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THINNABLE IDEALS AND INVARIANCE OF CLUSTER POINTS

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ABSTRACT. We define a class of so-called thinnable ideals $\mathcal{I}$ on the positive integers which includes several well-known examples, e.g., the collection of sets with zero asymptotic density, sets with zero logarithmic density, and several summable ideals. Given a sequence (x_n) taking values in a separable metric space and a thinnable ideal $\mathcal{I}$, it is shown that the set of $\mathcal{I}$-cluster points of (x_n) is equal to the set of $\mathcal{I}$-cluster points of almost all of its subsequences, in the sense of Lebesgue measure. Lastly, we obtain a characterization of ideal convergence, which improves the main result in **[15]**.

1. Introduction. It is well known that the set of ordinary limit points of "almost every" subsequence of a real sequence (x_n) coincides with the set of ordinary limit points of the original sequence, in the sense of Lebesgue measure, see Buck **[5]**. In the same direction, we prove its analogue for ideal cluster points.

Towards this aim, let $\mathcal{I}$ be an ideal on the positive integers $\mathbf{N}$, that is, a family of subsets of $\mathbf{N}$ closed under taking finite unions and subsets of its elements. It is assumed that $\mathcal{I}$ contains the collection Fin of finite subsets of $\mathbf{N}$, and it is different from the entire power set of $\mathbf{N}$. Note that the collection of subsets with zero asymptotic

$$\mathcal{I}_0 := \left\{ S \subseteq \mathbf{N} : \lim_{n\to\infty} \frac{|S \cap [1,n]|}{n} = 0 \right\},$$

is an ideal. Also, let $x = (x_n)$ be a sequence taking values in a topological space X. We denote by $\Gamma_x(\mathcal{I})$ the set of $\mathcal{I}$-cluster points of

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x, that is, the set of all $\ell \in X$ such that

$$\{n : x_n \in U\} \notin \mathcal{I}$$

for all neighborhoods U of ℓ. Statistical cluster points (that is, $\mathcal{I}_0$-cluster points) of real sequences were introduced by Fridy [**8**], cf., also [**7, 9, 11**]. However, it is worth noting that ideal cluster points have been studied much before under a different name. Indeed, as it follows by [**11,** Theorem 4.2], they correspond to classical "cluster points" of a filter $\mathscr{F}$ on $\mathbf{R}$ (depending upon x), cf., [**4,** page 69, Definition 2].

As anticipated, the main question addressed here is to find suitable conditions on X and $\mathcal{I}$ such that the set of $\mathcal{I}$-cluster points of a sequence (x_n) is equal to the set of $\mathcal{I}$-cluster points of "almost all" of its subsequences. Finally, we obtain a characterization of ideal convergence. Related results were obtained in [**1, 6, 15, 16, 17, 18**].

2. Thinnability. Given $k \in \mathbf{N}$ and *infinite* sets $A, B \subseteq \mathbf{N}$ with canonical enumeration $\{a_n : n \in \mathbf{N}\}$ and $\{b_n : n \in \mathbf{N}\}$, respectively, we write $A \leq B$ if $a_n \leq b_n$ for all $n \in \mathbf{N}$ and define

$$A_B := \{a_b : b \in B\} \quad \text{and} \quad kA := \{ka : a \in A\}.$$

Definition 2.1. An ideal $\mathcal{I}$ is said to be *weakly thinnable* if $A_B \notin \mathcal{I}$ whenever $A \subseteq \mathbf{N}$ admits non-zero asymptotic density and $B \notin \mathcal{I}$.

If, in addition, $B_A \notin \mathcal{I}$ and $X \notin \mathcal{I}$ whenever $X \leq Y$ and $Y \notin \mathcal{I}$, then $\mathcal{I}$ is said to be *thinnable.*

Definition 2.2. An ideal $\mathcal{I}$ is said to be *stretchable* if $kA \notin \mathcal{I}$ for all $k \in \mathbf{N}$ and $A \notin \mathcal{I}$.

The terminology has been suggested from the related properties of finitely additive measures on $\mathbf{N}$ studied in [**21**]. In this regard, Fin is thinnable and stretchable.

This is the case of several other ideals:

Proposition 2.3. *Let $f : \mathbf{N} \to (0, \infty)$ be a definitively non-increasing function such that $\sum_{n\ge 1} f(n) = \infty$. Define the summable ideal*

$$\mathcal{I}_f := \left\{ S \subseteq \mathbf{N} : \sum_{n \in S} f(n) < \infty \right\}.$$

Then $\mathcal{I}_f$ is thinnable, provided $\mathcal{I}_f$ is stretchable.

In addition, suppose that

$$\liminf_{n\to\infty} \frac{\sum_{i\in[1,n]} f(i)}{\sum_{i\in[1,kn]} f(i)} \neq 0 \quad \text{for all } k \in \mathbf{N}, \tag{2.1}$$

and define the Erdős-Ulam ideal

$$\mathscr{E}_f := \left\{ S \subseteq \mathbf{N} : \lim_{n\to\infty} \frac{\sum_{i\in S\cap[1,n]} f(i)}{\sum_{i\in[1,n]} f(i)} = 0 \right\}.$$

Then, $\mathscr{E}_f$ is thinnable, provided $\mathscr{E}_f$ is stretchable.

Proof. We suppose that $A = \{a_n : n \in \mathbf{N}\}$ admits asymptotic density $c > 0$ and $B = \{b_n : n \in \mathbf{N}\} \notin \mathcal{I}_f$, that is, $\sum_{n\ge 1} f(b_n) = \infty$. Define the integer $k := \lfloor 1/c \rfloor + 1 \ge 2$, and note that $\sum_{n\ge 1} f(kb_n) = \infty$ by the fact that $\mathcal{I}_f$ is stretchable. Then, $a_n = (1/c)n(1 + o(1))$ as $n \to \infty$, which implies

$$\sum_{n\ge 1} f(a_{b_n}) \ge O(1) + \sum_{n\ge 1} f(kb_n) = \infty, \tag{2.2}$$

i.e., $A_B \notin \mathcal{I}_f$; hence, $\mathcal{I}_f$ is weakly thinnable. Moreover, observe that

$$\begin{aligned} \sum_{n\equiv 1 \bmod k} f(b_n) \ge \sum_{n\equiv 2 \bmod k} f(b_n) \ge \cdots \\ \ge \sum_{n\equiv 0 \bmod k} f(b_n) \ge \sum_{\substack{n\equiv 1 \bmod k \\ n\neq 1}} f(b_n), \end{aligned} \tag{2.3}$$

and note that the first sum is finite if and only if the last sum is finite. Since $I \notin \mathcal{I}_f$, then all of the above sums are infinite, which implies that

$$\sum_{n\ge 1} f(b_{a_n}) \ge O(1) + \sum_{n\ge 1} f(b_{kn}) = \infty,$$

i.e., $B_A \notin \mathcal{I}_f$. Lastly, given infinite sets $X, Y \subseteq \mathbf{N}$ with $X \le Y$ and $X \in \mathcal{I}_f$, we have $\sum_{y \in Y} f(y) \le \sum_{x \in X} f(x) < \infty$. Therefore, $\mathcal{I}_f$ is thinnable.

The proof of the second part is similar, where (2.2) is replaced by

$$\sum_{a_{b_n} \le x} f(a_{b_n}) \ge O(1) + \sum_{b_n \le x/k} f(kb_n).$$

Moreover, $B \notin \mathscr{E}_f$ implies that $kB \notin \mathscr{E}_f$ by the hypothesis of strechability, i.e.,

$$\sum_{b_n \le x/k} f(kb_n) \ne o\Bigg(\sum_{i \le x/k} f(i) \Bigg);$$

due to (2.1), we conclude that

$$\sum_{b_n \le x/k} f(kb_n) \ne o\Bigg(\sum_{i \le x} f(i) \Bigg);$$

hence, $A_B \notin \mathscr{E}_f$, which shows that $\mathscr{E}_f$ is weakly thinnable. In addition, we obtain

$$\frac{f(b_{a_1}) + \cdots + f(b_{a_n})}{f(1) + \cdots + f(b_{a_n})} \ge \frac{O(1) + f(b_k) + \cdots + f(b_{kn})}{f(1) + \cdots + f(b_{kn})} \not\longrightarrow 0,$$

so that $B_A \notin \mathscr{E}_f$, where the last $\not\to$ comes from reasoning similar to (2.3). Finally, given infinite subsets $X, Y \subseteq \mathbf{N}$ with canonical enumeration $\{x_n : n \in \mathbf{N}\}$ and $\{y_n : n \in \mathbf{N}\}$, respectively, such that $X \le Y$ and $X \in \mathscr{E}_f$, the following holds:

$$\frac{f(x_1) + \cdots + f(x_n)}{f(1) + \cdots + f(x_n)} \ge \frac{f(y_1) + \cdots + f(y_n)}{f(1) + \cdots + f(y_n)}$$

for all $n \in \mathbf{N}$; therefore, $Y \in \mathscr{E}_f$. □

Given a real $\alpha \ge -1$, let $\mathcal{I}_\alpha$ be the collection of subsets with zero α-density, that is,

$$\mathcal{I}_\alpha := \{S \subseteq \mathbf{N} : \mathrm{d}^\star_\alpha(S) = 0\}, \tag{2.4}$$

$$\text{where } \mathrm{d}^\star_\alpha(S) = \limsup_{n \to \infty} \frac{\sum_{i \in S \cap [1,n]} i^\alpha}{\sum_{i \in [1,n]} i^\alpha}.$$

Proposition 2.4. *All ideals $\mathcal{I}_\alpha$ are thinnable.*

Proof. If $\alpha \in [-1, 0]$, the claim follows from Proposition 2.3 (we omit the details). Hence, we suppose hereafter that $\alpha > 0$. Fix infinite sets $X, Y \subseteq \mathbf{N}$ with canonical enumerations $\{x_n : n \in \mathbf{N}\}$ and $\{y_n : n \in \mathbf{N}\}$, respectively, such that $Y \notin \mathcal{I}_\alpha$. Then, there exists an infinite set S such that $|Y \cap [1, y_n]| \geq \lambda y_n$ for all $n \in S$, where

$$\lambda := 1 - \left(1 - \frac{1}{2}\,\mathrm{d}^\star_\alpha(Y)\right)^{1/(\alpha+1)} > 0.$$

Indeed, in the opposite case, we would have that

$$\begin{aligned}\frac{\alpha+1}{y_n^{\alpha+1}} \sum_{i \leq n} y_i^\alpha &\leq \frac{\alpha+1}{y_n^{\alpha+1}} \sum_{i \in ((1-\lambda)y_n, y_n]} i^\alpha \\ &\leq \left(1 - (1-\lambda)^{\alpha+1}\right)(1 + o(1)) < \frac{2}{3}\,\mathrm{d}^\star_\alpha(Y)\end{aligned}$$

for all sufficiently large n. Since $|Y \cap [1, n]| \leq |X \cap [1, n]|$ for all n, we conclude that

$$\frac{1}{x_n^{\alpha+1}} \sum_{i \leq n} x_i^\alpha \geq \frac{1}{x_n^{\alpha+1}} \sum_{i \leq \lambda y_n} i^\alpha \geq \frac{1}{x_n^{\alpha+1}} \sum_{i \leq \lambda x_n} i^\alpha \geq \frac{\lambda^{\alpha+1}}{2}$$

for all large $n \in S$, so that $X \notin \mathcal{I}_\alpha$.

At this point, fix sets $A, B \subseteq \mathbf{N}$ with canonical enumerations $\{a_n : n \in \mathbf{N}\}$ and $\{b_n : n \in \mathbf{N}\}$, respectively, such that A admits asymptotic densities $c > 0$ and $B \notin \mathcal{I}_\alpha$. Also, fix $\varepsilon > 0$ sufficiently small, and note that there exists an $n_0 = n_0(\varepsilon) \in \mathbf{N}$ such that

$$\left(\frac{1}{c} - \varepsilon\right) n \leq a_n \leq \left(\frac{1}{c} + \varepsilon\right) n$$

for all $n \geq n_0$. In particular, it follows that

$$\frac{1}{a_{b_n}^{\alpha+1}} \sum_{k \leq n} (a_{b_k})^\alpha \geq \frac{1}{(1/c + \varepsilon)^{\alpha+1} {b_n}^{\alpha+1}} \left(O(1) + \sum_{n_0 \leq k \leq n} \left(\frac{1}{c} - \varepsilon\right)^\alpha {b_k}^\alpha\right).$$

Therefore, setting

$$\kappa := \min\left\{\left(\frac{1}{c} + \varepsilon\right)^{-\alpha-1}, \left(\frac{1}{c} - \varepsilon\right)^\alpha\right\} > 0,$$

we obtain

$$\begin{aligned}\frac{\mathrm{d}^{\star}_{\alpha}(A_B)}{\alpha+1} &= \limsup_{n\to\infty} \frac{1}{a_{b_n}^{\alpha+1}} \sum_{k\le n} (a_{b_k})^{\alpha} \\ &\ge \limsup_{n\to\infty} \frac{\kappa}{{b_n}^{\alpha+1}} \left(O(1) + \sum_{n_0\le k\le n} \kappa {b_k}^{\alpha} \right) \\ &= \kappa^2 \limsup_{n\to\infty} \frac{1}{{b_n}^{\alpha+1}} \sum_{n_0\le k\le n} {b_k}^{\alpha} \\ &= \kappa^2 \frac{\mathrm{d}^{\star}_{\alpha}(B)}{\alpha+1} > 0.\end{aligned}$$

This proves that $A_B \notin \mathcal{I}_{\alpha}$. Finally, let k be an integer greater than $1/c$, and note that $B_A \le B_{k\mathbf{N}} \setminus S$, for some finite set S. By the previous observation, it is sufficient to show that $B_{k\mathbf{N}} \notin \mathcal{I}_{\alpha}$ and this is straightforward by an analogous argument of (2.3). □

To mention another example, let $\mathcal{I}_{\mathfrak{p}}$ be the *Pólya ideal*, i.e., $\mathcal{I}_{\mathfrak{p}} := \{S \subseteq \mathbf{N} : \mathfrak{p}^{\star}(S) = 0\}$, where

$$\mathfrak{p}^{\star}(S) = \lim_{s\to 1^-} \limsup_{n\to\infty} \frac{|S \cap [ns, n]|}{(1-s)n}.$$

Among other things, the upper Pólya density $\mathfrak{p}^*$ has been used in a number of remarkable applications in analysis and economic theory, see e.g., [**13, 14, 19**].

Corollary 2.5. *The Pólya ideal $\mathcal{I}_{\mathfrak{p}}$ is thinnable.*

Proof. The upper Pólya density $\mathfrak{p}^*$ is the pointwise limit of the real net of the upper α-densities $\mathrm{d}^{\star}_{\alpha}$ defined in (2.4), see [**12,** Theorem 4.3].

Fix infinite sets $X, Y \subseteq \mathbf{N}$ with canonical enumerations $\{x_n : n \in \mathbf{N}\}$ and $\{y_n : n \in \mathbf{N}\}$, respectively, such that $Y \notin \mathcal{I}_{\mathfrak{p}}$. Then, there exists an $\alpha > 0$ such that $\mathrm{d}^{\star}_{\alpha}(Y) > 0$ and, due to Proposition 2.4, we obtain $\mathrm{d}^{\star}_{\alpha}(X) > 0$ as well. This implies that $X \notin \mathcal{I}_{\mathfrak{p}}$. Other properties can be similarly shown. □

Lastly, it is worth noting that there exist summable ideals which are not weakly thinnable; for instance, let $\mathcal{I}_f$ be the ideal defined by

$f(2n) = 1$ and $f(2n-1) = 0$ for all $n \in \mathbf{N}$, so that

$$\mathcal{I}_f = \{I \subseteq \mathbf{N} : I \cap 2\mathbf{N} \in \mathrm{Fin}\}.$$

Set $A := \mathbf{N} \setminus \{1\}$ and $B := 2\mathbf{N}$. Then, A has asymptotic density 1, $B \notin \mathcal{I}_f$ and $A_B = 2\mathbf{N}+1 \in \mathcal{I}_f$. Therefore, $\mathcal{I}_f$ is not weakly thinnable.

3. Main results. Consider the natural bijection between the collection of all subsequences (x_{n_k}) of (x_n) and real numbers $\omega \in (0,1]$ with non-terminating dyadic expansion

$$\sum_{i\geq 1} d_i(\omega)2^{-i},$$

where $d_i(\omega) = 1$ if $i = n_k$, for some integer k, and $d_i(\omega) = 0$ otherwise, cf., [**3,** Appendix A31], [**15**]. Accordingly, for each $\omega \in (0,1]$, denote by $x \upharpoonright \omega$ the subsequence of (x_n) obtained by omitting x_i if and only if $d_i(\omega) = 0$.

Moreover, let $\lambda : \mathscr{M} \to \mathbf{R}$ denote the Lebesgue measure, where $\mathscr{M}$ stands for the completion of the Borel σ-algebra on $(0,1]$. Our main result follows:

Theorem 3.1. *Let $\mathcal{I}$ be a thinnable ideal and (x_n) a sequence taking values in a first countable space X where all closed sets are separable. Then*:

$$\lambda\left(\{\omega \in (0,1] : \Gamma_x(\mathcal{I}) = \Gamma_{x\upharpoonright\omega}(\mathcal{I})\}\right) = 1.$$

Proof. Let Ω be the set of normal numbers, that is,

$$\Omega := \left\{\omega \in (0,1] : \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d_i(\omega) = \frac{1}{2}\right\}. \tag{3.1}$$

It follows from Borel's normal number theorem [**3,** Theorem 1.2] that $\Omega \in \mathscr{M}$ and $\lambda(\Omega) = 1$. Then, it is claimed that

$$\Gamma_{x\upharpoonright\omega}(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I}) \quad \text{for all } \omega \in \Omega. \tag{3.2}$$

Towards this aim, fix $\omega \in \Omega$, and denote by (x_{n_k}) the subsequence $x \upharpoonright \omega$. Let us suppose, for the sake of contradiction, that $\Gamma_{x\upharpoonright\omega}(\mathcal{I}) \setminus \Gamma_x(\mathcal{I}) \neq \emptyset$ and fix a point ℓ therein. Then, the set of indices $\{n_k : k \in \mathbf{N}\}$ has asymptotic density $1/2$ and, for each neighborhood U of ℓ, it holds that

$\{k : x_{n_k} \in U\} \notin \mathcal{I}$. This implies that

$$\{n : x_n \in U\} \supseteq \{n_k : x_{n_k} \in U\} \notin \mathcal{I},$$

by the hypothesis that $\mathcal{I}$ is, in particular, weakly thinnable. Therefore, $\{n : x_n \in U\} \notin \mathcal{I}$, which is a contradiction since ℓ would also be a $\mathcal{I}$-cluster point of x. This proves (3.2).

To complete the proof, it is sufficient to show that

$$\lambda\left(\{\omega \in (0,1] : \Gamma_x(\mathcal{I}) \subseteq \Gamma_{x\upharpoonright\omega}(\mathcal{I})\}\right) = 1. \tag{3.3}$$

This is clear if $\Gamma_x(\mathcal{I})$ is empty. Otherwise, note that $\Gamma_x(\mathcal{I})$ is closed by **[11,** Lemma 3.1(iv)]; hence, there exists a non-empty countable dense subset L. Fix $\ell \in L$, and let (U_m) be a decreasing local base of neighborhoods at ℓ. Also fix $m \in \mathbf{N}$, and define $I := \{n : x_n \in U_m\}$, which does not belong to $\mathcal{I}$; in particular, I is infinite, and we let $\{i_n : n \in \mathbf{N}\}$ be its enumeration. Again, by Borel's normal number theorem,

$$\Theta(\ell, U_m) := \left\{\omega \in (0,1] : \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} d_{i_j}(\omega) = \frac{1}{2}\right\}$$

belongs to $\mathscr{M}$ and has Lebesgue measure 1. Fix ω in the above set, and denote by (x_{n_k}) the subsequence $x \upharpoonright \omega$. Hence, the set $J := \{n : i_n \in \{n_k : k \in \mathbf{N}\}\}$ admits asymptotic density 1/2 and, by the thinnability of $\mathcal{I}$, we obtain $I_J \notin \mathcal{I}$. Lastly, note that

$$\{k : x_{n_k} \in U_m\} = \{k : n_k \in I\} \leq \{n_k : n_k \in I\} = I_J.$$

Therefore, $\{k : x_{n_k} \in U_m\} \notin \mathcal{I}$. In addition, $\Theta(\ell) := \bigcap_{m\geq 1} \Theta(\ell, U_m)$ belongs to $\mathscr{M}$ and has Lebesgue measure 1. This implies that

$$\lambda\left(\{\omega \in (0,1] : \ell \in \Gamma_{x\upharpoonright\omega}(\mathcal{I})\}\right) = 1.$$

(Also, see **[20,** Theorem 1] for the case $\mathcal{I} = \text{Fin}$.) At this point, since L is countable, we get $\lambda(\{\omega \in (0,1] : L \subseteq \Gamma_{x\upharpoonright\omega}(\mathcal{I})\}) = 1$. Claim (3.3) follows from the fact that $\Gamma_{x\upharpoonright\omega}(\mathcal{I})$ is also closed by **[11,** Lemma 3.1(iv)], so that each of these $\Gamma_{x\upharpoonright\omega}(\mathcal{I})$ contains the closure of L, i.e., $\Gamma_x(\mathcal{I})$. □

Note added in proof. It turns out that the topological analogue of Theorem 3.1 is quite different, providing a non-analogue between measure and category. Indeed, it has been shown **[10]** that, if x is a sequence in a separable metric space, then $\{\omega \in (0,1] : \Gamma_x(\mathcal{I}_0) =$

$\Gamma_{x\restriction\omega}(\mathcal{I}_0)\}$ is not a first Baire category set if and only if every ordinary limit point of x is also a statistical cluster point of x, that is, $\Gamma_x(\mathrm{Fin}) = \Gamma_x(\mathcal{I}_0)$.

Remark 3.2. Separable metric spaces X satisfy the hypotheses of Theorem 3.1. Indeed, X is first countable, and every closed subset F of X is separable. In order to prove the latter, let A be a countable dense subset of X, and note that

$$\mathscr{F} := \{B(a,r) \cap F : a \in A, 0 < r \in \mathbf{Q}\} \setminus \{\emptyset\}$$

is a base for F, where $B(a,r)$ is the open ball with center a and radius r. Then, a set where one point is chosen for every set in $\mathscr{F}$ is a countable dense subset of F.

As a consequence of Proposition 2.4, Theorem 3.1 and Remark 3.2, we obtain:

Corollary 3.3. *Let x be a sequence taking values in a separable metric space. Then, the set of statistical cluster points of x is equal to the set of statistical cluster points of almost all its subsequences* (*in the sense of Lebesgue measure*).

Similarly, setting $\mathcal{I} = \mathrm{Fin}$, we recover Buck's result **[5]**:

Corollary 3.4. *Let x be a sequence taking values in a separable metric space. Then, the set of ordinary limit points of x is equal to the set of ordinary limit points of almost all of its subsequences* (*in the sense of Lebesgue measure*).

Lastly, we recall that a sequence $x = (x_n)$ taking values in topological space X converges (with respect to an ideal $\mathcal{I}$) to $\ell \in X$, shortened as $x \to_{\mathcal{I}} \ell$, if

$$\{n : x_n \notin U\} \in \mathcal{I}$$

for all neighborhoods U of ℓ. In this regard, Miller **[15,** Theorem 3] proved that a real sequence x statistically converges to ℓ, i.e., $x \to_{\mathcal{I}_0} \ell$, if and only if almost all of its sequences statistically converge to ℓ.

This is extended in the following result. Here, we say that an ideal $\mathcal{I}$ is *invariant* if, for each $A \subseteq \mathbf{N}$ with positive asymptotic density, $A_B \notin \mathcal{I}$ holds if and only if $B \notin \mathcal{I}$ (in particular, $\mathcal{I}$ is weakly thinnable). This condition is strictly related with the so-called "property (G)" defined in [**2**].

Theorem 3.5. *Let $\mathcal{I}$ be an invariant ideal and x a sequence taking values in a topological space. Then, $x \to_{\mathcal{I}} \ell$ if and only if*

$$\lambda\left(\{\omega \in (0,1] : x \restriction \omega \to_{\mathcal{I}} \ell\}\right) = 1.$$

Proof. First, we suppose that $x \to_{\mathcal{I}} \ell$, and let U be a neighborhood of ℓ. Let Ω be set of normal numbers defined in (3.1), fix $\omega \in \Omega$, and denote by (x_{n_k}) the subsequence $x \restriction \omega$. Then, $I := \{n : x_n \notin U\} \in \mathcal{I}$, and $A := \{n_k : k \in \mathbf{N}\}$ has asymptotic density $1/2$. Define $B := \{k : x_{n_k} \notin U\} = \{k : n_k \in I\}$. Since $\mathcal{I}$ is, in particular, weakly thinnable and $A_B = \{n_k : x_{n_k} \notin U\} \in \mathcal{I}$, it follows that $B \in \mathcal{I}$, i.e., $x \restriction \omega \to_{\mathcal{I}} \ell$.

Conversely, note that $\lambda(\Omega \cap (1-\Omega)) = 1$. Hence, there exists an $\omega \in \Omega$ such that $x \restriction \omega \to_{\mathcal{I}} \ell$ and $x \restriction (1-\omega) \to_{\mathcal{I}} \ell$. It easily follows that $x \to_{\mathcal{I}} \ell$. Indeed, denoting by (x_{n_k}) and (x_{m_r}) the subsequences $x \restriction \omega$ and $x \restriction (1-\omega)$, respectively, we have that, for each neighborhood U of ℓ, the following hold: $\{k : x_{n_k} \notin U\} \in \mathcal{I}$ and $\{r : x_{m_r} \notin U\} \in \mathcal{I}$. Since $\{n_k : k \in \mathbf{N}\}$ and $\{m_r : r \in \mathbf{N}\}$ form a partition of $\mathbf{N}$, then

$$\{n : x_n \notin U\} = \{n_k : x_{n_k} \notin U\} \cup \{m_r : x_{m_r} \notin U\}.$$

The claim follows from the hypothesis that $\mathcal{I}$ is invariant. □

It is impossible to extend Theorem 3.5 on the class of all ideals: indeed, it has been shown [**2,** Example 2] that there exist an ideal $\mathcal{I}$ and a real sequence x such that $x \to_{\mathcal{I}} \ell$ and, on the other hand, $\lambda(\{\omega \in (0,1] : x \restriction \omega \to_{\mathcal{I}} \ell\}) = 0$.

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