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# ON THE BOUNDEDNESS OF MINIMIZERS OF SOME INTEGRAL FUNCTIONALS WITH DEGENERATE ANISOTROPIC INTEGRANDS

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ABSTRACT. In this paper, we obtain the boundedness of minimizers for a class of integral functionals, defined in a weighted anisotropic space.

**1. Introduction.** In this paper, we consider the following higher order integral functional:

$$I(u) = \int_{\Omega} \{A(x, \nabla_2 u) + A_0(x, u)\}\, dx \tag{1.1}$$

defined in a weighted space $\mathring{W}^{1,q}_{2,p}(\nu, \mu, \Omega)$, where $\Omega$ is an open bounded set of $\mathbb{R}^n$, and $\nu = \{\nu_\alpha : |\alpha| = 1\}$ and $\mu = \{\mu_\alpha : |\alpha| = 2\}$ are sets of positive functions in $\Omega$ satisfying some hypotheses specified later; $\nabla_2 u = \{D^\alpha u : |\alpha| = 1, 2\}$. Working with the functional $I(u)$ instead of working with its Euler equation, we derive the boundedness of function $u(x)$ minimizing functional (1.1). The proof is based on the application of a modification of the Moser method (see Lemma 3.3) which essentially consists of obtaining uniform $L^r$-estimates (at $r \to +\infty$) for an auxiliary function $\varphi(u)$ (see, also, **[19]**, or more recently, **[11, 18]**).

It is supposed that $A(x, \xi)$ is a Carathéodory function, convex with respect to $\xi = \{\xi_\alpha : |\alpha| = 1, 2\}$, and, for almost every $x \in \Omega$ and every $\xi$, satisfying the following inequality:

$$\begin{aligned} c_1 \Big\{ \sum_{|\alpha|=1} \nu_\alpha(x) |\xi_\alpha|^{q_\alpha} + \sum_{|\alpha|=2} \mu_\alpha(x) |\xi_\alpha|^{p_\alpha} \Big\} - f(x) &\le A(x, \xi) \\ &\le c_2 \Big\{ \sum_{|\alpha|=1} \nu_\alpha(x) |\xi_\alpha|^{q_\alpha} + \sum_{|\alpha|=2} \mu_\alpha(x) |\xi_\alpha|^{p_\alpha} \Big\} + f(x), \end{aligned} \tag{1.2}$$

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where $c_1$ and $c_2$ are positive constants, $f(x)$ is a nonnegative function, $f \in L^{t_*}(\Omega)$, $t_* > 1$, $q_\alpha$ and $p_\alpha$ are real numbers such that $q_\alpha \in ]1, n[$, if $|\alpha| = 1$, $p_\alpha \in ]1, n/2[$, if $|\alpha| = 2$ ($q = \{q_\alpha : |\alpha| = 1\}$, $p = \{p_\alpha : |\alpha| = 2\}$) and $1/q_\gamma + 1/q_\beta < 1/p_{\gamma+\beta}$, if $|\gamma| = |\beta| = 1$.

Moreover, $A_0(x, \eta)$ is a Carathéodory function, convex with respect to $\eta$, and, for almost every $x \in \Omega$ and every $\eta \in \mathbb{R}$, satisfying

$$-c_4|\eta|^{q_-} - f_0(x) \leq A_0(x, \eta) \leq c_3|\eta|^{q_-} + f_0(x),$$

where $c_3 > 0$, $c_4 \in [0, c_1/c_0[$, $q_- = \min_{|\alpha|=1} q_\alpha$, $f_0(x)$ is a nonnegative function with summability in $\Omega$ to be made more specific later on.

A similar result was established in **[5]**; however, condition (1.2) is more general than the corresponding condition in **[5]** by the presence of the set of exponents $q_\alpha$, $p_\alpha$ and of the sets of weighted functions.

We recall that a strengthened coercivity condition such as that provided on the left side of inequality (1.2) goes back to the pioneering paper **[19]**, wherein the authors established, for $q > mp$, the boundedness and the Hölder continuity of generalized solutions from the class $W^{m,p}(\Omega) \cap W^{1,q}(\Omega)$ for nonlinear elliptic equations of the divergent form

$$\sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \mathcal{A}_\alpha(x, u, \dots, D^m u) = 0 \quad \text{in } \Omega.$$

Moreover, the study of regularity for solutions of a class of higher order degenerate elliptic equations and variational inequalities in the anisotropic case was treated in **[4, 12]**. Finally, for the non degenerate case, the problem of regularity of minimizers of integral functionals was studied in **[6, 8, 13, 15]** and, more recently, in **[2, 3, 7, 14]**.

**2. Preliminaries.** We shall suppose that $\mathbb{R}^n$, $n > 2$, is the $n$-dimensional Euclidian space with elements $x = (x_1, x_2, \dots, x_n)$. Let $\Omega$ be a bounded open set of $\mathbb{R}^n$. Let, for every multiindex $\alpha$, $|\alpha| = 1$, $q_\alpha$ be numbers such that $1 < q_\alpha < n$, and let, for every multiindex $\alpha$, $|\alpha| = 2$, $p_\alpha$ be numbers such that $1 < p_\alpha < n/2$; we denote

$$q = \{q_\alpha : |\alpha| = 1\}, \qquad p = \{p_\alpha : |\alpha| = 2\}.$$

We assume that, for every multiindex $\beta$, $|\beta| = 1$ and $\gamma$, $|\gamma| = 1$,

$$\frac{1}{q_\gamma} + \frac{1}{q_\beta} < \frac{1}{p_{\gamma+\beta}}. \tag{2.1}$$

**Hypothesis 2.1.** *Let, for every multiindex* $\alpha, |\alpha| = 1$, $\nu_\alpha$ *be a positive measurable function in* $\Omega$ *such that*

$$\nu_\alpha(x) \in L^1_{\text{loc}}(\Omega), \qquad \left(\frac{1}{\nu_\alpha(x)}\right)^{1/(q_\alpha - 1)} \in L^1_{\text{loc}}(\Omega).$$

For more details, cf., [**1, 9, 10, 16, 17**].

We set $\nu = \{\nu_\alpha : |\alpha| = 1\}$, $q_- = \min_{|\alpha|=1} q_\alpha$, $q_+ = \max_{|\alpha|=1} q_\alpha$, and denote by $W^{1,q}(\nu, \Omega)$ the set of all functions $u \in L^{q_-}(\Omega)$, such that the distribution derivatives $D^\alpha u$, $|\alpha| = 1$, satisfy

$$\nu_\alpha |D^\alpha u|^{q_\alpha} \in L^1(\Omega).$$

$W^{1,q}(\nu, \Omega)$ is a Banach space with respect to the norm

$$\|u\|_{1,q,\nu} = \left(\int_\Omega |u|^{q_-}\, dx\right)^{1/q_-} + \sum_{|\alpha|=1} \left(\int_\Omega \nu_\alpha |D^\alpha u|^{q_\alpha}\, dx\right)^{1/q_\alpha}.$$

$\mathring{W}^{1,q}(\nu, \Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,q}(\nu, \Omega)$.

**Hypothesis 2.2.** *There exist numbers* $\widetilde{c} > 0$ *and* $\widetilde{q} > q_+$ *such that, for every* $u \in \mathring{W}^{1,q}(\nu, \Omega)$,

$$\left(\int_\Omega |u|^{\widetilde{q}} dx\right)^{1/\widetilde{q}} \le \widetilde{c} \sum_{|\alpha|=1} \left(\int_\Omega \nu_\alpha |D^\alpha u|^{q_\alpha}\, dx\right)^{1/q_\alpha}. \tag{2.2}$$

Consequently,

$$\|u\|_{1,q,\nu} \le \Gamma \sum_{|\alpha|=1} \left(\int_\Omega \nu_\alpha |D^\alpha u|^{q_\alpha}\, dx\right)^{1/q_\alpha}, \tag{2.3}$$

and

$$\int_\Omega |u|^{q_-}\, dx \le c_0 \sum_{|\alpha|=1} \left(\int_\Omega \nu_\alpha |D^\alpha u|^{q_\alpha}\, dx\right) + \overline{c}, \tag{2.4}$$

where $\Gamma, c_0, \overline{c}$ are positive constants depending only upon $n, \widetilde{c}, q_-, q_+, \widetilde{q}$ and $\operatorname{meas}\Omega$.

**Lemma 2.3.** *If Hypothesis* 2.2 *is satisfied, then the imbedding of* $\mathring{W}^{1,q}(\nu,\Omega)$ *in* $L^{q-}(\Omega)$ *is compact.*

*Proof.* Let $\{u_n\}$ be a sequence of functions of $\mathring{W}^{1,q}(\nu,\Omega)$ with equi-bounded norms, and let $\{\Pi_k\}$ be a sequence of pluri-intervals in $\Omega$ such that:

(a) $\Pi_k \subset \Pi_{k+1}$, for any $k \in \mathbb{N}$;

(b) $\lim_{k\to+\infty} \mathring{\Pi}_k = \Omega$;

(c) for any $C$ closed, bounded set of $\Omega$, there exists a $\overline{k} : C \subset \mathring{\Pi}_k$, $k \geq \overline{k}$.

Then the norms of $\{u_n\}$ in $W^{1,1}(\mathring{\Pi}_1)$ are equi-bounded. We can extract from $\{u_n\}$ a subsequence $\{u_{1,n}\}$ that converges almost everywhere in $\mathring{\Pi}_1$. Arguing as above, we can extract from $\{u_{1,n}\}$ a subsequence $\{u_{2,n}\}$ that converges almost everywhere in $\mathring{\Pi}_2$, etc. By the diagonal method, we obtain that $\{u_{n,n}\}$ converges almost everywhere in $\Omega$ and, from (2.2), in $L^{q-}(\Omega)$. □

**Hypothesis 2.4.** *Let, for every multiindex* $\alpha$, $|\alpha| = 2$ , $\mu_\alpha$ *be a positive function in* $\Omega$ *such that*

$$\mu_\alpha(x) \in L^1_{\text{loc}}(\Omega), \qquad \left(\frac{1}{\mu_\alpha(x)}\right)^{1/(p_\alpha-1)} \in L^1_{\text{loc}}(\Omega).$$

We set $\mu = \{\mu_\alpha : |\alpha| = 2\}$ and denote by $W^{1,q}_{2,p}(\nu,\mu,\Omega)$ the function space of all real-valued functions $u \in W^{1,q}(\nu,\Omega)$ such that distribution derivatives $D^\alpha u$ and $|\alpha| = 2$ satisfy

$$\mu_\alpha |D^\alpha u|^{p_\alpha} \in L^1(\Omega).$$

$W^{1,q}_{2,p}(\nu,\mu,\Omega)$ is a Banach space with the norm

$$\|u\| = \|u\|_{1,q,\nu} + \sum_{|\alpha|=2} \left(\int_\Omega \mu_\alpha |D^\alpha u|^{p_\alpha} dx\right)^{1/p_\alpha}.$$

We denote by $\mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$ the closure in $W^{1,q}_{2,p}(\nu,\mu,\Omega)$ of the set $C_0^\infty(\Omega)$.

**Hypothesis 2.5.** *There exists a positive constant c such that, for every multiindex $\beta$, $|\beta| = 1$ and $\gamma$, $|\gamma| = 1$*

$$\mu_{\beta+\gamma}^{1/p_{\beta+\gamma}} \le c\nu_\beta^{1/q_\beta}\nu_\gamma^{1/q_\gamma} \quad \text{in } \Omega.$$

We note that our functional spaces are specific cases of the spaces introduced in [**12**].

**3. Auxiliary results.** Let $h \in C^\infty(\mathbb{R})$ be a non-decreasing function such that $h = 0$ on $]-\infty, 0]$ and $h = 1$ on $[1, +\infty[$.

We set

$$\widetilde{c}_1 = \max_{\mathbb{R}} |h'|, \qquad \widetilde{c}_2 = 2\max_{\mathbb{R}} |h'| + \max_{\mathbb{R}} |h''|.$$

Let, for every $s \in \mathbb{R}$, $h_s : \mathbb{R} \to \mathbb{R}$ be the function such that

$$h_s(\eta) = \eta + (s+1-\eta)h(\eta - s) - (s+1+\eta)h(-\eta - s), \quad \eta \in \mathbb{R}.$$

We have $\{h_s\} \subseteq C^\infty(\mathbb{R})$ and, for every $s \in \mathbb{R}$, the following property holds:

$$h_s(\eta) = \eta \quad \text{if } |\eta| \le s$$
$$h_s(\eta) = -s-1 \quad \text{if } \eta \le -s-1$$
$$h_s(\eta) = s+1 \quad \text{if } \eta \ge s+1.$$

Moreover, for every $s \in \mathbb{N}$ and $\eta \in \mathbb{R}$, we have

$$|h_s(\eta)| \le 2|\eta|, \quad 0 \le h_s'(\eta) \le \widetilde{c}_1,$$
$$|\eta| h_s'(\eta) \le 2\widetilde{c}_1 |h_s(\eta)|, \qquad |h_s''(\eta)| \le \widetilde{c}_2,$$
$$|\eta||h_s''(\eta)| \le 2\widetilde{c}_2 |h_s(\eta)|.$$

For more details concerning the functions $h$ and $h_s$, see [**11**].

Due to the assumptions of Section 2 and the properties of the function $h_s$, we have the following:

**Lemma 3.1.** *Let $u \in \mathring{W}^{1,q}(\nu, \Omega)$, $s \in \mathbb{N}$, $r > 0$. Let*

$$\varphi = u[1 + h_s^2(u)]^r$$
$$\psi = [1 + h_s^2(u)]^r + 2r[1 + h_s^2(u)]^{r-1} h_s(u) h_s'(u) u.$$

*Then,* $\varphi \in \mathring{W}^{1,q}(\nu,\Omega)$ *and, for every multiindex* $\alpha$, $|\alpha| = 1$, $D^\alpha\varphi = \psi D^\alpha u$ *almost everywhere in* $\Omega$.

Using Hypothesis 2.4 and properties of the function $h_s$, we establish the following:

**Lemma 3.2.** *Let* $u \in \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$, $s \in \mathbb{N}$, $r > 0$. *Let* $\varphi$ *and* $\psi$ *be defined as in Lemma* 3.1. *Then,* $\varphi \in \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$ *and*:

(a) *for every multiindex* $\alpha$, $|\alpha| = 1$, $D^\alpha\varphi = \psi D^\alpha u$ *almost everywhere in* $\Omega$;

(b) *for every multiindex* $\beta$, $|\beta| = 1$, *and* $\gamma, |\gamma| = 1$,

$$|D^{\beta+\gamma}\varphi - \psi D^{\beta+\gamma}u| \le 6\widetilde{c}_2(r+1)^2[1+h_s^2(u)]^r|D^\beta u||D^\gamma u|$$

*almost everywhere in* $\Omega$.

We refer to **[4]** for more details concerning the proof of Lemmas 3.1 and 3.2. Finally, under Hypotheses 2.1 and 2.2, we shall prove the following:

**Lemma 3.3.** *Let* $m_1, m_2 > 0$, $t > \widetilde{q}/(\widetilde{q} - q_+)$, $\Phi \in L^t(\Omega)$, *and let* $u \in \mathring{W}^{1,q}(\nu,\Omega)$. *Let, for every* $s \in \mathbb{N}$ *and* $r > 0$,

$$\begin{aligned} &\int_\Omega \Big\{ \sum_{|\alpha|=1} \nu_\alpha |D^\alpha u|^{q_\alpha} \Big\} [1+h_s^2(u)]^r\,dx \\ &\qquad \le m_1(1+r)^{m_2} \int_\Omega \{|u|^{q_-} + \Phi\}[1+h_s^2(u)]^r\,dx. \end{aligned} \tag{3.1}$$

*Then,*

$$\operatorname*{ess\,sup}_\Omega |u| \le M_0 \tag{3.2}$$

*where the positive constant* $M_0$ *depends only upon* $n$, $\widetilde{c}$, $\widetilde{c}_1$, $\widetilde{q}$, $q_-$, $m_1$, $m_2$, $\|u\|_{1,q,\nu}$, $\|\Phi\|_{L^t(\Omega)}$ *and* meas $\Omega$.

*Proof.* We set $t' = t/(t-1)$ and $\overline{q} \in\, ]q_+, \widetilde{q}[$. By $d_i$, $i = 1, 2, \dots$, we shall denote positive constants which depend only upon $n$, $q_-$, $q_+$, $\widetilde{q}$, $\widetilde{c}$, $\widetilde{c}_1$ and meas $\Omega$.

For all $s \in \mathbb{N}$, $r > 0$, let

$$T_s(r) = 1 + \int_\Omega [|u|^{\overline{q}} + g][1 + h_s^2(u)]^r dx,$$

where $g = \Phi + 1$. It results in

$$T_s(r) \le 1 + \int_\Omega |\widetilde{u}|^{\overline{q}} dx + ||g||_{L^t(\Omega)} \left( \int_\Omega (\widetilde{w}_1 + 1)^{\widetilde{q}} dx \right)^{1/t'},$$

where

$$\widetilde{u} = u[1 + h_s^2(u)]^{r/\overline{q}},$$
$$\widetilde{w} = [1 + h_s^2(u)]^{rt'/\widetilde{q}} - 1.$$

Hence,

$$\begin{aligned} T_s(r) \le 1 &+ \left( \int_\Omega |\widetilde{u}|^{\widetilde{q}} dx \right)^{\overline{q}/\widetilde{q}} (\operatorname{meas}\Omega)^{(\widetilde{q}-\overline{q})/\widetilde{q}} \\ &+ \|g\|_{L^t(\Omega)} 2^{\widetilde{q}/t'} \left( \int_\Omega \widetilde{w}^{\widetilde{q}} dx \right)^{1/t'} \\ &+ \|g\|_{L^t(\Omega)} 2^{\widetilde{q}/t'} (\operatorname{meas}\Omega)^{1/t'}. \end{aligned}$$

The last inequality and Hypothesis 2.2 give:

$$\begin{aligned} T_s(r) \le d_1 &+ d_2 \left[ \sum_{|\alpha|=1} \left( \int_\Omega \nu_\alpha |D^\alpha \widetilde{u}|^{q_\alpha} dx \right)^{1/q_\alpha} \right]^{\overline{q}} \\ &+ d_3 \left[ \sum_{|\alpha|=1} \left( \int_\Omega \nu_\alpha |D^\alpha \widetilde{w}|^{q_\alpha} dx \right)^{1/q_\alpha} \right]^{\widetilde{q}/t'}. \end{aligned} \tag{3.3}$$

Next, simple computations imply:

$$|D^\alpha \widetilde{u}| \le d_4(r+1)[1 + h_s^2(u)]^{r/\overline{q}} |D^\alpha u|, \tag{3.4}$$

$$|D^\alpha \widetilde{w}| \le d_5 r [1 + h_s^2(u)]^{rt'/\widetilde{q}} |D^\alpha u|. \tag{3.5}$$

From (3.3)–(3.5), we get

(3.6)

$$T_s(r) \le d_1 + d_6(r+1)^{\overline{q}}\Bigg[\sum_{|\alpha|=1}\bigg(\int_\Omega \nu_\alpha |D^\alpha u|^{q_\alpha}[1+h_s^2(u)]^{rq_\alpha/\overline{q}}dx\bigg)^{1/q_\alpha}\Bigg]^{\overline{q}}$$

$$+ d_7(r+1)^{\widetilde{q}}\Bigg[\sum_{|\alpha|=1}\bigg(\int_\Omega \nu_\alpha |D^\alpha u|^{q_\alpha}[1+h_s^2(u)]^{t'rq_\alpha/\widetilde{q}}dx\bigg)^{1/q_\alpha}\Bigg]^{\widetilde{q}/t'}.$$

We set $\theta \in \mathbb{R}$:

$$1 < \theta < \min\left(\frac{\overline{q}}{q_+}, \frac{\widetilde{q}}{t' q_+}\right).$$

From the Hölder inequality and (3.6), for all $s \in \mathbb{N}$, $r > 0$, we obtain:

$$T_s(r) \le d_1 + d_8(r+1)^{\widetilde{q}} \sum_{|\alpha|=1}\bigg(\int_\Omega \nu_\alpha |D^\alpha u|^{q_\alpha}[1+h_s^2(u)]^{r/\theta}dx\bigg)^{\theta},$$

where the positive constant $d_8$ depends on known parameters and $\|u\|_{1,q,\nu}$. Choosing $r = r/\theta$ in (3.1), from the last inequality, we have

$$T_s(r) \le d_9(r+1)^{m_3}[T_s(r/\theta)]^{\theta} \quad \text{for all } r > 0, \tag{3.7}$$

where $m_3 = \widetilde{q}(1 + m_2/q_+)$. We introduce a sequence $\{\rho_j\}$ such that

$$\rho_j = \sigma\theta^{j+1} \quad \text{for all } j \in \mathbb{N}_0,$$

where

$$\sigma = \frac{1}{2\theta}\min\left(\widetilde{q} - \overline{q}, \frac{\widetilde{q}}{t'}\right).$$

We have $\rho_j/\theta = \rho_{j-1}$. This and (3.7) yield

$$T_s(\rho_j) \le d_9(\rho_j + 1)^{m_3}[T_s(\rho_{j-1})]^{\theta}.$$

Recursion relation and the inequality

$$T_s(\rho_0) \le d_{10} + d_{11}\int_\Omega |u|^{\widetilde{q}}dx$$

lead to the conclusion that, for all $j = 1, 2, \ldots,$

$$T_s(\rho_j) \le d_{12}^{\theta^j},$$

where $d_{12}$ depends upon the known parameters, $\|\Phi\|_{L^t(\Omega)}$ and $\|u\|_{1,q,\nu}$.

Now, noting that $h_s(u) \to u$ as $s \to \infty$, from the definition of $T_s(r)$ and Fatou's lemma, it follows that

$$\int_\Omega |u|^{\overline{q}+\rho_j}\, dx \le (d_{12}^{1/(\sigma\theta)} + 1)^{\overline{q}+\rho_j}, \quad j = 1, 2, \ldots,$$

and, thus, the inequality (3.2) is shown. □

**4. Hypotheses and statement of the main result.** In this section, we give structural hypotheses on integrands in order to guarantee the existence of integrals. The set of all multiindex $\alpha$ such that $|\alpha| = 1$ or $|\alpha| = 2$ is $\Lambda$; $\mathbb{R}^{n,2}$ is the space of all sets $\xi = \{\xi_\alpha : \alpha \in \Lambda\}$ of real numbers; if $u \in W^{1,q}_{2,p}(\nu, \mu, \Omega)$, then $\nabla_2 u = \{D^\alpha u : \alpha \in \Lambda\}$. We shall study the boundedness of minimizers for the class of functionals of higher order:

$$I(u) = \int_\Omega \big\{A(x, \nabla_2 u) + A_0(x, u)\big\} dx, \tag{4.1}$$

defined in the weighted space $\overset{\circ}{W}{}^{1,q}_{2,p}(\nu, \mu, \Omega)$.

**Hypothesis 4.1.** *Let the principal part $A : \Omega \times \mathbb{R}^{n,2} \to \mathbb{R}$ of the functional be a Carathéodory function, convex with respect to $\xi \in \mathbb{R}^{n,2}$ almost everywhere $x \in \Omega$; we suppose that there exist real positive constants $c_1$ and $c_2$, $t_* > \widetilde{q}/(\widetilde{q} - q_+)$ and a nonnegative function $f \in L^{t_*}(\Omega)$ such that almost everywhere in $\Omega$ and for all $\xi \in \mathbb{R}^{n,2}$, the following inequality holds*:

$$\begin{aligned} c_1\bigg\{\sum_{|\alpha|=1} \nu_\alpha(x)|\xi_\alpha|^{q_\alpha} + \sum_{|\alpha|=2} \mu_\alpha(x)|\xi_\alpha|^{p_\alpha}\bigg\} - f(x) &\le A(x, \xi) \\ \le c_2\bigg\{\sum_{|\alpha|=1} \nu_\alpha(x)|\xi_\alpha|^{q_\alpha} + \sum_{|\alpha|=2} \mu_\alpha(x)|\xi_\alpha|^{p_\alpha}\bigg\} &+ f(x). \end{aligned} \tag{4.2}$$

**Hypothesis 4.2.** *Let $A_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that, for all $\eta \in \mathbb{R}$, the function $A_0(\cdot, \eta)$ is measurable in $\Omega$, and $A_0(x, \cdot)$ is convex in $\mathbb{R}$ for almost all $x \in \Omega$. Also, there exist $c_3 > 0$, $c_4 \in [0, c_1/c_0[$, $\widehat{t} > \widetilde{q}/(\widetilde{q} - q_+)$ and $f_0 \in L^{\widehat{t}}(\Omega)$ nonnegative such that almost everywhere in $\Omega$ and for all $\eta \in \mathbb{R}$, the following inequality holds*:

$$-c_4|\eta|^{q_-} - f_0(x) \le A_0(x, \eta) \le c_3|\eta|^{q_-} + f_0(x). \tag{4.3}$$

We observe that the functional in (4.1) is well defined due to the inequalities (4.2) and (4.3). Moreover, using well-known results of the existence of convex and coercive functionals, due to the properties of the functions $A(x,\xi)$ and $A_0(x,\eta)$, as well as to the inequalities (2.2), (4.2) and (4.3), for all closed and convex $V \subset \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$, there exists a function $u \in V$ which is a minimizer for the functional $I$ in $V$.

**Hypothesis 4.3.** *Let be $V$ a nonempty, closed, convex set in $\mathring{W}^{1,q}_{2,p}$ $(\nu,\mu,\Omega)$, satisfying the following property: if $v \in V$, $\varphi : \Omega \to \mathbb{R}, 0 \le \varphi \le 1$ in $\Omega$ and $\varphi v \in \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$, then $v - \varphi v \in V$.*

**Remark 4.4.** If, for example, $V = \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$ or

$$V = \{u \in \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega) : |u| \le 1\},$$

then Hypothesis 4.3 is satisfied.

We shall prove the following:

**Theorem 4.5.** *Let Hypotheses* 2.1, 2.2, 2.4, 2.5, 4.1–4.3 *be satisfied. If $u$ is a minimizer of the functional $I$ in $V$, then*

$$\operatorname*{ess\,sup}_{\Omega} |u| \le M,$$

*where $M$ depends upon known constants,* meas $\Omega$ *and* $\|u\|$.

**5. Construction of a minimizer for $I$.** In the general hypothesis, $V \subset \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$, a closed and convex set, we want to construct function $u(x)$, a minimizer for $I$ in $V$, using direct methods.

Let $v \in \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$. We set $p_- = \min_{|\alpha|=2} p_\alpha$. From (2.1), it follows that $p_- < q_\alpha$ for every $q_\alpha$, $|\alpha| = 1$. This fact and (2.3) imply

$$\|v\|^{p_-} \le \Gamma_1\Bigg(1 + \sum_{|\alpha|=1} \int_\Omega \nu_\alpha |D^\alpha v|^{q_\alpha}\,dx + \sum_{|\alpha|=2} \int_\Omega \mu_\alpha |D^\alpha v|^{p_\alpha}\,dx\Bigg), \tag{5.1}$$

where $\Gamma_1 > 0$ depends only upon $p_-$, $n$ and $\Gamma$.

On the other hand, from (2.4), (4.2) and (4.3) we have:

$$I(v) \ge (c_1 - c_4 c_0)\Bigg(\sum_{|\alpha|=1} \int_\Omega \nu_\alpha |D^\alpha v|^{q_\alpha}\,dx + \sum_{|\alpha|=2} \int_\Omega \mu_\alpha |D^\alpha v|^{p_\alpha}\,dx\Bigg) - \|f + f_0\|_{L^1(\Omega)} - c_4\overline{c}. \tag{5.2}$$

Then, from (5.1) and (5.2), we derive:

$$I(v) \geq \Gamma_2 \|v\|^{p_-} - \Gamma_3. \tag{5.3}$$

Here, and in the sequel, $\Gamma_i$, $i = 2, 3, 4$, denotes a positive constant dependent upon known parameters. We set

$$d = \inf_{v \in V} I(v). \tag{5.4}$$

From (5.3), we obtain:

$$d \geq -\Gamma_3.$$

Let $\{v_k\}$ be such that

$$\lim_{k \to \infty} I(v_k) = d. \tag{5.5}$$

We wish to prove that $v_k$ is bounded in $\mathring{W}^{1,q}_{2,p}(\Omega, \nu, \mu)$. From (5.3) and (5.5), we obtain:

$$\|v_k\|^{p_-} \leq \Gamma_4 \quad \text{for all } k \geq k_0. \tag{5.6}$$

Then, we can extract from $\{v_k\}$ a sequence, which we call $\{v_{k_i}\}$, that converges in $L^{q_-}(\Omega)$ (cf., Lemma 2.3), almost everywhere in $\Omega$ and weakly, to a function $u \in \mathring{W}^{1,q}_{2,p}(\nu, \mu, \Omega)$. Next, as is well known, the convexity of $A(x, \xi)$ with respect to $\xi$ is a sufficient condition for the sequential weak lower semicontinuity of $I$. Hence,

$$\liminf_{i \to \infty} I(v_{k_i}) \geq I(u). \tag{5.7}$$

From (5.4) and (5.7), we have:

$$I(u) = \inf_{v \in V} I(v).$$

We have found that $u$ is a minimizer for $I$ in $V$. If $I$ is strictly convex, then the minimizer is unique.

**6. Proof of Theorem 4.5.** In this section, we prove the boundedness of the minimizing function.

*Proof of Theorem* 4.5. We fix $s \in \mathbb{N}$ and $r > 0$, and define the following functions:

$$\omega = [1 + h_s^2(u)]^r u,$$

$$z = [1 + h_s^2(u)]^r + 2r[1 + h_s^2(u)]^{r-1} h_s(u) h_s'(u) u.$$

It is simple to prove that, in $\Omega$:

$$[1+h_s^2(u)]^r \le z \le (1+4\widetilde{c}_1 r)[1+h_s^2(u)]^r. \tag{6.1}$$

We observe that Lemma 3.2 shows that the function $\omega \in \mathring{W}^{1,q}_{2,p}(\nu,\mu,\Omega)$; moreover,

$$D^\alpha \omega = z D^\alpha u, \quad \text{almost everywhere in } \Omega,\ |\alpha| = 1 \tag{6.2}$$

and

$$|D^{\beta+\gamma}\omega - zD^{\beta+\gamma}u| \le 6\widetilde{c}_2(r+1)^2[1+h_s^2(u)]^r |D^\beta u||D^\gamma u| \quad \text{almost everywhere in } \Omega, \tag{6.3}$$

for every multiindex $\beta$, $|\beta| = 1$, and $\gamma, |\gamma| = 1$.

Let $G_\alpha = 0$ if $|\alpha| = 1$ and $G_\alpha = D^\alpha w - zD^\alpha u$ if $|\alpha| = 2$, $G = \{G_\alpha : |\alpha| = 1, 2\}$. From (6.2) and (6.3), it follows that:

$$\nabla_2 \omega = z\nabla_2 u + G \tag{6.4}$$

and

$$|G_\alpha| \le 6\widetilde{c}_2(r+1)^2[1+h_s^2(u)]^r |D^\beta u||D^\gamma u| \quad \text{almost everywhere in } \Omega, \tag{6.5}$$

for every multiindex $\alpha$, $\alpha = \beta + \gamma$, with $|\beta| = |\gamma| = 1$. Here, and in the sequel, with $k_i$, $i = 1, 2, \ldots$, we intend to use positive constants dependent only upon $n$, $p_\alpha$, $q_\alpha$, $\widetilde{q}$, c, $c_1$, $c_2$, $c_3$, $c_4$, $\widetilde{c}$, $\widetilde{c}_1$ and $\widetilde{c}_2$. Defining

$$\lambda = \frac{1}{(1+4\widetilde{c}_1 r)[1+(s+1)^2]^r},$$

from (6.1), we deduce:

$$0 < \lambda z \le 1. \tag{6.6}$$

We choose

$$\varphi = \frac{[1+h_s^2(u)]^r}{[1+(s+1)^2]^r(1+4\widetilde{c}_1 r)}.$$

We have $\varphi u = \lambda \omega$. From Hypothesis 4.3, we have:

$$u - \lambda\omega \in V.$$

Since

$$I(u) \le I(u - \lambda\omega),$$

we have

$$\int_\Omega A(x,\nabla_2 u)\,dx \le \int_\Omega A(x,\nabla_2 u-\lambda\nabla_2\omega)\,dx + \int_\Omega A_0(x,u-\lambda\omega)\,dx - \int_\Omega A_0(x,u)\,dx. \tag{6.7}$$

Due to (6.4) and (6.6) as well as to the convexity of $A(x,\xi)$, we have:

$$A(x,\nabla_2 u-\lambda\nabla_2\omega) \le (1-\lambda z)A(x,\nabla_2 u) + \lambda z A\Big(x,-\frac{G}{z}\Big). \tag{6.8}$$

From (4.2), we have:

$$A\Big(x,-\frac{G(x)}{z(x)}\Big) \le c_2\sum_{|\alpha|=2}\mu_\alpha\Big|\frac{G_\alpha(x)}{z(x)}\Big|^{p_\alpha} + f(x). \tag{6.9}$$

We fix an arbitrary multiindex $\alpha$, $|\alpha| = 2$, and let $\beta$ and $\gamma$ be multiindexes such that $|\beta| = |\gamma| = 1$ and $\alpha = \beta+\gamma$. From Hypothesis 2.5, inequalities (6.1) and (6.5), we obtain

$$\mu_\alpha\Big|\frac{G_\alpha(x)}{z(x)}\Big|^{p_\alpha} \le [6c\widetilde{c}_2(r+1)^2]^{p_\alpha}\nu_\beta^{p_\alpha/q_\beta}|D^\beta u|^{p_\alpha}\nu_\gamma^{p_\alpha/q_\gamma}|D^\gamma u|^{p_\alpha}. \tag{6.10}$$

We set

$$\rho = \frac{q_\beta q_\gamma}{q_\beta q_\gamma - p_\alpha(q_\gamma+q_\beta)}, \tag{6.11}$$

and take $\epsilon\in(0,1)$. Using (2.1), (6.11) and the Young inequality, from (6.10) we derive

$$\mu_\alpha\Big|\frac{G_\alpha(x)}{z(x)}\Big|^{p_\alpha} \le \frac{\epsilon}{(r+1)}\{\nu_\beta|D^\beta u|^{q_\beta} + \nu_\gamma|D^\gamma u|^{q_\gamma}\} + \epsilon^{1-\rho}[6c\widetilde{c}_2(r+1)^{2+1/q_\beta+1/q_\gamma}]^{\rho p_\alpha}.$$

The last inequality and (6.9) imply

$$A\Big(x,-\frac{G(x)}{z(x)}\Big) \le \frac{2c_2n^2\epsilon}{(r+1)}\sum_{|\chi|=1}\nu_\chi|D^\chi u|^{q_\chi} + c_2n^2\epsilon^{-m_0}(1+6c\widetilde{c}_2)^{m_0}(r+1)^{3m_0} + f(x), \tag{6.12}$$

where

$$m_0 = \max_{|\beta|=|\gamma|=1}\frac{p_{\beta+\gamma}q_\beta q_\gamma}{q_\beta q_\gamma - p_{\beta+\gamma}(q_\gamma+q_\beta)}.$$

From (6.1), (6.8) and (6.12) we obtain:

$$\begin{aligned} A(x,\nabla_2 u-\lambda\nabla_2\omega) &\le (1-\lambda z)A(x,\nabla_2 u) \\ &\quad + k_1\lambda\epsilon\sum_{|\chi|=1}\nu_\chi|D^\chi u|^{q_\chi}[1+h_s^2(u)]^r \\ &\quad + k_2\lambda(1+r)^{(3m_0+1)}\epsilon^{-m_0}[1+h_s^2(u)]^r[1+f(x)]. \end{aligned} \tag{6.13}$$

Using the convexity of the function $A_0(x,\eta)$ and (4.3), we have:

$$A_0(x,u-\lambda\omega)\le A_0(x,u)+\lambda[1+h_s^2(u)]^r[c_3|u|^{q^-}+f_0(x)]. \tag{6.14}$$

Taking into account inequalities (6.13) and (6.14), from (6.7) we derive:

$$\begin{aligned} \int_\Omega zA(x,\nabla_2 u)\,dx &\le k_3\epsilon\int_\Omega\sum_{|\chi|=1}\nu_\chi|D^\chi u|^{q_\chi}[1+h_s^2(u)]^r dx \\ &\quad + k_4(1+r)^{(3m_0+1)}\epsilon^{-m_0}\int_\Omega[1+f(x)][1+h_s^2(u)]^r dx \\ &\quad + \int_\Omega[c_3|u|^{q_-}+f_0(x)][1+h_s^2(u)]^r dx. \end{aligned}$$

From Hypothesis 4.1, (6.1) and the previous inequality, we have:

$$\begin{aligned} &\int_\Omega\sum_{|\alpha|=1}\nu_\alpha|D^\alpha u|^{q_\alpha}[1+h_s^2(u)]^r dx \\ &\quad\le k_5(1+r)^{(3m_0+1)}\int_\Omega[|u|^{q_-}+1+f(x)+f_0(x)][1+h_s^2(u)]^r dx. \end{aligned}$$

Next, we set $\Phi=\{1+f+f_0\}$. Taking into account that $\Phi\in L^t(\Omega)$, $t=\min(t_*,\widehat{t})>\widetilde{q}/(\widetilde{q}-q_+)$, we can apply Lemma 3.3 and obtain that $u\in L^\infty(\Omega)$. □

**7. Example.** We show an example where all of the assumptions on weight functions are satisfied. Towards this aim, we use some ideas of [**12,** Example 6.2].

Let $n>5$, and let $\Omega=\{x\in\mathbb{R}^n : |x|<1\}$. Let $q_\alpha, |\alpha|=1$, be numbers such that

$$\frac{3n}{n-2}<q_-,\quad q_+<n, \tag{7.1}$$

$$\frac{q_+ - q_-}{q_-(q_+ - 1)} < \frac{1}{n}. \tag{7.2}$$

We set a number $\sigma$ such that $2n/((n-2)q_-) < \sigma < 1$ and define, for every multiindex $\alpha$, $|\alpha| = 2$,

$$p_\alpha = \sigma \frac{q_\beta q_\gamma}{q_\beta + q_\gamma}, \tag{7.3}$$

where $\beta$ and $\gamma$ are multiindices such that $|\beta| = |\gamma| = 1$ and $\beta + \gamma = \alpha$. Since $1 < q_- \leq q_+ < n$ and $2n/((n-2)q_-) < \sigma < 1$, the numbers $q_\alpha, |\alpha| = 1$, and $p_\alpha, |\alpha| = 2$, satisfy inequality (2.1) with $p_\alpha > 1$.

Note that, by virtue of the inequality $q_+ < n$, we have

$$\sum_{|\alpha=1} \frac{1}{q_\alpha} > 1,$$

and, by (7.2), we obtain

$$\frac{1}{n}\left(\sum_{|\alpha|=1} \frac{1}{q_\alpha} - 1\right) < \frac{q_- - 1}{q_-(q_+ - 1)}.$$

Let $\lambda_\alpha, |\alpha| = 1$, be positive numbers such that

$$\frac{1}{n} \max_{|\alpha|=1} \frac{\lambda_\alpha}{q_\alpha} < \frac{q_- - 1}{q_-(q_+ - 1)} - \frac{1}{n}\left(\sum_{|\alpha|=1} \frac{1}{q_\alpha} - 1\right). \tag{7.4}$$

For every multiindex $\alpha, |\alpha| = 2$, we set

$$\tau_\alpha = \sigma \frac{q_\beta \lambda_\gamma + q_\gamma \lambda_\beta}{q_\beta + q_\gamma}, \tag{7.5}$$

where $\beta$ and $\gamma$ are multiindices such that $|\beta| = |\gamma| = 1$ and $\beta + \gamma = \alpha$.

Now, for every multiindex $\alpha, |\alpha| = 1$, let $\nu_\alpha$ be the function in $\Omega$ defined by $\nu_\alpha(x) = |x|^{\lambda_\alpha}$, and let, for every multiindex $\alpha, |\alpha| = 2$, $\mu_\alpha$ be the function in $\Omega$ defined by $\mu_\alpha(x) = |x|^{\tau_\alpha}$. Using (7.1), (7.3)–(7.5) and the inequality $2n/((n-2)q_-) < \sigma$, we obtain that Hypotheses 2.1 and 2.4 are fulfilled. Moreover, from (7.3) and (7.5), it follows that Hypothesis 2.5 holds.

Finally, there exist real numbers $\widetilde{c} > 0$ and $\widetilde{q}$, $\widetilde{q} > (q_-(q_+ - 1))/(q_- - 1)$ such that, for every $u \in \overset{\circ}{W}{}^{1,q}(\nu, \Omega)$,

$$\left( \int_\Omega |u|^{\widetilde{q}} dx \right)^{1/\widetilde{q}} \le \widetilde{c} \sum_{|\alpha|=1} \left( \int_\Omega \nu_\alpha |D^\alpha u|^{q_\alpha} dx \right)^{1/q_\alpha}.$$

For more details concerning the above assertion, see **[12]**. Taking into account that

$$\frac{q_-(q_+ - 1)}{q_- - 1} \ge q_+,$$

Hypothesis 2.2 is satisfied.

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