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ON CONJUGACY OF ABSTRACT ROOT BASES OF ROOT SYSTEMS OF COXETER GROUPS

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ABSTRACT. We introduce and study a combinatorially defined notion of the root basis of a (real) root system of a possibly infinite Coxeter group. Known results on conjugacy up to sign of root bases of certain irreducible finite rank real root systems are extended to abstract root bases, to a larger class of real root systems and, with a short list of (genuine) exceptions, to infinite rank irreducible Coxeter systems.

1. Introduction. It is well known that any two root bases (simple systems of roots) for a root system of a finite Weyl group (or finite Coxeter group) W are W-conjugate. This result has been extended to W-conjugacy up to sign of root bases of root systems of certain reflection representations of finite rank, irreducible Coxeter systems in [**16, 20**], see also [**19,** Proposition 5.9].

In this paper, we extend these results in three ways. First, we reformulate the results of [**16**] by asserting conjugacy up to sign of suitably defined *abstract root bases* of *abstract root systems* of irreducible, finite rank Coxeter systems (W, S); the main novel feature is that we provide a characterization of root bases which does not require the linear structure of the ambient real vector space. Second, we use the abstract result to extend the above conjugacy result to (real) root systems of more general reflection representations of finite rank irreducible Coxeter systems. Third, we prove that two root bases for any (real or abstract) root system of an irreducible Coxeter system of possibly infinite rank are *locally W-conjugate* up to sign, with a small number of types of exceptions.

In order to explain some of these results in more detail, let (W, S) be a Coxeter system, and let Φ be the root system of the standard

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reflection representation of (W, S) (see **[3,** Chapter 5], **[17,** Chapter 5]), with a standard set of positive roots Φ_+ and corresponding standard root basis Π. For $\alpha \in \Phi$, the reflection s_α permutes Φ, and W identifies with the group of permutations of Φ generated by the restrictions of the s_α to Φ (which we still denote as s_α). We let the sign group $\{\pm 1\}$ act on Φ by $(\pm 1)\alpha = \pm\alpha$ for $\alpha \in \Phi$. The map $\alpha \mapsto s_\alpha$ is a two-fold covering of its image (the set T of reflections of (W, S)) with the orbits of the sign group as fibers. We call the set Φ with its action by $\{\pm 1\}$ and the map

$$\alpha \longmapsto s_\alpha \colon \Phi \longrightarrow \mathrm{Sym}(\Phi)$$

the standard abstract root system of (W, S).

We say that a subset Ψ_+ of Φ is a quasi-positive system if it is a set of orbit representatives for the sign group acting on Φ. Then, $\alpha \in \Psi_+$ is said to be a simple root of Ψ_+ if s_α permutes $\Psi_+ \setminus \{\alpha\}$. Let Π' be the set of simple roots of Ψ_+,

$$S' = \{s_\alpha \mid \alpha \in \Pi'\}$$

the set of simple reflections for Ψ_+ and $W' = \langle S' \rangle$ the subgroup generated by S'. It is easy to see that (W', S') is a Coxeter system (we prove this here as Proposition 2.7, although it also follows from **[9,** 1.8]). We say that Ψ_+ is a generative quasi-positive system if $W' = W$. In this case, it need not be true that (W, S) is isomorphic to (W, S') as a Coxeter system; indeed, some (but not all) examples of non-isomorphic finite rank, irreducible Coxeter systems (W, S) and (W, S') related by diagram twisting in the sense of [**4**] arise in this way (see subsection 2.12, Corollary 2.21 and Example 2.22). Questions of conjugacy or isomorphism of generative quasi-positive systems are of interest in relation to reflection rigidity and strong reflection rigidity (see [**4, 6**]) of Coxeter systems.

We say that $\gamma \in \Phi$ is between $\alpha \in \Phi$ and $\beta \in \Phi$ if and only if $\gamma = a\alpha + b\beta$ for some $a, b \in \mathbb{R}_{\geq 0}$. Say that $\Psi \subseteq \Phi$ is closed if $\alpha, \beta \in \Psi$, and $\gamma \in \Phi$ with γ between α and β implies $\gamma \in \Psi$. The closed sets of roots are those which are closed for a closure operator considered in [**12, 21**]. We define $\Psi_+ \subseteq \Psi$ to be an abstract positive system for Ψ if it is a generative, closed, quasi-positive system. The above notions of abstract root system, betweenness, abstract positive system, etc., may all be reformulated purely combinatorially and algebraically in terms of (W, S).

Recall the Coxeter system of type $A_{\infty,\infty}$ has, as an underlying Coxeter group, the group of all permutations of $\mathbb{Z}$ leaving all but finitely many elements fixed, with Coxeter generators given by the adjacent transpositions $(n, n+1)$ for $n \in \mathbb{Z}$. The Coxeter system of type A_∞ is defined in the same manner, except with $\mathbb{Z}$ replaced by $\mathbb{N}$. These Coxeter systems are non-isomorphic, but any bijection $\mathbb{N} \to \mathbb{Z}$ induces a reflection preserving isomorphism of the underlying Coxeter groups $W(A_\infty) \xrightarrow{\cong} W(A_{\infty,\infty})$. The following results are proven in this paper.

Theorem 1.1. *Let (W, S) be an irreducible Coxeter system which is not necessarily of finite rank. Let Ψ_+ be an abstract positive system of Φ, and let Π', S' denote, respectively, the sets of simple roots and simple reflections of Ψ_+. Then*:

(a) *(W, S) and (W, S') have the same finite rank parabolic subgroups.*

(b) *The Coxeter system (W, S) is isomorphic to (W, S'), unless perhaps one is of type A_∞ and the other is of type $A_{\infty,\infty}$.*

(c) *If (W, S) and (W, S') are isomorphic, then there is a permutation σ of Φ and a sign $\epsilon \in \{\pm 1\}$ with $\Psi_+ = \epsilon\sigma(\Phi_+)$ and $\Pi' = \epsilon\sigma(\Pi)$ such that, for any subgroup W' of W which is generated by a finite subset of T, there exists $w = w(W') \in W$ such that $\sigma(\alpha) = w(\alpha)$ for all $\alpha \in \Phi$ with $s_\alpha \in W'$.*

(d) *If (W, S) is of finite rank, then, in (c), $\sigma \in W$, and moreover, σ and ϵ are uniquely determined, provided $\epsilon = 1$ if W is finite.*

(e) *A subset Δ of Φ is a root basis of Φ if and only if it is the set of abstract simple roots of some abstract positive system of Φ.*

In the paper, we extend the theorem to a more general class of root systems (see Section 3) roughly parameterized by what we call possibly non-integral generalized Cartan matrices (NGCMs). The same class is also considered in [**13**]. Most (reduced, real) root systems usually considered in the literature (for instance, in [**3, 11, 16, 17, 19**]) are in this class, and it has the important technical advantages of being closed under passage to root subsystems for arbitrary reflection subgroups and including the real root systems of Kac-Moody Lie algebras. In general, for some of the reflection representations considered, it is not possible to choose a "reduced" root system; however, we do not restrict to the subclass of "reduced" root systems since this would amount to imposing an unnatural condition on the NGCM (compare subsections 3.3, 3.5 and [**15**]). Since, for non-reduced root systems, a given set of simple reflections may correspond to several different sets of simple roots differing by "rescaling" (multiplying each simple root by a positive scalar), the statement of the theorems needs minor modification for non-reduced root systems; for instance, conjugacy results hold only up to sign and rescaling. The restriction to irreducible Coxeter systems in (a)–(d) is merely a matter of convenience since those parts of the theorem can be applied separately to the irreducible components; the situation for (e), in general, is more complicated (see Theorem 4.2 and Example 4.8).

The arrangement of the paper is as follows. Section 2 contains definitions of, and basic facts about, abstract root systems of Coxeter systems. Some of the facts involving the relation of the root system to the "reflection cocycle" extend to similar structures (which we call quasi-root systems, to avoid confusion) attached to other groups. Section 3 collects basic properties of the real reflection representations of Coxeter groups, and corresponding root systems, which we consider in this paper. The main results, involving the connection between the abstract root systems of Section 2 and the real root systems of Section 3, and including the statements and proof of more general versions of the parts of Theorem 1.1, are given in Section 4.

There are two appendices. Appendix A recalls some definitions and basic properties of possibly non-abelian cohomology, which provides the natural setting for some of the special results proven in Section 2 involving reflection cocycles. In particular, the generalities in Appendix A are relevant to the classification of abstract root systems, although this is not pursued here. Appendix B gives some examples and further results involving general quasi-root systems, of which we make no essential use in the body of the paper. We observe, in particular, that the definition of Bruhat orders (and their twisted versions [**9**]) in terms of the reflection cocycle of a Coxeter system extends to other groups with quasi-root systems, linearly realized over real vector spaces, such as real orthogonal groups.

The main results of this paper have applications to the study of initial sections A of reflection orders of T, see [**12**], the twisted Bruhat orders $\leq_A$ on W, see [**9**], and related structures. In particular, our principal motivation for proving Theorem 1.1 was to clarify the relationship between two different *natural* sets of Coxeter generators of a reflection subgroup W_A, defined in [**9**], which plays an important role in the study of the order $\leq_A$. Many of the refinements discussed in [**12**] of long-standing conjectures and questions on initial sections involve the closure operator on root systems considered in this paper. Unfortunately, this closure operator has some quite unfavorable properties, even in the case of a finite Weyl group, see [**21**]. Another of our motivations has been to establish some first favorable properties of this closure operator for a general Coxeter group, as one tool for study of these refined conjectures.

2. Abstract root systems.

2.1. Let Ψ be a set with a given function

$$F\colon \Psi \longrightarrow \mathrm{Sym}(\Psi),$$

where $\mathrm{Sym}(\Psi)$ is the symmetric group consisting of all permutations of Ψ. Set $s_\alpha := F(\alpha)$. We make the following assumptions:

(i) there is a fixed point free action of the sign group $\{\pm 1\}$ on Ψ such that, for $\alpha \in \Psi$, $(-1)\alpha = -\alpha := s_\alpha(\alpha)$ and $s_{-\alpha} = s_\alpha$;

(ii) $s_{s_\alpha(\beta)} = s_\alpha s_\beta s_\alpha$ for all $\alpha, \beta \in \Psi$.

These conditions imply that s_α is an involution, that is, it is a permutation of order exactly 2, commuting with the action of the sign group.

To avoid any possible confusion with abstract root systems of Coxeter groups, which may naturally be regarded as examples of this formalism, see Proposition 2.5, we call such a pair (Ψ, F) a *quasi-root system*; other examples of quasi-root systems are mentioned in Appendix B. By abuse of notation, we sometimes call Ψ itself a quasi-root system.

An arbitrary family $((\Psi^i, F^i))_{i\in I}$ of quasi-root systems has a *union* (Ψ, F), where $\Psi = \coprod_{i\in I} \Psi^i$ is the coproduct, that is, the disjoint union, in the category of sets, and, for $\alpha \in \Psi^i \subseteq \Psi$,

$$F(\alpha)_{|\Psi^i} = F^i(\alpha),$$

while

$$F(\alpha)_{|\Psi^j} = \mathrm{Id}_{\Psi^j} \quad \text{for } j \neq i.$$

2.2. A morphism $(\Psi^1, F^1) \to (\Psi^2, F^2)$ of quasi-root systems is defined to be a function $\theta\colon \Psi^1 \to \Psi^2$ such that $\theta(s_\alpha(\beta)) = s_{\theta(\alpha)}(\theta(\beta))$ for all $\alpha, \beta \in \Psi^1$. Taking $\alpha = \beta$ shows that $\theta(-\alpha) = -\theta(\alpha)$ for $\alpha \in \Psi_1$.

With the obvious definition of composition of morphisms, the quasi-root systems form a category. In particular, this gives rise to the usual (categorical) notions of an isomorphism of quasi-root systems, and of the action of a group G on a quasi-root system; an isomorphism is a bijective morphism of quasi-root systems, and an action of a group G on a quasi-root system Ψ is a homomorphism from G to the group $\mathrm{Aut}(\Psi)$ of automorphisms of Ψ. Observe that $s_\alpha \in \mathrm{Aut}(\Psi)$ for all $\alpha \in \Psi$.

2.3. Up to Corollary 2.4, we fix a quasi-root system (Ψ, F) and let G be a group acting on Ψ by means of a homomorphism $\iota\colon G \to G_\Psi := \mathrm{Aut}(\Psi)$. Set

$$T = T_\Psi := \{s_\alpha \mid \alpha \in \Psi\} \subseteq G_\Psi,$$

and let $W = W_\Psi := \langle T \rangle$ be the subgroup of G_Ψ generated by T. We call W the group associated to Ψ.

Regard the power set $\mathcal{P}(X)$ of any set X as an abelian group under symmetric difference:

$$A + B := (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) \quad \text{for } A, B \subseteq X.$$

Then, $\mathcal{P}(T)$ becomes a G-module with G acting by conjugation:

$$\sigma \cdot A = \sigma A \sigma^{-1} := \{\sigma t \sigma^{-1} \mid t \in A\}$$

for $\sigma \in G$ and $A \subseteq T$. Define a map

$$\tau = \tau_\Psi\colon \Psi \longrightarrow T$$

by $\tau(\alpha) = s_\alpha$. We call τ the *reflection map* of Ψ. The sets $\tau^{-1}(t)$ for $t \in T$ are called the *fibers* of τ. Note that the set of fibers of τ is a system of imprimitivity for the action of G on Ψ.

2.4. Consider two sets Ψ_+, Ψ'_+ of orbit representatives for the sign group on Ψ, that is, Ψ_+ is a subset of Ψ with $-\Psi_+ = \Psi \setminus \Psi_+$, and similarly for Ψ'_+. We say that Ψ_+ and Ψ'_+ are *compatible* if, for each

$t \in T$, $\Psi_+ \cap \tau^{-1}(t) = \pm(\Psi'_+ \cap \tau^{-1}(t))$, or equivalently, if T is the disjoint union

$$T = \{s_\alpha \mid \alpha \in \Psi_+ \cap -\Psi'_+\} \cup \{s_\alpha \mid \alpha \in \Psi_+ \cap \Psi'_+\}.$$

Note that Ψ_+ and $-\Psi_+$ are compatible. Compatibility is an equivalence relation on the family of sets of orbit representatives for $\{\pm 1\}$ on Ψ.

We say that Ψ_+ is a *quasi-positive system* for G on Ψ if Ψ_+ and $\sigma(\Psi_+)$ are compatible for all $\sigma \in G$, or, equivalently, $\sigma(\Psi^+ \cap \tau^{-1}(t)) = \pm(\Psi_+ \cap \tau^{-1}(\sigma t \sigma^{-1}))$ for all $\sigma \in G$ and $t \in T$. Thus, Ψ_+ is a quasi-positive system if and only if

$$\{\epsilon\Psi_+ \cap \tau^{-1}(t) \mid t \in T, \epsilon = \pm 1\}$$

is a system of imprimitivity for the action of G on Ψ. In particular, if $\{\pm 1\}$ acts simply transitively on each fiber of the reflection map, then, any set of orbit representatives for $\{\pm 1\}$ is a quasi-positive system for G on Ψ, and any two quasi-positive systems for G on Ψ are compatible.

Lemma 2.1. *Let Ψ_+ and Ψ'_+ be compatible quasi-positive systems, $\rho, \rho' \in G$ and $\epsilon, \epsilon' \in \{\pm 1\}$. Then,*

$$\tau^{-1}(\{s_\alpha \mid \alpha \in \rho(\epsilon\Psi_+) \cap \rho'(\epsilon'\Psi'_+)\}) = \rho(\epsilon\Psi_+) + \rho'(-\epsilon'\Psi'_+).$$

Proof. Since $\rho(\epsilon\Psi_+)$ and $\rho'(\epsilon'\Psi'_+)$ are compatible quasi-positive systems, it follows that, if $\alpha \in \rho(\epsilon\Psi_+) \cap \rho'(\epsilon'\Psi'_+)$, then $\tau^{-1}(s_\alpha) \cap \rho(\epsilon\Psi_+) = \tau^{-1}(s_\alpha) \cap \rho'(\epsilon'\Psi'_+)$ and $\tau^{-1}(s_\alpha) \cap -\rho(\epsilon\Psi_+) = \tau^{-1}(s_\alpha) \cap -\rho'(\epsilon'\Psi'_+)$, whence

$$\tau^{-1}(s_\alpha) \in \big(\rho(\epsilon\Psi_+) \cap \rho'(\epsilon'\Psi'_+)\big) \cup -\big(\rho(\epsilon\Psi_+) \cap \rho'(\epsilon'\Psi'_+)\big).$$

Thus,

$$\tau^{-1}\big(\{s_\alpha \mid \alpha \in \rho(\epsilon\Psi_+) \cap \rho'(\epsilon'\Psi'_+)\}\big) \subseteq \\ (\rho(\epsilon\Psi_+) \cap \rho'(\epsilon'\Psi'_+)) \cup -(\rho(\epsilon\Psi_+) \cap \rho'(\epsilon'\Psi'_+)).$$

Since the reverse inclusion is clear, and the right hand side is $(\rho(\epsilon\Psi_+) \setminus \rho'(-\epsilon'\Psi'_+)) \cup (\rho'(-\epsilon'\Psi'_+) \setminus \rho(\epsilon\Psi_+))$, the result follows. $\square$

Proposition 2.2. *Let Ψ_+ be a quasi-positive system for G on Ψ. Define the function $N_{\Psi_+}: G \to \mathcal{P}(T)$ by*

$$N_{\Psi_+}(\sigma) = \{s_\alpha \mid \alpha \in \Psi_+ \cap \sigma(-\Psi_+)\}.$$

(a) *The function N_{Ψ_+} is a cocycle on G with values in $\mathcal{P}(T)$.*

(b) *$T_{\Psi_+} := T \times \{\pm 1\}$ is a G-set with the action*

$$\sigma(t,\epsilon) = (\sigma t \sigma^{-1}, \eta(\sigma^{-1}, t)\epsilon),$$
$$\eta(\sigma, t) := \begin{cases} 1, & t \notin N_{\Psi_+}(\sigma) \\ -1, & t \in N_{\Psi_+}(\sigma). \end{cases}$$

(c) *Define the map $F_{\Psi_+}: T_{\Psi_+} \to \mathrm{Sym}(T_{\Psi_+})$ by*

$$(t,\epsilon) \longmapsto (\alpha \longmapsto t(\alpha)),$$

regarding T_{Ψ_+} as a G-set via (b)*. Then, (T_{Ψ_+}, F_{Ψ_+}) is a quasi-root system with G-action. Furthermore, $\{\pm 1\}$ acts simply transitively on each fiber of the reflection map for T_{Ψ_+}, and $T \times \{1\}$ is a quasi-positive system for G on T_{Ψ_+}*

(d) *The map*

$$\rho\colon \Psi \longrightarrow T_{\Psi_+}$$

defined by $\alpha \mapsto (s_\alpha, \epsilon)$ for $\epsilon \in \{\pm 1\}$ and $\alpha \in \epsilon\Psi_+$ is a surjective morphism of G-sets. Furthermore, the map

$$\Psi'_+ \longmapsto \rho(\Psi'_+)$$

is a bijection between the set of quasi-positive systems of Ψ compatible with Ψ_+ and the set of quasi-positive systems for T_{Ψ_+}. Finally, ρ is an isomorphism if $\{\pm 1\}$ acts simply transitively on each fiber of the reflection map for Ψ.

Proof. Write N for N_{Ψ_+}. For (a), we have to check the cocycle condition (see A.1)

$$N(\sigma\rho) = N(\sigma) + \sigma N(\rho)\sigma^{-1}, \quad \sigma, \rho \in G.$$

Define $M\colon G \to \mathcal{P}(\Psi)$ by $M(\sigma) = \tau^{-1}(N(\sigma))$ for $\sigma \in G$, where τ is the reflection map. Since τ is a G-equivariant surjection, it will suffice to check that $M(\sigma\tau) = M(\sigma) + \sigma M(\tau)$ for all $\sigma, \tau \in G$, that is, we must show that M is a cocycle for the action of G on $\mathcal{P}(\Psi)$ induced by the

G-action on Ψ. However, Lemma 2.1 and the definition of N imply that, for any $\sigma \in G$, $M(\sigma) = \Psi_+ + \sigma(\Psi_+)$. This shows that M is a coboundary, and hence, a cocycle, as required to prove (a).

Part (b) follows from (a) by straightforward calculation (more generally, see Proposition A.1 and its proof, noting the identification

$$\mathcal{P}(T) \cong \prod_{t \in T} \{\pm 1\}$$

of groups with G-action). Finally, (c)–(d) follow from (b) and the definitions. $\square$

Proposition 2.3. *Let Ψ_+ and Ψ'_+ be compatible quasi-positive systems for Ψ.*

(a) *Let $A := \{s_\alpha \mid \alpha \in \Psi_+ \cap -\Psi'_+\}$. Then, for $\sigma \in G$,*

$$N_{\Psi_+}(\sigma) + \sigma A \sigma^{-1} = \{s_\alpha \mid \alpha \in \Psi_+ \cap \sigma(-\Psi'_+)\} = N_{\Psi'_+}(\sigma) + A.$$

(b) *The cocycles N_{Ψ_+} and $N_{\Psi'_+}$ are cohomologous, that is, the cohomology classes of N_{Ψ_+} and $N_{\Psi'_+}$ in $H^1(G, \mathcal{P}(T))$ are equal.*

(c) *The map $T_{\Psi_+} \to T_{\Psi'_+}$, given by*

$$(s_\alpha, \epsilon) \longmapsto (s_\alpha, \epsilon\epsilon')$$

for $\epsilon, \epsilon' \in \{\pm 1\}$ and $\alpha \in \Psi_+ \cap \epsilon' \Psi'_+$, is a G-equivariant isomorphism of quasi-root systems.

Proof. For (a), it suffices to show that the three sets in the displayed equation all have the same inverse image under the reflection map τ. Using Lemma 2.1,

$$\begin{aligned}\tau^{-1}(N_{\Psi_+}(\sigma) + \sigma A \sigma^{-1}) &= \tau^{-1}(N_{\Psi_+}(\sigma)) + \tau^{-1}(\sigma A \sigma^{-1})) \\ &= (\Psi_+ + \sigma(\Psi_+)) + (\sigma(\Psi_+) + \sigma(\Psi'_+)) \\ &= \Psi_+ + \sigma(\Psi'_+).\end{aligned}$$

Similar calculations show that $\Psi_+ + \sigma(\Psi'_+)$ is also the inverse image of the other two sets.

Note that (a) provides an explicit cohomology between N_{Ψ_+} and $N_{\Psi'_+}$, proving (b) (see A.1). Then, (c) follows from (b) by simple calculation (more generally, see A.2). $\square$

Corollary 2.4. *Let X denote the set of quasi-positive systems for the quasi-root system T_{Ψ_+}, with G-action $(\sigma, \Psi'_+) \mapsto \sigma(\Psi'_+)$. Then, the map $X \to \mathcal{P}(T)$, given by*

$$\Psi'_+ \longmapsto \{t \in T \mid (t, -1) \in \Psi'_+\}$$

is a G-equivariant bijection, where $\mathcal{P}(T)$ has the G-action

$$(g, A) \longmapsto N_{\Psi_+}(g) + gAg^{-1},$$

see Appendix A.2.

Proof. This follows using Proposition 2.3. $\square$

2.5. Now, we recall some useful characterizations of Coxeter systems which are all either implicit or explicit in the literature. For general references on Coxeter groups, see [**2, 3, 17**].

Consider a pair (W, S) consisting of a group W and a set S of involutions generating W. The function $l\colon W \to \mathbb{N}$, defined by

$$l(w) = \min\{n \in \mathbb{N} \mid w = s_1 \cdots s_n \text{ for some } s_1, \ldots, s_n \in S\}$$

is called the *length function* of (W, S). Let

$$T = \bigcup_{w \in W} wSw^{-1}$$

and $\mathcal{P}(T)$ denote the power set of T, regarded as an abelian group under the symmetric difference. We let W act on $\mathcal{P}(T)$ by conjugation and the group $\{\pm 1\}$ act on $\widehat{T} := T \times \{\pm 1\}$ by multiplication on the second factor. Set $\widehat{T}_+ = T \times \{1\} \subseteq \widehat{T}$.

Proposition 2.5. *The following conditions are equivalent*:

(i) *the pair (W, S) is a Coxeter system.*

(ii) *There is an action of the group W by permutations on the set $\widehat{T}$ such that $s(t, \epsilon) = (sts, (-1)^{\delta_{s,t}}\epsilon)$ for $s \in S$, $t \in T$, $\epsilon \in \{\pm 1\}$.*

(iii) *There is a function* $N\colon W \to \mathcal{P}(T)$, *satisfying*

$$N(xy) = N(x) + xN(y)x^{-1} \quad \text{for } x, y \in W,$$
$$N(s) = \{s\} \quad \text{for } s \in S.$$

Suppose that these conditions hold. Then, for $w \in W$, $t \in T$ *and* $\epsilon = \pm 1$, *we have:*

(a) $N(w) = \{t \in T \mid l(tw) < l(w)\} = \{t \in T \mid w^{-1}(t, 1) \in -\widehat{T}_+\}$.

(b) $|N(w)| = l(w)$.

(c) *Write*

$$\eta(w,t) = \begin{cases} 1 & \text{if } t \notin N(w) \\ -1 & \text{if } t \in N(w). \end{cases}$$

Then, $w^{-1}(t, \epsilon) = (w^{-1}tw, \eta(w,t)\epsilon)$

(d) *The action of* W *by permutations on* $\Psi := \widehat{T}$ *is faithful.*

(e) *Set* $s_\alpha = r \in \mathrm{Sym}(\Psi)$ *for* $\alpha = (r, \mu) \in \widehat{T}$, *and let* $F\colon \Psi \to \mathrm{Sym}(\Psi)$ *be the function* $\alpha \mapsto s_\alpha$. *Then,* (Ψ, F) *is a quasi-root system for* W *with* $\widehat{T}_+$ *as a quasi-positive system.*

Proof. For completeness, we sketch direct proofs of all the implications amongst (i)–(iii). For (i) implies (ii), see [**3,** Chapter 4, subsection 1.4]; for the converse, note that the arguments of [**3,** Chapter 4, subsections 1.4–1.5] work with minor changes under assumption (ii) instead of (i). If (ii) holds, it is easy to see that (Ψ, F) is a quasi-root system with $T \times \{1\}$ as a quasi-positive system, and then, (ii) implies (iii) follows readily from Proposition 2.2 (a). A proof of the equivalence of (i) and (iii) can be found in [**8, 11**], although the proof that (i) implies (iii) found there proceeds via (ii) (or a similar result). A direct proof that (i) implies (iii) can be given by noting that a cocycle on a group G with values in an abelian group can be arbitrarily specified on a set of generators of G, provided it preserves (in an obvious sense) a set of defining relations for G in terms of those generators; it is then straightforward to deduce existence of the cocycle N, as in (iii), from the standard presentation of W.

A direct proof of the equivalence of (ii) and (iii) is implicit as a special case of the proof of Proposition A.1. Specifically, there is a bijection between the set of functions

$$N\colon W \longrightarrow \mathcal{P}(T)$$

and the set of functions

$$\eta\colon W \times T \longrightarrow \{\pm 1\},$$

given by sending N to η as defined by the formula in the statement of the Proposition 2.5. It can be immediately verified that N is a cocycle if and only if $\eta(xy,t) = \eta(x,t)\eta(y,x^{-1}tx)$ for all $x,y \in W$, $t \in T$. This holds if and only if $w(t,\epsilon) = (wtw^{-1}, \eta(w^{-1},t)\epsilon)$ defines a representation of W on $\widehat{T}$. Furthermore, for $s \in S$, $N(s) = \{s\}$ if and only if $s(t,\epsilon) = (sts, (-1)^{\delta_{s,t}}\epsilon)$, where $\delta_{s,t}$ is the Kronecker delta.

The remaining assertions (a)–(e) are clear from the above references and the preceding arguments. $\square$

2.6. Whenever the conditions of the proposition hold, we call l the *standard length function*, T the set of *reflections* and N the *reflection cocycle* of (W,S). We call the quasi-root system $(\widehat{T},F)$ the *standard abstract root system* of (W,S). The sets

$$\widehat{T}_+ := T \times \{1\} \subseteq \widehat{T} \quad \text{and} \quad \widehat{S} := S \times \{1\} \subseteq \widehat{T}_+$$

are called the *standard set of positive roots* and *standard root basis* of $\widehat{T}$, respectively. When convenient, we identify W with a group of permutations of $\widehat{T}$, in the natural way.

Here, we recall a well-known fact from **[3]**, which will be used several times in the sequel.

Lemma 2.6. *For any Coxeter system* (W,S)*,* S *is a minimal* (*under inclusion*) *set of generators of* W*.*

2.7. Consider a quasi-root system (Ψ,F) with a specified quasi-positive system Ψ_+. If $t = s_\beta$ with $\beta \in \Psi_+$, then $s_\beta(\beta) = -\beta \in -\Psi_+$. We say that $\beta \in \Psi_+$ is a *simple* quasi-root for Ψ_+ if, for each $\alpha \in \Psi_+$ with $s_\beta(\alpha) \notin \Psi_+$, we have $s_\beta = s_\alpha$. Note that we may have distinct simple quasi-roots β, γ for Ψ_+ with $s_\beta = s_\gamma$. The reflection s_β in a simple quasi-root β for Ψ_+ is called a *simple reflection* for Ψ_+.

Let S' be any subset of the set of all simple reflections for Ψ_+, and let

$$\Pi' := \{\beta \in \Psi_+ \mid s_\beta \in S'\}$$

be the set of all simple quasi-roots β for Ψ_+ with $s_\beta \in S'$. Let $W' = \langle S' \rangle$ denote the subgroup of $W = \langle s_\alpha \mid \alpha \in \Psi \rangle$, generated by S',

$$\Phi' := \{w(\alpha) \mid w \in W', \alpha \in \Pi'\},$$

$$T' := \{s_\alpha \mid \alpha \in \Phi'\}$$

and

$$\Phi'_+ := \Phi' \cap \Psi_+.$$

From the definitions of Π' and T',

$$\Phi'_+ = \{\alpha \in \Psi_+ \mid s_\alpha \in T'\}.$$

Proposition 2.7.

(a) *For $\alpha \in \Phi'$, s_α restricts to a permutation s'_α of Φ'.*

(b) *Define the map $F' \colon \Phi' \to \mathrm{Sym}(\Phi')$ by $\alpha \mapsto s'_\alpha$. Then, (Φ', F') is a quasi-root system with Φ'_+ as a positive system.*

(c) *The pair (W', S') is a Coxeter system with reflection cocycle*

$$N' \colon W' \longrightarrow \mathcal{P}(T'),$$

given by

$$N'(w) = \{s_\alpha \mid \alpha \in \Psi_+ \cap w(-\Psi_+)\} = \{s_\alpha \mid \alpha \in \Phi'_+ \cap w(-\Phi'_+)\}.$$

(d) *Restriction induces an isomorphism*

$$W' = \langle s_\alpha \mid \alpha \in \Phi' \rangle \xrightarrow{\cong} \langle s'_\alpha \mid \alpha \in \Phi' \rangle.$$

(e) *Π' is the set of all simple roots for Φ'_+ in Φ'.*

(f) *Let $(\widehat{T'}, F'')$ be the standard abstract root system of (W', S'). Then, there is a W'-equivariant surjection $\Phi' \to \widehat{T'}$, given by $\alpha \mapsto (s_\alpha, \epsilon)$ for $\alpha \in \epsilon\Phi'_+$, where $\epsilon \in \{\pm 1\}$. If $\{\pm 1\}$ acts simply transitively on the fibers of the reflection map of (Φ', F'), this surjection is an isomorphism*

$$(\Phi', F') \longrightarrow (\widehat{T'}, F'')$$

of quasi-root systems.

Proof. Part (a) follows from the definition of Φ' if $\alpha \in \Pi$ and from (B.1), in general.

Part (b) readily follows from the corresponding facts for Ψ. Define the function

$$N_{\Psi_+} : W \longrightarrow \mathcal{P}(T)$$

by

$$N_{\Psi_+}(w) := \{s_\alpha \mid \alpha \in \Psi_+ \cap w(-\Psi_+)\}.$$

By Proposition 2.2 (a), N_{Ψ_+} is a cocycle, and from the definition of S', $N_{\Psi_+}(s) = \{s\}$ for $s \in S'$. The cocycle condition implies $N_{\Psi_+}(w) \subseteq T'$ for all $w \in W'$; thus, N_{Ψ_+} restricts to a cocycle

$$N' \colon W' \longrightarrow \mathcal{P}(T'),$$

satisfying $N'(s) = \{s\}$ for all $s \in S'$. By Proposition 2.5, (W', S') is a Coxeter system with reflection cocycle N', where $N'(w) = \{s_\alpha \mid \alpha \in \Psi_+ \cap w(-\Psi_+)\}$.

To prove (c), it will suffice to show that, for $w \in W'$,

$$\Psi_+ \cap w(-\Psi_+) = \Phi'_+ \cap w(-\Phi'_+).$$

Clearly, the right hand side is included in the left. However, if $\alpha \in \Psi_+$ with $w^{-1}(\alpha) \in -\Psi_+$, then, from above, $s_\alpha \in T'$, so $\alpha \in \Phi'_+$ and $w^{-1}(\alpha) \in -\Phi'_+$, which proves equality.

Now for (d), suppose that $w \in W'$ has restriction $w_{|\Phi'} = \mathrm{Id}_{\Phi'}$. Then,

$$\Phi'_+ \cap w(-\Phi'_+) = \Phi'_+ \cap w_{|\Phi'}(-\Phi'_+) = \emptyset;$$

thus, $N(w) = \emptyset$ and $w = \mathrm{Id}_\Psi$ by Proposition 2.5.

For (e), it is clear that Π' is a subset of the set of all simple roots for Φ'_+ in Φ'. On the other hand, if $\alpha \in \Phi'$ is a simple root for Φ'_+ in Φ', then the definitions readily give that $N'(s_\alpha) = \{s_\alpha\}$, and thus, $s_\alpha \in S$, which implies $\alpha \in \Pi$.

Finally, (f) follows from Proposition 2.2 (d). $\square$

Definition 2.8. A quasi-positive system Ψ_+ for a quasi-root system (Ψ, F) will be said to be *generative* if, for some set S' of simple reflections for Ψ_+, we have $\Phi' = \Psi$ (or equivalently, $W' = W$) in subsection 2.7 above. From Lemma 2.6, S' is necessarily the set of all simple reflections for Ψ_+.

We then call (W, S') the Coxeter system attached to Ψ_+. As we observe later, the isomorphism type of (W, S') as a Coxeter system in general depends upon the choice of Ψ_+. The sets of quasi-positive systems and of generative quasi-positive systems are clearly both stable under the W-action.

Example 2.9. Let (W, S) be a Coxeter system of type B_2 with $S := \{r, s\}$ as its set of simple reflections (so rs has order 4). The set of reflections is $T = \{r, rsr, srs, s\}$. We consider the standard abstract root system $\widehat{T} = T \times \{\pm 1\}$ of (T, S), which has standard positive system $\widehat{T}_+ := T \times \{1\}$. It can readily be verified that

$$\Psi_+ := \{(s, 1), (srs, 1), (r, 1), (rsr, -1)\}$$

is a generative quasi-positive system for $\widehat{T}$, with $\Delta := \{(s, 1), (srs, 1)\}$ as the corresponding set of abstract simple roots and $S' := \{s, srs\}$ as the corresponding set of abstract simple reflections. Observe that $\widehat{T}_+$ and Ψ_+ are not W-conjugate.

Definition 2.10. An *abstract root system* is a quasi-root system (Ψ, F) for which there exists some generative quasi-positive system. Elements of Ψ will then be called *roots* instead of quasi-roots.

Note the pair (W, T), where $T = \{s_\alpha \mid \alpha \in \Psi\}$ depends only on (Ψ, F) and not on the choice of generative quasi-positive system.

2.8. Fix a Coxeter system (W, S). For any $J \subseteq S$, let W_J denote the subgroup of W generated by J. The subgroups W_J are called *standard parabolic subgroups* of (W, S), and their W-conjugates are called *parabolic subgroups.* A *reflection subgroup* of W is a subgroup W' which is generated by $W' \cap T$. A *dihedral reflection subgroup* is a reflection subgroup which may be generated by two distinct reflections of (W, S).

2.9. Here, we recall some properties of reflection subgroups; for proofs, see **[8, 11]**. For any reflection subgroup $W' = \langle W' \cap T \rangle$ of W,

$$S' = \chi(W') = \chi_{(W,S)}(W') := \{t \in T \mid N(t) \cap W' = \{t\}\} \tag{2.1}$$

is a set of Coxeter generators for W'. The corresponding set of reflections and reflection cocycle for (W', S') are

$$T' := W' \cap T$$

and

$$N' \colon w \longrightarrow N(w) \cap W',$$

respectively. If $R \subseteq T$ is any set of reflections of W with $W' = \langle R \rangle$, then

$$\bigcup_{w \in W'} wRw^{-1} = T'$$

and $|S'| \leq |R|$. In particular, if R is a set of Coxeter generators for W with $R \subseteq T$, then

$$T = \bigcup_{w \in W} wRw^{-1}$$

and $|R| = |S|$. We call $|S|$ the rank of (W, S); it depends only on the pair (W, T). It is known that any dihedral reflection subgroup is contained in a unique maximal (under inclusion) dihedral reflection subgroup.

The following result follows directly from the above facts and Proposition 2.5.

Proposition 2.11. *Let $\widehat{T'} = T' \times \{\pm 1\}$ and, for $\alpha \in \widehat{T'}$, let s'_α denote the restriction of s_α to a permutation of $\widehat{T'}$. Let*

$$F' \colon \widehat{T'} \longrightarrow \mathrm{Sym}(\widehat{T'})$$

denote the map $\alpha \mapsto s'_\alpha$. Then, $(\widehat{T'}, F')$ is equal to the standard abstract root system of (W', S').

Proposition 2.12. *Let $\Delta \subseteq \widehat{T}_+$. Then, Δ is the standard set of simple roots of the standard abstract root system $T' \times \{\pm 1\}$ of some reflection subgroup W' of W, with $T' = T \cap W'$, if and only if, for each $\alpha \neq \beta$ in Δ, $\{\alpha, \beta\}$ is the set of simple roots of the standard positive system of the standard abstract root system $(\langle s_\alpha, s_\beta \rangle \cap T) \times \{\pm 1\}$ of $\langle s_\alpha, s_\beta \rangle$.*

Proof. This is a direct translation using [**11,** subsection 3.5, Proposition 2.5]. $\square$

2.10. If W' is a reflection subgroup of (W,S) and $w \in W$, there is a unique $y \in wW'$ with $N(y^{-1}) \cap W' = \emptyset$. Then,

$$\chi(wW'w^{-1}) = y\chi(W')y^{-1},$$

by [**10,** Lemma 1]. As in subsection 2.9, regard the (underlying sets of) abstract root systems attached to $(W', \chi(W'))$ and $(wW'w^{-1}, \chi(wW'w^{-1}))$ as subsets of $\widehat{T}$.

Proposition 2.13. *For W', w and y as in subsection* 2.10, *the map*

$$\alpha \longmapsto y(\alpha) \colon \widehat{T} \longrightarrow \widehat{T}$$

restricts to an isomorphism from the standard abstract root system of $(W', \chi(W'))$ to the standard abstract root system of $(wW'w^{-1}, \chi(wW'w^{-1}))$. This bijection restricts to a bijection between corresponding sets of abstract positive roots, and, similarly, for abstract simple roots.

Proof. This follows easily using Proposition 2.5 and the definitions. □

We record the following lemma for use in Section 4.

Lemma 2.14. *Let $J \subseteq S$ be such that (W_J, J) is an infinite irreducible Coxeter system. If $w \in W$ with $l(wr) > l(w)$ for all $r \in J$ and $wJw^{-1} \subseteq S$, then $w \in W_K$, where*

$$K := \{r \in S \setminus J \mid rs = sr \textit{ for all } s \in J\}.$$

Proof. This directly follows from a result of Deodhar [**5,** Proposition 2.3] on conjugacy of parabolic subgroups and is even implicit as a very special case of the discussion after the statement of Theorem A in *loc cit.* In order to prove it explicitly, we use induction on $l(w)$. If $w \neq 1_W$, choose $a \in S \cap N(w^{-1})$. From Proposition 2.3 of *loc cit.*, $a \notin J$, and the irreducible component of $(W_{J\cup\{a\}}, J \cup \{a\})$ containing a is a finite Coxeter system, that is, in the notation of *loc cit.*, the element $v[a, J]$ is defined. Since W_J is infinite and irreducible, this implies that this irreducible component has a as its only simple reflection, and hence, that $a \in K$. Now, $l(war) > l(wa)$ for all $r \in J$, and $(wa)J(wa)^{-1} = wJw^{-1} \subseteq S$; thus, by induction, $wa \in W_K$, and the proof is complete. □

2.11. Next, we more closely examine the result Proposition 2.7 in the special case that (Ψ, F) is the standard abstract root system $(\widehat{T}, F)$ of a Coxeter system (W, S). For a quasi-positive system $\Psi_+ \subseteq \widehat{T}$, let Δ_{Ψ_+} and S_{Ψ_+} denote the sets of simple roots and simple reflections for Ψ_+, that is,

$$\Delta_{\Psi_+} := \{\alpha \in \Psi_+ \mid s_\alpha(\Psi_+ \setminus \{\alpha\}) \subseteq \Psi_+\} \tag{2.2}$$

$$S_{\Psi_+} := \{s_\alpha \mid \alpha \in \Delta_{\Psi_+}\}. \tag{2.3}$$

Let $W_{\Psi_+} := \langle S_{\Psi_+} \rangle$ denote the subgroup of W generated by S_{Ψ_+}. (The fact in Proposition 2.7 (c) that (W_{Ψ_+}, S_{Ψ_+}) is a Coxeter system also follows from [**9,** subsection 1.8] using Proposition 2.5.) For example, if $\Psi_+ = \widehat{T}_+$, then $\Delta_{\Psi_+} = S \times \{1\}$ and $S_{\Psi_+} = S$, as is well known.

Definition 2.15. A subset S' of T is an *abstract set of simple reflections* for $\widehat{T}$ if $S' = S_{\Psi_+}$ for some generative quasi-positive system Ψ_+. A subset Δ of $\widehat{T}$ is an *abstract set of simple roots* for $\widehat{T}$ if $\Delta = \Delta_{\Psi_+}$ for some generative quasi-positive system Ψ_+.

Note that any abstract set of simple reflections of $\widehat{T}$ is a set of Coxeter generators of W contained in the set of reflections of (W, S), but that the converse does not hold, see Example 2.22.

Lemma 2.16. *The map*

$$\Psi_+ \longmapsto \Delta_{\Psi_+}$$

is a bijection between the set of generative quasi-positive systems and the set of abstract sets of simple roots of $(\widehat{T}, F)$.

Proof. Clearly, Ψ_+ determines Δ_{Ψ_+}, for any quasi-positive system Ψ_+. On the other hand, if Ψ_+ is a generative quasi-positive system, then, by Proposition 2.5,

$$S_{\Psi_+} = \{s_\alpha \mid \alpha \in \Delta_{\Psi_+}\},$$
$$W = \langle S_{\Psi_+} \rangle, \quad \text{and}$$

$$\Psi_+ = \{w(\alpha) \mid w \in W, \alpha \in \Delta_{\Psi_+}, l_{\Psi_+}(ws_\alpha) > l(w)\},$$

where l_{Ψ_+} is the standard length function of (W, S_{Ψ_+}). This shows that Δ_{Ψ_+} determines Ψ_+. $\square$

The next lemma will be crucial in Section 4.

Lemma 2.17. *Let Ψ_+ be a generative quasi-positive system for the standard abstract root system $(\widehat{T}, F)$ of (W,S) and W' a reflection subgroup of (W,S). Set $S' = \chi(W')$. Regard the abstract root system $\widehat{T'}$ of (W',S') as a subset of $\widehat{T}$, as in Proposition* 2.11. *Then, $\Psi'_+ := \Psi_+ \cap \widehat{T'}$ is a generative quasi-positive system for $\widehat{T'}$, and the set of simple reflections of Ψ'_+ is the set $\chi_{(W,S_{\Psi_+})}(W')$ of canonical generators of W' with respect to the Coxeter system (W, S_{Ψ_+}).*

Proof. Clearly, Ψ'_+ is a quasi-positive system for $\widehat{T'}_+$. By Lemma 2.6, it suffices to show that, if $\alpha \in \Psi'_+$ with $s_\alpha \in \chi_{(W,S_{\Psi_+})}(W')$, then $\alpha \in \Pi'$, where Π' is the set of simple roots for Ψ'_+, that is, $s_\alpha(\Psi'_+ \setminus \{\alpha\}) \subseteq \Psi'_+$. Now by subsection 2.9, $N_{\Psi_+}(s_\alpha) \cap W' = \{s_\alpha\}$, that is, if $\gamma \in \Psi_+$ with $s_\gamma \in W'$ and $s_\alpha(\gamma) \in -\Psi_+$, then $\gamma = \alpha$. Since, for $\gamma \in \Psi_+$, we have $s_\gamma \in W'$ if and only if $\gamma \in \Psi'_+$, this gives the desired conclusion. $\square$

Example 2.18. Let $\{C_i\}_{i \in I}$ be the conjugacy classes of reflections in W. For each $i \in I$, choose $\epsilon_i \in \{\pm 1\}$. Then,

$$\Phi_+ := \bigcup_{i \in I} (C_i \times \{\epsilon_i\})$$

is a quasi-positive system. It is easy to verify that $S \subseteq S_{\Phi_+}$, so $W_{\Phi_+} = W$ and $S_{\Phi_+} = S$ by Lemma 2.6. The non-standard generative quasi-positive system Ψ_+ for type B_2 in Example 2.9 arises by conjugation of one of the quasi-positive systems arising as above.

Proposition 2.20 below shows, in particular, that any generative quasi-positive system with S as the corresponding set of simple reflections arises, as in the preceding example. For the proof, we use the following fact about conjugacy of simple reflections, which readily follows from the exchange condition and is a special case of the more general results of Deodhar and Brink-Howlett [**5**] on conjugacy of parabolic subgroups.

Lemma 2.19. *Let (W,S) be a Coxeter system and $r, s \in S$, $w \in W$ satisfy $wrw^{-1} = s$ and $l(wr) = l(w) + 1$. Then, there exist $k \in \mathbb{N}$ and* *sequences* $w_1, \dots, w_k$ *in* W, $J_1, \dots, J_k \subseteq S$, $a_0, a_1, \dots, a_k$ *in* W, *with the following properties*:

(i) $a_0 = r$, $a_k = s$;
(ii) $a_{i-1}, a_i \in J_i$ *for* $i = 1, \dots, k$;
(iii) $|J_i| = 2$ *for* $i = 1, \dots, k$;
(iv) $w_i \in W_{J_i}$, $w_i a_i = a_{i+1} w_i$ *and* $l(w_i a_i) = l(w_i) + 1$ *for* $i = 1, \dots, k$;
(v) $w = w_k \cdots w_2 w_1$ *with* $l(w) = l(w_1) + l(w_2) + \cdots + l(w_k)$.

Proposition 2.20. *Let* $S' \subseteq T$ *be such that* (W, S') *is a Coxeter system. Consider a family of roots*

$$\Pi' = \{\alpha_r\}_{r \in S'} \subseteq \widehat{T},$$

such that $s_{\alpha_r} = r$ *for all* $r \in S'$. *Then,* Π' *is the set of simple roots of some* (*generative*) *quasi-positive system* Ψ_+ *for* $\widehat{T}$ *if and only if* $(rt)^m(\alpha_r) = \alpha_t$ *whenever* $r \neq t$ *are in* S' *with the order of* rt *an odd integer* $2m+1$.

Proof. The "only if" direction is clear, since, under the above conditions on r and t, if we write $\beta = (rt)^m(\alpha_r)$, we have $s_\beta = (rt)^m r (rt)^{-m} = t$ so $\beta \in \pm\{\alpha_t\}$; however, $\beta \in \Psi_+$ since $l'((rt)^m r) > l'((rt)^m)$, where l' is the standard length function of (W, S').

For the "if" direction, we shall first show that, for any $x, y \in W$, $r, t \in S'$ with $xrx^{-1} = yty^{-1}$, $l'(xr) > l'(x)$ and $l'(yt) > l'(y)$, we have $x(\alpha_r) = y(\alpha_t)$. In fact, using Proposition 2.5, the conditions imply that $zrz^{-1} = t$ and $l'(zr) > l'(z)$ with $z = y^{-1}x$. By Lemma 2.19, the proof that $z(\alpha_r) = \alpha_t$ under these conditions reduces to its special case in which $z \neq 1$ and z, r, t all lie in some finite rank two standard parabolic subgroup of (W, S'), say, of order $2m$. If m is even, $z(\alpha_r) = \alpha_t$ is automatic, whereas, if m is odd, $z(\alpha_r) = \alpha_t$ by the assumptions.

Next, we set

$$\Psi_+ := \{x(\alpha_r) \mid x \in W, r \in S', l'(xr) > l'(x)\}.$$

The above implies that Ψ_+ is a quasi-positive system for $\widehat{T}$. Using the exchange condition for (W, S'), we see that Π' is contained in the set of simple roots of Ψ_+, and equality follows by Lemma 2.6. $\square$

2.12. We illustrate the preceding result in relation to isomorphisms of Coxeter groups obtained by diagram twisting, as introduced in [**4**].

Let (W,S) be a Coxeter system with Coxeter matrix $(m_{r,s})_{r,s\in S}$. Thus, for $r,s\in S$, $m_{r,s}\in\mathbb{N}_{\geq 1}\cup\{\infty\}$ denotes the order of the product $rs\in W$. Suppose that S is the disjoint union $S=J\cup K\cup L\cup M$, where W_K is finite with longest element w_K,

$$M\subseteq\{s\in S\mid m_{s,r}=\infty \text{ for all } r\in J\}$$

and $m_{r,s}=2$ for all $r\in L$, $s\in K$. We let

$$J'=w_K J w_K \quad\text{and}\quad S':=J'\cup K\cup L\cup M\subseteq T.$$

Then, [**4**] (W,S') is a Coxeter system, said to be obtained from (W,S) by a *diagram twist.* The Coxeter matrix $(m'_{r,s})_{r,s\in S'}$ of (W,S') is given as follows. If $r,s\in K\cup L\cup M$, then $m'_{r,s}=m_{r,s}$. If $r\in J'$ and $s\in K\cup L\cup M$, then $m'_{r,s}=m'_{s,r}=m_{w_K r w_K,s}$. Finally, if $r,s\in J'$, then $m'_{r,s}=m_{w_K r w_K,w_K s w_K}$.

Corollary 2.21. *Suppose, in Lemma* 2.20*, that* (W,S') *is obtained by twisting* (W,S) *with notation as in subsection* 2.12*. Set* $\alpha_r=(r,\epsilon_r)$ *with* $\epsilon_r\in\{\pm 1\}$ *for all* $r\in S'$*. Then,* $\Pi':=\{\alpha_r\}_{r\in S'}$ *is the set of simple roots of some generative quasi-positive system for* $\widehat{T}$ *if and only if the following conditions* (i)–(iii) *hold*:

(i) $\epsilon_r=\epsilon_s$ *whenever* $r\neq s$ *are in* $K\cup L\cup M$ *and* $m'_{r,s}$ *is finite and odd*;
(ii) $\epsilon_r=-\epsilon_s$ *when* $r\in J'$ *and* $s\in K$ *with* $m'_{r,s}$ *finite and odd*;
(iii) $\epsilon_r=\epsilon_s$ *whenever* $r\in J'$, $s\in L$ *and* $m'_{r,s}$ *is finite and odd.*

Proof. The corollary readily follows on applying Lemma 2.20 to both S' and $S\times\{1\}\subseteq\widehat{T}$ and noting that each pair of distinct elements of S' which are not both in S and whose product has finite order is conjugated to a pair of elements of S by w_K. $\square$

Example 2.22. Consider the Coxeter system (W,S) with $S=\{r,s,t,u\}$ and Coxeter graph, as at left in Figure 1. Twisting, as in 2.12 using $J=\{r\}$, $K=\{s,t\}$, $M=\{u\}$ and $L=\emptyset$, gives the Coxeter system (W,S'), where $S'=\{r',s,t,u\}$ with $r'=stsrsts$ and with Coxeter graph as at right:

FIGURE 1.

In this case, Corollary 2.21 implies that $\{(r', -1), (s, 1), (t, 1), (u, 1)\}$ is the set of abstract simple roots of some generative quasi-positive system, with S' as the corresponding set of simple reflections. In particular, this shows that the non-isomorphic (irreducible, finitely-generated) Coxeter systems (W, S) and (W, S') have isomorphic standard abstract root systems.

On the other hand, take (W, S) with $S = \{a, b, c, d\}$ and the Coxeter graph as on left hand side of Figure 2. Twisting with $J = \{d\}$, $K = \{a\}$, $L = \{b\}$ and $M = \{c\}$ give the isomorphic Coxeter system (W, S') with $S' = \{a, b, c, d'\}$ where $d' = ada$, and Coxeter graph as on right hand side of Figure 2.

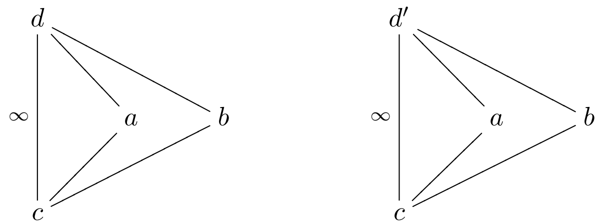


FIGURE 2.

Here, Corollary 2.21 shows that there is no generative quasi-positive system with S' as its corresponding set of simple reflections. This shows that arbitrary diagram twists do not necessarily extend to twists of the standard abstract root system.

Question 2.23. *The above suggests the problem of determining when (finitely generated, irreducible) Coxeter systems have isomorphic standard abstract root systems. We might also ask whether there is some* *natural way of attaching an abstract root system* $\widehat{\Phi}(W,S)$ *to each Coxeter system* (W,S) *so that two (say, finitely generated, irreducible) Coxeter systems* (W_i,S_i) *are isomorphic if and only if they have isomorphic abstract root systems* $\widehat{\Phi}(W_i,S_i)$.

In order to limit the, somewhat pathological, behavior seen in examples such as Examples 2.9, 2.18 and 2.22, we now combinatorially introduce a notion which corresponds to that of betweenness considered in the introduction, Proposition 4.1 (c).

Definition 2.24. Let $\alpha,\beta,\gamma \in \widehat{T}$. We say that γ is *between* α and β if one of the following conditions holds:

(i) $\alpha = \pm\beta$ and $\gamma \in \{\alpha,\beta\}$.

(ii) $\alpha \neq \pm\beta$ and, for any maximal dihedral reflection subgroup W' of (W,S), all $w \in W'$ and all $\epsilon \in \{\pm 1\}$ with $\epsilon w(\alpha), \epsilon w(\beta) \in \widehat{T'}_+ := (W' \cap T) \times \{1\}$, we have $\epsilon w(\gamma) \in \widehat{T'}_+$.

We let $[\alpha,\beta]$ denote the set of all $\gamma \in \widehat{T}$ such that γ is between α and β. We shall refer to $[\alpha,\beta]$ as an *interval of roots* with *endpoints* α and β.

If $\alpha \neq \pm\beta$, the only possible maximal dihedral reflection subgroup W' as in (ii) is that containing s_α and s_β (although, possibly, several pairs (w,ϵ) may satisfy the conditions of (ii)). On the other hand, using subsection 2.10, Proposition 2.13 and the fact that the set of maximal dihedral reflection subgroups of W is closed under conjugation by elements of W, we see that an equivalent condition to (ii) would be obtained by replacing "$w \in W'$" in (ii) by "$w \in W$." From this, it is clear, for any $w \in W$, $\epsilon \in \{\pm 1\}$ and α,β,γ in $\widehat{T}$, that $\gamma \in [\alpha,\beta]$ if and only if $\epsilon w(\gamma) \in [\epsilon w(\alpha), \epsilon w(\beta)]$.

2.13. We provide here, without proof, a more concrete description of betweenness, compare Proposition 4.1 (c). Let $\alpha,\beta \in \widehat{T}$ with $\alpha \neq \pm\beta$. Let W' be the maximal dihedral reflection subgroup W' of W containing $\{s_\alpha, s_\beta\}$, and set

$$T' = W' \cap T.$$

Let $\chi(W') = \{r,s\}$ and $m = |T'|$. Consider the diagram in Figure 3 of m straight lines through the origin $\bullet = (0,0)$ in the plane $\mathbb{R}^2$:

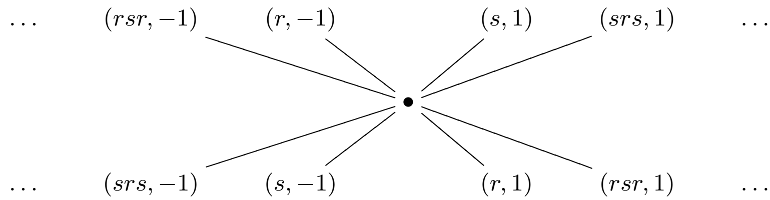


FIGURE 3.

If m is finite, the lines are supposed to pass, say, through the vertices of some regular k-gon with centroid at $\bullet$, where $k = m$ if m is odd and $k = 2m$ if m is even, while, if m is infinite, the lines are taken as those passing through, say, the points $(n, \pm 1)$ for $n \in \mathbb{Z} \setminus \{0\}$. The resulting $2m$ (closed) rays with $\bullet$ as endpoint are labeled by the elements of $\widehat{T'} \times \{\pm 1\}$, as suggested by the diagram, so that each line is the union of rays with labels $(t, 1)$ and $(t, -1)$ for some (unique) $t \in T$.

Now, for $\gamma \in \widehat{T}$, we have $\gamma \in [\alpha, \beta]$ if and only if the ray labelled γ is in the convex closure of the union of the rays labelled α and β. (This is easy to see either directly from the definition, or using Proposition 4.1 (c).)

Lemma 2.25. *For any $\alpha, \beta \in \widehat{T}$, $s_\beta(\alpha)$ is completely determined by α, β, the function*

$$(\gamma, \delta) \longmapsto [\gamma, \delta] \colon \widehat{T} \times \widehat{T} \longrightarrow \mathcal{P}(\widehat{T}),$$

and the action of $\{\pm 1\}$ on $\widehat{T}$.

Proof. First, if $\beta = \pm\alpha$, then $s_\beta(\alpha) = -\alpha$, so we suppose henceforward that $\beta \neq \pm\alpha$. Consider the smallest (under inclusion) subset $\widehat{T'}$ of $\widehat{T}$ containing $\{\alpha, \beta\}$, closed under the action of $\{\pm 1\}$ and such that, if $\gamma, \delta \in \widehat{T'}$, then $[\gamma, \delta] \subseteq \widehat{T'}$. It is easy to see that $\widehat{T'} = T' \times \{\pm 1\}$, where W' is the maximal dihedral reflection subgroup of W containing $\{s_\alpha, s_\beta\}$ and $T' = W' \cap T$, that is, $\widehat{T'}$ is the standard root system of W'.

Now, there is a unique element $\gamma \in \widehat{T'}$ such that $\alpha \in [\beta, \gamma]$ and $[\gamma, -\beta] = \{\gamma, -\beta\}$. In fact, $\{\beta, \gamma\}$ is the set of simple roots of some quasi-positive system Φ_+ of $\widehat{T'}$, given by $\Phi_+ = \epsilon w'(T' \times \{1\})$, for some $\epsilon \in \{\pm 1\}$ and $w' \in W'$. Furthermore, $[\beta, \gamma] = \epsilon w'(T' \times \{1\})$. Note that

$[\gamma,\beta]\setminus\{\beta\}=[\gamma,s_\beta(\gamma)]$ contains α and is s_β-stable. We claim that the action of s_β on the interval of roots $[\gamma,s_\beta(\gamma)]$ containing α is uniquely determined by the facts that s_β interchanges the endpoints γ and $s_\beta(\gamma)$ and preserves betweenness.

To prove the claim, observe that both sets

$$\{[\gamma,\alpha')]\mid \alpha'\in[\gamma,s_\beta(\gamma)]\}$$

and

$$\{[\alpha',s_\beta(\gamma)]\mid \alpha'\in[\gamma,s_\beta(\gamma)]\}$$

are totally ordered by inclusion. For any $\alpha'\in[\gamma,s_\beta(\gamma)]$, attach the ordered pair $(m_{\alpha'},n_{\alpha'})$ of cardinalities $m_{\alpha'}=|[\alpha',s_\beta(\gamma))|$ and $n_{\alpha'}=|[\gamma,\alpha']|$. Note that, at most, one of $m_{\alpha'}$ and $n_{\alpha'}$ is infinite. The preceding observation involving total orders, therefore, shows that α' is uniquely determined by $(m_{\alpha'},n_{\alpha'})$. Take $\alpha':=s_\beta(\alpha)\in[\gamma,s_\beta(\gamma)]$. Since s_β preserves betweenness, it follows from the definitions that $(m_{\alpha'},n_{\alpha'})=(n_\alpha,m_\alpha)$. Hence, $\alpha'=s_\beta(\alpha)$ is determined by α in the required manner. $\square$

Definition 2.26. A subset P of $\widehat{T}$ is said to be *closed* (in $\widehat{T}$) if, for any $\alpha,\beta\in P$ and $\gamma\in\widehat{T}$ such that γ is between α and β, we have $\gamma\in P$. We say that P is *biclosed* (in $\widehat{T}$) if P and $\widehat{T}\setminus P$ are both closed in $\widehat{T}$. Similarly, for a reflection subgroup W' of W, we may say that a subset P of $\widehat{T'}$ is closed or biclosed in $\widehat{T'}=T'\times\{\pm1\}$.

There should be no confusion with the notion of a closed subset of the root system of a finite Weyl group as defined in **[3]**, which will not be used in this paper.

Example 2.27. Suppose that (W,S) is a dihedral Coxeter system, say $S=\{r,s\}$, where $r\neq s$. Let B denote the set of all biclosed quasi-positive systems Ψ_+ for $\widehat{T}$. It is easy to see that, if W is finite, then $B=\{w(\widehat{T}_+)\mid w\in W\}$. If W is infinite,

$$B=\{\epsilon w(\widehat{T}_+)\mid w\in W,\epsilon\in\{\pm1\}\}\cup\{\Psi_+,\widehat{T}\setminus\Psi_+\},$$

where

$$\Psi_+=\{(t,1)\mid t\in T, r\in N(t)\}\cup\{(t,-1)\mid t\in T, s\in N(t)\}.$$

Hence, if $P\in B$ is generative, then P is W-conjugate to $\pm\widehat{T}_+$.

Definition 2.28. A subset Ψ_+ of $\widehat{T}$ is an *abstract positive system* for (W, S) if Ψ_+ is a biclosed, generative, quasi-positive system of $\widehat{T}$. Then, Δ_{Ψ_+} is called the corresponding *abstract root basis* for Ψ_+, and S_{Ψ_+} is called the set of *simple reflections* of W corresponding to Ψ_+. Recall from subsection 2.11 that (W, S_{Ψ_+}) is a Coxeter system.

When confusion with the notion of a positive system of roots of a real root system (as considered in Section 3) seems unlikely, we may call abstract positive systems merely positive systems.

2.14. Let (W, S) be a Coxeter system with irreducible components (W_i, S_i) for $i \in I$. Then, the standard abstract root system $(\widehat{T}, F)$ of (W, S) identifies (as quasi-root system) with the union as in subsection 2.2 of the (quasi-root systems arising as) standard abstract root systems $(\widehat{T}_i, F_i)$ of (W_i, S_i), for $i \in I$. Under this identification, the standard root basis of $(\widehat{T}, F)$ is the union of the standard root bases of $(\widehat{T}_i, F_i)$. The analogous statement for standard sets of positive roots, instead of standard root bases, also holds.

Lemma 2.29. *Let $\Psi_+ \subseteq \widehat{T}$ and other notation be as in subsection* 2.14.

(a) Ψ_+ *is a generative quasi-positive system of $\widehat{T}$ if and only if for each i, $\Psi_+^i := \Psi_+ \cap \widehat{T}_i$ is a generative quasi-positive system of $\widehat{T}_i$. In that case, the corresponding set of simple roots is*

$$\Delta_{\Psi_+} = \bigcup_i \Delta_+^i,$$

a disjoint union, where Δ_+^i is the set of simple roots of Ψ_+^i in $\widehat{T}_i$.

(b) *In* (a), Ψ_+ *is a positive system of $\widehat{T}$ if and only if, for each i, $\Psi_+^i := \Psi_+ \cap \widehat{T}_i$ is a positive system of $\widehat{T}_i$.*

Proof. Observe that, if $t \in W_i \cap T$ and $s \in W_j \cap T$ with $i \neq j$ are reflections of W which are contained in distinct irreducible components of W, then $st = ts$ and the maximal dihedral reflection subgroup containing s and t is merely $\{1, s, t, st\}$. Using this, the lemma is easily proven from the definitions. Further details are omitted. $\square$

Lemma 2.30. *Let W' be a reflection subgroup of (W,S). Set $S'=\chi(W')$. Regard the abstract root system $\widehat{T'}$ of (W',S') as a subset of $\widehat{T}$. Fix $\alpha,\beta,\gamma\in\widehat{T'}$ and $P\subseteq\widehat{T}$.*

(a) *γ is between α and β in $\widehat{T'}$ if and only if γ is between α and β in $\widehat{T}$.*

(b) *If P is closed in $\widehat{T}$, then $P\cap\widehat{T'}$ is closed in $\widehat{T'}$. Similarly, if P is biclosed in $\widehat{T}$, then $P\cap\widehat{T'}$ is biclosed in $\widehat{T'}$.*

(c) *If Ψ_+ is a positive system for $\widehat{T}$, then $\Psi_+\cap\widehat{T'}$ is a positive system for $\widehat{T'}$.*

(d) *Suppose that W' is a dihedral reflection subgroup. If Ψ_+ is a positive system for $\widehat{T}$, then $\Psi_+\cap W'$ is conjugate in W' to $T'\times\{\epsilon\}$ for some $\epsilon\in\{\pm1\}$, where $T'=W'\cap T$. Furthermore, the abstract root basis of the abstract root system $\widehat{T'}$ of W' corresponding to the positive system $\Psi_+\cap\widehat{T'}$ is then conjugate by an element of W' to $\chi(W')\times\{\epsilon\}$.*

Proof. Note that every maximal dihedral reflection subgroup of W' is contained in a maximal dihedral reflection subgroup of W, and any maximal dihedral reflection subgroup of W which contains at least two reflections of W' intersects W' in a maximal dihedral reflection subgroup of W'. Now, (a) follows from these remarks, together with Propositions 2.11, 2.12, 2.13 and subsection 2.10 (or by other arguments involving subsection 2.13). Then (b) follows from (a), and (c) follows from (b) and Lemma 2.17. Using (c), (d) reduces to the case in which $W=W'$ is dihedral and $\chi(W')=S$, when it follows using Example 2.27. $\square$

Lemma 2.31. *For $i=1,2$, let (W_i,S_i) be a Coxeter system with abstract root system $\Phi(W_i,S_i)$. Let*

$$\theta\colon\Phi(W_1,S_1)\longrightarrow\Phi(W_2,S_2)$$

be a bijection. Consider the following conditions (i)–(iii):

(i) *for $\alpha,\beta,\gamma\in\Phi(W_1,S_1)$, γ is between α and β if and only if $\theta(\gamma)$ is between $\theta(\alpha)$ and $\theta(\beta)$ in $\Phi(W_2,S_2)$;*

(ii) *$\theta(\pm\alpha)=\pm\theta(\alpha)$ for all $\alpha\in\Phi_1(W,S)$;*

(iii) *θ is an isomorphism of quasi-root systems, that is,* $\theta(s_\alpha(\beta)) = s_{\theta(\alpha)}(\theta(\beta))$ *for all* $\alpha, \beta \in \Phi(W_1, S_1)$.

Then:

(a) *Conditions* (i) *and* (ii) *above hold if and only if* (i) *and* (iii) *hold. In that case, θ is an isomorphism of abstract root systems.*

(b) *Condition* (iii) *implies that there is a group isomorphism* $W_1 \to W_2$ *mapping* $s_\alpha \mapsto s_{\theta(\alpha)}$.

We say that θ preserves betweenness if conditions (i)–(iii) *above are satisfied.*

Proof. By taking $\beta = \alpha$ in (iii), we see that (iii) implies (ii). That (i) and (ii) implies (iii) is a direct consequence of Lemma 2.25. The remainder of (a) is trivial. Part (b) may be proven using the following fact, which is well known and easily verified: if (W, S) is a Coxeter system with set of reflections T, then W is isomorphic to the group generated by generators $\bar{t}$ for $t \in T$ subject to relations $\bar{t}\,\overline{t'}\,\bar{t} = \overline{tt't}$ for all $t, t' \in T$. $\square$

Proposition 2.32. *Let (W, S) be a Coxeter system with standard abstract root system $(\widehat{T}, F)$. Let Ψ_+ be any generative quasi-positive system for $(\widehat{T}, F)$, and define a map*

$$\theta\colon \widehat{T} \longrightarrow \widehat{T}$$

by $\theta(\alpha) = (s_\alpha, \epsilon)$ *for* $\alpha \in \epsilon\Psi_+$ *and* $\epsilon \in \{\pm 1\}$.

(a) *Let* $S' := S_{\Psi+}$ *be the set of abstract simple reflections for* Ψ_+. *Then,* (W, S') *is a Coxeter system with reflections* $T' = T$.

(b) *Let* $(\widehat{T'}, F'')$ *be the standard abstract root system of* (W, S'). *Then,* $\widehat{T'} = \widehat{T}$, *and the map θ is an isomorphism* $(\widehat{T}, F) \to (\widehat{T'}, F'')$ *of quasi-root systems.*

(c) *We have* $\theta^2 = \mathrm{Id}_{\widehat{T}}$ *and* $\theta(\widehat{T}_+) = \Psi_+$.

(d) *If Ψ_+ is a positive system of $(\widehat{T}, F)$, then θ preserves betweenness.*

(e) *In general, Ψ_+ is a generative quasi-positive system of $(\widehat{T}, F'')$. If Ψ_+ is a positive system of $(\widehat{T}, F)$, it is also a positive system of $(\widehat{T}, F'')$.*

Proof. We write $\Psi = \widehat{T}$ and use the results (and notation) of subsection 2.7 and Proposition 2.7. Since Ψ_+ is generative, we have $W' := \langle S' \rangle = W$, and (a) follows. Hence, $\Phi' := W\Pi' = \Psi$ (where Π' is the set of abstract simple roots of Ψ_+), $\Phi'_+ := \Phi' \cap \Psi_+ = \Psi_+$ and $\widehat{T'} = T' \times \{\pm 1\} = T \times \{\pm 1\} = \widehat{T}$. In Proposition 2.7, we, therefore, have $(\Phi', F') = (\Psi, F)$ and Proposition 2.7 (e) proves (b). For (c), we verify that, if $\beta = (t, \epsilon) \in \widehat{T}$, then

$$\theta(t,\epsilon) = \begin{cases} (t,\epsilon) & \text{if } (t,1) \in \Psi_+, \\ (t,-\epsilon) & \text{if } (t,1) \in -\Psi_+, \end{cases}$$

$$\theta(\beta) = \begin{cases} \beta & \text{if } \beta \in \pm(\Psi_+ \cap \widehat{T}_+), \\ -\beta & \text{if } \beta \in \pm(\Psi_+ \setminus \widehat{T}_+). \end{cases}$$

Now, we prove (d). Assume that Ψ is a positive system for $(\widehat{T}, F)$. We show that, for $\alpha, \beta \in \widehat{T}$,

$$\theta([\alpha,\beta]) = [\theta(\alpha), \theta(\beta)]''$$

in $(\widehat{T}, F'')$, where $[\alpha, \beta]$ and $[\theta(\alpha), \theta(\beta)]''$ are the evident intervals of roots in the standard abstract root systems $(\widehat{T}, F)$ and $(\widehat{T}, F'')$ of (W, S) and (W, S'), respectively. If $\alpha = \pm\beta$, this is trivial; thus, henceforward, we assume that $\alpha \neq \pm\beta$. Note that (W, S) and (W, S') have the same maximal dihedral reflection subgroups (since those subgroups depend only on (W, T)). Let W_d be the maximal dihedral reflection subgroup of (W, S) (and (W, S')) containing s_α and s_β, and let $T_d = W_d \cap T$ be its set of reflections. Regard the abstract root system $\widehat{T}_d = T_d \times \{1\}$ as a subset of $\widehat{T}$; more precisely, denote the corresponding quasi-root system as $(\widehat{T}_d, F_d)$. This abstract root system has a standard positive system $\widehat{T}_{d,+} = \widehat{T}_d \times \{1\}$, giving rise to a Coxeter system (W_d, S_d), where S_d is the set of simple reflections of $\widehat{T}_{d,+}$. By Lemma 2.30 (c), there is another positive system $\Psi_{d,+} = \Psi_+ \cap T_d$. Attached to this data, as in (a)–(c), we have another Coxeter system (W_d, S'_d), where $S_{d'}$ is the set of simple reflections of $\Psi_{d,+}$, the standard abstract root system $(\widehat{T}_d, F''_d)$ of (W_d, S_d), and a bijection

$$\theta_d \colon \widehat{T}_d \longrightarrow \widehat{T}_d,$$

inducing an isomorphism of quasi-root systems

$$(\widehat{T}_d, F_d) \longrightarrow (\widehat{T}_d, F''_d).$$

Now, the definitions readily imply that θ_d is merely the evident restriction of θ. Moreover, by the definition of betweenness, $[\alpha,\beta] = [\alpha,\beta]_d$ (where the right hand side is an interval in $(\widehat{T}_d, F_d)$) and $[\theta(\alpha),\theta(\beta)]'' = [\theta(\alpha),\theta(\beta)]''_d$ (where the right hand side is an interval in $(\widehat{T}_d, F''_d)$). Thus, to prove $\theta([\alpha,\beta]) = [\theta(\alpha),\theta(\beta)]''$, we need only prove $\theta_d([\alpha,\beta]_d) = [\theta_d(\alpha),\theta_d(\beta)]''_d$. This reduces the proof of (d) to the case in which (W,S) is dihedral. In that case, Lemma 2.30 (d) assures us that Ψ_+ is W-conjugate up to sign to $\widehat{T}_+$, and the result is easily verified. This completes the proof of (d). Part (e) readily follows from (b)–(d) upon noting that $\widehat{T}_+$ is a positive system of both $(\widehat{T}, F)$ and $(\widehat{T}, F'')$. $\square$

Theorem 2.33. *Let $\Delta \subseteq \widehat{T}$, and set $S' := \{s_\alpha \mid \alpha \in \Delta\}$. Consider the conditions* (i)–(iii) *below*:

(i) $\langle S' \rangle = W$;

(ii) *for all $\alpha,\beta \in \Delta$ with $\alpha \neq \beta$, $\{\alpha,\beta\}$ is an abstract root basis for the root system $(W_{\alpha,\beta} \cap T) \times \{\pm 1\}$ of $W_{\alpha,\beta} = \langle s_\alpha, s_\beta\rangle$*;

(iii) *Δ is contained in some biclosed quasi-positive system Ψ_+ of $\widehat{T}$.*

Then:

(a) *If* (i)–(ii) *hold, Δ is the set of simple reflections of some quasi-positive system of $\widehat{T}$; in particular, (W,S') is a Coxeter system.*

(b) *Conditions* (i)–(iii) *all hold if and only if Δ is an abstract root basis of Ψ. In that case, Ψ_+ in* (iii) *is the unique positive system with Δ as its set of simple roots.*

Proof. Observe that (ii) holds if and only if, for all $\alpha,\beta \in \Delta$ with $\alpha \neq \beta$, we have

$$\{\alpha,\beta\} = w(\chi(W_{\alpha,\beta}) \times \{\epsilon\})$$

for some $w \in W_{\alpha,\beta}$ and $\epsilon \in \{\pm 1\}$. First, we prove (a). Assume that (i)–(ii) hold. From (ii), $\Delta \cap -\Delta = \emptyset$. To show (W,S') is a Coxeter system, we shall use a characterization of Coxeter systems in **[14]**. Define the (left) W-set $\Omega := W$ with W-action by left translation. We regard

W as a group of permutations of $\widehat{T}$. For $\alpha \in \widehat{T}$, define an "abstract halfspace" $H_\alpha \subseteq \Omega$ by

$$H_\alpha := \{w \in W \mid \alpha \in w(\Psi_+)\}.$$

We have $\Omega = H_\alpha \cup H_{-\alpha}$ and $H_\alpha \cap H_{-\alpha} = \emptyset$. Also, $1 \in \cap_{\alpha \in \Psi_+} H_\alpha$, and $w(H_\alpha) = H_{w(\alpha)}$ for all $w \in W$. Furthermore, for $\alpha, \beta \in \Psi$,

$$H_\alpha \cap H_\beta = \bigcap_{\gamma \in [\alpha, \beta]} H_\gamma,$$

since $w(\Psi_+)$ is closed in $\widehat{T}$.

We claim that $X := \{(s_\alpha, H_\alpha) \mid \alpha \in \Delta\}$ is a pairwise proper system of reflections for W acting on Ω in the sense of **[14]** (but, note that we use left group actions instead of right actions as in *loc cit.*). The main result of **[14]** is that this implies that (W, S') is a Coxeter system, with $S' = \{s_\alpha \mid \alpha \in \Delta\}$, and then (a) follows from (ii) and Lemma 2.20. According to the definition of system of reflections in *loc cit*, X is a system of reflections since, for $\alpha \in \Delta$, s_α is an involution, $H_\alpha \subseteq \Omega$, $H_\alpha \cap s_\alpha(H_\alpha) = \emptyset$ and $H_\Delta := \cap_{\beta \in \Delta} H_\beta \neq \emptyset$ (the latter since $1 \in H_\Delta$ by (ii) and the above). To show X is pairwise proper, fix $\alpha \neq \beta$ in Δ. Set $W_{\alpha,\beta} = \langle S_{\alpha,\beta} \rangle$, where $S_{\alpha,\beta} = \{s_\alpha, s_\beta\}$, and let $l_{\alpha,\beta}$ denote the length function of the (dihedral) Coxeter system $(W_{\alpha,\beta}, S_{\alpha,\beta})$. Set $H_{\alpha,\beta} = H_\alpha \cap H_\beta$, and note that it is non empty. According to the definition of the pairwise proper system of reflections, we must show, for all $w \in W_{\alpha,\beta}$, that either $w(H_{\alpha,\beta}) \subseteq H_\alpha$, or that $w(H_{\alpha,\beta}) \subseteq s_\alpha(H_\alpha)$ and $l_{\alpha,\beta}(s_\alpha w) < l_{\alpha,\beta}(w)$. However, by (ii), $\{\alpha, \beta\}$ is an abstract root basis for the abstract root system $\widehat{T_{\alpha,\beta}} = (W_{\alpha,\beta} \cap T) \times \{\pm 1\}$ of $W_{\alpha,\beta}$. The abstract positive system corresponding to $\{\alpha, \beta\}$ is $\Psi_{\alpha,\beta} = [\alpha, \beta] \cap \widehat{T_{\alpha,\beta}}$. Hence, by Proposition 2.5, for $w \in W$,

$$\{\gamma \in \Psi_{\alpha,\beta} \mid l_{\alpha,\beta}(s_\gamma w) < l_{\alpha,\beta}(w)\} = \Psi_{\alpha,\beta} \cap w(-\Psi_{\alpha,\beta}). \tag{2.4}$$

Note that, from (2.4),

$$H_{\alpha,\beta} = \bigcap_{\gamma \in \Psi_{\alpha,\beta}} H_\gamma$$

and

$$w(H_{\alpha,\beta}) = \bigcap_{\gamma \in w\Psi_{\alpha,\beta}} H_\gamma.$$

Now, let $\gamma \in \Psi_{\alpha,\beta}$. If $l_{\alpha,\beta}(s_\gamma w) < l_{\alpha,\beta}(w)$, (2.4) gives $-\gamma \in w(\Psi_{\alpha,\beta})$, so $w(H_{\alpha,\beta}) \subseteq H_{-\gamma} = s_\gamma(H_\gamma)$. Otherwise, $l_{\alpha,\beta}(s_\gamma w) > l_{\alpha,\beta}(w)$, so (2.4) gives $\gamma \in w(\Psi_{\alpha,\beta})$ and $w(H_{\alpha,\beta}) \subseteq H_\gamma$. For $\gamma = \alpha$, this is what must be shown to prove that X is pairwise proper.

Next, we prove (b) by an argument independent of (a). The conditions (i)–(iii) are clearly necessary for Δ to be an abstract root basis. Now, assume that (i)–(iii) hold. Let l' be the length function of (W, S). We first claim that, if $w \in W$ and $\alpha \in \Delta$ with $l'(ws_\alpha) \geq l'(w)$, then $w(\alpha) \in \Psi_+$ (this is an abstract version of a well-known fact, Lemma 3.1 (b), regarding real root systems, and the following proof is essentially the same). The claim is easily verified if (W, S) is dihedral, and we reduce to that case by induction on $l'(w)$. The claim is trivial if $l'(w) = 0$. Otherwise, write $w = w's_\beta$, where $\beta \in \Delta$ and $l'(w') = l(w) - 1$. Necessarily, $\beta \neq \alpha$. Write $w = xy$, where $x \in W$, $y \in \langle s_\alpha, s_\beta \rangle$ and $l'(x)$ is minimal. Then, $l'(xs_\alpha) \geq l'(x)$, $l'(xs_\beta) \geq l'(x)$ and $l'(x) < l'(w)$. By induction, $x(\alpha) \in \Psi_+$ and $x(\beta) \in \Psi_+$. Note that $l''(ys_\alpha) \geq l''(y)$, where l'' is the length function of $\langle s_\alpha, s_\beta \rangle$ with respect to its Coxeter generators $\{s_\alpha, s_\beta\}$. From the claim for the dihedral case, it follows that $y(\alpha)$ is between α and β, that is, $y(\alpha) \in [\alpha, \beta]$. Then, $w(\alpha) = xy(\alpha) \in [x(\alpha), x(\beta)]$. Since $x(\alpha), x(\beta) \in \Psi_+$ and Ψ_+ is closed, $w(\alpha) \in \Psi_+$, as required to prove the claim. From the claim, it follows that if $w \in W$ and $\alpha \in \Delta$ with $l'(ws_\alpha) \leq l'(w)$, then $w(\alpha) \in -\Psi_+$. It is easy to deduce from this by standard arguments that (W, S') satisfies the exchange condition and is a Coxeter system (as, of course, also follows from (a)). Using (ii), Lemma 2.20 and its proof imply that

$$\Psi'_+ := \{w(\alpha) \mid w \in W,\ \alpha \in \Pi,\ l'(ws_\alpha) \geq l'(w)\}$$

is a generative, quasi-positive system with Δ as its set of simple roots. From the above, $\Psi'_+ \subseteq \Psi_+$. Since Ψ'_+ and Ψ_+ are both quasi-positive systems, it follows that $\Psi_+ = \Psi'_+$. By the definitions, Ψ_+ is a positive system of $\widehat{T}$, and we have seen it has Δ as its set of simple roots. This completes the proof of (b). $\square$

Remark 2.34. For $\gamma \in \Psi_+$, we have in the above proof of (a) that

$$H_{\epsilon\gamma} = \{w \in W \mid \epsilon l'(s_\gamma w) < \epsilon l'(w)\}$$

for $\epsilon \in \{\pm 1\}$, where l' is the length function of the Coxeter system (W, S'). The sets H_γ for $\gamma \in \widehat{T}$ are the sets of chambers of the "roots" of (W, S'), as defined in the study of the chamber system attached to (W, S'), see, for example, [**1**].

If $|S|$ is finite, condition (iii) can be replaced in Theorem 2.33 (b) by the assumption that $|\Delta| = |S|$, see Corollary 4.7.

Example 2.35. Consider the standard abstract root system $\widehat{T}$ of the Coxeter system (W, S) of type $\widetilde{B}_2$ with Coxeter graph

$$r \overset{4}{=\!=\!=} s \overset{4}{=\!=\!=} t\,.$$

Define the subset $\Delta := \{(s, 1), (srs, 1), (t, 1)\}$ of $\widehat{T}$ and the corresponding set $S' := \{s_\alpha \mid \alpha \in \Delta\} = \{s, srs, t\}$ of reflections. Note that S' generates W and that, for each distinct $\alpha, \beta \in \Delta$, $\{\alpha, \beta\}$ is the abstract set of simple roots of some generative quasi-positive system for the standard abstract root system $(\langle s_\alpha, s_\beta\rangle \cap T) \times \{\pm 1\}$ of $\langle s_\alpha, s_\beta\rangle$. In fact, for $\{s_\alpha, s_\beta\} = \{s, srs\}$, this follows from Proposition 2.2, and, for the other pairs, $\{\alpha, \beta\} = \chi(\langle s_\alpha, s_\beta\rangle) \times \{1\}$. However, (W, S') is not a Coxeter system, showing that assumptions (ii) in Theorem 2.33 (a) cannot be weakened in an obvious way.

3. Real root systems. This section describes the real reflection representations and corresponding root systems which we consider in this paper. The results are variants of standard facts and can be proven in a similar way or deduced from the standard versions, [**3, 7, 11, 16, 17**], with a few minor differences which we indicate. A very similar notion is considered in [**13**].

3.1. Call a subset Π of a real vector space V *positively independent* if $\sum_{\alpha\in\Pi} c_\alpha \alpha = 0$, with $c_\alpha \in \mathbb{R}$ almost all zero and all c_α non-negative implies that all $c_\alpha = 0$. Similarly, say that a subset Π of V *strongly positively independent* if $\sum_{\alpha\in\Pi} c_\alpha \alpha = 0$, with $c_\alpha \in \mathbb{R}$ almost all zero and at most one c_α negative implies that all $c_\alpha = 0$. Thus, Π is positively independent if all of its elements are non-zero and, for any finite subset of Π, the set of non-negative linear combination of elements of Π

is a pointed polyhedral cone in V. In addition, Π is strongly positively independent if any finite subset of Π is a set of representatives of the extreme rays of some pointed polyhedral cone in V.

3.2. We consider two $\mathbb{R}$-vector spaces V, V', with a given fixed $\mathbb{R}$-bilinear pairing

$$\langle\ ,\ \rangle\colon V \times V' \longrightarrow \mathbb{R}.$$

For any $\alpha \in V$, $\alpha' \in V'$ with $\langle \alpha, \alpha' \rangle = 2$, we let $s_{\alpha,\alpha'} \in \mathrm{GL}(V)$ be the linear map (a pseudoreflection) given by

$$v \longmapsto v - \langle v, \alpha' \rangle \alpha,$$

and define $s_{\alpha',\alpha} \in \mathrm{GL}(V')$ similarly. Assume, given subsets $\Pi \subseteq V$, $\Pi^\vee \subseteq V'$ and a bijection $\iota\colon \Pi \to \Pi^\vee$ denoted $\alpha \mapsto \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ for $\alpha \in \Pi$. Let

$$S := \{s_{\alpha,\alpha^\vee} \mid \alpha \in \Pi\},$$

W denote the subgroup of $\mathrm{GL}(V)$ generated by S,

$$\Phi := \bigcup_{w \in W} w(\Pi), \quad \text{and} \quad \Phi_+ := \Phi \cap \sum_{\alpha \in \Pi} \mathbb{R}_{\geq 0}\alpha.$$

Define S', W', $\Phi^\vee$ and $\Phi^\vee_+$ similarly using $\Pi^\vee \subseteq V'$ instead of $\Pi \subseteq V$.

Lemma 3.1. *Define* $P := \{4\cos^2(\pi/m) \mid m \in \mathbb{N}_{\geq 2}\} \cup [4, \infty) \subseteq \mathbb{R}_{\geq 0}$. *Consider the following conditions* (i)–(iii):

(i) $\Phi = \Phi_+ \cup (-\Phi_+)$;

(ii) $\Phi^\vee = \Phi^\vee_+ \cup (-\Phi^\vee_+)$;

(iii) *for* $\alpha \neq \beta$ *in* Π, *we have* $c_{\alpha,\beta} := \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in P$ *and* $\langle \alpha, \beta^\vee \rangle \leq 0$; *moreover,* $\langle \alpha, \beta^\vee \rangle = 0$ *if and only if* $\langle \beta, \alpha^\vee \rangle = 0$.

Then:

(a) *If* Π *is positively independent and* (iii) *holds, then* Π *is strongly positively independent.*

(b) *If* Π *and* $\Pi^\vee$ *are strongly positively independent,* (i)–(iii) *are equivalent. In that case, for* $w \in W$ *and* $\alpha \in \Pi$, *we have* $w(\alpha) \in \Phi_+$ *if and only if* $l(ws_\alpha) \geq l(w)$, *where* l *is the length function of* (W, S).

Proof. For (a), if

$$x = \sum_{\beta \in \Pi} c_\beta \beta = 0$$

with $c_\alpha < 0$ and $c_\beta \geq 0$ for $\beta \neq \alpha$, then $0 = \langle x, \alpha^\vee \rangle < 0$, a contradiction. Part (b) can be verified by direct calculation in the dihedral case $|\Pi| = 2$ (the calculations, in general, easily reduce to those in [**11**], see subsection 3.6). The proof of (b), in general, is by reduction to the dihedral case by a standard argument [**7,** Proposition 2.1] (cf., the proof of Theorem 2.33 (b)). ☐

Remark 3.2. See [**15**] for other results in which the set P (or $P \setminus \{0\}$) naturally appear.

Proposition 3.3. *Assume that* Π *and* $\Pi^\vee$ *are positively independent and that conditions of Lemma* 3.1 (i)–(iii) *hold. For* $\alpha, \beta \in \Pi$, *define* $m_{\alpha,\beta} = 1$ *if* $\alpha = \beta$, $m_{\alpha,\beta} = \infty$ *if* $c_{\alpha,\beta} \geq 4$ *and* $m_{\alpha,\beta} = m$ *if* $c_{\alpha,\beta} = 4\cos^2(\pi/m)$ *with* $m \in \mathbb{N}_{\geq 2}$. *Then*:

(a) (W, S) *and* (W', S') *are isomorphic Coxeter systems with Coxeter matrix* $(m_{\alpha,\beta})_{\alpha,\beta \in \Pi}$, *an isomorphism being given by*

$$\theta : s_{\alpha,\alpha^\vee} \longmapsto s_{\alpha^\vee,\alpha} \quad \textit{for } \alpha \in \Pi.$$

(b) *Regarding* θ *as an identification, we have* $\langle w\alpha, \beta \rangle = \langle \alpha, w^{-1}\beta \rangle$, *that is, the representations of* W *on* V *and* V' *are "contragredient."*

(c) *The bijection* $\iota \colon \Pi \mapsto \Pi^\vee$ *extends to a* W*-equivariant bijection*

$$\widehat{\iota} \colon \Phi \longmapsto \Phi^\vee,$$

which we still denote as $\alpha \mapsto \alpha^\vee$, *and which restricts to a bijection* $\Phi_+ \to \Phi_+^\vee$. *Furthermore,* $w(\alpha) = c\beta$ *with* $w \in W$, $\alpha, \beta \in \Phi$, $c \in \mathbb{R}$ *implies* $w(\alpha^\vee) = c^{-1}\beta^\vee$.

Proof. In (c), in the most common situations, we necessarily have $c = 1$, and the result is either trivial, as in [**16**], or the proof uses extra structure not present here, as in [**19**]. In general, we can proceed as follows. It suffices to show that $w(\alpha) = c\beta$ with $w \in W$, $\alpha, \beta \in \Pi$, $c \in \mathbb{R}_{>0}$ implies $w(\alpha^\vee) = c^{-1}\beta^\vee$. This can be directly verified in the dihedral case (c.f., subsection 3.6 again), and then, the general case reduces to the result in the dihedral case by an argument similar to the proof of [**7,** Proposition 2.1]. In fact, if $w \neq 1$, write $w = ws_\gamma$, where $\gamma \in \Pi$ and $l(ws_\gamma) < l(w)$. Then, write $w = w'w''$, where $w'' \in \langle s_\alpha, s_\gamma \rangle$ and $l(w) = l(w') + l(w'')$, and $l(w')$ is minimal amongst all such expressions. We have $l(w''s_\alpha) > l(w'')$, so $w''(\alpha) = p\alpha + q\gamma$ for some $p, q \in \mathbb{R}_{\geq 0}$. In addition, $l(w's_\alpha) \geq l(w)$ and $l(w's_\gamma) \geq l(w')$; thus, $w'(\alpha), w'(\gamma) \in \Phi_+$. Now,

$$\Pi \ni \beta = c^{-1}w'w''(\alpha) = c^{-1}pw'(\alpha) + c^{-1}qw'(\beta),$$

which implies that at most one of p, q is non-zero. The argument is easily completed using the result for the dihedral case and induction on $l(w)$. Once we have (c) available, we may abbreviate $s_{\alpha,\alpha^\vee}$ as s_α and note that

$$s_{w(\alpha)} = ws_\alpha w^{-1};$$

and the remainder of the proofs of (a) and (b) are then standard. □

3.3. Maintain the assumptions of subsection 3.2, Lemma 3.1, Remark 3.2 and Proposition 3.3. The reflection $s_{\alpha,\alpha^\vee} \in \mathrm{GL}(V)$ in a root $\alpha \in \Phi$ or the reflection $s_{\alpha^\vee,\alpha} \in \mathrm{GL}(V')$ in the corresponding coroot $\alpha^\vee$ will merely be denoted as s_α and regarded as an element of W. Then, $s_{w(\alpha)} = ws_\alpha w^{-1}$, so the set of reflections of (W, S) is

$$T := \{s_\alpha \mid \alpha \in \Phi^+\} = \{wsw^{-1} \mid w \in W, s \in S\}.$$

Using Proposition 3.3 (c), we see that, for $\alpha, \beta \in \Phi$, we have $s_\alpha = s_\beta$ if and only if $\alpha = c\beta$ for some $c \in \mathbb{R}_{\neq 0}$.

We say that the ordered pair

$$E = (\langle -, - \rangle \colon V \times V' \longrightarrow \mathbb{R}, \hat{\iota} \colon \Phi \longrightarrow \Phi^\vee)$$

is a *root datum* and that

$$B = (\langle -, - \rangle \colon V \times V' \longrightarrow \mathbb{R}, \iota \colon \Pi \longrightarrow \Pi^\vee)$$

is a *based root datum*, with E as the underlying root datum. Note that E is determined by B, while B is determined by E and the subset Π of Φ, since ι is given by restriction of $\hat{\iota}$. We call Φ the (*real*) *root system* associated with E, and we call a subset Π of Φ a *root basis* of E if it arises from some based root datum B with underlying root datum E in this way. As is standard, we call

$$\Pi \subseteq \Phi_+ \subseteq \Phi$$

the set of *simple roots*, *positive roots* and *roots*, respectively, and

$$\Pi^\vee \subseteq \Phi_+^\vee \subseteq \Phi^\vee$$

the sets of *simple coroots*, *positive coroots* and *coroots*, respectively. The matrix

$$\langle \alpha, \beta^\vee \rangle_{\alpha,\beta\in\Pi}$$

will be called the *possibly non-integral generalized Cartan matrix* (NGCM) of B. The NGCMs with entries in $\mathbb{Z}$ and finite index sets are the generalized Cartan matrices (GCMs) of **[19]**.

3.4. We say that a based root datum, as above, is a *standard based root datum* if $V = V'$, $\langle -,- \rangle$ is a symmetric bilinear form on V, $\Pi = \Pi^\vee$ and $\iota = \mathrm{Id}_\Pi$. Every Coxeter system is isomorphic to a Coxeter system (W, S) associated to a standard root datum, with Π and $\Pi^\vee$ linearly independent. The class of standard-based root data affords the same class of root systems and reflection representation of Coxeter groups as considered in **[16]**.

3.5. We say that a root system (or a corresponding root datum or based root datum) is *reduced* if, for any $\alpha \in \Phi$ and $c \in \mathbb{R}$ with $c\alpha \in \Phi$, we have $c = \pm 1$, that is, the root system, root datum or based root datum is reduced if

$$\alpha \longmapsto s_\alpha \colon \Phi^+ \longrightarrow T$$

is a bijection whenever Φ_+ is the set of positive roots associated to some root basis Π. The root datum is reduced if and only if, for any root basis Π, we have $\langle \alpha, \beta^\vee \rangle = \langle \beta, \alpha^\vee \rangle$ for all $\alpha, \beta \in \Pi$ with $m_{\alpha,\beta}$ finite and odd. By Lemma 2.19, the proof of this reduces to the case of dihedral Coxeter systems, where it follows by simple computation using the remarks at the end of subsection 3.6. In particular, a root datum for which the NGCM is symmetric or is a GCM is reduced in this sense.

3.6. Let B be a based root datum with simple roots Π, and let c_α for $\alpha \in \Pi$ be non-negative scalars. Then, there is a new based datum

$$B' = (\langle -,- \rangle \colon V \times V' \longrightarrow \mathbb{R}, \iota' \colon \Pi' \longrightarrow \Pi'^\vee),$$

where $\Pi' = \{c_\alpha \alpha \mid \alpha \in \Pi\}$ and $\iota'(c_\alpha\alpha) = c_\alpha^{-1}\iota(\alpha)$. We say B' is obtained by *rescaling* B (or by rescaling Π). If B' can be chosen so as to have a symmetric NGCM, we say B is *symmetrizable*.

It is easy to see that there are root data which cannot be rescaled to any reduced root datum (thus, in particular, they are not symmetrizable). However, if the Coxeter graph of (W, S) is a tree, then any root datum is symmetrizable. In particular, this applies if (W, S) is dihedral. Since the root systems of dihedral groups with symmetric NGCM are described in **[11]**, we easily obtain a description of root systems of arbitrary-based root data affording dihedral Coxeter systems.

The following fact will play an important role in the proof of the main result of this paper.

Theorem 3.4.

(a) *Let W' be any reflection subgroup of W. There is a based root datum*

$$B' = (\langle -, - \rangle \colon V \times V' \longrightarrow \mathbb{R}, \iota' \colon \Pi' \longrightarrow {\Pi'}^\vee),$$

with associated Coxeter system (W', S') such that $\Pi' \subseteq \Phi_+$ and ι' is the restriction of $\hat{\iota} \colon \Phi \to \Phi^\vee$.

(b) *For any B' satisfying* (a), *$S' = \chi(W')$. Hence, B' is unique up to rescaling, and it is unique if B is a reduced root datum.*

(c) *A subset Π' of Φ_+ arises as in* (a) *from some reflection subgroup W' if and only if the conditions of Lemma* 3.1 (iii) *hold for all $\alpha \neq \beta \in \Pi'$.*

Furthermore, if these conditions hold, $W' = \langle s_\alpha \mid \alpha \in \Pi' \rangle$.

Proof. In the case of standard based root data, this is proven in **[8, 11]**. Either proof extends mutatis mutandis to the more general situation here, using Lemma 3.1. □

In the above setting, we call B' a *based root subdatum* of B corresponding to W'.

4. Comparison of real and abstract root systems.

Proposition 4.1. *Let*

$$B = (\langle -, - \rangle \colon V \times V' \longrightarrow \mathbb{R}, \iota \colon \Pi \longrightarrow \Pi^\vee)$$

be a root-based datum with associated Coxeter system (W,S) and real root system Φ, and let $(\widehat{T},F)$ be the standard abstract root system of (W,S). Let

$$F'\colon \Phi \longrightarrow \mathrm{Sym}(\Phi)$$

be defined by $F'(\alpha)=(s_\alpha)_{|\Phi}$, the restriction of s_α to Φ.

(a) *(Φ,F') is an abstract root system with Φ_+ as a generative quasi-positive system and with an associated Coxeter system naturally isomorphic to (W,S).*

(b) *There is a surjective W-equivariant map*

$$\theta\colon \Phi \longrightarrow \widehat{T},$$

determined by $\alpha \mapsto (s_\alpha,\epsilon)$ for $\alpha\in\epsilon\Phi_+$ and $\epsilon\in\{\pm 1\}$. Furthermore, θ determines an isomorphism $(\Phi,F')\cong(\widehat{T},F)$ if and only if Φ is reduced.

(c) *Let $\alpha,\beta,\gamma\in\Phi$. Then, $\gamma\in\mathbb{R}_{\geq 0}\alpha+\mathbb{R}_{\geq 0}\beta$ if and only if $\theta(\gamma)$ is between $\theta(\alpha)$ and $\theta(\beta)$.*

Proof. Parts (a)–(b) are clear from the definitions and the results in Sections 2–3. For part (c), the definitions and Theorem 3.4 immediately reduce the proof in general to that in the case of dihedral Coxeter systems. In the dihedral case, there is an obvious bijection from the set of rays spanned by the roots to the set of rays in the diagram in Figure 3, mapping $\mathbb{R}_{\geq 0}\alpha$ to the ray labeled $\theta(\alpha)$ for $\alpha\in\Phi$. From direct calculations for dihedral root systems (simplified by using the remarks at the end of subsection 3.6), we verify that this bijection preserves the sets of rays in the convex closure of a union of two rays. $\square$

Theorem 4.2. *Let the notation be as in the preceding proposition. Let (W_i,S_i) for $i\in I$ be the irreducible components of*

$$(W,S), \Pi_i=\{\alpha\in\Pi \mid s_\alpha\in S_i\},$$

and let $V_i:=\mathbb{R}\Pi_i$ denote the $\mathbb{R}$-vector space spanned by Π_i. Assume that the sum $\sum_i \mathbb{R}V_i$ of subspaces of V is direct (for example, (W,S) is irreducible or Π is linearly independent) and that the dual condition, with V replaced by V', also holds. Then, a subset Π' of Φ is a root basis of E, affording a root-based datum B', say, if and only if the restriction $\theta_{|\Pi'}$ is injective and $\theta(\Pi')$ is an abstract root basis of $(\widehat{T},F)$. In that *case, the set of positive roots of B' is $\theta^{-1}(\Psi_+)$, where Ψ_+ is the positive system of $(\widehat{T}, F)$ corresponding to $\theta(\Pi')$.*

Proof. If Π' is a root basis, then $\theta_{|\Pi'}$ is injective, $\theta(\Pi')$ is an abstract root basis of $(\widehat{T}, F)$ and the positive roots of B' are $\theta^{-1}(\Psi_+)$, using Proposition 4.1.

Suppose now that $\theta_{|\Pi'}$ is injective and $\theta(\Pi')$ is an abstract root basis of $(\widehat{T}, F)$. For any $\alpha, \beta \in \Pi'$, let

$$W_{\alpha,\beta} := \langle s_\alpha, s_\beta \rangle \quad \text{and} \quad \widehat{T}_{\alpha,\beta} = (W_{\alpha,\beta} \cap T) \times \{\pm 1\} \subseteq \widehat{T}.$$

Using Lemma 2.30 (c), $\theta(\{\alpha, \beta\})$ is an abstract set of simple roots for $W_{\alpha,\beta}$ in the abstract root system $T_{\alpha,\beta}$ of $W_{\alpha,\beta}$. This implies, by direct calculation for the dihedral groups, Example 2.27, that there exists an element (w, ϵ) of $W_{\alpha,\beta} \times \{\pm 1\}$ such that $\epsilon w\{\theta(\alpha), \theta(\beta)\} = \{s_\gamma, s_\delta\} \times \{1\}$, where $\gamma, \delta \in \Phi_+$ are such that $\{s_\gamma, s_\delta\} = \chi(W_{\alpha,\beta})$. Interchanging γ and δ, if necessary, we may assume, without loss of generality, that $w(\alpha) = c\gamma$ and $w(\beta) = d\delta$ for some $c, d \in \mathbb{R}$ with $cd > 0$ (since c, d can be taken of the same sign as ϵ). Now, $\langle \gamma, \delta^\vee \rangle$ and $\langle \delta, \gamma^\vee \rangle$ are non positive real numbers whose product is in the set P of Lemma 3.1, and such that both are zero if either is zero. Since $w(\alpha^\vee) = c^{-1}\gamma^\vee$ and $w(\beta^\vee) = d^{-1}\delta^\vee$, it follows that $\langle \alpha, \beta^\vee \rangle$ and $\langle \beta, \alpha^\vee \rangle$ have the same properties. Since $W = \langle s_\alpha \mid \alpha \in \Pi' \rangle$, we conclude from the definitions in subsection 3.3 that B' is a root-based datum, provided that Π' is positively independent. Then, ${\Pi'}^\vee$ will be positively independent by symmetry.

We prove positive independence of Π' first only under the additional hypothesis that Π is finite and linearly independent; this extra hypothesis will be replaced by the more general one of the theorem in subsection 4.2. Thus, assume that Π is finite and linearly independent. Let $S' = \{s_\alpha \mid \alpha \in \Pi'\}$. Then, (W, S) and (W, S') are Coxeter systems with $S' \subseteq T$; hence, $|S| = |S'|$ from subsection 2.9, and thus, $|\Pi| = |\Pi'|$. Since $\langle S' \rangle = W$ with $S' \subseteq T$, every reflection of W, in particular, any element of S, is equal to a reflection in some element of $\langle S' \rangle \Pi'$. It follows that $\Pi \subseteq \mathbb{R}\Pi'$, and hence, Π' is an $\mathbb{R}$-basis of $\mathbb{R}\Pi$. Since Π' is linearly independent, it is positively independent, as required. $\square$

It will be helpful to keep in mind the following. Let (W, S) and (W, S') be two Coxeter systems with $S' \subseteq T$. If (W, S) is of finite rank, then (W, S') is of finite rank (and the ranks are the same; see subsection 2.9). In addition, (W, S) is irreducible if and only if (W, S') is irreducible.

Theorem 4.3. *Suppose that*

$$B = (\langle -, - \rangle : V \times V' \longrightarrow \mathbb{R}, \iota : \Pi \longrightarrow \Pi^\vee)$$

and

$$B' = (\langle -, - \rangle : V \times V' \longrightarrow \mathbb{R}, \iota' : \Delta \longrightarrow \Delta^\vee)$$

are two root-based data with the same underlying root datum $(\langle -, - \rangle : V \times V' \to \mathbb{R}, \hat{\iota} : \Phi \to \Phi^\vee)$. *Assume that the associated Coxeter systems* (W, S) *and* (W, S'), *respectively, are both of finite rank and irreducible. Then, there exist a* $w \in W$, $\epsilon \in \{\pm 1\}$ *and scalars* $c_\alpha \in \mathbb{R}_{>0}$ *for* $\alpha \in \Pi$ *such that* $\Delta = \{\epsilon c_\alpha w(\alpha) \mid \alpha \in \Pi\}$. *In particular,* $S' = wSw^{-1}$, *and hence,* (W, S) *is isomorphic to* (W, S').

Proof. Suppose that B is a standard-based root datum. It is clear that B' must be a standard-based root datum also. In this case, the conclusion is one of the main results of **[16]** (with all $c_\alpha = 1$, necessarily, in this case). If B is not a standard-based root datum, the result will be proven in subsection 4.1. □

Remark 4.4. Slightly more generally, we could assume in Theorem 4.3 that the underlying root data of B and B' are not necessarily equal but differ by rescaling. This version immediately reduces to that above by rescaling B', say, appropriately.

Theorem 4.5. *Suppose that* (W, S) *is an irreducible Coxeter system of finite rank. If* Ψ_+ *is an abstract system of positive roots for* $\widehat{T}$, *then there exist a* $w \in W$ *and* $\epsilon \in \{\pm 1\}$ *with* $\Psi_+ = \epsilon w(\widehat{T}_+)$. *In particular,* $\Delta_{\Psi_+} = \epsilon w(S_+)$, $S_{\Psi_+} = wSw^{-1}$ *and* (W, S_{Ψ_+}) *is isomorphic to* (W, S) *as a Coxeter system.*

Proof. We may suppose, without loss of generality, that (W, S) is the Coxeter system associated to a standard based root datum B such that Π is linearly independent. Note that both Theorems 4.2 and 4.3 have already been proved for B of this special type. Set $\Pi' = \theta^{-1}(\Delta_{\Psi_+})$.

Now, θ is a bijection. By Theorem 4.2, Π' is a root basis of the root datum E underlying B. From Theorem 4.3, $\Pi' = \epsilon w(\Pi)$ for some $w \in W$ and $\epsilon \in \{\pm 1\}$. Then,

$$\Delta_{\Psi_+} = \theta(\Pi') = \epsilon w \theta(\Pi) = \epsilon w(S \times \{1\}),$$

as required. $\square$

For subsequent use, we record the following facts regarding abstract positive and abstract simple systems in the case of irreducible W; with a more technical statement, (a)–(b) could be combined and extended to any, possibly reducible, W of finite rank, using subsection 2.14 and Lemma 2.29.

Corollary 4.6. *Let (W, S) be an irreducible, finite rank, Coxeter system with abstract root system $\widehat{T}$. Let P and Q denote, respectively, the set of all abstract positive systems of $\widehat{T}$ and the set of all abstract simple systems of $\widehat{T}$, each endowed with its natural $W \times \{\pm 1\}$-action.*

- (a) *If W is finite, then W acts simply transitively both on P and on Q.*
- (b) *Suppose that W is infinite and irreducible. Then, $W \times \{\pm 1\}$ acts simply transitively both on P and on Q.*
- (c) *If W is infinite irreducible, then, for all $\Psi_+, \Phi_+ \in P$, the following conditions are equivalent:*
  - (i) *$\Psi_+ \cap \Phi_+$ is infinite;*
  - (ii) *$\Psi_+ \cap -\Phi_+$ is finite;*
  - (iii) *Ψ_+ and Φ_+ are in the same W-orbit on P;*
  - (iv) *Ψ_+ and $-\Phi_+$ are in different W-orbits on P.*

Proof. Since P and Q are canonically isomorphic as $W \times \{\pm 1\}$-sets, it suffices to prove the claims concerning P. From Theorem 4.5, the $W \times \{\pm 1\}$-action on P is transitive. Therefore, $P = W\widehat{T}_+ \cup W(-\widehat{T}_+)$ contains either one or two W-orbits. Now, if $w \in W$, then, for $w \in W$, we have

$$|\widehat{T}_+ \cap w(-\widehat{T}_+)| = |N(w)| = l(w) < \infty,$$

by Proposition 2.5 (a)–(b). If W is infinite, then so is $|T_+|$, and therefore, there is no $w \in W$ with $w(-T_+) = T_+$; in this case, there are two W-orbits on P. On the other hand, if W is finite, then its longest element maps T_+ to $-T_+$, and so, there is a single W-orbit on P. The last displayed equation also shows that, for either finite or infinite W, if $w \in W$ satisfies $w(\widehat{T}_+) = \widehat{T}_+$, then $l(w) = 0$, and so, $w = 1$. This implies that W acts simply transitively on any W-orbit $W(\pm\widehat{T}_+)$ on P. This completes the proof of (a)–(b).

It remains to prove (c). It may easily be verified that assertions (i)–(iv) for (Ψ_+, Φ_+) are equivalent to assertions (i)–(iv) for $(\epsilon u\Psi_+, \pm\epsilon u\Phi_+)$, where $(u, \epsilon) \in W \times \{\pm 1\}$, and the sign is arbitrary. By Theorem 4.5 and (b), we are, therefore, reduced to showing that (i)–(iv) are equivalent if $(\Psi_+, \Phi_+) = (\widehat{T}_+, w(\widehat{T}_+))$ for some $w \in W$. However, in that case, (iii)–(iv) hold from the above, (ii) holds by the last displayed equation and (i) also holds, following from (ii) since $|\Psi_+|$ is infinite. Hence, (i)–(iv) are equivalent in this case since they are all true, and their equivalence follows in general. □

4.1. Completion of the proof of Theorem 4.3 in general. Let θ be as in Proposition 4.1. From (an already proven) part of Theorem 4.2, $\theta(\Pi')$ and $\theta(\Pi)$ are two abstract root bases of $\widehat{T}$. By Theorem 4.5, there are $w \in W$ and $\epsilon \in \{\pm 1\}$ such that $\theta(\Pi') = \epsilon\theta(\Pi)$. This implies that Π' and $\epsilon w(\Pi)$ are the same up to rescaling, which is what is required.

4.2. Completion of the proof of Theorem 4.2 in general. Assume that $\sum_i \mathbb{R}\Pi_i$ is a direct sum and that the dual condition also holds. Let Π' be a subset of Φ such that $\theta_{|\Pi'}$ is injective and $\theta(\Pi')$ is an abstract root basis of $\widehat{T}$. To complete the proof, it remains to show that Π' is positively independent. Using subsection 2.14 and the hypothesis that $\sum_i \mathbb{R}\Pi_i$ is direct, this readily reduces to the case that (W, S) is irreducible, as we assume henceforward.

Let R be any finite subset of S' such that the Coxeter system $(\langle R\rangle, R)$ is irreducible. It will suffice to show that $\Delta = \{\alpha \in \Pi' \mid s_\alpha \in R\}$ is positively independent for any such R (since Π' is positively independent if all of its finite subsets are positively independent). Now, let D be a root-based subdatum of B corresponding to $W' := \langle R\rangle$, as in Theorem 3.4. By rescaling D, if necessary, we may further assume that some root basis Δ' of D is contained in $W'\Delta$. By a further rescaling of Δ, if necessary, we may assume that $\Psi := W'\Delta' = W'\Delta''$ is the root system of D. Let $\Psi_+ = \Psi \cap \Phi_+$ denote the positive roots of D. Let $\widehat{T'}$ be the abstract root system of W' and $\theta' \colon \Psi \to \widehat{T'}$ the analogue for D of the map $\theta \colon \Phi \to \widehat{T}$ for B (clearly, θ' is given by restriction of θ). Since

$\theta(\Pi')$ is an abstract root basis of $\widehat{T}$, it easily follows that $\theta'(\Delta)$ is an abstract root basis of $\widehat{T'}$. However, by Theorem 3.4 and Proposition 4.1, $\theta'(\Delta')$ is also an abstract root basis of $\widehat{T'}$. Since (W', R) is irreducible and of finite rank, by assumption, and $R \subseteq T$, $(W', \chi(W'))$ is also irreducible of finite rank. Hence, $\theta'(\Delta) = \epsilon w \theta'(\Delta')$ for some $\epsilon \in \{\pm 1\}$ and $w \in W'$, by Theorem 4.5. Rescaling Δ' again, if necessary, we may assume that $\Delta = \epsilon w(\Delta')$. Since Δ' is positively independent, so is Δ.

Corollary 4.7. *Suppose that* (W, S) *is of finite rank. Let* $\Delta \subseteq \widehat{T}$*. Then,* Δ *is an abstract root basis of* $\widehat{T}$ *if and only if the conditions Theorem* 2.33 (i)–(ii) *hold and* $|\Delta| = |S|$*.*

Proof. If Δ is an abstract root basis, then Theorem 2.33 (i)–(ii) hold and $|\Delta| = |S|$. For the converse, assume that Theorem 2.33 (i)–(ii) hold and $|\Delta| \leq |S|$. We may assume that (W, S) is realized as the Coxeter system associated to a standard root-based datum B with linearly independent root basis Π and root system Φ. Let θ be as in Proposition 4.1 (b), and $\Pi' = \theta^{-1}(\Delta) \subseteq \Phi$. In order to show that Δ is an abstract root basis of $\widehat{T}$, it is sufficient by Theorem 4.2 to show that Π' is a root basis of B. Using Theorem 3.4, we see from Theorem 4.2, assumptions (i)–(ii), that it would be sufficient to show that Π' is positively independent. Since $\langle S' \rangle = W$, where $S' = \{s_\alpha \mid \alpha \in \Pi'\}$, we have $\langle S' \rangle \Pi' = \Phi$, and a similar argument to that in the last paragraph of the proof of Theorem 4.2 shows that Π' is a basis of $\mathbb{R}\Pi$, and hence, Π' is positively independent. ☐

Example 4.8. We show that Theorem 4.2 may fail without the hypothesis that $\sum_i V_i$ is direct. Consider a real vector space V with basis $\{e_i\}_{i \in I} \cup \{\delta\}$, equipped with a symmetric bilinear form

$$\langle -, - \rangle \colon V \times V \longrightarrow \mathbb{R},$$

determined by $\langle \delta, V \rangle = 0$ and $\langle e_i, e_j \rangle = 2\delta_{i,j}$. Set $\Pi = \{e_i, \delta - e_i \mid i \in I\}$, $\Pi' = \Pi$, $\iota = \mathrm{Id}_\Pi \colon \Pi \to \Pi'$. These data determine a standard root-based datum B as in subsection 3.4. Let (W, S) be the corresponding Coxeter system; thus, $S = \bigcup_{i \in I} S_i$ (disjoint union) where $S_i := \{s_\alpha \mid \alpha \in \Pi_i\}$ and $\Pi_i := \{e_i, \delta - e_i\}$. In fact, the irreducible components of (W, S) are (W_i, S_i) (of type $\widetilde{A}_1$) for $i \in I$, where $W_i := \langle S_i \rangle$. Now, for any signs $\epsilon_i \in \{\pm 1\}$,

$$\Gamma := \bigcup_{i \in I} S_i \times \{\epsilon_i\}$$

is clearly an abstract root basis of $\widehat{T}$ by Example 2.18. However, $\theta\colon \Phi \to \widehat{T}$ is bijective, and

$$\Pi' = \theta^{-1}(\Gamma) = \bigcup_{i\in I} \epsilon \Pi_i$$

is not a root basis for B if $i \mapsto \epsilon_i$ is not a constant function. For, if $i, j \in I$ with $\epsilon_i = -\epsilon_j$, then

$$\epsilon_i e_i + \epsilon_i(\delta - e_i) + \epsilon_j e_j + \epsilon_j(\delta - e_j) = 0,$$

and Π' is not positively independent.

We remark that, if $|I| = 4$, the above root-based datum B is isomorphic to a root-based subdatum of the standard, as in **[3, 17]**, root-based datum of the affine Weyl group of type $\widetilde{D}_4$ **[11]**.

Example 4.9. Prior to studying the conjugacy of root bases of infinite rank Coxeter systems, we discuss in more detail the two Coxeter systems which play an exceptional role in Theorem 1.1. Note that these two Coxeter systems, of types $A_{\infty,\infty}$ and A_∞, have Coxeter graphs

$$\cdots \text{———} \bullet \text{———} \bullet \text{———} \bullet \text{———} \cdots \qquad\qquad \bullet \text{———} \bullet \text{———} \bullet \text{———} \cdots,$$

respectively, so are obviously non-isomorphic.

Fix a Coxeter system (W, S) of type $A_{\infty,\infty}$. We may identify W with the group of all permutations of $\mathbb{Z}$ which fix all but finitely many integers so that

$$S = \{s_n = (n, n+1) \mid n \in \mathbb{Z}\}$$

(the set of adjacent transpositions). Let $K = \{s_n \mid n \in \mathbb{N}\} \subseteq S$. Then, the standard parabolic subsystem (W_K, K) is a Coxeter system of type A_∞. Clearly, by restriction of its action to $\mathbb{N} \subseteq \mathbb{Z}$, W_K identifies with the group of all permutations of $\mathbb{N}$ which fix all but finitely many integers, with Coxeter generators

$$\{s_n = (n, n+1) \mid n \in \mathbb{N}\}.$$

Then, the set of reflections of (W, S) is equal to the set $T = \{(i, j) \mid i < j \text{ in } \mathbb{Z}\}$ of transpositions. Similarly, the set of reflections of (W_K, K) is equal to the set of transpositions $T' = \{(i, j) \mid i < j \text{ in } \mathbb{N}\}$.

Clearly, any bijection $\sigma : \mathbb{N} \xrightarrow{\cong} \mathbb{Z}$ induces a group isomorphism $\sigma'\colon W_K \to W$, defined by $w \mapsto \sigma w \sigma^{-1}$. Note that σ' restricts to a bijection $T \to T'$. Also, any permutation τ of $\mathbb{Z}$ induces an (in general outer) automorphism τ' of W, defined by $w \mapsto \tau w \tau^{-1}$ such that τ' restricts to a permutation of T. Similarly, a permutation τ of $\mathbb{N}$ induces an (outer) automorphism τ' of W_K, defined by $w \mapsto \tau w \tau^{-1}$ such that τ' restricts to a permutation of T'.

Let V be a real vector space on basis e_n for $n \in \mathbb{Z}$, with (positive definite) symmetric bilinear form, defined by

$$\left\langle \sum_n a_n e_n, \sum_n b_n e_n \right\rangle = \sum_n a_n b_n.$$

Let $\Pi = \{e_n - e_{n+1} \mid n \in \mathbb{Z}\} \subseteq V$. Then,

$$(\langle -, - \rangle \colon V \times V \longrightarrow \mathbb{R}, \mathrm{Id}_\Pi \colon \Pi \longrightarrow \Pi)$$

is a standard-based root datum affording a Coxeter system (W, S) of type $A_{\infty,\infty}$. (The set $X = \{e_n \mid n \in \mathbb{Z}\} \subseteq V$ is stable under the W-action. Restriction of the W-action to X and identifying X with $\mathbb{Z}$ via the bijection $n \mapsto e_n \colon \mathbb{Z} \to X$ provides the description of W as a subgroup of $\mathrm{Sym}(\mathbb{Z})$ from above. Similarly, $Y = \{e_n \mid n \in \mathbb{N}\} \subseteq V$ is stable under the action of W_K and affords, in a similar way, the embedding $W_K \subseteq \mathrm{Sym}(\mathbb{N})$.)

For any $J \subseteq S$, let $\Pi_J := \{\alpha \in \Pi \mid s_\alpha \in J\}$, $V_J := \mathbb{R}\Pi_J$ denote the $\mathbb{R}$-span of Π_J, $W_J := \langle J \rangle$, $\Phi_J := W_J \Pi_J \subseteq V_J$ and $\langle -, - \rangle_J$ denote the restriction of $\langle -, - \rangle$ to a bilinear form on V_J. Then,

$$B_J := (\langle -, - \rangle_J, \mathrm{Id}_{\Pi_J} \colon \Pi_J \longrightarrow \Pi_J)$$

is a standard-based root datum for the standard parabolic subsystem (W_J, J) of (W, S). Let E_J denote the root datum underlying the based root datum B_J, with root system Φ_J. Now, fix a bijection $\sigma \colon \mathbb{N} \to \mathbb{Z}$. This induces an isomorphism $\widetilde{\sigma} \colon \mathbb{R}Y \to \mathbb{R}X$ of vector spaces mapping basis elements by $e_n \mapsto e_{\sigma(n)}$. It is straightforward to check that $\widetilde{\sigma}(\Pi_K)$ is a root basis of E_S, with respect to which the associated Coxeter system is of type A_∞. In addition, $\widetilde{\sigma}^{-1}(\Pi_S)$ is a root basis of E_K, with respect to which the associated Coxeter system is of type $A_{\infty,\infty}$.

Similarly, a permutation τ of $\mathbb{Z}$ induces an $\mathbb{R}$-linear automorphism $\widetilde{\tau}$ of $\mathbb{R}X$, determined by $e_n \mapsto e_{\tau(n)}$, and $\widetilde{\tau}(\Pi_S)$ is a root basis of E_S which is not, in general, W-conjugate to Π_S up to sign, although the Coxeter system associated to this root basis is still of type $A_{\infty,\infty}$.

Equally well, a permutation τ of $\mathbb{N}$ induces an $\mathbb{R}$-linear automorphism $\widetilde{\tau}$ of $\mathbb{R}Y$, determined by $e_n \mapsto e_{\tau(n)}$, and $\widetilde{\tau}(\Pi_K)$ is a root basis of E_K which is not, in general, W-conjugate to Π_K up to sign, although the Coxeter system associated to this root basis is still of type A_∞.

Remark 4.10. The above example comes from a similar isomorphism of infinite-rank Kac-Moody Lie algebras with Dynkin diagrams, as above, namely, such a Lie algebra of type $A_{\infty,\infty}$ may be realized as the complex Lie algebra of all $\mathbb{Z} \times \mathbb{Z}$-indexed complex matrices with only finitely many non-zero entries, and trace zero. Similarly, such a Lie algebra of type A_∞ may be realized as the complex Lie algebra of all $\mathbb{N} \times \mathbb{N}$-indexed complex matrices with only finitely many non-zero entries, and trace zero. A bijection $\mathbb{Z} \to \mathbb{N}$ induces an isomorphism between these Lie algebras in the obvious way, inducing the above isomorphism of root systems in the (restricted dual) of the corresponding Cartan subalgebras.

Definition 4.11. We say that a map $\lambda\colon \widehat{T} \to \widehat{T}$ is *given locally by the action of* W if, for each finitely generated reflection subgroup W' of W, there is an element $w = w(W')$ of W such that $\lambda(\alpha) = w(\alpha)$ for all $\alpha \in (W' \cap T) \times \{\pm 1\}$. Such a map λ is injective and preserves betweenness; if λ is invertible, its inverse is also given locally by the action of W. We let $\widehat{W}$ denote the group of all permutations of $\widehat{T}$, which are given locally by the action of W, under composition. Then, $W \subseteq \widehat{W}$, and equality holds if (W, S) is of finite rank.

Theorem 4.12. *Let (W, S) be an irreducible Coxeter system with standard abstract root system $\widehat{T}$, and let Ψ_+ be a positive system of $\widehat{T}$.*

(a) *(W, S_{Ψ_+}) is isomorphic to (W, S) unless one of them is of type A_∞ and the other is of type $A_{\infty,\infty}$;*

(b) *if $(W, S_{\Psi_+}) \cong (W, S)$, there are $\widehat{w} \in \widehat{W}$ and $\epsilon \in \{\pm 1\}$ with $\Psi_+ = \epsilon\widehat{w}(\widehat{T}_+)$;*

(c) *any betweenness-preserving automorphism σ of the abstract root system of (W, S) is expressible (not necessarily uniquely) as $\sigma = \epsilon'\widehat{w}d'$ where $\widehat{w} \in \widehat{W}$, d' is an automorphism induced by a diagram automorphism of (W, S) and ϵ' is the automorphism induced by the action of $\epsilon \in \{\pm 1\}$.*

4.3. The proof of Theorem 4.12 requires some generalities on an (possibly infinite rank) irreducible Coxeter system (W, S). Fix such a Coxeter system with reflections T and standard abstract root system $\widehat{T}$. Following, we distinguish two cases. We say that (W, S) is *locally finite*, or of type (LocFin), if every finite rank standard parabolic subgroup of (W, S) is finite. We say that (W, S) is *locally infinite*, or of type (LocInf), if it is not of type (LocFin).

Lemma 4.13. *Let (W, S) be an irreducible Coxeter system.*

(a) *The following conditions* (i)–(v) *are equivalent*:
  (i) (W, S) *is of type* (LocInf);
  (ii) (W, S) *has an infinite, finite rank, standard parabolic subgroup W'*;
  (iii) (W, S) *has an infinite, finite rank, parabolic subgroup W'*;
  (iv) (W, S) *has an infinite, finite rank, reflection subgroup W'*;
  (v) W *has a finitely generated infinite subgroup*;

(b) *each of* (a) (ii)–(iv) *is equivalent to its variant* (a) (ii)$'$–(iv)$'$, *in which W' is required, in addition, to be irreducible*;

(c) *the type,* (LocFin), *or* (LocInf), *of (W, S) depends only on the group W.*

Proof. We prove (a). By the definitions, (i) is equivalent to (ii). It is trivial that (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v). Finally, (v) implies (ii) since an infinite, finitely generated subgroup of W is contained in some finite rank, (necessarily infinite) standard parabolic subgroup W' of W. For (b), note that, if W' satisfies (ii), then W' has an infinite irreducible component, which satisfies (ii)$'$, and, similarly, for (iii) or (iv) in place of (ii). Part (c) directly follows from the equivalence of (i) and (v) in (a). $\square$

4.4. Continue to assume that (W, S) is irreducible. If (W, S) is of type (LocFin), we let $\mathcal{S}'$ denote the set of all irreducible, finite rank reflection subgroups of (W, S), ordered by inclusion. If (W, S) is of type (LocInf), we let $\mathcal{S}'$ denote the set of all infinite, finite rank irreducible reflection subgroups of (W, S), again ordered by inclusion. The following facts may be checked separately for (W, S) of either type.

Note that $\mathcal{S}'$ depends only upon the pair (W, T). To allow greater convenience of application of some of the following results, we work below, not only with $\mathcal{S}'$, but, more generally, with any cofinal subset $\mathcal{S}$ of $\mathcal{S}'$. (Recall that a subset X of a poset Y is said to be cofinal in Y if, for any $y \in Y$, there is an $x \in X$ with $y \leq x$.) Since (W, S) is irreducible, $\mathcal{S}$ is cofinal in the (inclusion-ordered) family of all finite rank reflection subgroups of (W, S). Also, $\mathcal{S}$ is directed; given $W_1, W_2 \in \mathcal{S}$, there is a $W_3 \in \mathcal{S}$ with

$$W_3 \supseteq W_1 \cup W_2.$$

For any reflection subgroup W' of W, let $\widehat{T'} = (W' \cap T) \times \{\pm 1\}$. For any subset Ψ_+ of $\widehat{T}$, set $\Psi_+(W') := \widehat{T'} \cap \Psi_+$ so that

$$\Psi_+ = \bigcup_{W' \in \mathcal{S}} \Psi_+(W').$$

For example, in this notation, Lemma 2.30 (c) asserts that, if Ψ_+ is an abstract positive system for $\widehat{T}$, then $\Psi_+(W')$ is an abstract positive system for $\widehat{T'}$. Note that, if $\Psi_+^0 = \widehat{T}_+$ is the standard abstract positive system for $\widehat{T}$, then $\Psi_+^0(W')$ is the standard abstract positive system for $\widehat{T'}$. If F is a function from $\mathcal{S}$ to the power set of some set, we define

$$\lim_{W' \in S} F(W') := \bigcup_{W' \in \mathcal{S}} \operatorname{core}(F, W'),$$

where

$$\operatorname{core}(F, W') := \bigcap_{\{W'' \in \mathcal{S} \mid W'' \supseteq W'\}} F(W'').$$

Lemma 4.14. *With (W, S) and $\mathcal{S}'$ as above, let $\mathcal{S}$ be an arbitrary cofinal subset of $\mathcal{S}'$. Let $\Psi_+^0 = \widehat{T}_+$. Then, a subset $\Psi_+ \subseteq \widehat{T}$ is an abstract system of positive roots for $\widehat{T}$ if and only if, for all $W' \in \mathcal{S}$, $\Psi_+(W') = \epsilon a_{W'} \Psi_+^0(W')$ for some sign $\epsilon \in \{\pm 1\}$ and function $W' \mapsto a_{W'} : \mathcal{S} \to W$ satisfying the following conditions*:

(i) *$a_{W'} \in W'$;*
(ii) *for $W', W'' \in \mathcal{S}$ with $W' \subseteq W''$, $a_{W'}^{-1} a_{W''}$ is the unique element of minimal length of the coset $W' a_{W''}$, with respect to the standard length function of (W, S);*
(iii) *$S' := \lim_{W' \in \mathcal{S}} a_{W'} \chi(W') a_{W'}^{-1}$ generates W.*

Here, the $a_{W'}$ *and* ϵ *are uniquely determined, provided* $\epsilon = 1$ *in case* (LocFin). *If the conditions hold, the abstract root basis corresponding to* Ψ_+ *is* $\Delta_{\Psi_+} = \lim_{W'\in\mathcal{S}} \epsilon a_{W'}(\chi(W') \times \{1\})$*, and* $S' = S_{\Psi_+}$*.*

Proof. Let Ψ_+^1 be an abstract positive system for $\widehat{T}$, and recall that $\Psi_+^0 = \widehat{T}_+$ is also an abstract positive system for $\widehat{T}$. We show that there exists an $\epsilon \in \{\pm 1\}$ and that, for all $W' \in \mathcal{S}$, there exists an $a_{W'} \in W'$ such that $\Psi_+^1(W') = \epsilon a_{W'}(\Psi_+^0(W'))$ for all $W' \in \mathcal{S}$ and such that $\epsilon = 1$ in case (LocFin). Furthermore, the function $\mathcal{S} \to W$, given by $W' \mapsto a_{W'}$, and the sign ϵ are uniquely determined by these conditions.

In the case (LocFin), this immediately follows from Theorem 4.5 and Corollary 4.6. In the case (LocInf), we also have from Theorem 4.5 and Corollary 4.6 that, for $W' \in \mathcal{S}$, $\Psi_+^1(W') = \epsilon_{W'} a_{W'}(\Psi_+^0)$ for a unique sign $\epsilon_{W'}$, and $a_{W'} \in W'$, and we must show that $\epsilon_{W'}$ is independent of W'. Since $\mathcal{S}$ is directed, it suffices to show that $\epsilon_{W_1'} = \epsilon_{W_2'}$ if $W_1' \subseteq W_2' \in \mathcal{S}$. Since $\Psi_+^i(W_1') \subseteq \Psi_+^i(W_2')$ if $W_1' \subseteq W_2'$, this follows from the facts, from Corollary 4.6 (b), that $\epsilon_{W'} = 1$ if and only if $\Psi_+^1(W') \cap -\Psi_+^0(W')$ is finite, and $\epsilon_{W'} = -1$ if and only if $\Psi_+^1(W') \cap \Psi_+^0(W')$ is finite.

Now, suppose that $W' \subseteq W''$ in $\mathcal{S}$. Since $\Psi_+^1(W'') \supseteq \Psi_+^1(W')$, we have

$$a_{W''}^{-1} a_{W'} \Psi_+^0(W') \subseteq \Psi_+^0(W'') \subseteq \widehat{T}_+, \tag{4.1}$$

which gives (ii).

Next, we show $\Delta_{\Psi_+^1} = \Delta'$, where

$$\Delta' := \lim_{W'\in\mathcal{S}} \epsilon a_{W'}(\chi(W') \times \{1\}).$$

Note that $\Delta' \cap -\Delta' = \emptyset$ and that S', defined as in (iii), also satisfies $S' = \{s_\alpha \mid \alpha \in \Delta'\}$. Now, if $\alpha \in \Delta_{\Psi_+^1}$, then, for any $W'' \in \mathcal{S}$ with $\{1, s_\alpha\} \subseteq W''$, α is an abstract simple root for $\Psi_+^1(W'')$ (since it is one for Ψ_+^1), so $\alpha \in \epsilon a_{W''}(\chi(W'') \times \{1\})$. This shows that $\Delta_{\Psi_+^1} \subseteq \Delta'$. Hence, $S_{\Psi_+} \subseteq S'$ both generate W. Since any relation on S' in W involves elements of only a finite subset of S', it is easy to see that S' is a set of Coxeter generators for W; thus, $S' = S_{\Psi_+^1}$ by Lemma 2.6. Since $\Delta' \cap -\Delta' = \emptyset$, this gives $\Delta_{\Psi_+^1} = \Delta'$.

Conversely, suppose, given $\epsilon \in \{\pm 1\}$ and $a_{W'} \in W$, for $W' \in \mathcal{S}$ satisfies (i)–(iii). Define the sets

$$X_{W'} := \epsilon a_{W'} \Psi_+^0(W') \quad \text{and} \quad \Psi_+ := \bigcup_{W' \in \mathcal{S}} X_{W'}.$$

From (i) and (ii), we deduce that $X_{W'} \subseteq (W' \cap T) \times \{\pm 1\}$, $X_{W'} \subseteq X_{W''}$ for $W' \subseteq W''$ in $\mathcal{S}$, see (4.1), and $X_{W'} = \Psi_+(W')$. Clearly, Ψ_+ is a quasi-positive system. Since any dihedral refection subgroup of (W, S) is contained in W' for some $W' \in \mathcal{S}$, it easily follows that Ψ_+ is biclosed. Now, let Δ' be as in the paragraph immediately above. As before, $\Delta' \cap -\Delta' = \emptyset$ and $S' = \{s_\alpha \mid \alpha \in \Delta'\}$. If $\alpha \in \Delta'$, then there is some $W' \in \mathcal{S}$ such that α is a simple root for $\Psi_+(W'')$ for all $W'' \supseteq W'$ in $\mathcal{S}$. Since

$$\Psi_+ = \bigcup_{\{W'' \in \mathcal{S} \mid W'' \supseteq W'\}} \Psi_+(W''),$$

the definitions give $\Delta' \subseteq \Delta_{\Psi_+}$. On the other hand, if $\alpha \in \Delta_{\Psi_+}$, then α is a simple reflection of $\Psi_+(W'')$ for all w'' with $s_\alpha \in W''$, so $\alpha \in \Delta_{\Psi_+}$. Hence, $\Delta' = \Delta_{\Psi_+}$, $S' = S_{\Psi_+}$ and, by (iii), Ψ_+ is generative. □

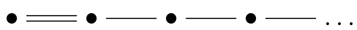
Corollary 4.15. *Let Ψ_+^i be abstract systems of positive roots for the arbitrary Coxeter system (W, S) for $i = 0, 1$. Then, $(W, S_{\Psi_+^1})$ and $(W, S_{\Psi_+^0})$ have the same finitely generated parabolic subgroups.*

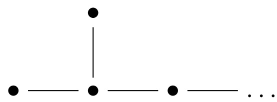
Proof. Using subsection 2.14, we may reduce to the case where (W, S) is irreducible. In that case, it will suffice to show that any finite subset Δ_1 of $\Delta^1_{\Psi_+}$ is W-conjugate up to sign to some finite subset Δ_0 of $\Delta_{\Psi_+^0}$. By symmetry, see Proposition 2.32, we may assume that $\Psi_+^0 = \widehat{T}_+$. Choose a finitely generated standard parabolic subgroup $W' = W_K \in \mathcal{S}'$ of (W, S) such that $\Delta_1 \subseteq \Psi_+^1(W')$. Then, as in the proof of Lemma 4.14, $\Delta_1 \subseteq \epsilon a_{W'}(K \times \{1\})$. We may take

$$\Delta_0 = \epsilon a_{W'}^{-1} \Delta_1 \subseteq (K \times \{1\}) \subseteq (S \times \{1\}) = \Delta_{\Psi_+^0}.$$

This proves the corollary. □

Example 4.16. Suppose that (W, S) is of type (LocFin) but is not of finite rank. Then, cf., **[19]**, by the classification of irreducible Coxeter systems, (W, S) is either of type A_∞ or $A_{\infty,\infty}$ as in Example 4.9, or of type B_∞ or D_∞ with Coxeter graphs as shown below, respectively:





Corollary 4.15 and the classification of finite Coxeter systems implies that, for a root-based datum X with associated Coxeter system (W,S) of type A_∞ or $A_{\infty.\infty}$, any abstract root basis of (W,S) is also of type A_∞ or $A_{\infty,\infty}$ (although not necessarily of the same type as X). For similar reasons, for a root-based datum X with associated Coxeter system (W,S) of type B_∞ or D_∞, any abstract root basis of X must be of exactly the same type, B_∞ or D_∞, respectively, as X.

For use in the proof of Theorem 4.12, we describe the various possible root bases in these cases. We realize root systems of type $A_{\infty,\infty}$ as E_S as in Example 4.9, and those of type A_∞ as E_K as in Example 4.9. We realize the root system of type B_∞ as the subset

$$\Phi_B := \{\pm(e_i+e_j), \pm(e_i-e_j), \pm e_i \mid i,j\in\mathbb{N}, i\neq j\}$$

of V_K, and that of type D_∞ as the subset

$$\Phi_D := \{\pm(e_i+e_j), \pm(e_i-e_j) \mid i,j\in\mathbb{N}, i\neq j\}$$

of V_K as in Example 4.9. As corresponding sets of simple roots, we take

$$\Pi_B = \{e_0\}\cup\{e_{n+1}-e_n \mid n\in\mathbb{N}\}$$

and

$$\Pi_D = \{e_0+e_1\}\cup\{e_{n+1}-e_n \mid n\in\mathbb{N}\}.$$

These determine standard-based root data E'_B and E'_D of types B_∞ and D_∞, respectively, on V_K with the restriction of the form $\langle -,-\rangle$ on V_S from Example 4.9.

Let $G(A_\infty)$ be the group consisting of all $\mathbb{R}$-linear operators on V_K which restrict to bijections of the set $\{e_n \mid n\in\mathbb{N}\}$, and let $G(A_{\infty,\infty})$ be the group consisting of all $\mathbb{R}$-linear operators on V_S which restrict to bijections of the set $\{e_n \mid n\in\mathbb{Z}\}$. In addition, let $G(B_\infty)=G(D_\infty)$ be the "infinite signed permutation group" consisting of all invertible linear operators on V_K which induce bijections on the set $\{\pm e_n \mid n \in \mathbb{N}\}$.

It is easy to directly verify by ad hoc arguments from the preceding paragraph that, up to sign, every root basis of E_S or E_K is one of those described in the last paragraph of Example 4.9. Furthermore, the group $G(A_\infty)$ acts transitively on the set of root bases of the root systems for E_K which are of type A_∞, and the group $G(A_{\infty,\infty})$ acts transitively on the set of root bases of the root systems for E_S which are of type $A_{\infty,\infty}$. Similarly, $G(B_\infty)$ acts transitively on the set of root bases of Φ_B, and $G(D_\infty)$ acts transitively on the set of root bases of Φ_D.

If (W, S) is an infinite irreducible Coxeter system of type (LocFin), its type X is one of A_∞, $A_{\infty,\infty}$, B_∞ or D_∞. Let G be the group $G(X)$ of corresponding type X acting on the real root system Φ of type X, as above. Identify $\Phi = \widehat{T}$ as in Proposition 4.1 (b). From above, G acts transitively on the set of abstract root bases of $\widehat{T}$ of the same type X as (W, S). It is easy to see that the action of G on $\widehat{T}$ is by elements of $\widehat{W}$; actually, there is a natural identification $G \cong \widehat{W}$ in each case.

Proof of Theorem 4.12. First, we prove (a)–(b). We assume that (W, S) is not of finite rank; otherwise, (a)–(b) follow from Theorem 4.5. If (W, S) is of type (LocFin), then (a)–(b) follow from Corollary 4.15. Now, suppose that (W, S) is of type (LocInf). Let ϵ and $a_{W'}$ be as in the statement of Lemma 4.14. We define a map

$$\lambda \colon \Psi \longrightarrow \Psi,$$

given locally by the action of W, as follows. Let R be a finite subset of S_{Ψ_+} such that the Coxeter system $(\langle R \rangle, R)$ is infinite and irreducible (note that the family of all such R is cofinal in the family of all finite subsets of S_{Ψ_+}). Let

$$\Delta := \{\alpha \in \Delta_{\Psi_+} \mid s_\alpha \in R\}.$$

Choose $J \subseteq S$ with $W_J \in \mathcal{S}'$ such that $\langle R \rangle \subseteq W_J$. Set $\lambda(\alpha) = \left(a_{W_J}\right)^{-1}(\alpha)$ for all $\alpha \in (\langle R \rangle \cap T) \times \{\pm 1\}$. To show λ is well defined, it will suffice to show that, if $J \subseteq K \subseteq S$ with $W_K \in \mathcal{S}'$, then $a_{W_K} a_{W_J}^{-1}$ fixes all $\alpha \in (\langle R \rangle \cap T) \times \{\pm 1\}$, or equivalently, that $p := a_{W_K}^{-1} a_{W_J}$ fixes $a_{W_J}^{-1}(\Delta)$ elementwise. However, from the proof of Lemma 4.14,

$\epsilon a_{W_J}^{-1}\Delta = J' \times \{1\} \subseteq J \times \{1\}$ and $\epsilon a_{W_K}^{-1}\Delta = K' \times \{1\} \subseteq K \times \{1\}$, where $J' \subseteq J$ and $K' \subseteq K$, so $p(J' \times \{1\}) = K' \times \{1\}$. Since $(\langle R\rangle, R)$ is infinite and irreducible, so are $(W_{J'}, J')$ and $(W_{K'}, K')$ (since they are isomorphic to $(\langle R\rangle, R)$). From Lemma 2.14 applied to (W, S), $J' = K'$, and p is in the subgroup of W_K generated by reflections in $S \setminus J'$ which commute with each element of J', giving the desired conclusion. Note that, from the construction, $\epsilon\lambda(\Psi_+) \subseteq \widehat{T}_+$ and, in fact, $\epsilon\lambda(\Delta_{\Psi_+}) \subseteq S \times \{1\}$. By symmetry, interchanging the roles of Ψ_+ and $\widehat{T}_+$, we also get a map $\lambda' \colon \Psi \to \Psi$ given locally by conjugation and with $\epsilon\lambda'(\widehat{T}') \subseteq \Psi_+$ and $\epsilon\lambda'(S \times \{1\}) \subseteq \Delta_{\Psi_+}$. We claim that the composite $\lambda'\lambda$ is the identity map on Ψ. In order to see this, let R and Δ be as above in the definition of λ. For some $w \in W$, we have $\lambda'\lambda(\alpha) = w(\alpha)$ for all $\alpha \in (\langle R\rangle \cap T) \times \{\pm 1\}$. Since $w(\Delta) \subseteq \Delta_{\Psi_+}$, and $(\langle R\rangle, R)$ is infinite irreducible, Lemma 2.14 applied to (W, S_{Ψ_+}) shows that w is a product of elements of $S_{\Psi_+} \setminus R$ which commute with each element of R; thus, $w(\alpha) = \alpha$ for all $\alpha \in (\langle R\rangle \cap T) \times \{\pm 1\}$. Similarly, $\lambda\lambda' = \mathrm{Id}_\Psi$ so $\lambda' = \lambda^{-1}$. Hence, $\widehat{w} := \lambda \in \widehat{W}$. Clearly, $\epsilon\widehat{w}$ maps Δ_{Ψ_+} bijectively to $S \times \{1\}$, maps Ψ_+ bijectively to $\widehat{T}_+$ and induces an isomorphism $(W, S_{\Psi_+}) \cong (W, S)$. This proves (a) and (b) in case (LocInf), and hence, in all cases.

Now, we prove (c), in general. Let $\Psi_+ := \sigma(\widehat{T})$. Then, Ψ_+ is an abstract positive system, and, by Lemma 2.31, σ induces an isomorphism

$$s_\alpha \longmapsto s_{\sigma(\alpha)} \colon (W, S) \cong (W, S_{\Psi_+}).$$

Let ϵ, $\widehat{w}$ be as in (b). Then, $d' = \widehat{w}^{-1}\epsilon'\sigma$ is an automorphism of $\widehat{T}$ fixing $\widehat{T}_+$ setwise, and therefore, fixing the standard root basis $S \times \{1\}$ setwise as well. It is clear that d' induces an automorphism $s_\alpha \mapsto s_{d'(\alpha)}$ of (W, S), that is, a diagram automorphism d of (W, S), and that d' is the automorphism of $\widehat{T}$ induced by d. This completes the proof of the theorem. $\square$

Corollary 4.17. *Let E be a root datum with root system Φ and Π, Π' two root bases for E affording root-based data B, B' with corresponding Coxeter systems (W, S) and (W, S'), respectively. Assume that (W, S) and (W, S') are isomorphic and irreducible. Then, after possibly rescaling B' (still keeping $\Pi' \subseteq \Phi$), there are linear maps $\sigma \colon V \to V$ and $\sigma' \colon V' \to V'$ and a sign $\epsilon \in \{\pm 1\}$ with the following properties*:

(i) *$\epsilon\sigma$ restricts to a bijection $\Pi' \to \Pi$, and $\epsilon\sigma'$ restricts to a bijection $\Pi^\vee \to \Pi'^\vee$;*

(ii) *for any subspaces $V_1 \subseteq V$ spanned by a finite set of roots and $V_1' \subseteq V'$ of V' spanned by a finite set of coroots, there is a $w = w_{V_1,V_1'} \in W$ such that $\sigma(v) = w(v)$ for any $v \in V_1$ and $\sigma(v') = w(v')$ for any $v' \in V_1'$;*

(iii) *σ restricts to a permutation of Φ, σ' restricts to a permutation of $\Phi^\vee$ and $\sigma(\alpha)^\vee = \sigma'(\alpha^\vee)$ for all $\alpha \in \Phi$;*

(iv) *$\langle \sigma(v), \sigma'(v') \rangle = \langle v, v' \rangle$ for all $v \in \mathbb{R}\Pi$, $v' \in \mathbb{R}\Pi'$.*

In particular, $\langle \alpha, \beta^\vee \rangle = \langle \sigma(\alpha), \sigma(\beta)^\vee \rangle$ for all $\alpha, \beta \in \Pi'$, in other words, Π and $\Pi^\vee$ afford the same NGCM (*up to rescaling and reindexing*).

Remark 4.18. There is an obvious notion of an isomorphism of root-based data. If $V = \mathbb{R}\Pi$ and $V' = \mathbb{R}\Pi'$, $\epsilon\sigma$ and $\epsilon\sigma'$ together define such an isomorphism of root-based data $B \xrightarrow{\cong} B'$ (after the possible rescaling of B').

Proof of Corollary 4.17. We assume, without loss of generality, that Π, and therefore, also Π', spans V, and similarly for V'. If (W, S) is of finite rank, the corollary follows from Theorem 4.3 with $\sigma = \sigma' \in W$. Assume now that (W, S) is of infinite rank. If (W, S) is of type (LocInf), then, since any set of simple roots for a finite standard parabolic subgroup of (W, S) must be linearly independent, Π must be linearly independent. Also, the Coxeter graph of (W, S) is a tree. Therefore, up to rescaling Π, B is of the type considered in Examples 4.9 and 4.16, and the desired conclusion readily follows. Finally, assume that (W, S) is of infinite rank of type (LocInf). We may define σ following the definition of λ in the proof of Theorem 4.12, namely, let $R \subseteq S'$ be such that the parabolic subsystem $(\langle R \rangle, R)$ is finitely generated and irreducible. Consider the abstract positive Ψ_+ in $\widehat{T}$ corresponding to the positive roots of the root datum B', and define $a_{W'}$, ϵ as in Lemma 4.14. Following the proof of Theorem 4.12, set $\sigma(\alpha) = a_{W_J}^{-1}(\alpha)$ for all α in the $\mathbb{R}$-span of $\{\beta \in \Phi \mid s_\beta \in R\}$. This is well defined since, for p, J', K' as in the proof of Theorem 4.12, p fixes

$$\{\alpha \in \Phi_+ \mid s_\alpha \in J'\}$$

elementwise. In a similar manner to that in the proof of Theorem 4.12, it is seen that σ is invertible (here, as a linear map). We may define σ' in a similar way on $\mathbb{R}\Pi^\vee$, and then, clearly, (ii)–(iv) hold. It is obvious that $\epsilon\sigma(\Pi')$ differs from Π simply by rescaling. Rescaling Π appropriately, we, therefore, assume that we have $\epsilon\sigma(\Pi') = \Pi$. It may be verified from (iii)–(iv) that this implies that $\epsilon\sigma'(\Pi'^\vee) = \Pi^\vee$, and this gives (i). $\square$

The analog of the following for a class of crystallographic reflection representations (arising from Kac-Moody Lie algebras) was proven in **[20]**.

Corollary 4.19. *Suppose that B is a root-based datum affording an irreducible Coxeter system (W, S) and that V is finite-dimensional. Then, any two root bases of B are W-conjugate up to sign and rescaling.*

Proof. Note that (W, S) cannot be of type (LocInf) but infinite, for then it would have an infinite, linearly independent set of simple roots contrary to the finite dimensionality of V. Hence, it is not of type A_∞ or $A_{\infty,\infty}$, and the desired conclusion follows from the preceding theorem. $\square$

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APPENDICES

A. Cocycles and extensions of group actions.

A.1. Let G be a (multiplicatively written) group and A a (multiplicatively written) group on which G acts on the left so that $g(a \cdot a') = g(a) \cdot g(a')$. We recall **[22,** Chapter 7 Appendix] that a (1-)*cocycle* of G in A is a map

$$s \longmapsto a_s \colon G \longmapsto A$$

satisfying $a_{st} = a_s \cdot s(a_t)$ for all $s, t \in G$. Two cocycles a, b are said to be *cohomologous* if there exists a $c \in A$ such that $b_s = c^{-1} \cdot a_s \cdot s(c)$ for all $s \in G$. This defines an equivalence relation on the set $Z^1(G, A)$ of cocycles of G in A, and the set $H^1(G, A)$ of equivalence classes is called the *first cohomology set* of G with values in A. In general, $H^1(G, A)$ is only a pointed set (with base point given by the equivalence class of the trivial cocycle a defined by $a_s = 1$ for all s).

If A is abelian, $Z^1(G,A)$ has a natural structure of abelian group, with pointwise product of functions as multiplication. A map

$$s \longrightarrow a_s \colon G \longrightarrow A$$

is called a *coboundary* (more precisely, a 1-coboundary) if there is some $b \in A$ such that $a_s = b(sb)^{-1}$ for all $s \in G$. The set $B^1(G,A)$ of coboundaries is a subgroup of $Z^1(G,A)$, and two cocycles in $Z^1(G,A)$ are cohomologous if and only if they lie in the same coset of $B^1(G,A)$. The first cohomology set, therefore, acquires a natural structure of abelian group, called the *first cohomology group*:

$$H^1(G,A) = Z^1(G,A)/B^1(G,A).$$

A.2. Let S be a set with a left action of the group A, denoted $(x,y) \mapsto x \cdot y$ for $x \in G, s \in S$) of A and a left action of the group G, denoted $(g,y) \mapsto g(y)$, compatible with the left action of G on A from A.1 in the sense that $g(x \cdot y) = g(x) \cdot g(y)$ for $g \in G$, $x \in A$, $y \in S$. This applies, in particular, with $S = A$ with its given left G-action and natural left A-action by left translation. For a cocycle a, as in A.1, we let S_a denote the G-set S with the left action $\times_a$ of G on A_a defined by $s \times_a y = a_s \cdot s(y)$ for $s \in G$ and $y \in A$. If b is another cocycle, cohomologous to a by the element $c \in A$ as in A.1, then the map

$$y \longmapsto c \cdot y \colon S_b \longrightarrow S_a$$

defines an isomorphism of G-sets $S_b \to S_a$.

A.3. Let G be a group acting on a set X. For $x \in X$, let $[x] := Gx$ denote the orbit of x. Assume, given for each orbit $[x]$, a multiplicative group $A_{[x]}$. We consider a category C in which an object is a G-set $\widetilde{X}$ with a given (surjective) G-equivariant map $\pi \colon \widetilde{X} \to X$ and, for each orbit $[x]$, a right action of the group $A_{[x]}$ on $\pi^{-1}([x])$ commuting with the G-action on this set and such that, for each $y \in [x]$, $A_{[x]}$ acts simply transitively on $\pi^{-1}(y)$. Morphisms are given by G-equivariant maps between the corresponding G-sets commuting with the projections π and the $A_{[x]}$-actions on the inverse images $\pi^{-1}([x])$, and the composition of morphisms is by composition of the underlying maps of G-sets. (If there is a group H with $A_{[x]} = H$ for all x, this notion is an analog in the category of G-sets of a principal H-bundle **[18]**.) Any morphism in C is clearly an isomorphism. Next, we indicate how the objects of C are classified up to isomorphism by an appropriate first cohomology group $H^1(G, A)$.

For each $x \in X$, define a multiplicative abelian group $A_x = A_{[x]}$. Form the product group

$$A := \prod_{x \in X} A_x,$$

with projections $\pi_x \colon A \to A_x$ for $x \in X$. For $a \in A$, we write a as $a = (a_x)_{x \in X}$ or, for short, $a = (a_x)$, where $a_x = \pi_x(a)$. The group G acts naturally on the left of A by $g(a) = b$ where $b_x = a_{g^{-1}(x)}$, or, for short, $g(a_x) = (a_{g^{-1}(x)})$.

Proposition A.1. *There is a natural bijection between the isomorphism classes of objects of the category C defined in* A.3 *and* $H^1(G, A)$.

Proof. Arbitrary functions $a \mapsto a_g \colon G \to A$ correspond bijectively to functions

$$\eta \colon G \times X \longrightarrow \bigcup_{x \in X} A_{[x]}$$

with $\eta(g, x) \in A_x$ for all $x \in X$, $g \in G$, by the correspondence $a_g = (\eta(g, x))_{x \in X}$. We check that the formula

$$g(x, y_x) := (gx, \eta(g^{-1}, x)^{-1} y_x)$$

for $x \in X$, $g \in G$, $y_x \in A_x$, defines a left G-action on the set

$$\widetilde{X}_a = \bigcup_{x \in X} \{x\} \times A_x$$

if and only if the function $g \mapsto a_g$ is a cocycle a of G with values in A. In this case, $\widetilde{X}_a$ has an obvious structure as an object of C. Clearly, every object of C is isomorphic to that obtained from this construction from some function η. It is straightforward to verify that, for two cocycles a and b, $\widetilde{X}_a$ is isomorphic in C to $\widetilde{X}_b$ if and only if a and b are cohomologous. Hence, the map sending the cocycle a to the isomorphism class of $\widetilde{X}_a$ gives the required bijection. $\square$

A.4. Although we shall not need it, we also describe an analog in the above setting of the construction of an associated bundle of a principal bundle. For $x \in X$, let S_x be a left $A_{[x]}$-set with $S_x = S_y$ whenever

$[x] = [y]$. Give the product set $S := \prod_{x \in X} S_x$ the left A-action with $(a_x) \cdot (y_x) = (a_x \cdot y_x)$. There is an action of G on S defined by $g(y_x) = (y_{g^{-1}(x)})$, compatible with the G-action on A in the sense of A.2. If $a\colon G \to A$ is a cocycle and η the corresponding function as in the proof of Proposition A.1, the formula

$$g(x, y_x) := (gx, \eta(g^{-1}, x)^{-1} y_x)$$

for $x \in X$, $g \in G$, $y_x \in S_x$, defines a left G-action on the set $X' := \bigcup_{x \in X} \{x\} \times S_x$.

Any G-set X' with given equivariant map $\pi'\colon X' \to X$ such that, for each G-orbit $[x]$, there is some set $S_{[x]}$ such that the fibers $(\pi')^{-1}(y)$ for $y \in [x]$ are all in bijection with $S_{[x]}$, arises from the above construction with $A_{[x]} = \mathrm{Sym}(S_{[x]})$, the symmetric group acting on $S_{[x]}$ in the usual way, and $S_x := S_{[x]}$.

Remark A.2. We remark that most results in this appendix extend to groupoid actions in place of group actions. Such results find application in the study of various groupoids with abstract root systems, such as the groupoids studied in **[5]** (although their root systems are not constructed there).

B. Quasi-root systems. In this appendix, we show how the definition and some elementary properties of Bruhat order and weak order on Coxeter groups extend to groups with a suitable quasi-root system, linearly realized in a real vector space. The principal example, other than Coxeter groups, is provided by real orthogonal groups.

B.1. Let (Ψ, F) be a quasi-root system. We use notation as in subsection 2.3. Suppose that Ψ_+ is a quasi-positive system for Ψ, and let

$$N := N_{\Psi_+} : W \longrightarrow \mathcal{P}(T)$$

be the corresponding reflection cocycle.

Note that G acts on the left on T by conjugation:

$$(g, t) \longmapsto \iota(g) t \iota(g)^{-1}.$$

By abuse of notation, we write this as $(g, t) \mapsto gtg^{-1}$. Then,

$$s_{\sigma(\alpha)} = \sigma s_\alpha \sigma^{-1}, \quad \sigma \in G,\ \alpha \in \Psi. \tag{B.1}$$

We define certain pre-orders (reflexive, transitive relations) on W as follows. The *weak pre-order* $\leq_w$ is defined by $x \leq_w y$ if and only if $N(x) \subseteq N(y)$; this is a partial order if and only if $N(x) = N(y)$ for $x, y \in W$ implies $x = y$, or equivalently, by the cocycle condition if and only if $N(x) = \emptyset$ with $x \in W$ implies $x = \emptyset$. Let $A \subseteq T$. The *twisted Bruhat pre-order* $\leq_A$ on W is defined by $x \leq_A y$ if and only if there exist $x = x_0, x_1, \ldots, x_n \in W$ and $t_i \in x_i \cdot A$ with $x_{i-1} = t_i x_i$, where $w \cdot A = N(w) + wAw^{-1}$. It is easy to see that, for $x, y \in W$,

$$x \leq_A y \iff xw^{-1} \leq_{w \cdot A} yw^{-1} \iff y \leq_{T+A} x. \tag{B.2}$$

For $A = \emptyset$, the order $\leq_A$ is denoted merely as $\leq$ and is called the *Bruhat pre-order* on W.

In the case where (Ψ, F) is the standard abstract root system of a Coxeter system (W, S), $\leq_w$ and $\leq$ are partial orders, called *weak right order* and *Bruhat order*, respectively, and $\leq_A$ is a twisted Bruhat order in the sense of **[9]** if A is an *initial section of a reflection order* in the sense of *loc cit.*

B.2. We define a tuple $(\langle -, - \rangle, R, M, M^\vee, \iota, \iota^\vee)$ to be a *linear realization* of (Ψ, F) if the following conditions hold:

(i) R is a ring, M is a left R-module, $M^\vee$ is a right R-module, and $\langle -, - \rangle \colon M \times M^\vee \to R$ is a R-bilinear form;

(ii) $\iota \colon \Psi \to M$ and $\iota^\vee \colon \Psi \to M^\vee$ are injective functions. We identify Ψ with $\iota(\Psi)$; thus, ι becomes an inclusion, set $\Psi^\vee := \iota^\vee(\Psi^\vee)$ and set $\alpha^\vee = \iota^\vee(\alpha)$ for $\alpha \in \Psi \subseteq M$ so $\nu := (\alpha \mapsto \alpha^\vee) \colon \Psi \mapsto \Psi^\vee$ is a bijection;

(iii) $\langle \alpha, \alpha^\vee \rangle \alpha = 2\alpha$, $\alpha^\vee \langle \alpha, \alpha^\vee \rangle = 2\alpha^\vee$ for $\alpha \in \Psi$. For $\alpha \in \Psi$, define $s_\alpha \in \mathrm{GL}_R(M)$ by $m \mapsto m - \langle m, \alpha^\vee \rangle \alpha$ and $s_{\alpha^\vee} \in \mathrm{GL}_R(M^\vee)$ by $m \mapsto m - \alpha^\vee \langle \alpha, m \rangle$;

(iv) for $\alpha \in \Psi$, $s_\alpha(\Psi) = \Psi$ and $F(\alpha) = (s_\alpha)_{|\Psi}$, while $s_{\alpha^\vee}(\Psi^\vee) = \Psi^\vee$ and $(s_{\alpha^\vee})_{|\Psi^\vee} = F(\alpha)$;

(v) $\langle s_\alpha \mid \alpha \in \Psi \rangle$ identifies with $W \subseteq \mathrm{Sym}(\Psi)$ by restriction and $\langle s_{\alpha^\vee} \mid \alpha \in \Psi \rangle$ identifies with $W \subseteq \mathrm{Sym}(\Psi^\vee)$, by restriction.

Here, we use ν to identify any action of a group, such as $W := \langle F(\alpha) \mid \alpha \in \Psi \rangle$, on $\Psi \subseteq M$ with an action of that group on $\Psi^\vee$. Note that (v) automatically follows from (i)–(iv) if $R\Psi = M$ and $\Psi^\vee R = M^\vee$.

B.3. The main examples of quasi-root systems and their linear realizations are provided by Coxeter groups. Any root datum E as in subsection 3.3 may be naturally regarded as providing a linear realization of some quasi-root system (in fact, an abstract root system in the sense of Definition 2.10). Other interesting examples of linear realizations of abstract root systems arise from natural reflection representations of Coxeter groups over certain commutative or non-commutative coefficient rings.

B.4. Certain orthogonal groups provide examples of (linearly realized) quasi-root systems as follows. Let K be a field of characteristic unequal to two, and let V be a finite-dimensional vector space over K equipped with a symmetric, non-degenerate bilinear form $(- \mid -)\colon V \times V \to K$. Let $O(V)$ be the orthogonal group of the quadratic space $(V, (- \mid -))$. By definition, $O(V)$ consists of invertible K-linear transformations of V preserving the form. For non-isotropic $\alpha \in V$, let $s_\alpha \in O(V)$ denote the orthogonal reflection, defined by

$$v \longmapsto v - 2(v \mid \alpha)/(\alpha \mid \alpha)\alpha.$$

We choose a subset Ψ of V such that Ψ is stable under $O(V)$, consists of non-isotropic vectors and each non-isotropic line l in $O(V)$ contains at least one element of Ψ. Such a set Ψ always exists; for example, Ψ could be taken to consist of all non-isotropic vectors in V. Alternatively, Ψ could be chosen so that each non-isotropic line contains exactly two elements of Ψ; for instance, if $K = \mathbb{R}$, Ψ could be taken to consist of all $\alpha \in V$ with $(\alpha \mid \alpha) = \pm 1$. For $\alpha \in \Psi$, let $F(\alpha) \in \mathrm{Sym}(\Psi)$ denote the restriction of s_α to Ψ. Let $\Psi^\vee := \{\alpha^\vee \mid \alpha \in \Psi\}$, where $\alpha^\vee := 2\alpha/(\alpha \mid \alpha)$. Then, clearly, (Ψ, F) is a quasi-root system. Moreover, $((-,-), K, V, V, \iota, \iota^\vee)$ is a linear realization of (Ψ, F), where $\iota\colon \Psi \to V$ is the inclusion and $\iota^\vee\colon \Psi \to V$ is the map $\alpha \mapsto \alpha^\vee$.

B.5. Assume that (Ψ, F) is a quasi-root system with a linear realization over $\mathbb{R}$ of the form $(\langle -,- \rangle, \mathbb{R}, V, V^\vee, \iota, \iota^\vee)$ so that V and $V^\vee$ are real vector spaces. We fix a *vector space total ordering* $\preceq$ of $V^\vee$, that is, $\preceq$ is a total ordering of $V^\vee$ such that the set of positive elements is closed under addition and multiplication by positive real numbers. We also assume, given a family $\{\omega_i\}_{i \in I}$ of elements of $V^\vee$, indexed by a well-ordered set I, such that, if $v \neq 0$ in V, there is an $i \in I$ with $\langle v, \omega_i \rangle \neq 0$.

For a totally ordered set X with total order $\leq$, let X^I be the family of I-tuples $(x_i)_{i\in I}$ of elements of X, totally ordered by the lexicographic order $\leq_{\mathrm{lex}}$ induced by $\leq$ on X and the well ordering of I. In particular, we have totally ordered sets $(V^\vee)^I$ with order $\preceq_{\mathrm{lex}}$ and $\mathbb{R}^I$ with order $\leq_{\mathrm{lex}}$ defined using the standard ordering $\leq$ of $\mathbb{R}$. The vector space embedding

$$v \longmapsto (\langle v, \omega_i\rangle)_{i\in I} \colon V \longrightarrow \mathbb{R}^I$$

gives rise, by restriction of the order $\leq_{\mathrm{lex}}$, to a vector space total ordering of V, which we also denote as $\leq_{\mathrm{lex}}$. (It is well known that, if V is finite-dimensional and $\langle -,-\rangle$ is a perfect pairing, then every vector space total order of V is equal to an order $\leq_{\mathrm{lex}}$ arising from some family $\{\omega_i\}$.)

It is easy to see that, for $\alpha, \beta \in \Psi$, $s_\alpha = s_\beta$ if and only if there is a $c \in \mathbb{R}_{\neq 0}$ such that $\beta = c\alpha$ and $\beta^\vee = c^{-1}\alpha^\vee$; it readily follows that

$$\Psi_+ := \{\alpha \in \Psi \mid 0 \preceq \alpha^\vee\} \quad \text{and} \quad \Phi_+ := \{\alpha \in \Psi \mid \alpha \leq_{\mathrm{lex}} 0\}$$

are compatible quasi-positive systems for (Ψ, F). Define the reflection cocycle $N = N_{\Psi_+} \colon G \to \mathcal{P}(T)$, and set

$$A = \{F(\alpha) \mid \alpha \in \Phi_+ \cap -\Psi_+\} \subseteq T.$$

Proposition B.1. *Let assumptions and notation be as immediately above. Then*:

(a) *for all $\alpha \in \Psi_+$ and $w \in W$,*
$$(s_\alpha w(\omega_i))_{i\in I} \preceq_{\mathrm{lex}} (w(\omega_i))_{i\in I}$$
in $(V^\vee)^I$ if and only if $s_\alpha \in w \cdot A = N(w) + wAw^{-1}$;

(b) *the twisted Bruhat pre-order $\leq_A$ is a partial order on W.*

Proof. For (a), we have $(s_\alpha w(\omega_i))_i \preceq_{\mathrm{lex}} (w(\omega_i))_i$ if and only if we have

$$(w(\omega_i) - \langle \alpha, w\omega_i\rangle \alpha^\vee)_i \preceq_{\mathrm{lex}} (w(\omega_i))_i$$

in $(V^\vee)^I$. This holds if and only if $(\langle \alpha, w(\omega_i)\rangle)_i \geq_{\mathrm{lex}} (0)_i$ in $\mathbb{R}^I$ (since $0 \prec \alpha^\vee$), and also if and only if $w^{-1}(\alpha) \in -\Phi_+$. Finally, this holds if and only if $s_\alpha \in w \cdot A$, by Proposition 2.3 (a). Now, we show that (a) implies that $\leq_A$ is transitive, which will prove (b). If $x \leq_A y$ and

$y \leq_A x$, then, by (a), we have

$$(x(\omega_i))_i \preceq_{\text{lex}} (y(\omega_i))_i \preceq_{\text{lex}} x(\omega_i))_i,$$

and hence, $(x(\omega_i))_i = (y(\omega_i))_i$. However, if $x \neq y$, we have $x^{-1}(v) \neq y^{-1}(v)$ for some $v \in V$; thus, $\langle x^{-1}(v), \omega_j\rangle \neq \langle y^{-1}(v), \omega_j\rangle$ for some $j \in I$ by the assumption on the family $(\omega_i)_i$. However, then $\langle v, x(\omega_j)\rangle \neq \langle v, y(\omega_j)\rangle$ and $x(\omega_j) \neq y(\omega_j)$, contrary to the above. $\square$

B.6. We conclude with some remarks about the above-defined orders in the special case of the quasi-root systems of real orthogonal groups defined in B.4. In this situation, every vector space total order on $V = V^\vee$ arises as the order $\leq_{\text{lex}}$ from some family $(\omega_i)_{i\in I} = (\omega_1, \ldots, \omega_n)$ which may, without loss of generality, be taken to be a basis of V. It is easy to see that $w(\Psi_+) = \Psi_+$ implies that $w = 1$, and it follows that the weak pre-order is actually a partial order. Since $-1 \in W$ with $N(-1) = \Psi_+$, it can readily be seen that $w \mapsto -w$ is an isomorphism

$$(W, \leq_A) \cong (W, \leq_A^{\text{op}})$$

in any of the twisted Bruhat orders $\leq_A$ and in the weak order $\leq_w$ (this is an analogue of the fact that multiplication by the longest element induces an order-reversing bijection of a finite Coxeter group in (any twisted) Bruhat order or weak order). However, there may be several non-isomorphic Bruhat orders (and several non-isomorphic weak orders) depending upon the choices of $\preceq$, $(\omega_i)_i$, etc.

Assume now that the form $(- \mid -)$ on V is positive definite. Then, without loss of generality, we may take $(\omega_i)_i$ as an orthonormal basis of V, and we see that the vector space total orders of V correspond bijectively to ordered orthonormal bases $(\omega_1, \ldots \omega_n)$ of V. In particular, $W = O(V)$ acts simply transitively on the set of vector space total orderings of V, and the choice of the particular ordering $\preceq$ affects neither the family of order types of posets arising as (W,$\leq_A$) for varying A nor the order type of $\leq_w$. Furthermore, from this, we see that $\Phi_+ = w(\Psi_+)$ for some $w \in W$. Hence, using Corollary 2.4 and (B.2), the posets $(W, \leq_A)$ are all isomorphic to $(W, \leq_\emptyset)$. Thus, for a positive definite form, there is, up to poset isomorphism, only one twisted Bruhat order $\leq_A$ on W (this is analogous to the fact that there is, up to isomorphism, only one twisted Bruhat order on a finite Coxeter group).

One might ask whether other properties of Bruhat order and weak order on finite Coxeter groups are shared by the corresponding orders on orthogonal groups. For example, does W under the weak order form a (complete) lattice?

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