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# SOME REFINEMENTS OF CLASSICAL INEQUALITIES

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ABSTRACT. We give some new refinements and reverses of Young inequalities in both additive and multiplicative-type for two positive numbers/operators. We show our advantages by comparing with known results. A few applications are also given. Some results relevant to the Heron mean are also considered.

**1. Introduction and preliminaries.** In this paper, an operator means a bound linear operator on a Hilbert space $\mathcal{H}$. An operator $X$ is said to be positive (denoted by $X \geq 0$) if $\langle Xy, y\rangle \geq 0$ for all $y \in \mathcal{H}$, and, in addition, an operator $X$ is said to be strictly positive (denoted by $X > 0$) if $X$ is positive and invertible. For convenience, we often use the following notation:

$$A!_v B \equiv ((1-v)A^{-1}+vB^{-1})^{-1}, \qquad A\sharp_v B \equiv A^{1/2}(A^{-1/2}BA^{-1/2})^v A^{1/2},$$
$$H_v(A,B) \equiv \frac{A\sharp_v B + A\sharp_{1-v} B}{2}, \qquad A\nabla_v B \equiv (1-v)A + vB,$$

where $A, B$ are strictly positive operators and $0 \leq v \leq 1$. When $v = 1/2$, we write $A!B$, $A\sharp B$, $H(A,B)$ and $A\nabla B$ for brevity, respectively.

A fundamental inequality between positive real numbers $a, b$ is the Young inequality, which states

$$a^{1-v}b^v \leq (1-v)a + vb, \quad 0 \leq v \leq 1,$$

with equality if and only if $a = b$. If $v = 1/2$, we obtain the arithmetic-geometric mean inequality $\sqrt{ab} \leq (a+b)/2$. Recently, considerable

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attention has been dedicated to the study of Young inequalities and their operator versions [**20, 21**].

It is well known that, cf., [**12**]:

$$A!_v B \le A\sharp_v B \le A\nabla_v B, \quad 0 \le v \le 1, \tag{1.1}$$

where the second inequality in (1.1) is known as the operator arithmetic-geometric mean inequality (or the operator Young inequality).

Based on the refined scalar Young inequality, Kittaneh and Manasrah [**14**] obtained that

$$r(A+B-2A\sharp B)+A\sharp_v B \le A\nabla_v B \le R(A+B-2A\sharp B)+A\sharp_v B, \tag{1.2}$$

where $r = \min\{v, 1-v\}$ and $R = \max\{v, 1-v\}$.

Zou et al., [**24**] refined the operator Young inequality with the Kantorovich constant $K(x) \equiv (x+1)^2/4x$, $x > 0$, and proposed the following result:

$$K^r(h)A\sharp_v B \le A\nabla_v B, \tag{1.3}$$

where

$$0 < \alpha' I \le A \le \alpha I \le \beta I \le B \le \beta' I$$

or

$$0 < \alpha' I \le B \le \alpha I \le \beta I \le A \le \beta' I,$$

$h = \beta/\alpha$ and $h' = \beta'/\alpha'$. Note also that the inequality (1.3) improves Furuichi's result from [**9**], which includes the well-known Specht's ratio instead of the Kantorovich constant.

As for the reverse of the operator Young inequality, under the same conditions, Liao et al., [**15**] gave the following inequality:

$$A\nabla_v B \le K^R(h)A\sharp_v B. \tag{1.4}$$

For more related inequalities and applications, see e.g., [**8, 11, 20, 21**].

This paper gives some refinements and reverses for the operator Young inequality via the Hermite-Hadamard inequality, that is, the following theorem is one of the main results in this paper.

**Theorem A.** *Let $A, B$ be strictly positive operators such that $0 < h'I \le A^{-1/2}BA^{-1/2} \le hI \le I$ for some positive scalars $h$ and $h'$.*

*Then, for each* $0 \le v \le 1$*,*

$$m_v(h)A\sharp_v B \le A\nabla_v B \le M_v(h')A\sharp_v B, \tag{1.5}$$

*where*

$$m_v(x) \equiv 1 + \frac{2^v v(1-v)(x-1)^2}{(x+1)^{v+1}},$$

*and*

$$M_v(x) \equiv 1 + \frac{v(1-v)(x-1)^2}{2x^{v+1}}.$$

The proof of Theorem A is given in Section 2, and its advantage for previously known results is given by Proposition 3.1 in Section 3.

To state our second main result, we recall that the family of the Heron mean [**1**] for two positive numbers $a$ and $b$ is defined as

$$F_{r,v}(a,b) \equiv ra^{1-v}b^v + (1-r)\{(1-v)a + vb\}, \quad 0 \le v \le 1,\ r \in \mathbb{R}.$$

More recently, Furuichi [**10**] showed that, if $r \le 1$, then

$$\left((1-v)a^{-1} + vb^{-1}\right)^{-1} \le F_{r,v}(a,b), \quad 0 \le v \le 1. \tag{1.6}$$

**Theorem B.** *Let* $a, b \ge 0$*,* $r \in \mathbb{R}$*,* $0 \le v \le 1$*. Define*

$$g_{r,v}(a,b) \equiv v\left(\frac{b-a}{a}\right)\left\{r\left(\frac{a+b}{2a}\right)^{v-1} + (1-r)\right\} + 1,$$

$$G_{r,v}(a,b) \equiv \frac{v}{2}\left(\frac{b-a}{a}\right)\{ra^{1-v}b^{v-1} + 2 - r\} + 1.$$

(i) *If either* $a \le b$*,* $r \ge 0$ *or* $b \le a$*,* $r \le 0$*, then*

$$g_{r,v}(a,b) \le F_{r,v}(a,b) \le G_{r,v}(a,b).$$

(ii) *If either* $a \le b$*,* $r \le 0$ *or* $b \le a$*,* $r \ge 0$*, then*

$$G_{r,v}(a,b) \le F_{r,v}(a,b) \le g_{r,v}(a,b).$$

The proof of Theorem B along with its advantages is shown in Section 4 using four propositions.

**2. On refined Young inequalities and reverse inequalities.** To achieve our results, we need the well known Hermite-Hadamard inequality which asserts that, if $f : [a,b] \to \mathbb{R}$ is a convex (concave) function, then the following chain of inequalities hold:

$$f\left(\frac{a+b}{2}\right) \le (\ge) \frac{1}{b-a}\int_a^b f(x)\,dx \le (\ge) \frac{f(a)+f(b)}{2}. \tag{2.1}$$

Our first attempt, which is a direct consequence of [**18,** Theorem 1], gives an additive-type improvement and reverse for the operator Young inequality via (2.1).

To obtain inequalities for bounded self-adjoint operators in Hilbert space, we shall use the following monotonicity property for operator functions: if $X \in \mathcal{B}(\mathcal{H})$ is a self-adjoint operator with a spectrum $Sp(X)$ and $f, g$ are continuous real-valued functions on $Sp(X)$, then

$$f(t) \le g(t), \quad t \in Sp(X) \Longrightarrow f(X) \le g(X).$$

The next lemma provides a technical result which will be needed in the sequel.

**Lemma 2.1.** *Let* $0 < v \le 1$.

(i) *For each* $t > 0$, *the function* $f_v(t) = v(1 - t^{v-1})$ *is concave.*

(ii) *The function* $g_v(t) = v(1-v)(t-1)/t^{v+1}$ *is concave if* $t \le 1 + 2/v$, *and convex if* $t \ge 1 + 2/v$.

*Proof.* The function $f_v(t)$ is twice differentiable and $f_v''(t) = v(1-v)(v-2)t^{v-3}$. According to the assumptions $t > 0$, $0 \le v \le 1$, so $f_v''(t) \le 0$. The function $g_v(t)$ is also twice differentiable and $g_v''(t) = v(1-v)(v+1)((vt - v - 2)/t^{v+3})$, which implies (ii). $\square$

Using this lemma, together with (2.1), we have the following proposition.

**Proposition 2.2.** *Let* $A, B$ *be strictly positive operators such that* $A \le B$. *Then, for each* $0 \le v \le 1$,

$$\begin{aligned} &v(B-A)A^{-1}\left(\frac{A - A\natural_{v-1}B}{2}\right) + A\sharp_v B \le A\nabla_v B \\ &\le v(B-A)A^{-1}\left(A - A^{1/2}\left(\frac{I + A^{-1/2}BA^{-1/2}}{2}\right)^{v-1} A^{1/2}\right) + A\sharp_v B. \end{aligned} \tag{2.2}$$

*Proof.* In order to prove (2.2), we firstly prove the corresponding scalar inequalities. As was shown in Lemma 2.1 (i), the function $f_v(t) = v(1 - t^{v-1})$, where $t \geq 1$ and $0 \leq v \leq 1$ is concave. Moreover, it is readily verified that

$$\int_1^x f_v(t)\,dt = (1-v) + vx - x^v.$$

From inequality (2.1) for the concave function, we infer that

$$\begin{aligned} v(x-1)\left(\frac{1-x^{v-1}}{2}\right) + x^v &\leq (1-v) + vx \\ &\leq v(x-1)\left(1 - \left(\frac{1+x}{2}\right)^{v-1}\right) + x^v, \end{aligned} \tag{2.3}$$

where $x \geq 1$ and $0 \leq v \leq 1$. With $X = A^{-1/2}BA^{-1/2}$, and thus $Sp(X) \subseteq (1, +\infty)$, relation (2.3) holds for any $x \in Sp(X)$. Therefore,

$$\begin{aligned} v(X-I)\left(\frac{I-X^{v-1}}{2}\right) + X^v &\leq (1-v)I + vX \\ &\leq v(X-I)\left(I - \left(\frac{I+X}{2}\right)^{v-1}\right) + X^v. \end{aligned}$$

Finally, multiplying both sides by $A^{1/2}$, we obtain (2.2). □

By virtue of Proposition 2.2, we can improve the first inequality in (1.1).

**Remark 2.3.** It is worthwhile remarking that the left-hand side of inequality (2.2) is a refinement of the operator Young inequality in the sense of $v(x-1)(1-x^{v-1}/2) \geq 0$ for each $x \geq 1$ and $0 \leq v \leq 1$, i.e.,

$$A\sharp_v B \leq v(B-A)A^{-1}\left(\frac{A - A\natural_{v-1}B}{2}\right) + A\sharp_v B \leq A\nabla_v B. \tag{2.4}$$

Replacing $A$ and $B$ by $A^{-1}$ and $B^{-1}$, respectively, in (2.4), we obtain

$$\begin{aligned} A^{-1}\sharp_v B^{-1} &\leq v(B^{-1} - A^{-1})A\left(\frac{A^{-1} - A^{-1}\natural_{v-1}B^{-1}}{2}\right) + A^{-1}\sharp_v B^{-1} \\ &\leq A^{-1}\nabla_v B^{-1}. \end{aligned} \tag{2.5}$$

Taking the inverse in (2.5), we get

$$A!_vB \le \left\{ v(B^{-1}-A^{-1})A\left(\frac{A^{-1}-A^{-1}\natural_{v-1}B^{-1}}{2}\right)+A^{-1}\sharp_v B^{-1}\right\}^{-1} \le A\sharp_v B.$$

In order to give a proof of our first main result, we need the following, essential result.

**Proposition 2.4.** *For each $0 < x \le 1$, $0 \le v \le 1$, the functions $m_v(x)$ and $M_v(x)$ defined in Theorem* A *are decreasing. Moreover, $1 \le m_v(x) \le M_v(x)$.*

*Proof.* The function $m_v(x)$ is differentiable, and

$$m_v{}'(x) = \frac{v(v-1)2^v}{(x+1)^{v+2}}((v-1)x^2+v+3-2(v+1)x).$$

By assumption, we can easily find that $m_v{}'(x) \le 0$, for any $0 < x \le 1$, $0 \le v \le 1$. In addition, $m_v(1) = 1$, so $m_v(x) \ge 1$.

Similarly, the function $M_v(x)$ is differentiable, and

$$M_v{}'(x) = \frac{v(v-1)(x-1)((v-1)x-v-1)}{2x^{v+2}}.$$

Therefore, $M_v{}'(x) \le 0$ for any $0 < x \le 1$, $0 \le v \le 1$. We also have $M_v(1) = 1$, i.e., $M_v(x) \ge 1$.

It remains to prove $m_v(x) \le M_v(x)$. Suppose that

$$\mathfrak{M}_v(x) \equiv M_v(x) - m_v(x), \quad 0 < x \le 1,\ 0 \le v \le 1.$$

In a manner similar to what was done above, we can calculate $\mathfrak{M}_v'(x)$ in the following:

$$\mathfrak{M}_v'(x) = \frac{v(1-v)(1-x)}{2(x+1)^2x^{v+2}}\mathfrak{h}_v(x),$$

where

$$\mathfrak{h}_v(x) \equiv 2x^2\{(1-v)x+v+3\}\left(\frac{2x}{x+1}\right)^v - \{(1-v)x^3+(3-v)x^2+(v+3)x+(v+1)\}.$$

Since $0 < x \leq 1$, $2x/(x+1)^v \leq 1$. Thus, $\mathfrak{M}'_v(x)$ is bounded from the above:

$$\mathfrak{M}'_v(x) \leq \frac{v(1-v)(1-x)}{2(x+1)^2 x^{v+2}} \mathfrak{k}_v(x),$$

where

$$\mathfrak{k}_v(x) \equiv (1-v)x^3 + 3(v+1)x^2 - (v+3)x - (v+1).$$

By elementary calculations, we find that

$$\mathfrak{k}''_v(x) = 6(1-v)x + 6(v+1) \geq 0, \quad \mathfrak{k}_v(0) = -(v+1) < 0, \ \mathfrak{k}_v(1) = 0.$$

Thus, we have $\mathfrak{k}_v(x) \leq 0$, which implies $\mathfrak{M}'_v(x) \leq 0$ so that $\mathfrak{M}_v(x) \geq \mathfrak{M}_v(1) = 0$. Therefore, the proposition follows. □

We are now in a position to prove Theorem A, which is a multiplicative-type refinement and reverse for the operator Young inequality.

*Proof of Theorem* A. It is routine to verify that the function

$$f_v(t) = \frac{v(1-v)(t-1)}{t^{v+1}},$$

where $0 < t \leq 1$, $0 \leq v \leq 1$, is concave. We can also verify that

$$\int_x^1 f_v(t)\, dt = 1 - \frac{(1-v) + vx}{x^v}.$$

Hence, from inequality (2.1), we can write

$$m_v(x)x^v \leq (1-v) + vx \leq M_v(x)x^v, \tag{2.6}$$

for each $0 < x \leq 1$, $0 \leq v \leq 1$.

Now, we shall use the same procedure as in [**9,** Theorem 2]. Inequality (2.6) implies that

$$\min_{h' \leq x \leq h \leq 1} m_v(x)x^v \leq (1-v) + vx \leq \max_{h' \leq x \leq h \leq 1} M_v(x)x^v.$$

Based on this inequality, it can easily be seen that, for $X$,

$$\min_{h' \leq x \leq h \leq 1} m_v(x)X^v \leq (1-v)I + vX \leq \max_{h' \leq x \leq h \leq 1} M_v(x)X^v. \tag{2.7}$$

By substituting $A^{-1/2}BA^{-1/2}$ for $X$ and taking into account that $m_v(x)$ and $M_v(x)$ are decreasing, relation (2.7) implies

$$m_v(h)(A^{-1/2}BA^{-1/2})^v \le (1-v)I + vA^{-1/2}BA^{-1/2} \le M_v(h')(A^{-1/2}BA^{-1/2})^v. \tag{2.8}$$

Multiplying $A^{1/2}$ on both sides in inequality (2.8), we have inequality (1.5). □

**Remark 2.5.** Note that, the condition $0 < h'I \le A^{-1/2}BA^{-1/2} \le hI \le I$ in Theorem A can be replaced by $0 < \alpha' I \le B \le \alpha I \le \beta I \le A \le \beta' I$. In this case, we have

$$m_v(h)A\sharp_v B \le A\nabla_v B \le M_v(h')A\sharp_v B,$$

where $h = \alpha/\beta$ and $h' = \alpha'/\beta'$.

It is well known that, for each strictly positive operator $A$, $B$ (see, e.g., [**13,** Proposition 3.3.11]),

$$H_v(A,B) \le A\nabla B, \quad 0 \le v \le 1. \tag{2.9}$$

A counterpart to inequality (2.9) is as follows:

**Remark 2.6.** Assume the conditions of Theorem A hold. Then,

$$A\nabla B \le \sqrt{M_v(h'^2)}H_v(A,B).$$

Theorem A can be used to infer the following remark:

**Remark 2.7.** Assume the conditions of Theorem A hold. Then,

$$m_v(h)A!_v B \le A\sharp_v B \le M_v(h')A!_v B.$$

The left-hand side of inequality (1.5) can be squared by a similar method as in [**16, 17**].

**Corollary 2.8.** *Let* $0 < \alpha' I \leq B \leq \alpha I \leq \beta I \leq A \leq \beta' I$. *Then, for every normalized positive linear map* $\Phi$,

$$\Phi^2(A\nabla_v B) \leq \left(\frac{K(h')}{m_v(h)}\right)^2 \Phi^2(A\sharp_v B) \tag{2.10}$$

*and*

$$\Phi^2(A\nabla_v B) \leq \left(\frac{K(h')}{m_v(h)}\right)^2 (\Phi(A)\sharp_v \Phi(B))^2, \tag{2.11}$$

*where* $h = \alpha/\beta$ *and* $h' = \alpha'/\beta'$.

*Proof.* According to the assumptions

$$(\alpha' + \beta')I \geq \alpha'\beta' A^{-1} + A, \qquad (\alpha' + \beta')I \geq \alpha'\beta' B^{-1} + B,$$

since $(t - \alpha')(t - \beta') \leq 0$ for $\alpha' \leq t \leq \beta'$. From these, we can write

$$(\alpha' + \beta')I \geq \alpha'\beta'\Phi(A^{-1}\nabla_v B^{-1}) + \Phi(A\nabla_v B), \tag{2.12}$$

where $\Phi$ is a normalized positive linear map. We have

$$\begin{aligned}
&\|\Phi(A\nabla_v B)\alpha'\beta' m_v(h)\Phi^{-1}(A\sharp_v B)\| \\
&\quad \leq \frac{1}{4}\|\Phi(A\nabla_v B) + \alpha'\beta' m_v(h)\Phi^{-1}(A\sharp_v B)\|^2 \quad \text{(by [3])} \\
&\quad \leq \frac{1}{4}\|\Phi(A\nabla_v B) + \alpha'\beta' m_v(h)\Phi(A^{-1}\sharp_v B^{-1})\|^2 \\
&\qquad\qquad \text{(by Choi's inequality [2, page 41])} \\
&\quad \leq \frac{1}{4}\|\Phi(A\nabla_v B) + \alpha'\beta'\Phi(A^{-1}\nabla_v B^{-1})\|^2 \quad \text{(by Remark 2.5)} \\
&\quad \leq \frac{1}{4}(\alpha' + \beta')^2 \quad \text{(by (2.12))}.
\end{aligned}$$

This is the same as stating

$$\|\Phi(A\nabla_v B)\Phi^{-1}(A\sharp_v B)\| \leq \frac{K(h')}{m_v(h)}, \tag{2.13}$$

where $h = \alpha/\beta$ and $h' = \alpha'/\beta'$. It is not difficult to see that (2.13) is equivalent to (2.10). The proof of inequality (2.11) proceeds likewise, and we omit the details. □

**Remark 2.9.** Obviously, the bounds in (2.10) and (2.11) are tighter than those in [**17,** Theorem 2.1], under the conditions $0 < \alpha' I \le B \le \alpha I \le \beta I \le A \le \beta' I$ with $h = \alpha/\beta$ and $h' = \alpha'/\beta'$.

**3. Connection with known results.** In this section, we point out connections between our results given in Section 2 and some inequalities proven in other contexts, that is, we now explain the advantages of our results. Let $0 \le v \le 1$, $r = \min\{v, 1-v\}$, $R = \max\{v, 1-v\}$ and $m_v(\cdot)$, $M_v(\cdot)$ be defined as in Theorem A. As we will show in Appendix A, the next proposition explains the advantages of our results.

**Proposition 3.1.** *The following statements are true.*

(I)

(i) *The lower bound of Proposition* 2.2 *improves the first inequality in* (1.2), *when* $3/4 \le v \le 1$ *with* $0 < A \le B$.

(ii) *The upper bound of Proposition* 2.2 *improves the second inequality in* (1.2), *when* $2/3 \le v \le 1$ *with* $0 < A \le B$.

(iii) *The upper bound of Proposition* 2.2 *improves the second inequality in* (1.2), *when* $0 \le v \le 1/3$ *with* $0 < A \le B$.

(II) *The upper bound of Theorem* A *improves the inequality*

$$(1-v) + vx \le x^v K(x),$$

*when* $x^v \ge 1/2$.

(III) *The upper bound of Theorem* A *improves the inequality given by Dragomir in* [**4,** Theorem 1],

$$(1-v) + vx \le \exp(4v(1-v)(K(x)-1))x^v, \quad x > 0, \tag{3.1}$$

*when* $0 \le v \le 1/2$ *and* $0 < x \le 1$.

(IV) *There is no ordering between Theorem* A *and the inequalities* (1.3) *and* (1.4).

Therefore, we conclude that Proposition 2.2 and Theorem A are not trivial results. The proofs of the above-mentioned are given in Appendix A.

**4. Inequalities related to the Heron mean.** This section aims to prove new inequalities containing (1.6). These inequalities were given in Theorem B. Our main idea and technical tool are closely related to inequality (2.1).

*Proof of Theorem* B. Consider the function $f_{r,v}(t) \equiv rvt^{v-1} + (1-r)v$, where $t > 0$, $r \in \mathbb{R}$, $0 \le v \le 1$. Since the function $f_{r,v}(t)$ is twice differentiable, it can easily be seen that

$$\frac{df_{r,v}(t)}{dt} = r(v-1)vt^{v-2},$$
$$\frac{d^2 f_{r,v}(t)}{dt^2} = r(v-2)(v-1)vt^{v-3}.$$

It is not difficult to verify that

$$\begin{cases} \frac{d^2 f_{r,v}(t)}{dt^2} \ge 0 & \text{for } r \ge 0, \\ \frac{d^2 f_{r,v}(t)}{dt^2} \le 0 & \text{for } r \le 0. \end{cases}$$

Utilizing inequality (2.1) for the function $f_{r,v}(t)$, we infer that

$$g_{r,v}(x) \le rx^v + (1-r)((1-v)+vx) \le G_{r,v}(x), \tag{4.1}$$

where

$$g_{r,v}(x) \equiv v(x-1)\left\{r\left(\frac{1+x}{2}\right)^{v-1} + (1-r)\right\} + 1, \tag{4.2}$$

$$G_{r,v}(x) \equiv \frac{v(x-1)}{2}(rx^{v-1} + 2 - r) + 1, \tag{4.3}$$

for each $x \ge 1$, $r \ge 0$, $0 \le v \le 1$. Similarly, for each $0 < x \le 1$, $r \ge 0$, $0 \le v \le 1$, we get

$$G_{r,v}(x) \le rx^v + (1-r)((1-v)+vx) \le g_{r,v}(x). \tag{4.4}$$

If $x \ge 1$ and $r \le 0$, we obtain

$$G_{r,v}(x) \le rx^v + (1-r)((1-v)+vt) \le g_{r,v}(x), \tag{4.5}$$

for each $0 \le v \le 1$. For the case $0 < x \le 1$, $r \le 0$, we have

$$g_{r,v}(t) \le rx^v + (1-r)((1-v)+vt) \le G_{r,v}(x), \tag{4.6}$$

for each $0 \le v \le 1$. □

Note that we equivalently obtain the operator inequalities from the scalar inequalities given in Theorem B. We omit such expressions here for simplicity.

Closing this section, we prove the ordering

$$\{(1-v)+vt^{-1}\}^{-1} \le g_{r,v}(t) \quad \text{and} \quad \{(1-v)+vt^{-1}\}^{-1} \le G_{r,v}(t)$$

under some assumptions, for the purpose of showing the advantages of our lower bounds given in Theorem B. It is known that

$$\left\{(1-v)+vt^{-1}\right\}^{-1} \le t^v, \quad 0 \le v \le 1,\ t>0,$$

so that we also have interests in the ordering $g_{r,v}(t)$ and $G_{r,v}(t)$ with $t^v$, that is, we can show the following four propositions. The proofs are given in Appendix B.

**Proposition 4.1.** *For $t \ge 1$, $0 \le v, r \le 1$, we have*

$$\{(1-v)+vt^{-1}\}^{-1} \le g_{r,v}(t). \tag{4.7}$$

**Proposition 4.2.** *For $0 < t \le 1$, $0 \le v, r \le 1$, we have*

$$\{(1-v)+vt^{-1}\}^{-1} \le t^v \le g_{r,v}(t). \tag{4.8}$$

**Proposition 4.3.** *For $0 \le r, v \le 1$, and $c \le t \le 1$ with $c \equiv (2^7-1)/5^4$, we have*

$$\{(1-v)+vt^{-1}\}^{-1} \le G_{r,v}(t). \tag{4.9}$$

**Proposition 4.4.** *For $0 \le v \le 1$, $r \le 1$, $t \ge 1$, we have*

$$\{(1-v)+vt^{-1}\}^{-1} \le t^v \le G_{r,v}(t). \tag{4.10}$$

**Remark 4.5.** Propositions 4.1–4.4 show that the lower bounds given in Theorem B are tighter than the known bound (harmonic mean) for the cases given in Propositions 4.1–4.4. If $r=1$ in Proposition 4.1, then $g_{r,v}(t) \le t^v$, for $t \ge 1$, $0 \le v \le 1$. If $r=1$ in Proposition 4.3, then $G_{r,v}(t) \le t^v$, for $c \le t \le 1$, $0 \le v \le 1$. We, thus, find that Propositions 4.1 and 4.3 make sense for the purpose of finding the functions between $\{(1-v)+vt^{-1}\}^{-1}$ and $t^v$.

**Remark 4.6.** In the process of the proof of Proposition 4.3, we find the inequality:

$$\frac{t^v + t}{2} \le \{(1-v) + vt^{-1}\}^{-1},$$

for $0 \le v \le 1$, $c \le t \le 1$. Then, we have the following inequalities:

$$\frac{A\sharp_v B + B}{2} \le A!_v B \le A\sharp_v B,$$

for $0 < cA \le B \le A$ with $c = (2^7 - 1)/5^4$, $0 \le v \le 1$.

In the process of the proof of Proposition 4.2, we also find the inequality:

$$t\left(\frac{t+1}{2}\right)^{v-1} \le \{(1-v) + vt^{-1}\}^{-1},$$

for $0 \le v \le 1$, $0 \le t \le 1$. Then, we have the following inequalities:

$$BA^{-1/2}\left(\frac{A^{-1/2}BA^{-1/2} + I}{2}\right)^{v-1} A^{1/2} \le A!_v B \le A\sharp_v B,$$

for $0 < B \le A$, $0 \le v \le 1$.

**5. Concluding remark.** Several refinements and generalizations of inequality (2.1) have been given (see, e.g., [**5, 6, 19, 22**]). Of course, if we apply them with similar considerations as those discussed above, we can find new results concerning mean inequalities. We leave the details of this idea to the interested reader, as it is merely an application of our main results.

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## APPENDICES

**A.** For the purpose of giving a proof of Proposition 3.1, we need the following lemma.

**Lemma A.1.** *For each $x \ge 1$, we have*

$$\left(\frac{x+1}{2}\right)^{2/3} \ge \left(\frac{\sqrt{x}+1}{2\sqrt{x}}\right)\left(1 + \log\left(\frac{x+1}{2}\right)\right). \tag{A.1}$$

*Proof.* We firstly prove

$$\left(\frac{x+1}{2}\right)^{2/3} \geq \left(\frac{1}{2}+\frac{x+1}{4x}\right)\left(1+\log\left(\frac{x+1}{2}\right)\right), \tag{A.2}$$

for $x \geq 1$. Setting $t = (x+1)/2 \geq 1$, the inequality (A.2) is equivalent to the inequality

$$t^{2/3} \geq \frac{(3t-1)}{2(2t-1)}(1+\log t),$$

which is equivalent to

$$2s^2(2s^3-1) \geq (3s^3-1)(1+3\log s),$$

where $s = t^{1/3} \geq 1$. In order to prove the above inequality, we set

$$\mathfrak{F}(s) \equiv 4s^5 - 3s^3 - 2s^2 + 1 - 9s^3 \log s + 3\log s, \quad s \geq 1.$$

By simple calculations, we have $\mathfrak{F}(s) \geq \mathfrak{F}(1) = 0$. Hence, we have inequality (A.2). For any $a > 0$, we have $2a/(1+a) \leq \sqrt{a}$, that is, $(a+1)/(2a) \geq 1/\sqrt{a}$. Therefore, for any $a > 0$, we have

$$\frac{1}{2}+\frac{a+1}{4a} \geq \frac{1}{2}+\frac{1}{2\sqrt{a}} = \frac{\sqrt{a}+1}{2\sqrt{a}},$$

which implies the following, second inequality:

$$\begin{aligned}\left(\frac{x+1}{2}\right)^{2/3} &\geq \left(\frac{1}{2}+\frac{x+1}{4x}\right)\left(1+\log\left(\frac{x+1}{2}\right)\right)\\ &\geq \left(\frac{\sqrt{x}+1}{2\sqrt{x}}\right)\left(1+\log\left(\frac{x+1}{2}\right)\right).\end{aligned}$$

This completes the proof. □

*Proof of Proposition* 3.1.

(I) Assume that $x \geq 1$.

(i) Consider the function

$$u_v(x) \equiv v(x-1)\left(\frac{1-x^{v-1}}{2}\right) - r\left(1-\sqrt{x}\right)^2.$$

For $3/4 \le v \le 1$, we have $u_v(x) \ge 0$. Let us prove this statement. Since $u_1(x) = 0$ and

$$\frac{d^2 u_v(x)}{dv^2} = \frac{1}{2}(1-x)x^{v-1}\{2\log x + v(\log x)^2\} \le 0$$

for $x \ge 1$, we have only to prove $u_{3/4}(x) \ge 0$ for $x \ge 1$. Since

$$u_{3/4}(x) = \frac{x^{5/4} - 3x + 4x^{3/4} - 5x^{1/4} + 3}{8x^{1/4}},$$

we set the function $\mathfrak{v}(x) \equiv x^{5/4} - 3x + 4x^{3/4} - 5x^{1/4} + 3$. Some calculations show $\mathfrak{v}(x) \ge \mathfrak{v}(x) = 0$, which implies $u_{3/4}(x) \ge 0$. Hence, our claim follows.

In this case, the first inequality in (2.2) can be considered as a refinement of the first inequality in (1.2).

(ii) Consider the function

$$w_v(x) \equiv R(1-\sqrt{x})^2 - v(x-1)\left(1 - \left(\frac{x+1}{2}\right)^{v-1}\right).$$

For $2/3 \le v \le 1$, we have $w_v(x) \ge 0$. In order to prove this inequality, let

$$\mathfrak{x}_v(x) = (1-\sqrt{x})^2 - (x-1)\left(1 - \left(\frac{x+1}{2}\right)^{v-1}\right).$$

For $x \ge 1$, we then have

$$\frac{d\mathfrak{x}_v(x)}{dv} = (x-1)\left(\frac{x+1}{2}\right)^{v-1}\left\{\log\left(\frac{x+1}{2}\right)\right\} \ge 0.$$

We have only to prove $\mathfrak{x}_{2/3}(x) \ge 0$ for $x \ge 1$. By slightly complicated calculations, we have

$$\mathfrak{x}_{2/3}(x) = \frac{2^{4/3}(\sqrt{x}-1)}{(x+1)^{1/3}}\left\{\frac{\sqrt{x}+1}{2} - \left(\frac{x+1}{2}\right)^{1/3}\right\} \ge 0.$$

Indeed, for $t \ge 1$, we have $(t-1)(t^2+3) \ge 0$ which is equivalent to $(t+1)^3 \ge 4(t^2+1)$. Setting $t = \sqrt{x}$, we obtain

$$\frac{(\sqrt{x}+1)^3}{8} \ge \frac{x+1}{2},$$

which yields

$$\frac{\sqrt{x}+1}{2} \geq \left(\frac{x+1}{2}\right)^{1/3}.$$

Thus, our assertion follows.

(iii) In addition, for $0 \leq v \leq 1/3$, we have $w_v(x) \geq 0$. In fact, since $v(x+1)/2^{v-1}$ is increasing for $v$, we estimate the first derivative of $w_v(x)$ as

$$\begin{aligned}
\frac{dw_v(x)}{dv} &= -(\sqrt{x}-1)^2-(x-1)+(x-1)\left(\frac{x+1}{2}\right)^{v-1}\left(1+v\log\left(\frac{x+1}{2}\right)\right) \\
&\leq -(\sqrt{x}-1)^2-(x-1) \\
&\quad + (x-1)\left(\frac{x+1}{2}\right)^{-2/3}\left(1+\frac{1}{3}\log\left(\frac{x+1}{2}\right)\right) \\
&= -\frac{2^{5/3}\sqrt{x}(\sqrt{x}-1)}{(x+1)^{2/3}}\left\{\left(\frac{x+1}{2}\right)^{2/3}\right. \\
&\qquad \left. - \frac{\sqrt{x}+1}{2\sqrt{x}}\left(1+\frac{1}{3}\log\left(\frac{x+1}{2}\right)\right)\right\} \leq 0.
\end{aligned}$$

The last inequality is due to Lemma A.1. Consequently, $w_v(x) \geq w_{1/3}(x)$. Thus, we prove $w_{1/3}(x) \geq 0$.

$$w_{1/3}(x) = \frac{\sqrt{x}-1}{3}\left(\frac{x+1}{2}\right)^{-2/3}\left\{(\sqrt{x}-3)\left(\frac{x+1}{2}\right)^{2/3}+\sqrt{x}+1\right\}.$$

Now, we set the function $\mathfrak{y}(t) \equiv (t-3)((t^2+1)/2)^{2/3}+t+1$ for $t \geq 1$. With some calculations, we get $\mathfrak{y}(t) \geq \mathfrak{y}(1) = 0$. Therefore, we have $w_v(t) \geq w_{1/3}(t) \geq 0$, as required.

In these cases, the second inequality in (2.2) provides an improvement for the second inequality in (1.2).[1]

(II) Let $x > 0$. It is clear that, if $x^v \geq 1/2$, then $M_v(x) \leq K(x)$. Indeed, by simple calculations, the inequality $M_v(x) \leq K(x)$ is equivalent to the inequality $2v(1-v) \leq x^v$. Since $v(1-v) \leq 1/4$, we have $x^v \geq 1/2 \geq 2v(1-v)$ under the condition $x^v \geq 1/2$.

(III) Dragomir [**4,** Theorem 1] obtained the inequality (3.1) for $x > 0$. However, for $0 \le v \le 1/2$, $0 < x \le 1$, we show

$$M_v(x) \le \exp(4v(1-v)(K(x)-1)). \tag{A.3}$$

Our upper bound of Theorem A is tighter than that given in [**4,** Theorem 1], when $0 \le v \le 1/2$.

Now, we prove the above inequality (A.3), which is identical to the inequality

$$1 + \frac{1}{2x^v} \frac{v(1-v)(x-1)^2}{x} \le \exp\left(\frac{v(1-v)(x-1)^2}{x}\right).$$

We use the inequality

$$\exp(y) \ge 1 + y + \frac{1}{2}y^2, \quad y \ge 0,$$

with $y = (v(1-v)(x-1)^2)/x \ge 0$. Then, we calculate

$$\begin{aligned} \text{(A.4)} \quad \exp&\left(\frac{v(1-v)(x-1)^2}{x}\right) - 1 - \frac{1}{2x^v}\frac{v(1-v)(x-1)^2}{x} \\ &\ge \frac{v(1-v)(x-1)^2}{x}\left(1 - \frac{1}{2x^v} + \frac{v(1-v)(x-1)^2}{2x}\right) \\ &= \frac{v(1-v)(x-1)^2}{x}\left(\frac{2x^v - 1 + v(1-v)x^{v-1}(x-1)^2}{2x^v}\right). \end{aligned}$$

Thus, we have only to prove $2x^v - 1 + v(1-v)x^{v-1}(x-1)^2 \ge 0$ for $0 < x \le 1$, $0 \le v \le 1/2$. By setting $t = 1/x$, the above inequality becomes

$$t^{-v-1}(2t - t^{v+1} + v(1-v)(t-1)^2) \ge 0.$$

Therefore, it is sufficient to prove the inequality

$$\mathfrak{g}_v(t) \equiv 2t - t^{v+1} + v(1-v)(t-1)^2 \ge 0,$$

for $t \ge 1$, $0 \le v \le 1/2$. With some calculations, we have $\mathfrak{g}_v(t) \ge \mathfrak{g}_{1/2}(t) \ge \mathfrak{g}_{1/2}(1) = 1 > 0$. Thus, the proof of inequality (A.3) is complete.

It should be mentioned here that inequality (A.3)holds for $0 \le v \le 1$ and $x \ge 1/2$ from (A.4).

(IV) It is natural to consider $m_v(x)$ and $M_v(x)$ as better than $K^r(x)$ and $K^R(x)$ under the assumption $0 < x \le 1$.

(i) In general, there is no ordering between $K^r(x)$ and $m_v(x)$. For this purpose, taking $v = 0.3$ and $x = 0.7$, then

$$m_v(x) - K^r(x) \approx 0.002.$$

On the other hand, taking $v = 0.7$ and $x = 0.1$, we have

$$m_v(x) - K^r(x) \approx -0.15.$$

(ii) In addition, we have no ordering between $K^R(x)$ and $M_v(x)$. In order to see this, putting $v = 0.2$ and $x = 0.4$, observe that

$$K^R(x) - M_v(x) \approx 0.08.$$

However, if we choose $v = 0.6$ and $x = 0.3$, we obtain

$$K^R(x) - M_v(x) \approx -0.17. \qquad \square$$

**B.** We begin by proving Proposition 4.1.

*Proof of Proposition* 4.1. Since $g_{r,v}(t)$ is decreasing in $r$, $g_{r,v}(t) \geq g_{1,v}(t)$ so that we only must prove, for $t \geq 1$ and $0 \leq v \leq 1$, the inequality $g_{1,v}(t) \geq \{(1-v) + vt^{-1}\}^{-1}$, which is equivalent to the inequality by $v(t-1) \geq 0$,

$$\left(\frac{t+1}{2}\right)^{v-1} \geq \frac{1}{(1-v)t+v}. \tag{B.1}$$

Since $t \geq 1$ and $0 \leq v \leq 1$, we have $t((t+1)/2)^{v-1} \geq t^v$. In addition, for $t > 0$, $0 \leq v \leq 1$, we have $t^v \geq \{(1-v) + vt^{-1}\}^{-1}$. Thus, we have $t((t+1)/2)^{v-1} \geq \{(1-v)+vt^{-1}\}^{-1}$, which implies the inequality (B.1). $\square$

*Proof of Proposition* 4.2. The first inequality is known for $t > 0$, $0 \leq v \leq 1$. Since $g_{r,v}(t)$ is deceasing in $r$, in order to prove the second inequality, we only need prove $g_{1,v}(t) \geq t^v$, that is,

$$v(t-1)\left(\frac{t+1}{2}\right)^{v-1} + 1 \geq t^v,$$

which is equivalent to the inequality

$$\frac{t^v - 1}{v} \leq (t-1)\left(\frac{t+1}{2}\right)^{v-1}.$$

With the use of the Hermite-Hadamard inequality along with a convex function $x^{v-1}$ for $0 \le v \le 1$, $x > 0$, the above inequality can be proven as

$$\left(\frac{t+1}{2}\right)^{v-1} \le \frac{1}{1-t}\int_t^1 x^{v-1}dx = \frac{1-t^v}{v(1-t)}. \qquad \square$$

*Proof of Proposition* 4.3. We firstly prove $\mathsf{h}(t) \equiv 2(t-1) - \log t \ge 0$ for $c \le t \le 1$. Since $\mathsf{h}''(t) \ge 0$,

$$\mathsf{h}(1) = 0 \quad \text{and} \quad \mathsf{h}(c) \approx -0.0000354367 < 0.$$

Thus, we have $\mathsf{h}(t) \le 0$ for $c \le t \le 1$. Secondly, we prove

$$\mathsf{l}_v(t) \equiv 2(t-1) - ((1-v)t + v)\log t \le 0.$$

Since

$$\frac{d\mathsf{l}_v(t)}{dv} = (t-1)\log t \ge 0,$$

we have

$$\mathsf{l}_v(t) \le \mathsf{l}_1(t) = \mathsf{h}(t) \le 0.$$

Since $G_{r,v}(t)$ is decreasing in $r$, we have $G_{r,v}(t) \ge G_{1,v}(t)$, so that we must only prove $G_{1,v}(t) \ge \{(1-v) + vt^{-1}\}^{-1}$, which is equivalent to the inequality by $v(t-1) \le 0$,

$$\frac{t^{v-1}+1}{2} \le \frac{1}{(1-v)t+v},$$

for $0 \le r, v \le 1$, $c \le t \le 1$. Towards this end, we set

$$\mathsf{f}_v(t) \equiv 2 - (t^{v-1}+1)((1-v)t+v).$$

Simple calculations imply $\mathsf{f}_v(t) \ge \mathsf{f}_1(t) = 0$. $\square$

*Proof of Proposition* 4.4. The first inequality is known for $t > 0$, $0 \le v \le 1$. Since $G_{r,v}(t)$ is deceasing in $r$, in order to prove the second inequality, we only need prove $G_{1,v}(t) \ge t^v$, which is equivalent to the inequality

$$\frac{1}{2}v(t-1)(t^{v-1}+1) + 1 \ge t^v.$$

Towards this end, we set

$$\mathsf{k}_v(t) \equiv v(t-1)(t^{v-1}+1) + 2 - 2t^v.$$

Simple calculations imply $\mathsf{k}_v(t) \ge \mathsf{k}_v(1) = 0$. $\square$

## ENDNOTES

1. It is interesting to note that, with computer calculations, we find that, if $v \geq 0.7$, then $u_v(x) \geq 0$ and, if $v \geq 0.6$ or $v \leq 0.4$, we have $w_v(x) \geq 0$. This means that we have a possibility of extending the range of $v$ to satisfy one of the conditions of (I) (i), (I) (ii) and (I) (iii) in Proposition 3.1.

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