset may not be a subgroup. For example, if the Sylow p-subgroup P of G is not a normal subgroup of G, then the union of conjugates of P cannot be a subgroup. However, if G is an abelian group, then, for any integer k, the subset X_k is actually a subgroup.

For a finite group G, we denote by $\mathrm{sfp}(G)$ the square-free part of $|G|$, i.e., $\mathrm{sfp}(G)$ is the product of distinct primes dividing $|G|$. Note that a collection of proper subgroups dominates the intersection graph if and only if their union contains X_t, where $t = \mathrm{sfp}(G)$.

Theorem 3.1. *Let G be a finite abelian group with proper non-trivial subgroups. Then*:

(i) $\gamma(G) = 1$ *if and only if* $\mathrm{sfp}(G) < \exp(G)$;
(ii) $\gamma(G) = 2$ *if and only if* $\mathrm{sfp}(G) = \exp(G)$ *and* $\mathrm{sfp}(G)$ *is not a prime number*;
(iii) $\gamma(G) = p + 1$ *if and only if* $p = \mathrm{sfp}(G) = \exp(G)$ *is a prime number.*

Proof. Let t be the square-free part of $|G|$ and m the exponent of G.

Assertion (i). Observe that $N_G = \{x \in G \colon x^t = 1\}$. By Lemma 2.2, $\gamma(G) = 1$ if and only if N_G is a proper subgroup, which is the case if and only if $t < m$.

For Assertions (ii) and (iii), it is enough to prove the sufficiency conditions since $2 \neq t + 1$ for any prime number t.

Assertion (ii). Suppose that $t = m$, and t is not a prime number. Then, there exist two distinct primes p and q dividing t. Clearly, the subgroups

$$H = \{h \in G \colon h^{t/p} = 1\}$$

and

$$K = \{k \in G \colon k^{t/q} = 1\}$$

dominate $\Gamma(G)$. Since there is no dominating set of cardinality one (by virtue of Assertion (i)), $\gamma(G) = 2$.

Assertion (iii). Suppose that $t = m$, and t is a prime number. We consider G to be a vector space over the field $\mathbb{F}_t$ of t elements of dimension $d \geq 2$ and fix a basis for this vector space in a canonical manner. Let $H_i = \langle h_i \rangle$, $1 \leq i \leq t+1$, where $h_i = (1, 0, \dots, 0, i)$ for $i \leq t$ and $h_{t+1} = (0, \dots, 0, 1)$. Then,

$$\{K_i := H_i^{\perp} \colon 1 \leq i \leq t+1\}$$

is a dominating set for $\Gamma(G)$. In order to see this, first observe that, for any $g \in G$, there exists an h_i such that $g \cdot h_i = 0$, i.e., $g \in K_i$ for some $1 \leq i \leq t+1$. Next, suppose that there exists a dominating set $\mathcal{D} = \{M_1, \dots, M_s\}$ with cardinality $s < t+1$. We want to derive a contradiction. Without loss of generality, elements of $\mathcal{D}$ can be taken as maximal subgroups. Let $A_j = M_j^{\perp}$, $1 \leq j \leq s$. Suppose that the A_j are generated by linearly independent vectors (if not, we may take a maximal subset of $\{A_1, \dots, A_s\}$ with this property and apply the same arguments). Using a change of basis, if necessary, we may take $A_j = H_j$. However, that means $g = (1, 0, \dots, 0)$ is not contained in any $M_i \in \mathcal{D}$ as $g \cdot h_j \neq 0$, for $1 \leq j \leq s$. This contradiction completes the proof. $\square$

Remark 3.2. We may prove the first and second assertions by regarding G as a $\mathbb{Z}$-module (compare with [**34,** Theorem 4.4]). In addition, for the third assertion, we may argue as follows. By Lemma 2.5,

$$\gamma(G) \leq t+1 \quad \text{as } \gamma(C_p \times C_p) = t+1,$$

and the rank of G, say r, is greater than or equal to two. On the other hand, any non-identity element of G belongs to exactly one minimal subgroup, and there are $t^r - 1$ of them. Any maximal subgroup of G contains $t^{r-1} - 1$ non-identity elements, and t maximal subgroups may cover at most $t^r - t$ elements; hence, there is no dominating set for G of size $< t+1$.

From the fundamental theorem of finite abelian groups, any finite abelian group can be written as the direct product of cyclic groups of prime power orders; thus, we may restate Theorem 3.1 as:

$$\gamma(C_{p_1^{\alpha_1}} \times \cdots \times C_{p_1^{\alpha_k}}) = \begin{cases} 1 & \text{if } \alpha_i \geq 2 \text{ for some } 1 \leq i \leq k; \\ 2 & \text{if } \alpha_i = 1 \text{ for all } 1 \leq i \leq k \text{ and} \\ & \quad p_{j_1} \neq p_{j_2} \text{ for some } j_1 \neq j_2; \\ p+1 & \text{if } \alpha_i = 1 \text{ and } p_i = p \text{ for all } 1 \leq i \leq k; \end{cases}$$

where $p_1, \ldots, p_k$ are prime numbers and $\alpha_1, \ldots, \alpha_k$ are positive integers with $k = 1$, implying $\alpha_1 > 1$.

4. Solvable groups. Although finite abelian groups can be classified by their domination number, in general this is not possible. Nevertheless, we may use the structural results to find upper bounds for the domination number of groups belonging to larger families.

Proposition 4.1. *Let G be a finite nilpotent group and p a prime number. Suppose that $G \not\cong C_p$. Then*:

(i) $\gamma(G) \leq p+1$ *if G is a p-group.*
(ii) $\gamma(G) \leq 2$ *if G is not a p-group.*

Proof.

Assertion (i). There is a normal subgroup N of G of index p^2 such that the quotient group G/N is isomorphic to either C_{p^2} or $C_p \times C_p$. The assertion follows from Lemma 2.5.

Assertion (ii). Let G be the internal direct product of P, Q and N, where P and Q are the non-trivial Sylow p- and q- subgroups of G. Clearly, NP and NQ form a dominating set of $\Gamma(G)$. $\square$

Let G be a finite group. We denote by R_G the intersection of the subgroups in the lower central series of G. This subgroup is the smallest subgroup of G in which the quotient group G/R_G is nilpotent. Obviously, the nilpotent residual R_G is a proper subgroup whenever G is a solvable group.

Corollary 4.2. *Let G be a finite group such that G/R_G has proper non-trivial subgroups. Then*:

(i) $\gamma(G) \leq p+1$ *if G/R_G is a p-group.*
(ii) $\gamma(G) \leq 2$ *if G/R_G is not a p-group.*

Let D_{2n} denote the dihedral group of order $2n$. Then, $R_{D_{2n}} = D'_{2n} \cong C_n$, and so, the quotient $D_{2n}/R_{D_{2n}}$ has no proper non-trivial subgroups; thus, Corollary 4.2 does not apply in this case. Nevertheless, the structure of dihedral groups is fairly specific, allowing us to determine the exact formulas for their domination number, depending upon the order $2n$.

Lemma 4.3. *Let p be the smallest prime dividing n. Then:*

$$\gamma(D_{2n}) = \begin{cases} p & \text{if } p^2 \mid n, \\ p+1 & \text{otherwise.} \end{cases}$$

Proof. Let $D_{2n} := \langle a, b \mid a^n = b^2 = 1, bab = a^{-1}\rangle$. Elements of D_{2n} can be listed as

$$\{1,\ a,\ a^2, \dots, a^{n-1},\ b,\ ab,\ a^2b, \dots, a^{n-1}b\}.$$

It can be easily observed that the latter half of these elements are all of order two, and hence, the minimal subgroups of D_{2n} consist of subgroups of $\langle a\rangle$ that are of prime order and subgroups $\langle a^jb\rangle$, where $j \in \{0, 1, \dots, n-1\}$. In addition, the maximal subgroups of D_{2n} consist of $\langle a\rangle$ and subgroups of the form $\langle a^t, a^rb\rangle$, where $t \mid n$ is a prime number. Fix a prime $t \mid n$. Observe that any element of the form a^jb is contained in exactly one of the maximal subgroups $T_r := \langle a^t, a^rb\rangle$ with $r \in \{0, 1, \dots, t-1\}$. Let p be the smallest prime dividing n. Obviously, the union of the subgroups $P_r := \langle a^p, a^rb\rangle$ with $r \in \{0, 1, \dots, p-1\}$ contains all minimal subgroups of the form $\langle a^jb\rangle$, $j \in \{0, 1, \dots, n-1\}$, and there is no possible way to cover them with fewer than p subgroups. Finally, if $p^2 \mid n$, the subgroups contain all minimal subgroups of $\langle a\rangle$; and, if $p^2 \nmid n$, we must take P_r, $r \in \{0, 1, \dots, p-1\}$, together with a subgroup containing $\langle a^{n/p}\rangle$, to form a dominating set with least cardinality. $\square$

Remark 4.4. It is difficult to find an n-sum group G such that $\gamma(G) = n$ and $\mathrm{sfp}(G) < \exp(G)$. One example satisfying those conditions is the dihedral group of order 36. By Lemma 4.3, $\gamma(D_{36}) = 3$. Moreover, any cyclic subgroup of D_{36} is contained by those three subgroups that are of index two, and hence, D_{36} is a 3-sum group.

Consider the *normal* series of subgroups

$$G = G_0 \geq G_1 \geq \cdots \geq G_k = 1.$$

From the third isomorphism theorem and Lemma 2.5,

$$\gamma\left(\frac{G_0/G_2}{G_1/G_2}\right) = \gamma(G_0/G_1) \geq \gamma(G_0/G_2).$$

In addition, by repeated applications, we have

$$\gamma(G) \leq \gamma(G/G_{k-1}) \leq \cdots \leq \gamma(G/G_1).$$

Proposition 4.5. *Let G be a finite supersolvable group with proper non-trivial subgroups. Then, $\gamma(G) \leq p+1$ for some prime divisor p of $|G|$.*

Proof. Since G is a finite supersolvable group, it has a normal subgroup N of index m, where m is either a prime square or a product of two distinct primes. In the first case, $\gamma(G/N)$ is at most $p+1$, where $p = \sqrt{m}$, and, in the latter case, $\gamma(G/N)$ is at most 2. The proof follows from Lemma 2.5. □

At this stage, it is tempting to conjecture that the assertion of Proposition 4.5 holds more generally for solvable groups. However, we may construct a counterexample in the following way. Let $G = NH$ be a Frobenius group with a complement H such that the kernel N is a minimal normal subgroup of G. Further, suppose that $H \cong C_q$ for some prime q. Note that, since Frobenius kernels are nilpotent, G must be a solvable group. On the other hand, since N is a minimal normal subgroup, N has no characteristic subgroup and, in particular, $N \cong C_p \times \cdots \times C_p$ for some prime p. Suppose that the rank r of N is greater than 1. Now, since $N_G(H) = H$, and N is a minimal normal subgroup, each conjugate of H is a maximal subgroup. This means that there are wholly $|G : H| = p^r$ isolated vertices in $\Gamma(G)$ which, in turn, implies that $\gamma(G) = p^r + 1$. Since $p^r = 1 + kq$ for some integer k by Sylow's theorem, the domination number $\gamma(G)$ is greater than both $p+1$ and $q+1$.

Proposition 4.6. *Let G be a finite solvable group and H, K a pair of maximal subgroups such that $(|G:H|, |G:K|) = 1$. Then, $\gamma(G) \leq |G : N_G(H)| + |G : N_G(K)|$.*

Note that Hall's theorem guarantees the existence of such a pair of maximal subgroups, provided that G is not a p-group. Also, note that one of the summands in the stated inequality can always be taken as 1.

Proof. Take a minimal subgroup A. If $|A| \mid |G : H|$, then, by Hall's theorem, a conjugate of K contains A. And, if $|A| \nmid |G : H|$, then, clearly, a conjugate of H contains A. $\square$

5. Permutation groups. We denote by S_X the group of permutations of the elements of X. If $X = [n] := \{1, 2, \ldots, n\}$, we simply write S_n, and A_n will be the group of even permutations on n letters. Note that $S_n/R_{S_n} = S_n/A_n \cong C_2$.

Lemma 5.1. *$\gamma(S_n) \neq 1$ and $\gamma(A_n) \neq 1$.*

Proof. Since $S_2 \cong C_2$, we see that $\gamma(S_2) \neq 1$. For $n \geq 3$, since any transposition generates a minimal subgroup of the symmetric group, and, since adjacent transpositions generate the entire group, $\gamma(S_n) \neq 1$ by Lemma 2.2. Similarly, $\gamma(A_3) \neq 1$, and, for $n \geq 4$, since 3-cycles generate A_n, $\gamma(A_n) \neq 1$ by the same lemma. $\square$

Let G be a permutation group faithfully acting on $X = \{1, 2, \ldots, n\}$. An element $g \in G$ is called *homogeneous* if the associated permutation has a cycle of type $(p^k, 1^{n-pk})$ with p a prime. Since any minimal subgroup of G is a cyclic group of prime order, in order for $\mathcal{D} \subseteq V(G)$ to be a dominating set of $\Gamma(G)$, the union of the elements of $\mathcal{D}$ must contain every homogeneous element of G, and vice versa.

Theorem 5.2. *Let $\vartheta\colon \mathbb{N} \to \mathbb{N}$ be the function given by*

$$\vartheta(n) = \begin{cases} n+1 & \textit{if } n = 2k+1, \\ n & \textit{if } n = 2k. \end{cases}$$

Then, $\gamma(S_n) \leq \vartheta(n)$.

Proof. Let $\mathcal{D}$ be a dominating set for $\Gamma(S_n)$ of minimal size. Without loss of generality, we may assume that the elements of $\mathcal{D}$ are maximal subgroups of S_n. We know that every homogeneous element of S_n must belong to some subgroup in $\mathcal{D}$, in particular, every involution (elements of order 2) must be covered by $\mathcal{D}$. Observe that, if some collection of subgroups covers all involutions of type $(2^l, 1^{n-2l})$, with l odd, but not all homogeneous elements, then, by adding A_n to this collection, we obtain a dominating set for $\Gamma(S_n)$, for A_n which contains all cycles of prime length > 2.

For $n = 2k+1$, any involution must fix a point and, hence, must belong to some point stabilizer. Thus, together with A_n, we have a dominating set of cardinality $n+1$.

For $n = 2k$, we consider the 2-set stabilizers of S_n. Let $X_j = \{1, j\}$ and $Y_j = [n] \setminus X_j$, where $2 \leq j \leq n$. Observe that the union of the subgroups $S_{X_j} \times S_{Y_j}$ of S_n contains all involutions. Therefore, together with A_n, we have a dominating set of cardinality n. □

Corollary 5.3. *Let G be a primitive subgroup of S_n containing an odd permutation, and let ϑ be the function defined in Theorem* 5.2. *Then, the inequality $\gamma(G) \leq \vartheta(n)$ holds.*

Proof. Since G is a primitive subgroup, it is not contained by any imprimitive subgroup. The proof follows from Lemma 2.4 and the fact that the only primitive subgroup used to form dominating sets in the proof of Theorem 5.2 is A_n. □

6. Intersection graphs of G-sets. As is well known, any transitive G-set Ω is equivalent to a (left) coset space G/G_x, where G_x is the stabilizer of a point $x \in \Omega$. Thus, for example, if Ω is a regular G-set, then it can be represented with $G/1$, and, if Ω is the trivial G-set, it is represented with G/G. Since any G-set is the disjoint union of transitive G-sets, given a G-set Ω, it can be represented as the sum of coset spaces. For a subgroup H of G, we denote by (H) the conjugacy class of H in G, and by $[G/H]$ the isomorphism class of the transitive G-set G/H. It is well known that $[G/H] = [G/K]$ if and only if $H = {}^gK$ for some $g \in G$.

The *Burnside ring* $B(G)$ of G is the ring generated by the isomorphism classes of G-sets, where addition is the disjoint union and product is the Cartesian product of G-sets. Therefore, a typical element of $B(G)$ is of the form

$$\sum_j a_j[G/H_j],$$

where a_j are the integers, and H_j are the representatives of the conjugacy classes of subgroups of G. Let A and B be normal subgroups of a group G. Then, the canonical map $gA \cap B \mapsto (gA, gB)$ from $G/(A \cap B)$ to $G/A \times G/B$ is an injective group homomorphism. Now, we consider two (not necessarily normal) arbitrary subgroups: H_1 and H_2 of G. In this case, G/H_1 and G/H_2 are still G-sets, and the diagonal action of G on the Cartesian product $G/H_1 \times G/H_2$ yields the map:

$$\begin{aligned} \phi\colon G/(H_1 \cap H_2) &\longrightarrow G/H_1 \times G/H_2 \\ gH_1 \cap H_2 &\longmapsto (gH_1, gH_2), \end{aligned}$$

which is an injective G-equivariant map, that is to say, subgroup intersection is related with the multiplication operator of the Burnside ring.

Let Ω_1 and Ω_2 be two G-sets, and let $\mathcal{R}_{(\Omega_1,\Omega_2)}$ be a set of representative elements for the orbits of $\Omega_1 \times \Omega_2$. The Cartesian product $\Omega_1 \times \Omega_2$ decomposes into the disjoint union of transitive G-sets in a non-trivial fashion. More precisely,

$$\Omega_1 \times \Omega_2 \cong \bigsqcup_{(x,y)\in\mathcal{R}_{(\Omega_1,\Omega_2)}} G/(G_x \cap G_y),$$

where G_x and G_y are stabilizers of the points $x \in \Omega_1$ and $y \in \Omega_2$, respectively. Let $\mathcal{R}_{(H,K)}$ be a set of representatives for the equivalence classes of (H, K)-double cosets. Setting $\Omega_1 = G/H$, $\Omega_2 = G/K$, and using sigma notation for disjoint union, we may write

$$[G/H][G/K] = \sum_{g\in\mathcal{R}_{(H,K)}} [G/(H \cap {}^gK)]$$

as the orbits of $G/H \times G/K$ parametrized by the (H, K)-double cosets. For more information, the reader may refer to [**32,** Chapter I].

Let G be a fixed finite group. The following definition was suggested by Yaraneri [**35**]. The *intersection graph* $\Gamma[\Omega]$ of a G-set Ω is the simple graph with a vertex set containing the proper non-trivial stabilizers of points in Ω, and there is an edge between two distinct stabilizers if and only if their intersection is non-trivial. In this notation, we used "brackets" instead of "parentheses" to emphasize that the argument is a G-set. Observe that, since G is a finite group, there are at most finitely many intersection graphs $\Gamma[\Omega]$ that can be associated with G-sets. Note that both $\Gamma[G/1]$ and $\Gamma[G/G]$ are empty graphs by definition.

Example 6.1. Take $\Omega := \{1, 2, \ldots, n\}$ as the vertex set of a regular n-gon on a plane with n an odd number. Since the automorphism group of this polygon is isomorphic to the dihedral group D_{2n}, considered a stabilizer of a point of Ω corresponding to a unique involution of D_{2n}. Those involutions form a single conjugacy class, and $\Gamma[\Omega]$ consists of n isolated vertices. Let G be a finite group and H a subgroup of G. As a general fact, $\Gamma[G/H]$ consists of $|G : H|$ isolated vertices if and only if G is a Frobenius group with a complement H.

We denote by $\mathcal{C}(G)$ the set of conjugacy classes of subgroups of G. From the next proposition, intersection graphs of groups can be seen as particular cases of intersection graphs of G-sets.

Proposition 6.2. *Let G be a finite group and*

$$\Sigma := \bigsqcup_{(H)\in\mathcal{C}(G)} [G/H].$$

Then:

$$\Gamma[\Sigma] = \Gamma(G).$$

Proof. Since there is an edge between two vertices of $\Gamma[\Sigma]$ if and only if their intersection is non-trivial, it is enough to show that the vertex set of $\Gamma[\Sigma]$ is $V(G)$. However, this is obvious since the set of subgroups of G can be partitioned into the conjugacy classes of subgroups:

$$V(G) \cup \{1, G\} = \{{}^{g}H \colon (H) \in \mathcal{C}(G),\ g \in G\}.$$

Note that no vertices associated with 1 and G are in $\Gamma[\Sigma]$, by definition. $\square$

Let G be a finite group. We denote by $\mathcal{A}(G)$ the set of minimal subgroups of G, and by $\mathcal{M}(G)$ the set of maximal subgroups. The following characterizations are easy to deduce:

- $N \lhd G \Leftrightarrow [G/K][G/N] = |G\!:\!NK|[G/(N\cap K)]$ for every $K \leq G$.
- $A \in \mathcal{A}(G) \cup \{1\} \Leftrightarrow$ for any $K \leq G$ there are non-negative integers a and b such that $[G/K][G/A] = a[G/A] + b[G/1]$.
- $H \in \mathcal{M}(G) \Leftrightarrow [G/K][G/H]$ does *not* contain $[G/H]$ as a summand unless $K = G$ or $K \in (H)$.

Let A be a minimal subgroup and H a maximal subgroup of G. Observe that, in the case where G is abelian, the following equality holds:

$$[G/H][G/A] = \begin{cases} |G:H|[G/A] & \text{if } A \leq H, \\ [G/1] & \text{if } A \nleq H. \end{cases}$$

Theorem 3.1 may be restated:

(i) $\gamma(G) = 1$ if and only if
  ($*$) there exists an $H \in \mathcal{M}(G)$ such that, for every $A \in \mathcal{A}(G)$, the equality $[G/H][G/A] = |G:H|[G/A]$ holds.

(ii) $\gamma(G) = 2$ if and only if
  ($**$) for every $H \in \mathcal{M}(G)$, there exists an $A \in \mathcal{A}(G)$ such that $|G:H| \nmid |[G/A]|$.

(iii) $\gamma(G) = p + 1$ if and only if neither ($*$) nor ($**$) holds and $p = |G:H| = |A|$, where $H \in \mathcal{M}(G)$ and $A \in \mathcal{A}(G)$.

Proposition 6.3. *Let G be a finite group, and let $\mathcal{H} = \{H_i \in V(G) : 1 \leq i \leq s\}$ be a set with the property that, for any $K \in V(G)$, there exists an $H_i \in \mathcal{H}$ such that $[G/K][G/H_i] \neq k[G/1]$ for every positive integer k. Then*:

$$\gamma(G) \leq \sum_{i=1}^{s} |G:N_G(H_i)|.$$

Moreover, $\gamma(G) = 1$ if and only if there exists an $H \lhd G$ such that $[G/K][G/H] \neq k[G/1]$ for every $K \in V(G)$.

Proof. Let $\mathcal{H}$ be a set, as in the statement of the proposition. We want to show that $\mathcal{D} := \{{}^gH_i\colon H_i \in \mathcal{H},\, g \in G\}$ is a dominating set for $\Gamma(G)$. However, this is obvious since, for any $K \in V(G)$, $[G/K][G/H_i] \neq k[G/1]$ implies that $K \cap {}^gH_i \neq 1$ for some $g \in G$. This completes the first part of the proof.

Now, suppose that $\gamma(G) = 1$. Let $L < G$ be such that $L \cap K \neq 1$ for every $K \in V(G)$. Obviously,

$$[G/K][G/L] = \sum [G/(K \cap {}^{g}L)] = \sum [G/({}^{g^{-1}}K \cap L)]$$

contains no regular summand since L non-trivially intersects every conjugate of K. Take a conjugate ${}^{a}L$ of L. Since ${}^{a}L$ also forms a dominating set,

$$[G/K][G/(L \cap {}^{a}L)] = \sum [G/({}^{g^{-1}}K \cap L \cap {}^{a}L)]$$

contains no regular summand, either. Let

$$H := \bigcap {}^{g}L.$$

By repeated applications of the above argument, we see that $[G/K][G/H]$ has no regular summand. However, H is a normal subgroup, which means that $[G/K][G/H] \neq k[G/1]$.

Conversely, suppose that there exists an $H \triangleleft G$ such that

$$[G/K][G/H] \neq k[G/1]$$

for every $K \in V(G)$. Since H is normal, this means that $[G/K][G/H]$ has no regular summand which, in turn, implies that $\{H\}$ is a dominating set for $\Gamma(G)$. This completes the proof. □

Remark 6.4. Clearly, the elements of the set $\mathcal{H}$ in the statement of Proposition 6.3 can be taken as maximal subgroups. Then, Proposition 4.6 states that there exists such a 2-element set if G is a solvable group but *not* a p-group.

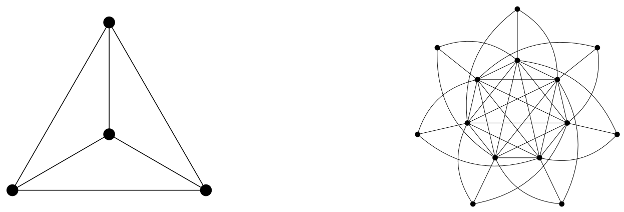
7. Intersection complexes. Recall that an (*abstract*) *simplicial complex* $\mathcal{S}$ over a set X is a finite collection of subsets of X such that the union of those subsets is X and, if σ is an element of $\mathcal{S}$, so is every subset of σ. The element σ of $\mathcal{S}$ is called a *simplex* of $\mathcal{S}$, and each subset of σ is called a *face* of σ. The *k-skeleton* of $\mathcal{S}$ is the subcollection of elements of $\mathcal{S}$ having cardinality at most $k+1$; hence, the 0-skeleton of $\mathcal{S}$ is the underlying set X plus the empty set. For a group G, we define the *intersection complex* $K(G)$ of G as the simplicial complex whose faces are the sets of proper subgroups of G which non-trivially intersect. As a graph, the 1-skeleton of $K(G)$ is isomorphic to the intersection graph $\Gamma(G)$. This notion can be compared with the other two notions in the literature, namely, the order complex and the clique complex. In the first case, we begin with a poset (poset of proper non-trivial subgroups of a group in our case) and construct its order complex by declaring chains of the poset as the simplices. And, in the latter case, we take a graph (i.e., the intersection graph of a group), and the underlying set of the corresponding clique complex is the vertex set of the graph with simplices being the cliques.

Example 7.1.

(1) The quaternion group Q_8 has three maximal subgroups, say $\langle i\rangle, \langle j\rangle$ and $\langle k\rangle$, of order four intersecting at the unique minimal subgroup $\{-1, 1\}$. Thus, $\Gamma(Q_8)$ is a complete graph K_4, depicted in Figure 1 (A). Moreover, $K(Q_8)$ is a tetrahedron since those four vertices form a simplex. However, the order complex of the poset of proper non-trivial subgroups of Q_8 is isomorphic to the star graph $K_{1,3}$ as a graph. Hence, order complexes and intersection complexes are different, in general. Note that the clique complex of $\Gamma(Q_8)$ is the same as $K(Q_8)$.

(2) The intersection graph of the elementary abelian group of order eight is represented in Figure 1 (B). Here, the vertices on the outer circle represent the minimal subgroups, and the vertices on the inner circle are the maximal subgroups. By the Product formula, any two maximal subgroups intersect at a subgroup of order 2. Therefore, the vertices in the inner circle form a complete subgraph, and those vertices form a simplex in the clique complex, whereas they do not in $K(C_2 \times C_2 \times C_2)$. Thus, intersection complexes and clique complexes are not the same, in general. Note that $\Gamma(C_2 \times C_2 \times C_2)$ is symmetrical enough to reflect the vector space structure of the group.

An important result of the subject for our purposes (see Lemma 7.2 below) uses the definitions of algebraic topology adapted to 'poset' context. By a *covering* C of a finite poset $\mathcal{P}$, we mean a finite collection $\{C_i\}_{i\in I}$ of subsets of $\mathcal{P}$ such that $\mathcal{P} = \cup_{i\in I} C_i$. The *nerve* $\mathcal{N}(C)$ of C is the simplicial complex whose underlying set is I, and the non-empty simplices are the $J \subseteq I$ such that $C_J := \cap_{i\in J} C_i \neq \varnothing$. The covering C is called *contractible* if each C_J is contractible, considered as an order complex, where J is a simplex in $\mathcal{N}(C)$. We say that C is a *downward closed* covering if each C_i, $i \in I$, is a closed subset of $\mathcal{P}$, i.e., for each



(A) $\Gamma(Q_8)$. (B) $\Gamma(C_2 \times C_2 \times C_2)$.

FIGURE 1. Intersection graphs of some groups of order 8.

C_i, if the condition: whenever

$$x \in C_i \text{ and } x' \leq_{\mathcal{P}} x, \text{ then } x' \in C_i$$

is satisfied. Of course, we may define *upward closed* coverings dually.

Lemma 7.2 ([**31,** Theorem 4.5.2])**.** *If C is an* (*upward or downward*) *closed contractible covering of a poset $\mathcal{P}$, then the order complex of $\mathcal{P}$ is homotopy equivalent to nerve $\mathcal{N}(C)$ of C.*

Let x be an element of a poset $\mathcal{P}$. We denote by $\mathcal{P}_{\leq x}$ the set of elements x' of $\mathcal{P}$ satisfying $x' \leq_{\mathcal{P}} x$. Similarly,

$$\mathcal{P}_{\geq x} := \{x' \in \mathcal{P} \colon x' \geq x\}.$$

The sets $\mathcal{P}_{\leq x}$ and $\mathcal{P}_{\geq x}$ are called *cones* in poset terminology, and it is a standard fact that they are contractible. Consider the set:

$$C := \{\mathcal{P}_{\geq x} \colon x \text{ is a minimal element in the ordering of } \mathcal{P}\}.$$

Clearly, C is a contractible, upward, closed covering of $\mathcal{P}$; hence, by Lemma 7.2 the nerve $\mathcal{N}(C)$ of C is homotopy equivalent to the order complex of $\mathcal{P}$. In Example 7.1 (1), we remarked that order complexes and intersection complexes are different in general. However, they are equivalent up to homotopy.

Proposition 7.3. *For a group G, the intersection complex $K(G)$ and the order complex of the poset of proper non-trivial subgroups of G are homotopy equivalent.*

Proof. Consider the face poset $\mathcal{F}$ of $K(G)$, i.e., the poset of simplices ordered by inclusion. For a proper non-trivial subgroup H of G, let C_H be the subset $\mathcal{F}_{\geq\{H\}}$ of $\mathcal{F}$. Then, the collection

$$C := \{C_H \colon 1 < H < G\}$$

is an upward, closed, contractible covering of $\mathcal{F}$, as the singletons $\{H\}$ are exactly the minimal elements of $\mathcal{F}$. Therefore, the order complex of $\mathcal{F}$ and the nerve of C are of the same homotopy type by Lemma 7.2. Observe that $\mathcal{N}(C)$ is exactly the intersection complex $K(G)$ of G. Since the order complex of $\mathcal{F}$ is the barycentric subdivision of $K(G)$, we see that $K(G)$ is homotopy equivalent to the order complex of the poset of proper non-trivial subgroups of G. □

Remark 7.4.

(1) Intersection complexes can be considered as a special instance of a more general construction in which, given a poset $\mathcal{P}$, we form a simplicial complex $K(\mathcal{P})$ by declaring simplices as the subsets of the poset having a well-defined meet. It is easy to see that the above proof can be adapted to work in this frame. Recall that, in Section 2, we defined $\mathcal{S}_p(G)$ as the set of proper non-trivial p-subgroups of G. Considered a poset order complex, $\mathcal{S}_p(G)$ shares the same homotopy type with $K(\mathcal{S}_p(G))$.

(2) An alternative argument to prove Proposition 7.3, due to Welker **[33]**, is as follows: consider the face poset $\mathcal{F}$ of $K(G)$. By the identification $H \mapsto \sigma_H$, the poset of proper non-trivial subgroups of G becomes a subposet (after reversing the order relation) of $\mathcal{F}$. We want to show that $\mathcal{F}$ and the poset of the proper non-trivial subgroups of G are of the same homotopy type as order complexes. Let f be the map taking a simplex σ in $\mathcal{F}$ to σ_K, where K is the intersection of all maximal subgroups containing the intersection of all of the elements in σ as subgroups. Then, f is a closure operator on $\mathcal{F}$. Let g be the map taking H to K, where K is the intersection of all maximal subgroups containing H. Then, g is a closure operator on the poset of proper non-trivial subgroups of G. Since closure operations on posets preserve the homotopy type of the order complex and, since the images of f and g are isomorphic by the identification $K \mapsto \sigma_K$, the proof is complete.

Let G be a finite group. For a proper non-trivial subgroup H of G, we denote by $V(G)_{\geq H}$ the set of proper subgroups of G containing H.

Similarly, $V(G)_{\leq H}$ is the set of non-trivial subgroups of G contained by H. Consider the following collections:

$$A := \{V(G)_{\geq H} : H \text{ is a minimal subgroup of } G\},$$
$$M := \{V(G)_{\leq H} : H \text{ is a maximal subgroup of } G\}.$$

Since A is an upward, closed, contractible covering of $V(G)$ considered as a poset under set theoretical inclusion, and M is a downward closed contractible covering, by Lemma 7.2, $\mathcal{N}(A)$, $\mathcal{N}(M)$ and the order complex of proper non-trivial subgroups of G share the same homotopy type. Recall that the Frattini subgroup $\Phi(G)$ of G is the intersection of all maximal subgroups of G.

Theorem 7.5. *Let G be a finite group. Then:*

(i) $\Phi(G) \neq 1$ *if and only if* $\mathcal{N}(M)$ *is a simplex.*

(ii) $\gamma(G) = 1$ *if and only if* $\mathcal{N}(A)$ *is a simplex.*

Proof.

Assertion (i). Clearly, $\mathcal{N}(M)$ is a simplex if and only if, for any subset $\mathcal{S}$ of $\mathcal{N}(M)$, the intersection of the subgroups in $\mathcal{S}$ is non-trivial, which is the case if and only if the intersection of all of the maximal subgroups of G is non-trivial.

Assertion (ii). Similar to the previous case, $\mathcal{N}(A)$ is a simplex if and only if N_G is a proper subgroup of G. The assertion follows from Lemma 2.2. □

Remark 7.6. Using Theorem 3.1, we may conclude that, for abelian groups, $\gamma(G) = 1$ *if and only if* $\Phi(G) \neq 1$. Let P be a p-group with p a prime number. It is also true that $\gamma(G) = 1$ implies $\Phi(G) \neq 1$; for, if P is a non-cyclic p-group and $\gamma(P) = 1$, then $P/\Phi(P)$ is elementary abelian of rank > 1, and $\gamma(P/\Phi(P)) = p + 1$ in that case. However, the converse statement is not true, even for p-groups. For example, $\Phi(D_8) \cong C_2$, but $\gamma(D_8) = 2$.

Since $\mathcal{N}(A)$ and $K(G)$ are of the same homotopy type by Lemma 7.2 and Proposition 7.3, as a consequence of Theorem 7.5, we have the following.

Corollary 7.7. *Let G be a finite group. If $\gamma(G) = 1$, then $K(G)$ is contractible.*

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